PHYSICAL REVIEW D **84,** 104033 (2011)

Discrete Wheeler-DeWitt equation

Herbert W. Hamber*

Institut des Hautes Etudes Scientifiques 35, route de Chartres 91440 Bures-sur-Yvette, France

Ruth M. Williams†

Department of Applied Mathematics and Theoretical Physics, Wilberforce Road, Cambridge CB3 0WA, United Kingdom and Girton College, Cambridge CB3 0JG, United Kingdom

(Received 12 September 2011; published 18 November 2011)

We present a discrete form of the Wheeler-DeWitt equation for quantum gravitation, based on the lattice formulation due to Regge. In this setup, the infinite-dimensional manifold of 3-geometries is replaced by a space of three-dimensional piecewise linear spaces, with the solutions to the lattice equations providing a suitable approximation to the continuum wave functional. The equations incorporate a set of constraints on the quantum wave functional, arising from the triangle inequalities and their higher-dimensional analogs. The character of the solutions is discussed in the strong-coupling (large-G) limit, where it is shown that the wave functional only depends on geometric quantities, such as areas and volumes. An explicit form, determined from the discrete wave equation supplemented by suitable regularity conditions, shows peaks corresponding to integer multiples of a fundamental unit of volume. An application of the variational method using correlated product wave functions suggests a relationship between quantum gravity in $n + 1$ dimensions, and averages computed in the Euclidean path integral formulation in n dimensions. The proposed discrete equations could provide a useful, and complementary, computational alternative to the Euclidean lattice path integral approach to quantum gravity.

DOI: 10.1103/PhysRevD.84.104033 PACS numbers: 04.60.−m, 04.60.Gw, 04.60.Nc, 98.80.Qc

I. INTRODUCTION

In this paper, we will present a lattice version of the Wheeler-DeWitt equation of quantum gravity. The approach used here will be rooted in the canonical formulation of quantum gravity, and can therefore be regarded as complementary to the Euclidean lattice version of the same theory discussed elsewhere. In the following, we will derive a discrete form of the Wheeler-DeWitt equation for pure gravity, based on the simplicial lattice formulation of gravity developed by Regge and Wheeler. It is expected that the resulting lattice equation will reproduce the original continuum equation in some suitable small lattice-spacing limit. In this formulation, the infinite-dimensional manifold of 3-geometries is replaced by the space of three-dimensional piecewise linear spaces, with solutions to the lattice equations then providing a suitable approximation to the continuum gravitational wave functional. The lattice equations will provide a new set of constraints on the quantum wave functional, which arise because of the imposition of the triangle inequalities and their higher-dimensional analogs. The equations are explicit enough to allow a number of potentially useful practical calculations in the quantum theory of gravity, such as the strong-coupling expansion, the weak-field expansion, mean-field theory, and the variational method. In this work, we will provide a number of sample calculations to illustrate the workings of the lattice theory, and what in our opinion is the likely physical interpretation of the results.

In the strong-coupling (large-G) limit, we will show that the wave functional depends on geometric quantities only, such as areas, volumes, and curvatures, and that in this limit the correlation length is finite in units of the lattice spacing. An explicit form of the wave functional, determined from the discrete equation supplemented by suitable regularity conditions, shows peaks corresponding to integer multiples of a fundamental unit of volume. Later, the variational method will be introduced, based here on correlated product (Jastrow-Slater-type) wave functions. This approach brings out a relationship between ground-state properties of quantum gravity in $n + 1$ dimensions, and certain averages computed in the Euclidean path integral formulation in n dimensions, i.e. in one dimension *less*. Because of its reliance on a different set of approximation methods, the $3 + 1$ lattice formulation presented here could provide a useful, and complementary, computational alternative to the Euclidean lattice path integral approach to quantum gravity in four dimensions. The equations are explicit enough that numerical solutions should be achievable in a number of simple cases, such as a toroidal regular lattice with N vertices in $3 + 1$ dimensions.

An outline of the paper is as follows. In Section II, as a background to the rest of the paper, we describe the formalism of classical gravity, as set up by Arnowitt, Deser, and Misner. In Section III, we introduce the continuum form of the Wheeler-DeWitt equation and, in Section IV, describe how it can be solved in the minisuperspace approximation.

*HHamber@uci.edu

†R. M. Williams@damtp.cam.ac.uk

Section V is the central core of the paper, where we transcribe the Wheeler-DeWitt equation to the lattice. Practical details of the lattice version are given in Section VI and the equation solved in the strong-coupling limit in both 2 + 1 and 3 + 1 dimensions. A general solution at the full range of couplings requires the inclusion of the curvature term, which was neglected in the strong-coupling expansion, and Sections VII and VIII outline methods of including this term, by perturbation theory and by the variational method. Section IX gives a short outline of the lattice weak-field expansion as it applies to the Wheeler-DeWitt equation. Section X concludes with a discussion.

II. ARNOWITT-DESER-MISNER (ADM) FORMALISM AND HAMILTONIAN

Since this paper involves the canonical quantization of gravity [1], we begin with a discussion of the classical canonical formalism derived by Arnowitt, Deser, and Misner [2]. While many of the results presented in this section are rather well known, it will be useful, in view of later applications, to recall the main results and formulas and provide suitable references for expressions used later in the paper.

The first step in developing a canonical formulation for gravity is to introduce a time slicing of spacetime, by introducing a sequence of spacelike hypersurfaces labeled by a continuous time coordinate t. The invariant distance is then written as

$$\begin{aligned} ds^2 &\equiv -d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \\ &= g_{ij} dx^i dx^j + 2 g_{ij} N^i dx^j dt - (N^2 - g_{ij} N^i N^j) dt^2, \end{aligned} \tag{1}$$

where x^i $(i = 1, 2, 3)$ are coordinates on a three-dimensional manifold and τ is the proper time, in units with $c = 1$. The relationship between the quantities $d\tau$, dt, dx_i, N, and N_i basically expresses the Lorentzian version of Pythagoras's theorem.

Indices are raised and lowered with $g_{ij}(\mathbf{x})$ $(i, j = 1, 2, 3)$, which denotes the three-metric on the given spacelike hypersurface, and $N(\mathbf{x})$ and $N^i(\mathbf{x})$ are the lapse and shift functions, respectively. These last two quantities describe the lapse of proper time (N) between two infinitesimally close hypersurfaces, and the corresponding shift in spatial coordinate (N^i). It is customary to mark four-dimensional quantities by the prefix^4, so that all unmarked quantities will refer to three dimensions (and are occasionally marked explicitly by a^3). In terms of the original four-dimensional metric ${}^4 g_{\mu\nu}$, one has

$$\begin{pmatrix} {}^4 g_{00} & {}^4 g_{0j} \\ {}^4 g_{i0} & {}^4 g_{ij} \end{pmatrix} = \begin{pmatrix} N_k N^k - N^2 & N_j \\ N_i & g_{ij} \end{pmatrix}, \tag{2}$$

and for its inverse

$$\begin{pmatrix} {}^4 g^{00} & {}^4 g^{0j} \\ {}^4 g^{i0} & {}^4 g^{ij} \end{pmatrix} = \begin{pmatrix} -1/N^2 & N^j/N^2 \\ N_i/N^2 & g^{ij} - N^i N^j/N^2 \end{pmatrix}, \tag{3}$$

which then gives for the spatial metric and the lapse and shift functions

$$g_{ij} = {}^4 g_{ij}, \qquad N = (-{}^4 g^{00})^{-1/2}, \qquad N_i = {}^4 g_{0i}. \tag{4}$$

For the volume element, one has

$$\sqrt{{}^4 g} = N \sqrt{g}, \tag{5}$$

where the latter involves the determinant of the three-metric, $g \equiv \det g_{ij}$. As usual, g^{ij} denotes the inverse of the matrix g_{ij}. In terms of these quantities, the Einstein-Hilbert Lagrangian of general relativity can then be written, up to an overall multiplicative constant, in the following (first-order) form:

$$\mathcal{L} = \sqrt{{}^4 g}\, {}^4 R = -g_{ij} \partial_t \pi^{ij} - N R^0 - N_i R^i, \tag{6}$$

up to boundary terms. Here, one has defined the following quantities:

$$\begin{aligned} \pi^{ij} &\equiv \sqrt{{}^4 g} ({}^4\Gamma^0_{kl} - g_{kl} {}^4\Gamma^0_{mn} g^{mn}) g^{ik} g^{lj}, \\ R^0 &\equiv -\sqrt{g} [{}^3 R + g^{-1} (\tfrac{1}{2}\pi^2 - \pi^{ij}\pi_{ij})], \\ R^i &\equiv -2 \nabla_j \pi^{ij}. \end{aligned} \tag{7}$$

The symbol ∇^i denotes covariant differentiation with respect to the index i using the spatial three-metric g_{ij}, and ${}^3 R$ is the scalar curvature computed from this metric. Also note that the affine connection coefficients Γ^k_{ij} have been eliminated in favor of the spatial derivatives of the metric $\partial_k g_{ij}$, and one has defined $\pi = \pi^i_i$. Since the quantities N and N^i do not appear in the $\pi^{ij} \partial_t g_{ij}$ part, they are interpreted as Lagrange multipliers, and the "Hamiltonian" density

$$\mathcal{H} \equiv N R^0 + N_i R^i \tag{8}$$

vanishes as a result of the constraints. Varying the first-order Lagrangian of Eq. (6) with respect to g_{ij}, N_i, N, and π_{ij}, one obtains a set of equations which are equivalent to Einstein's field equations. First, varying with respect to π_{ij} one obtains an equation which can be viewed as defining π_{ij},

$$\partial_t g_{ij} = 2 N g^{-1/2} (\pi_{ij} - \tfrac{1}{2} g_{ij} \pi) + \nabla_j N_i + \nabla_i N_j. \tag{9}$$

Varying with respect to the spatial metric g_{ij} gives the time evolution for π_{ij},

$$\begin{aligned} \partial_t \pi^{ij} = & -N\sqrt{g} ({}^3 R^{ij} - \tfrac{1}{2} g^{ij}\, {}^3 R) + \tfrac{1}{2} N g^{-1/2} g^{ij} (\pi^{kl}\pi_{kl} - \tfrac{1}{2}\pi^2) \\ & - 2 N g^{-1/2} (\pi^{ik} \pi^j_k - \tfrac{1}{2} \pi \pi^{ij}) \\ & + \sqrt{g} (\nabla^i \nabla^j N - g^{ij} \nabla^k \nabla_k N) + \nabla_k (\pi^{ij} N^k) \\ & - \nabla_k N^i \pi^{kj} - \nabla_k N^j \pi^{ki}. \end{aligned} \tag{10}$$

Finally, varying with respect to the lapse (N) and shift (N^i) functions gives

$$R^0(g_{ij}, \pi_{ij}) = 0, \qquad R^i(g_{ij}, \pi_{ij}) = 0, \tag{11}$$

which can be viewed as the four constraint equations ${}^4G^0_\mu = {}^4R^0_\mu - \frac{1}{2}\delta^0_\mu {}^4R = 0$, expressed for this choice of metric decomposition [1]. The above constraints can therefore be considered as analogous to Gauss's law $\partial_i F^{i0} = \nabla \cdot \mathbf{E} = 0$ in electromagnetism.

Some of the quantities introduced above (such as ${}^3R$) describe intrinsic properties of the spacelike hypersurface, while some others can be related to the extrinsic properties of such a hypersurface when embedded in four-dimensional space. If spacetime is sliced up (foliated) by a one-parameter family of spacelike hypersurfaces $x^\mu = x^\mu(x^i, t)$, then one has for the intrinsic metric within the spacelike hypersurface

$$g_{ij} = g_{\mu\nu} X^\mu_i X^\nu_j \quad \text{with} \quad X^\mu_i \equiv \partial_i x^\mu, \tag{12}$$

while the extrinsic curvature is given in terms of the unit normals to the spacelike surface, U^μ,

$$K_{ij}(x^k, t) = -(\nabla_\mu U_\nu) X^\nu_i X^\mu_j. \tag{13}$$

In this language, the lapse and shift functions appear in the expression

$$\partial_t x^\mu = N U^\mu + N^i X^\mu_i. \tag{14}$$

In the following, $K = g^{ij} K_{ij} = K^i_i$ is the trace of the matrix $\mathbf{K}$.

Now, in the canonical formalism, the momentum can be expressed in terms of the extrinsic curvature

$$\pi^{ij} = -\sqrt{g}(K^{ij} - K g^{ij}). \tag{15}$$

It is then convenient to define the quantities H and H_i as (here $\kappa = 8\pi G$)

$$\begin{aligned} H &\equiv 2\kappa g^{-1/2}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right) - \frac{1}{2\kappa}\sqrt{g}{}^3R, \\ H_i &\equiv -2\nabla_j \pi^j_i. \end{aligned} \tag{16}$$

The last two statements are essentially equivalent to the definitions in Eq. (7).

In this notation, the Einstein field equations in the absence of sources are equivalent to the initial value constraint

$$H(x) = H_i(x) = 0, \tag{17}$$

supplemented by the canonical evolution equations for π^{ij} and g_{ij}. The quantity

$$\mathbf{H} = \int d^3x [N(x) H(x) + N^i(x) H_i(x)] \tag{18}$$

should then be regarded as the Hamiltonian for classical general relativity.

When matter is added to the Einstein-Hilbert Lagrangian,

$$I[g, \phi] = \frac{1}{16\pi G} \int d^4x \sqrt{g}{}^4R(g_{\mu\nu}(x)) + I_\phi[g_{\mu\nu}, \phi], \tag{19}$$

where $\phi(x)$ are some matter fields, the action within the ADM parametrization of the metric coordinates needs to be modified to

$$\begin{aligned} I[g, \pi, \phi, \pi_\phi, N] = \int dt d^3x \Big(& \frac{1}{16\pi G}\pi^{ij}\partial_t g_{ij} \\ & + \pi_\phi \partial_t \phi - NT - N^i T_i \Big). \end{aligned} \tag{20}$$

One still has the same definitions as before for the (Lagrange multiplier) lapse and shift function, namely $N = (-{}^4g^{00})^{-1/2}$ and $N^i = g^{ij4}g_{0j}$. The gravitational constraints are modified as well, since now one defines

$$\begin{aligned} T &\equiv \frac{1}{16\pi G} H(g_{ij}, \pi^{ij}) + H^\phi(g_{ij}, \pi^{ij}, \phi, \pi_\phi), \\ T_i &\equiv \frac{1}{16\pi G} H_i(g_{ij}, \pi^{ij}) + H^\phi_i(g_{ij}, \pi^{ij}, \phi, \pi_\phi), \end{aligned} \tag{21}$$

with the first part describing the gravitational part given earlier in Eq. (16),

$$\begin{aligned} H(g_{ij}, \pi^{ij}) &= G_{ij,kl}\pi^{ij}\pi^{kl} - \sqrt{g}{}^3R + 2\lambda\sqrt{g}, \\ H_i(g_{ij}, \pi^{ij}) &= -2g_{ij}\nabla_k \pi^{jk}, \end{aligned} \tag{22}$$

here conveniently rewritten using the (inverse of the) DeWitt supermetric

$$G_{ij,kl} = \tfrac{1}{2} g^{-1/2}(g_{ik}g_{jl} + g_{il}g_{jk} + \alpha g_{ij}g_{kl}), \tag{23}$$

with parameter $\alpha = -1$. Note that in the previous expression a cosmological term (proportional to λ) has been added as well, for future reference. For the matter part, one has

$$\begin{aligned} H^\phi(g_{ij}, \pi^{ij}, \phi, \pi_\phi) &= \sqrt{g} T_{00}(g_{ij}, \pi^{ij}, \phi, \pi_\phi), \\ H^\phi_i(g_{ij}, \pi^{ij}, \phi, \pi_\phi) &= -\sqrt{g} T_{0i}(g_{ij}, \pi^{ij}, \phi, \pi_\phi). \end{aligned} \tag{24}$$

We note here that the (inverse of the) DeWitt supermetric in Eq. (23) is also customarily used to define a distance in the space of three-metrics $g_{ij}(x)$. Consider an infinitesimal displacement of such a metric $g_{ij} \to g_{ij} + \delta g_{ij}$, and define the natural metric G on such deformations by considering a distance in function space

$$\|\delta g\|^2 = \int d^3x N(x) G^{ij,kl}(x) \delta g_{ij}(x) \delta g_{kl}(x). \tag{25}$$

Here, the lapse $N(x)$ is an essentially arbitrary but positive function, to be taken equal to one in the following. The quantity $G^{ij,kl}(x)$ is the three-dimensional version of the DeWitt supermetric,

$$G^{ij,kl} = \tfrac{1}{2}\sqrt{g}(g^{ik}g^{jl} + g^{il}g^{jk} + \bar{\alpha} g^{ij}g^{kl}), \tag{26}$$

with the parameter α of Eq. (23) related to $\bar{\alpha}$ in Eq. (26) by $\bar{\alpha} = -2\alpha/(2+3\alpha)$, so that $\alpha = -1$ gives $\bar{\alpha} = -2$ (note that this is dimension dependent).

III. WHEELER-DEWITT EQUATION

Within the framework of the previous construction, a transition from a classical to a quantum description of gravity is obtained by promoting g_{ij}, π^{ij}, H, and H_i to quantum operators, with $\hat{g}_{ij}$ and $\hat{\pi}^{ij}$ satisfying canonical commutation relations. In particular, the classical constraints now select a physical vacuum state $|\Psi\rangle$, such that in the source-free case

$$\hat{H}|\Psi\rangle = 0, \qquad \hat{H}_i|\Psi\rangle = 0, \tag{27}$$

and in the presence of sources more generally

$$\hat{T}|\Psi\rangle = 0, \qquad \hat{T}_i|\Psi\rangle = 0. \tag{28}$$

As in ordinary nonrelativistic quantum mechanics, one can choose different representations for the canonically conjugate operators $\hat{g}_{ij}$ and $\hat{\pi}^{ij}$. In the functional *position representation*, one sets

$$\hat{g}_{ij}(\mathbf{x}) \to g_{ij}(\mathbf{x}), \qquad \hat{\pi}^{ij}(\mathbf{x}) \to -i\hbar \cdot 16\pi G \cdot \frac{\delta}{\delta g_{ij}(\mathbf{x})}. \tag{29}$$

In this picture, the quantum states become wave functionals of the three-metric $g_{ij}(\mathbf{x})$,

$$|\Psi\rangle \to \Psi[g_{ij}(\mathbf{x})]. \tag{30}$$

The two quantum constraint equations in Eq. (28) then become the Wheeler-DeWitt equation [3–5]

$$\left\{-16\pi G \cdot G_{ij,kl}\frac{\delta^2}{\delta g_{ij}\delta g_{kl}} - \frac{1}{16\pi G}\sqrt{g}({}^3R - 2\lambda) + \hat{H}^{\phi}\right\}\Psi[g_{ij}(\mathbf{x})] = 0, \tag{31}$$

with the inverse supermetric given in 3 + 1 dimensions by

$$G_{ij,kl} = \tfrac{1}{2}g^{-1/2}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}), \tag{32}$$

and the diffeomorphism (or momentum) constraint

$$\left\{2ig_{ij}\nabla_k\frac{\delta}{\delta g_{jk}} + \hat{H}_i^{\phi}\right\}\Psi[g_{ij}(\mathbf{x})] = 0. \tag{33}$$

This last constraint implies that the gradient of Ψ on the superspace of g_{ij}'s and ϕ's is zero along those directions that correspond to gauge transformations, i.e. diffeomorphisms on the three-dimensional manifold, whose points are labeled by the coordinates $\mathbf{x}$. The lack of covariance of the ADM approach has not gone away, and is therefore still part of the present formalism. Also note that the DeWitt supermetric is not positive definite, which implies that some derivatives with respect to the metric have the "wrong" sign. It is understood that these directions correspond to the conformal mode.

A number of basic issues need to be addressed before one can gain a full and consistent understanding of the dynamical content of the theory [6–10]. These include possible problems of operator ordering, and the specification of a suitable Hilbert space, which entails at some point a choice for the inner product of wave functionals, for example, in the Schrödinger form

$$\langle\Psi|\Phi\rangle = \int d\mu[g]\Psi^*[g_{ij}]\Phi[g_{ij}], \tag{34}$$

where $d\mu[g]$ is some appropriate measure over the three-metric g. Note also that the continuum Wheeler-DeWitt equation contains, in the kinetic term, products of functional differential operators, which are evaluated at the same spatial point $\mathbf{x}$. One would expect that such terms could produce $\delta^{(3)}(0)$-type singularities when acting on the wave functional, which would then have to be regularized in some way. The lattice cutoff discussed below is one way to provide such an explicit regularization.

A peculiar property of the Wheeler-DeWitt equation, which distinguishes it from the usual Schrödinger equation $H\Psi = i\hbar\partial_t\Psi$, is the absence of an explicit time coordinate. As a result, the right-hand side term of the Schrödinger equation is here entirely absent. The reason is of course diffeomorphism invariance of the underlying theory, which expresses now the fundamental quantum equations in terms of fields g_{ij}, and not coordinates. Consequently, the Wheeler-DeWitt equation contains no explicit time evolution parameter. Nevertheless, in some cases it seems possible to assign the interpretation of "time coordinate" to some specific variable entering the Wheeler-DeWitt equation, such as the overall spatial volume or the magnitude of some scalar field [9].

We shall not discuss here the connection between the Wheeler-DeWitt equation and the Feynman path integral for gravity. In principle, any solution of the Wheeler-DeWitt equation corresponds to a possible quantum state of the Universe. A similar situation already arises, of course in much simpler form, in nonrelativistic quantum mechanics [11]. The effects of the boundary conditions on the wave function will then act to restrict the class of possible solutions; in ordinary quantum mechanics, these are determined by the physical context of the problem and some set of external conditions. In the case of the Universe, the situation is far less clear, and in many approaches some suitable set of boundary conditions needs to be postulated, based on general arguments involving simplicity or economy. One proposal [12] is to restrict the quantum state of the Universe by requiring that the wave function Ψ be determined by a path integral over compact Euclidean metrics. The wave function would then be given by

$$\Psi[g_{ij}, \phi] = \int [dg_{\mu\nu}][d\phi] \exp(-\hat{I}[g_{\mu\nu}, \phi]), \tag{35}$$

where $\hat{I}$ is the Euclidean action for gravity plus matter

$$\hat{I} = -\frac{1}{16\pi G}\int d^4x\sqrt{g}(R - 2\lambda) - \frac{1}{8\pi G}\int d^3x\sqrt{g_{ij}}K - \int d^4x\sqrt{g}\mathcal{L}_m. \tag{36}$$

The semiclassical functional integral would then be restricted to those four-metrics which have the induced metric g_{ij} and the matter field ϕ as given on the boundary surface S. One would then expect (as in the case in nonrelativistic quantum mechanics, where the path integral with a boundary surface satisfies the Schrödinger equation), that the wave function constructed in this way would also automatically satisfy the Wheeler-DeWitt equation, and this is indeed the case.

IV. MINISUPERSPACE

Minisuperspace models can in part provide an additional motivation for our later work. The quantum state of a gravitational system is described, in the Wheeler-DeWitt framework just introduced, by a wave function Ψ which is a functional of the three-metric g_{ij} and the matter fields ϕ. In general, the latter could contain fields of arbitrary spins, but here we will consider for simplicity just one single component scalar field $\phi(x)$. The wave function Ψ will then obey the zero energy Schrödinger-like equation of Eqs. (31) and (33). The quantum state described by Ψ is then a functional on the infinite-dimensional manifold W consisting of all positive-definite metrics $g_{ij}(x)$ and matter fields $\phi(x)$ on a spacelike three-surface S. We note here that on this space there is a natural metric

$$ds^2[\delta g, \delta\phi] = \int \frac{d^3x d^3x'}{N(x)}[G^{ij,kl}(x, x')\delta g_{ij}\delta g_{kl}(x') + \sqrt{g}\delta^3(x - x')\delta\phi(x)\delta\phi(x')], \tag{37}$$

where

$$G^{ij,kl}(x, x') = G^{ij,kl}(x)\delta^3(x - x') \tag{38}$$

and

$$G^{ij,kl}(x) = \tfrac{1}{2}\sqrt{g}[g^{ik}(x)g^{jl}(x) + g^{il}(x)g^{jk}(x) - 2g^{ij}(x)g^{kl}(x)] \tag{39}$$

is the DeWitt supermetric.

In general, the wave function for all the dynamical variables of the gravitational field in the Universe is difficult to calculate, since an infinite number of degrees of freedom is involved: the infinitely many values of the metric at all spacetime points, and the infinitely many values of the matter field ϕ at the same points. One option is to restrict the choice of variable to a finite number of suitable degrees of freedom [13–17]. As a result, the overall quantum fluctuations are severely restricted, since these are now only allowed to be nonzero along the surviving dynamical directions. If the truncation is severe enough, the transverse-traceless nature of the graviton fluctuation is lost as well. Also, since one is not expanding the quantum solution in a small parameter, it can be difficult to estimate corrections.

In a cosmological context, it seems natural to consider initially a homogeneous and isotropic model, and restrict the function space to two variables, the scale factor $a(t)$ and a minimally coupled homogeneous scalar field $\phi(t)$ [16]. The spacetime metric is given by

$$d\tau^2 = N^2(t)dt^2 - g_{ij}dx^i dx^j. \tag{40}$$

The three-metric g_{ij} is then determined entirely by the scale factor $a(t)$,

$$g_{ij} = a^2(t)\tilde{g}_{ij}, \tag{41}$$

with $\tilde{g}_{ij}$ a time-independent reference three-metric with constant curvature,

$${}^3\tilde{R}_{ijkl} = k(\tilde{g}_{ij}\tilde{g}_{kl} - \tilde{g}_{il}\tilde{g}_{jk}), \tag{42}$$

and $k = 0, \pm 1$ corresponding to the flat, closed, and open Universe case, respectively. In this case, the minisuperspace W is two dimensional, with coordinates a and ϕ, and supermetric

$$ds^2[a, \phi] = N^{-1}(-a da^2 + a^3 d\phi^2). \tag{43}$$

From the above expression for $ds^2[a, \phi]$, one obtains the Laplacian in the above metric, required for the kinetic term in the Wheeler-DeWitt equation,[1]

$$-\frac{1}{2}\nabla^2(a, \phi) = \frac{N}{2a^2}\left\{\frac{\partial}{\partial a}a\frac{\partial}{\partial a} - \frac{1}{a}\frac{\partial^2}{\partial\phi^2}\right\}. \tag{44}$$

Since the space is homogeneous, the diffeomorphism constraint is trivially satisfied. Also, N is independent of g_{ij} so in the homogeneous case it can be taken to be a constant, conveniently chosen as $N = 1$.

It should be clear that in general the quantum behavior of the solutions is expected to be quite different from the classical one. In the latter case, one imposes some initial conditions on the scale factor at some time t_0, which then determines $a(t)$ at all later times. In the minisuperspace view of quantum cosmology, one has to instead impose a condition on the wave packet Ψ at $a = 0$. Because of their simplicity, in general it is possible to analyze the solutions

[1] The ambiguity regarding the operator ordering of $p^2/a = a^{-(q+1)}pa^q p$ in the Wheeler-DeWitt equation can in principle be retained by writing for the above operator ∇^2 the expression $-(N/a^{q+1})\{(\partial/\partial a)a^q(\partial/\partial a) - (\partial^2/\partial\phi^2)\}$, but this does not seem to affect the qualitative nature of the solutions. The case discussed in the text corresponds to $q = 1$, but $q = 0$ seems even simpler.

to the minisuperspace Wheeler-DeWitt equation in a rather complete way, given some sensible assumptions on how $\Psi(a, \phi)$ should behave, for example, when the scale factor a approaches zero.

In concluding the discussion on minisuperspace models as a tool for studying the physical content of the Wheeler-DeWitt equation, it seems legitimate though to ask the following question: to what extent can results for these very simple models, which involve such a drastic truncation of physical degrees of freedom, be ultimately representative of, and physically relevant to, what might, or might not, happen in the full quantum theory?

V. LATTICE HAMILTONIAN FOR QUANTUM GRAVITY

In constructing a discrete Hamiltonian for gravity, one has to decide first what degrees of freedom one should retain on the lattice. There are a number of possibilities, depending on which continuum theory one chooses to discretize, and at what stage. So, for example, one could start with a discretized version of Cartan's formulation, and define vierbeins and spin connections on a flat hypercubic lattice. Later, one could define the transfer matrix for such a theory, and construct a suitable Hamiltonian.

Another possibility, which is the one we choose to pursue here, is to use the more economical (and geometric) Regge-Wheeler lattice discretization for gravity [18,19], with edge lengths suitably defined on a random lattice as the primary dynamical variables. Even in this specific case, several avenues for discretization are possible. One could discretize the theory from the very beginning, while it is still formulated in terms of an action, and introduce for it a lapse and a shift function, extrinsic and intrinsic discrete curvatures, etc. Alternatively, one could try to discretize the continuum Wheeler-DeWitt equation directly, a procedure that makes sense in the lattice formulation, as these equations are still given in terms of geometric objects, for which the Regge theory is very well suited. It is the latter approach which we will proceed to outline here.

The starting point for the following discussion is therefore the Wheeler-DeWitt equation for pure gravity in the absence of matter, Eq. (31),

$$\left\{-(16\pi G)^2 G_{ij,kl}(\mathbf{x})\frac{\delta^2}{\delta g_{ij}(\mathbf{x})\delta g_{kl}(\mathbf{x})} - \sqrt{g(\mathbf{x})}(^3R(\mathbf{x}) - 2\lambda)\right\}\Psi[g_{ij}(\mathbf{x})] = 0 \tag{45}$$

and the diffeomorphism constraint of Eq. (33),

$$\left\{2ig_{ij}(\mathbf{x})\nabla_k(\mathbf{x})\frac{\delta}{\delta g_{jk}(\mathbf{x})}\right\}\Psi[g_{ij}(\mathbf{x})] = 0. \tag{46}$$

Note that these equations express a constraint on the state $|\Psi\rangle$ at *every* $\mathbf{x}$, each of the form $\hat{H}(\mathbf{x})|\Psi\rangle = 0$ and $\hat{H}_i(\mathbf{x})|\Psi\rangle = 0$.

On a simplicial lattice [20–24] (see for example [25], and references therein, for a more complete discussion of the lattice formulation for gravity), one knows that deformations of the squared edge lengths are linearly related to deformations of the induced metric. In a given simplex σ, take coordinates based at a vertex 0, with axes along the edges from 0. The other vertices are each at unit coordinate distance from 0 (see Figs. 1–3 for this labeling of a triangle and of a tetrahedron). In terms of these coordinates, the metric within the simplex is given by

$$g_{ij}(\sigma) = \tfrac{1}{2}(l_{0i}^2 + l_{0j}^2 - l_{ij}^2). \tag{47}$$

Note also that in the following discussion only edges and volumes along the spatial direction are involved. It follows that one can introduce in a natural way a lattice analog of the DeWitt supermetric of Eq. (26), by adhering to the following procedure. First, one writes for the supermetric in edge length space

$$\|\delta l^2\|^2 = \sum_{ij} G^{ij}(l^2)\delta l_i^2 \delta l_j^2, \tag{48}$$

with the quantity $G^{ij}(l^2)$ suitably defined on the space of squared edge lengths [26,27]. Through a straightforward exercise of varying the squared volume of a given simplex σ in d dimensions

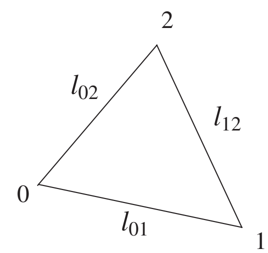

FIG. 1. A triangle with labels.

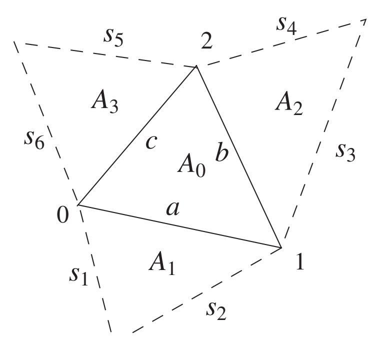

FIG. 2. Neighbors of a given triangle. The above picture is supposed to illustrate the fact that the Laplacian Δ_{l^2} appearing in the kinetic term of the lattice Wheeler-DeWitt equation (here in 2 + 1 dimensions) contains edges a, b, c that belong both to the triangle in question, as well as to several neighboring triangles (here three of them) with squared edges denoted sequentially by $s_1 = l_1^2 \dots s_6 = l_6^2$

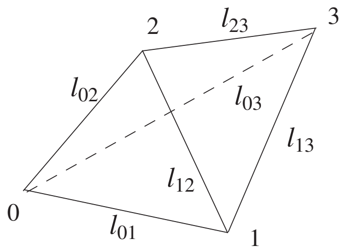

FIG. 3. A tetrahedron with labels.

$$V^2(\sigma) = \left(\frac{1}{d!}\right)^2 \det g_{ij}(l^2(\sigma)) \tag{49}$$

to quadratic order in the metric (on the right-hand side), or in the squared edge lengths belonging to that simplex (on the left-hand side), one finds the identity

$$\frac{1}{V(l^2)}\sum_{ij}\frac{\partial^2 V^2(l^2)}{\partial l_i^2 \partial l_j^2}\delta l_i^2 \delta l_j^2 = \frac{1}{d!}\sqrt{\det(g_{ij})}[g^{ij}g^{kl}\delta g_{ij}\delta g_{kl} - g^{ij}g^{kl}\delta g_{jk}\delta g_{li}]. \tag{50}$$

The right-hand side of this equation contains precisely the expression appearing in the continuum supermetric of Eq. (26) (for a specific choice of the parameter $\bar{\alpha} = -2$), while the left-hand side contains the sought-for lattice supermetric. One is therefore led to the identification

$$G^{ij}(l^2) = -d!\sum_{\sigma}\frac{1}{V(\sigma)}\frac{\partial^2 V^2(\sigma)}{\partial l_i^2 \partial l_j^2}. \tag{51}$$

It should be noted that in spite of the appearance of a sum over simplices σ, $G^{ij}(l^2)$ is quite local (in correspondence with the continuum, where it is ultralocal), since the derivatives on the right-hand side vanish when the squared edge lengths in question are not part of the same simplex. The sum over σ therefore only extends over those few tetrahedra which contain either the i or the j edge.

At this point, one is finally ready to write a lattice analog of the Wheeler-DeWitt equation for pure gravity, which reads

$$\left\{-(16\pi G)^2 G_{ij}(l^2)\frac{\partial^2}{\partial l_i^2 \partial l_j^2} - \sqrt{g(l^2)}[{}^3R(l^2) - 2\lambda]\right\} \times \Psi[l^2] = 0, \tag{52}$$

with $G_{ij}(l^2)$ the inverse of the matrix $G^{ij}(l^2)$ given above. The range of the summation over i and j and the appropriate expression for the scalar curvature, in this equation, are discussed below and made explicit in Eq. (53).

It should be emphasized that, just like there is one local equation for *each* spatial point $\mathbf{x}$ in the continuum, here too there is only one local (or semilocal, since strictly speaking more than one lattice vertex is involved) equation that needs to be specified at each simplex, or simplices, with G_{ij} defined in accordance with the definition in Eq. (51). On the other hand, and again in close analogy with the continuum expression, the wave function $\Psi[l^2]$ depends of course collectively on *all* the edge lengths in the lattice. The latter should therefore be regarded as a function of the whole simplicial geometry, whatever its nature might be, just like the continuum wave function $\Psi[g_{ij}]$ will be a function(al) of all metric variables, or more specifically of the overall geometry of the manifold, due to the built-in diffeomorphism invariance. On the side, we note here that the lattice supermetric is dimensionful, $G_{ij} \sim l^{4-d}$ and $G^{ij} \sim l^{d-4}$ in d spacetime dimensions, so it might be useful and convenient from now on to explicitly introduce a lattice spacing a (or a momentum cutoff $\Lambda = 1/a$) and express all dimensionful quantities (G, λ, l_i) in terms of this fundamental lattice spacing.

As noted, Eqs. (31) or (52) both express a constraint equation at each "point" in space. Here, we will elaborate a bit more on this point. On the lattice, points in space are replaced by a set of edge labels i, with a few edges clustered around each vertex, in a way that depends on the dimensionality and the local lattice coordination number. To be more specific, the first term in Eq. (52) contains derivatives with respect to edges i and j connected by a matrix element G_{ij} which is nonzero only if i and j are close to each other, essentially nearest neighbor. One would therefore expect that the first term could be represented by just a sum of edge contributions, all from within one $(d-1)$-simplex σ (a tetrahedron in three dimensions). The second term containing ${}^3R(l^2)$ in Eq. (52) is also local in the edge lengths: it only involves a handful of edge lengths which enter into the definition of areas, volumes, and angles around the point $\mathbf{x}$, and follows from the fact that the local curvature at the original point $\mathbf{x}$ is completely determined by the values of the edge lengths clustered around i and j. Apart from some geometric factors, it describes, through a deficit angle δ_h, the parallel transport of a vector around an elementary dual lattice loop. It should therefore be adequate to represent this second term by a sum over contributions over all $(d-3)$-dimensional hinges (edges in $3+1$ dimensions) h attached to the simplex σ, giving therefore in three dimensions

$$\left\{-(16\pi G)^2\sum_{i,j\subset\sigma}G_{ij}(\sigma)\frac{\partial^2}{\partial l_i^2 \partial l_j^2} - 2n_{\sigma h}\sum_{h\subset\sigma}l_h\delta_h + 2\lambda V_\sigma\right\}\Psi[l^2] = 0. \tag{53}$$

Here, δ_h is the deficit angle at the hinge h, l_h the corresponding edge length, $V_\sigma = \sqrt{g(\sigma)}$ the volume of the simplex (tetrahedron in three spatial dimensions) labeled by σ. $G_{ij}(\sigma)$ is obtained either from Eq. (51), or from the lattice transcription of Eq. (23),

$$G_{ij,kl}(\sigma) = \tfrac{1}{2}g^{-1/2}(\sigma)[g_{ik}(\sigma)g_{jl}(\sigma) + g_{il}(\sigma)g_{jk}(\sigma) - g_{ij}(\sigma)g_{kl}(\sigma)], \tag{54}$$

with the induced metric $g_{ij}(\sigma)$ within a simplex σ given in Eq. (47). The combinatorial factor $n_{\sigma h}$ ensures the correct normalization for the curvature term, since the latter has to give the lattice version of $\int \sqrt{g}^3 R = 2\sum_h \delta_h l_h$ (in three spatial dimensions) when summed over all simplices σ. The inverse of $n_{\sigma h}$ counts therefore the number of times the same hinge appears in various neighboring simplices, and consequently depends on the specific choice of underlying lattice structure; for a flat lattice of equilateral triangles in two dimensions $n_{\sigma h} = 1/6$.[2] The lattice Wheeler-DeWitt equation given in Eq. (53) is the main result of this paper.

It is in fact quite encouraging that the discrete equation in Eqs. (52) and (53) is very similar to what one would derive in Regge lattice gravity by doing the 3 + 1 split of the lattice metric carefully from the very beginning [28–30]. These authors also derived a lattice Hamiltonian in three dimensions, written in terms of lattice momenta conjugate to the edge length variables. In this formulation, the Hamiltonian constraint equations have the form

$$\begin{aligned} H_n &= \frac{1}{4}\sum_{\alpha\in n} G_{ij}^{(\alpha)} \pi^i \pi^j - \sum_{\beta\in n} \sqrt{g_\beta}\,\delta_\beta \\ &= \frac{1}{4}\sum_{\alpha\in n} \frac{1}{V_\alpha}\left[(\mathrm{tr}\pi^2)_\alpha - \frac{1}{2}(\mathrm{tr}\pi)_\alpha^2\right] - \sum_{\beta\in n}\sqrt{g_\beta}\,\delta_\beta = 0, \end{aligned} \tag{55}$$

with H_n defined on the lattice site n. The sum $\sum_{\alpha\in n}$ extends over neighboring tetrahedra labeled by α, whereas the sum $\sum_{\beta\in n}$ extends over neighboring edges, here labeled by β. $G_{ij}^{(\alpha)}$ is the inverse of the DeWitt supermetric at the site α, and δ_β the deficit angle around the edge β. $\sqrt{g_\beta}$ is the dual (Voronoi) volume associated with the edge β.

The lattice Wheeler-DeWitt equation of Eq. (52) has an interesting structure, which is in part reminiscent of the Hamiltonian for lattice gauge theories. The first, local kinetic term is the gravitational analog of the electric field term E_a^2. It contains momenta which can be considered as conjugate to the squared edge length variables. The second local term involving ${}^3R(l^2)$ is the analog of the magnetic $(\nabla \times A_a)^2$. In the absence of a cosmological constant term, the first and second term have opposite sign, and need to cancel out exactly on physical states in order to give $H(\mathbf{x})\Psi = 0$. On the other hand, the last term proportional to λ has no gauge theory analogy, and is, as expected, genuinely gravitational.

It seems important to note here that the squared edge lengths take on only positive values $l_i^2 > 0$, a fact that would seem to imply that the wave function has to vanish when the edge lengths do, $\Psi(l^2 = 0) \simeq 0$. This constraint will tend to select the regular solution close to the origin in edge length space, as will be discussed further below. In addition, one has some rather complicated further constraints on the squared edge lengths, due to the triangle inequalities. These ensure that the areas of triangles and the volumes of tetrahedra are always positive. As a result, one would expect an average soft local upper bound on the squared edge lengths of the type $l_i^2 \lesssim l_0^2$ where l_0 is an average edge length, $\langle l_i^2\rangle = l_0^2$. The term "soft" refers to the fact that while large values for the edge lengths are possible, these should nevertheless enter with a relatively small probability, due to the small phase space available in this region. In any case, the nature of the discrete Wheeler-DeWitt equation presented here is explicit enough so that these, and other related, issues can presumably be answered both satisfactorily and unambiguously.

The above considerations have some consequences already in the strong-coupling limit of the theory. For sufficiently strong coupling (large Newton constant G), the first term in Eq. (52) is dominant, which shows again some similarity with what one finds for non-Abelian gauge theories for large coupling g^2. One would then expect, both from the constraint $l_i > 0$ and the triangle inequalities, that the spectrum of this operator is discrete, and that the energy gap, the spacing between the lowest eigenvalue and the first excited state, is of the same order as the ultraviolet cutoff. Nevertheless, one important difference here is that one is not interested in the whole spectrum, but instead just in the zero mode.

Irrespective of its specific form, it is in general possible to simplify the lattice Hamiltonian constraint in Eqs. (52) and (53) by using scaling arguments, as one does often in ordinary nonrelativistic quantum mechanics (for a list of relevant dimensions see Table I and II). After setting for the scaled cosmological constant $\lambda = 8\pi G\lambda_0$ and dividing the equation out by common factors, it can be recast in the slightly simpler form

$$\begin{aligned} &\left\{-\alpha a^6 \cdot \frac{1}{\sqrt{g(l^2)}} G_{ij}(l^2)\frac{\partial^2}{\partial l_i^2 \partial l_j^2} - \beta a^2 \cdot {}^3R(l^2) + 1\right\} \\ &\quad \times \Psi[l^2] = 0, \end{aligned} \tag{56}$$

where one finds it useful to define a dimensionless Newton's constant, as measured in units of the cutoff $\bar{G} \equiv 16\pi G/a^2$, and a dimensionless cosmological constant

TABLE I. Dimension of the Laplacian term in d dimensions.

dimension	dimension of Laplacian Δ_g
d dimensions	$l^{4-d}/l^4 \sim l^{-d} \sim 1/V_d$
2 + 1 dimensions	$A/l^4 \sim 1/A$
3 + 1 dimensions	$l/l^4 \sim 1/l^3 \sim 1/V$

[2]Instead of the combinatorial factor $n_{\sigma h}$, one could insert a ratio of volumes $V_{\sigma h}/V_h$ (where V_h is the volume per hinge [23] and $V_{\sigma h}$ is the amount of that volume in the simplex σ), but the above form is simpler.

TABLE II. Dimensions of couplings in d dimensions.

dimension	G dimension	λ dimension	dimensional	dimensionless
d dimensions	l^{d-2}	$1/l^2$	$G/\sqrt{\lambda} \sim l^{d-1}$	$G^{2/(d-2)}\lambda$
$2+1$ dimensions	l	$1/l^2$	$G/\sqrt{\lambda} \sim A_\Delta$	$G^2\lambda$
$3+1$ dimensions	l^2	$1/l^2$	$G/\sqrt{\lambda} \sim V_T$	$G\lambda$

$\bar{\lambda}_0 \equiv \lambda_0 a^4$, so that in the above equation one has $\alpha = \bar{G}/\bar{\lambda}_0$ and $\beta = 1/\bar{G}\bar{\lambda}_0$. Furthermore, the edge lengths have been rescaled so as to be able to set $\lambda_0 = 1$ in lattice units (it is clear from the original gravitational action that the cosmological constant λ_0 simply multiplies the total spacetime volume, thereby just shifting around the overall scale for the problem). Schematically, Eq. (56) is therefore of the form

$$\left\{-\bar{G}\Delta_s - \frac{1}{\bar{G}}\,{}^3R(s) + 1\right\}\Psi[s] = 0, \tag{57}$$

with Δ_s a discretized form of the covariant super-Laplacian, acting locally on the function space of the $s = l^2$ variables.

We shall not discuss the lattice implementation of the diffeomorphism (or momentum) constraint in Eq. (46). It can be argued that this will be satisfied automatically for a regular or random homogeneous lattice. This will indeed be the case for the examples we will be discussing below.

VI. EXPLICIT SETUP FOR THE LATTICE WHEELER-DEWITT EQUATION

In this section, we shall establish our notation and derive the relevant terms in the discrete Wheeler-DeWitt equation for a simplex.

A. 2 + 1 dimensions

The basic simplex in this case is of course a triangle, with vertices and squared edge lengths labeled as in Fig. 1. We set $l_{01}^2 = a$, $l_{12}^2 = b$, $l_{02}^2 = c$.

The components of the metric for coordinates based at vertex 0, with axes along the 01 and 02 edges, are

$$g_{11} = a, \qquad g_{12} = \tfrac{1}{2}(a + c - b), \qquad g_{22} = c. \tag{58}$$

The area A of the triangle is given by

$$A^2 = \frac{1}{16}[2(ab + bc + ca) - a^2 - b^2 - c^2], \tag{59}$$

so the supermetric G^{ij}, according to Eq. (51), is

$$G^{ij} = \frac{1}{4A}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \tag{60}$$

with inverse

$$G_{ij} = -2A\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{61}$$

Thus, for the triangle we have

$$G_{ij}\frac{\partial^2}{\partial s_i \partial s_j} = -4A\left(\frac{\partial^2}{\partial a \partial b} + \frac{\partial^2}{\partial b \partial c} + \frac{\partial^2}{\partial c \partial a}\right), \tag{62}$$

and the Wheeler-DeWitt equation is

$$\left\{(16\pi G)^2 4A\left(\frac{\partial^2}{\partial a \partial b} + \frac{\partial^2}{\partial b \partial c} + \frac{\partial^2}{\partial c \partial a}\right) - 2n_{\sigma h}\sum_h \delta_h + 2\lambda A\right\}\Psi[s] = 0, \tag{63}$$

where the sum is over the three vertices h of the triangle. The combinatorial factor $n_{\sigma h}$ ensures the correct normalization for the curvature term, since the latter has to give the lattice version of $\int \sqrt{g}\,{}^2R = 2\sum_h \delta_h$ when summed over all simplices (triangles in this case) σ. The inverse of $n_{\sigma h}$ counts therefore the number of times the same vertex appears in various neighboring triangles, and consequently depends on the specific choice of underlying lattice structure.

Alternatively, we can evaluate $G_{ij,kl}\frac{\partial^2}{\partial g_{ij}\partial g_{kl}}$ directly, using

$$G_{ij,kl} = \frac{1}{2\sqrt{g}}(g_{ik}g_{jl} + g_{il}g_{jk} - 2g_{ij}g_{kl}) \tag{64}$$

(note the different coefficient of the last term in two dimensions), with the metric g_{ij} as found above. The derivatives with respect to the metric are expressed in terms of derivatives with respect to squared edge lengths by

$$\frac{\partial}{\partial g_{ij}(s)} = \sum_m \frac{\partial s_m}{\partial g_{ij}}\frac{\partial}{\partial s_m}. \tag{65}$$

This leads to

$$\frac{\partial}{\partial g_{11}} = \frac{\partial}{\partial a} + \frac{\partial}{\partial b}, \tag{66}$$

$$\frac{\partial}{\partial g_{12}} = \frac{\partial}{\partial g_{21}} = -\frac{\partial}{\partial b}, \tag{67}$$

and

$$\frac{\partial}{\partial g_{22}} = \frac{\partial}{\partial b} + \frac{\partial}{\partial c}. \tag{68}$$

This procedure gives exactly the same expression for the kinetic term.

B. 3 + 1 dimensions

In this case, both methods described for 2 + 1 dimensions can be followed, but one is much easier than the other.

For ease of notation, we define $l_{01}^2 = a$, $l_{12}^2 = b$, $l_{02}^2 = c$, $l_{03}^2 = d$, $l_{13}^2 = e$, $l_{23}^2 = f$. For the tetrahedron labeled as in Fig. 3, we have

$$g_{11} = a, \qquad g_{22} = c, \qquad g_{33} = d, \tag{69}$$

$$g_{12} = \tfrac{1}{2}(a + c - b), \qquad g_{13} = \tfrac{1}{2}(a + d - e), \qquad g_{23} = \tfrac{1}{2}(c + d - f), \tag{70}$$

and its volume V is given by

$$\begin{aligned} V^2 = \frac{1}{144}[&af(-a - f + b + c + d + e) \\ &+ bd(-b - d + a + c + e + f) \\ &+ ce(-c - e + a + b + d + f) \\ &- abc - ade - bef - cdf]. \end{aligned} \tag{71}$$

The matrix G^{ij} is then given by

$$G^{ij} = -\frac{1}{24V}\begin{pmatrix} -2f & e+f-b & b+f-e & d+f-c & c+f-d & p \\ e+f-b & -2e & b+e-f & d+e-a & q & a+e-d \\ b+f-e & b+e-f & -2b & r & b+c-a & a+b-c \\ d+f-c & d+e-a & r & -2d & c+d-f & a+d-e \\ c+f-d & q & b+c-a & c+d-f & -2c & a+c-b \\ p & a+e-d & a+b-c & a+d-e & a+c-b & -2a \end{pmatrix}, \tag{72}$$

where

$$\begin{aligned} p &= -2a - 2f + b + c + d + e, \\ q &= -2c - 2e + a + b + d + f, \\ r &= -2b - 2d + a + c + e + f. \end{aligned} \tag{73}$$

It is nontrivial to invert this (although it can be done), so instead of using $G_{ij}\frac{\partial^2}{\partial s_i \partial s_j}$, we evaluate

$$G_{ij,kl} = \frac{1}{2\sqrt{g}}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}), \tag{74}$$

with

$$\begin{aligned} &\frac{\partial}{\partial g_{11}} = \frac{\partial}{\partial a} + \frac{\partial}{\partial b} + \frac{\partial}{\partial e}, \qquad \frac{\partial}{\partial g_{22}} = \frac{\partial}{\partial b} + \frac{\partial}{\partial c} + \frac{\partial}{\partial f}, \\ &\frac{\partial}{\partial g_{33}} = \frac{\partial}{\partial d} + \frac{\partial}{\partial e} + \frac{\partial}{\partial f}, \qquad \frac{\partial}{\partial g_{12}} = -\frac{\partial}{\partial b}, \\ &\frac{\partial}{\partial g_{13}} = -\frac{\partial}{\partial e}, \qquad \frac{\partial}{\partial g_{23}} = -\frac{\partial}{\partial f}. \end{aligned} \tag{75}$$

The matrix representing the coefficients of the derivatives with respect to the squared edge lengths is given in the Appendix, and is the inverse of G^{ij} found earlier. This is a nontrivial result as it acts as confirmation of the Lund-Regge expression which was derived in a completely different way.

Then, in 3 + 1 dimensions, the discrete Wheeler-DeWitt equation is

$$\left\{-(16\pi G)^2 G_{ij}\frac{\partial^2}{\partial s_i \partial s_j} - 2n_{\sigma h}\sum_h \sqrt{s_h}\,\delta_h + 2\lambda V\right\}\Psi[s] = 0, \tag{76}$$

where the sum is over hinges (edges) h in the tetrahedron. Note the mild nonlocality of the equation in that the curvature term, through the deficit angles, involves edge lengths from neighboring tetrahedra. In the continuum, the derivatives also give some mild nonlocality.

The discrete Wheeler-DeWitt equation is hard to solve analytically, even in 2 + 1 dimensions, because of the complicated dependence on edge lengths in the curvature term, which involves arcsin or arccos of convoluted expressions. When the curvature term is negligible, the differential operators may be transformed into derivatives with respect to the area (in 2 + 1 dimensions) or the volume (in 3 + 1 dimensions) and solutions found for the wave function, Ψ. Figs. 4 and 5 give a pictorial representation of lattices that can be used for numerical studies of quantum gravity in 2 + 1 and 3 + 1 dimensions, respectively.

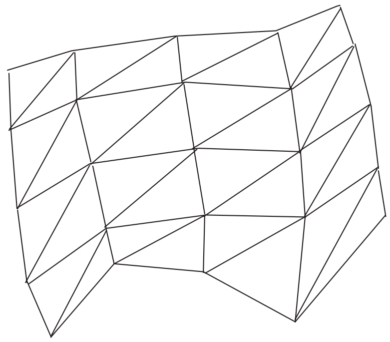

FIG. 4. A small section of a suitable spatial lattice for quantum gravity in 2 + 1 dimensions.

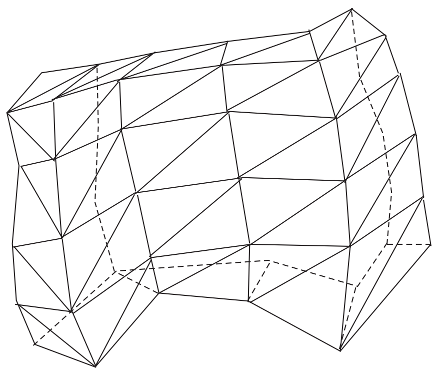

FIG. 5. A small section of a suitable spatial lattice for quantum gravity in 3 + 1 dimensions.

C. Solution of the triangle problem in 2 + 1 dimensions

In this section, we will consider the solution of the Wheeler-DeWitt equation for a single triangle. The present calculation is a necessary starting point and should provide a basic stepping stone for the strong-coupling expansion in $1/G$. In addition, it will show the physical nature of the wave-function solution deep in the strong-coupling regime. Note that for $1/G \to 0$ the coupling term between different simplices, which is due to the curvature term, disappears and one ends up with a completely decoupled problem, where the edge lengths in each simplex fluctuate independently. This is of course quite analogous to what happens in gauge theories on the lattice at strong coupling, the chromoelectric field fluctuates independently on each link, giving rise to short-range correlations, a mass gap, and confinement. Here, it is this single-simplex probability amplitude that we will set out to compute.

As in the Euclidean lattice theory of gravity, it will be convenient to factor out an overall scale from the problem, and set the (unscaled) cosmological constant equal to one [23] (see Table II). Recall that the Euclidean path integral weight contains a factor $P(V) \propto \exp(-\lambda_0 V)$ where $V = \int \sqrt{g}$ is the total volume on the lattice. The choice $\lambda_0 = 1$ then fixes this overall scale once and for all. Since $\lambda_0 = 2\lambda/16\pi G$, one then has $\lambda = 8\pi G$ in this system of units. In the following, we will also find it rather convenient to introduce the scaled coupling $\tilde{\lambda}$,

$$\tilde{\lambda} \equiv \frac{\lambda}{2}\left(\frac{1}{16\pi G}\right)^2, \tag{77}$$

so that for $\lambda_0 = 1$ (in units of the UV cutoff, or of the fundamental lattice spacing) one has $\tilde{\lambda} = 1/64\pi G$.

Moreover, it will often turn out to be desirable to avoid large numbers of factors of 16π's by the replacement, which we will follow from now on in this section, of $16\pi G \to G$. Then, $\tilde{\lambda} = 1/4G$ is the natural expansion parameter. Note that the coupling $\tilde{\lambda}$ has dimensions of length to the minus four, or inverse area squared, in 2 + 1 dimension, and length to the minus six, or inverse volume squared, in 3 + 1 dimensions.

Now, from Eq. (63), the Wheeler-DeWitt equation for a single triangle and constant curvature density R reads

$$\left\{(16\pi G)^2 4A_\Delta\left(\frac{\partial^2}{\partial a\partial b} + \frac{\partial^2}{\partial b\partial c} + \frac{\partial^2}{\partial c\partial a}\right) + (2\lambda - R)A_\Delta\right\}\Psi[s] = 0, \tag{78}$$

where a, b, c are the three squared edge lengths for the given triangle, and A_Δ is the area of the same triangle. In the following, we will take for simplicity $R = 0$. Equivalently, one needs to solve

$$\left\{\frac{\partial^2}{\partial a\partial b} + \frac{\partial^2}{\partial b\partial c} + \frac{\partial^2}{\partial c\partial a} + \tilde{\lambda}\right\}\Psi[a, b, c] = 0. \tag{79}$$

If one sets

$$\Psi[s] = \Phi[A_\Delta], \tag{80}$$

then one can show that

$$\frac{\partial^2}{\partial a\partial b}\Psi = \frac{1}{(16A_\Delta)^2}(b + c - a)(a + c - b) \times\left(\frac{d^2\Phi}{dA_\Delta^2} - \frac{1}{A_\Delta}\frac{d\Phi}{dA_\Delta}\right) + \frac{1}{16A_\Delta}\frac{d\Phi}{dA_\Delta}. \tag{81}$$

Summing the partial derivatives leads to the equation

$$A_\Delta\frac{d^2\Phi}{dA_\Delta^2} + 2\frac{d\Phi}{dA_\Delta} + 16\tilde{\lambda}A_\Delta\Phi = 0. \tag{82}$$

Solutions to the above equation are given by

$$\Psi[a,b,c] = \text{const.}\frac{1}{A_\Delta}\exp\left[\pm i\cdot 4A_\Delta\sqrt{\tilde{\lambda}}\right], \tag{83}$$

or alternatively by

$$\Psi[a,b,c] = \frac{1}{A_\Delta}\left[c_1\cos\left(4A_\Delta\sqrt{\tilde{\lambda}}\right) + c_2\sin\left(4A_\Delta\sqrt{\tilde{\lambda}}\right)\right]. \tag{84}$$

Note the remarkable, but not entirely unexpected, result that the wave function only depends on the area of the triangle $A_\Delta(a,b,c)$. In other words, it depends on the geometry only. Regularity of the wave function as the area of the triangle approaches zero, $A_\Delta \to 0$, requires for the constant $c_1 = 0$. Therefore, the correct quantum-mechanical solution is unambiguously determined,

$$\Psi[a,b,c] = \frac{1}{\sqrt{2\pi\sqrt{\tilde{\lambda}}}}\frac{1}{A_\Delta}\sin\left(4A_\Delta\sqrt{\tilde{\lambda}}\right). \tag{85}$$

The overall normalization constant has been fixed by the standard rule of quantum mechanics,

$$\int_0^\infty dA_\Delta|\Psi(A_\Delta)|^2 = 1. \tag{86}$$

Moreover, we note that a bare $\lambda < 0$ is not possible, and that the oscillatory nature of the wave function is seen here to give rise to well-defined peaks in the probability distribution for the triangle area, located at

$$(A_\Delta)_n = \frac{n\pi}{4\sqrt{\tilde{\lambda}}} \tag{87}$$

with n integer.

D. Solution of the tetrahedron problem in 3 + 1 dimensions

In this section, we will consider the nature of quantum-mechanical solutions for a single tetrahedron. Now, from Eq. (76), the Wheeler-DeWitt equation for a single tetrahedron with a constant curvature density term R reads

$$\left\{-(16\pi G)^2 G_{ij}\frac{\partial^2}{\partial s_i \partial s_j} + (2\lambda - R)V\right\}\Psi[s] = 0, \tag{88}$$

where now the squared edge lengths $s_1 \ldots s_6$ are all part of the same tetrahedron, and G_{ij} is given by a rather complicated, but explicit, 6×6 matrix given earlier.

As in the 2 + 1 case discussed in the previous section, here too it is found that, when acting on functions of the tetrahedron volume, the Laplacian term still returns some other function of the volume only, which makes it possible to readily obtain a full solution for the wave function. In terms of the volume of the tetrahedron V_T, one has the equivalent equation for $\Psi[s] = f(V_T)$ (we again replace $16\pi G \to G$ from now on),

$$\frac{7}{16}Gf'(V_T) + \frac{1}{16}GV_T f''(V_T) + \frac{1}{G}(2\lambda - R)V_T f(V_T) = 0 \tag{89}$$

with primes indicating derivatives with respect to V_T. From now on, we will set the constant curvature density $R = 0$; then, the solutions are Bessel functions J_m or Y_m with $m = 3$,

$$\psi_R(V_T) = \text{const.}J_3\left(4\sqrt{2}\frac{\sqrt{\lambda}}{G}V_T\right)/V_T^3, \tag{90}$$

or

$$\psi_S(V_T) = \text{const.}Y_3\left(4\sqrt{2}\frac{\sqrt{\lambda}}{G}V_T\right)/V_T^3. \tag{91}$$

Only $J_m(x)$ is regular as $x \to 0$, $J_m(x) \sim \Gamma(m+1)^{-1}\times (x/2)^m$. So, the only physically acceptable wave function is

$$\Psi(a,b,\ldots f) = \Psi(V_T) = \mathcal{N}\frac{J_3\left(4\sqrt{2}\frac{\sqrt{\lambda}}{G}V_T\right)}{V_T^3}, \tag{92}$$

with the normalization constant $\mathcal{N}$ given by

$$\mathcal{N} = \frac{45\sqrt{77\pi}}{10242^{3/4}}\left(\frac{G}{\sqrt{\lambda}}\right)^{5/2}. \tag{93}$$

The latter is obtained from the wave-function normalization requirement

$$\int_0^\infty dV_T|\Psi(V_T)|^2 = 1. \tag{94}$$

Consequently, the average volume of a tetrahedron is given by

$$\langle V_T\rangle \equiv \int_0^\infty dV_T\cdot V_T\cdot|\Psi(V_T)|^2 = \frac{31185\pi G}{262144\sqrt{2}\sqrt{\lambda}} = 0.2643\frac{G}{\sqrt{\lambda}}. \tag{95}$$

This last result allows us to define an average lattice spacing, by comparing it to the value for an equilateral tetrahedron which is $V_T = (1/6\sqrt{2})l_0^3$. One then obtains for the average lattice spacing at strong coupling

$$l_0 = 1.3089\left(\frac{G}{\sqrt{\lambda}}\right)^{1/3}. \tag{96}$$

Note that in terms of the parameter $\tilde{\lambda}$ defined in Eq. (77), one has in all the above expressions $\sqrt{\lambda}/G = \sqrt{2\tilde{\lambda}}$.

The above results further show that for strong gravitational coupling, $1/G \to 0$, lattice quantum gravity has a finite correlation length, of the order of 1 lattice spacing,

$$\xi \sim l_0. \tag{97}$$

This last result is simply a reflection of the fact that for large G the edge lengths, and therefore the metric, fluctuate more or less independently in different spatial regions due to the absence of the curvature term. The same is true in the Euclidean lattice theory as well, in the same limit [23]. It is the inclusion of the curvature term that later leads to a coupling of fluctuations between different spatial regions. Only at the critical point in G, if one can be found, is the correlation length, measured in units of the fundamental lattice spacing, expected to diverge [25]. This last circumstance should then allow the construction of a proper lattice continuum limit, as is done in the Euclidean lattice theory of gravity [31] (and in many other lattice field theories as well).

VII. PERTURBATION THEORY IN THE CURVATURE TERM

As shown in the previous section, in a number of instances it is not difficult to find the solution Ψ of the Wheeler-DeWitt equation in the strong-coupling (large-G) limit, where the curvature term is neglected, and only the kinetic and λ terms are retained. Then, the dynamics at different points decouples, and the wave function can be written as a product of relatively simple wave functions. It is then possible, at least in principle, to include the curvature term as a perturbation to the zeroth order solution. Accordingly, the unperturturbed Wheeler-DeWitt Hamiltonian is denoted by H_0,

$$H_0 \equiv -16\pi G \cdot G_{ij}(l^2)\frac{\partial^2}{\partial l_i^2 \partial l_j^2} + \frac{1}{16\pi G}\sqrt{g(l^2)} \cdot 2\lambda \tag{98}$$

and the perturbation by H_1,

$$H_1 \equiv -\frac{1}{16\pi G}\sqrt{g(l^2)^3}R(l^2). \tag{99}$$

The corresponding unperturbed wave function is denoted by Ψ_0, and satisfies

$$H_0\Psi_0 = 0. \tag{100}$$

To the next order in Raleigh- Schrödinger perturbation theory, one needs to solve

$$(H_0 + \epsilon H_1)\Psi = 0, \tag{101}$$

where for Ψ one sets as well

$$\Psi = \Psi_0 \exp\{\epsilon\Psi_1\}. \tag{102}$$

The sought-after first-order correction Ψ_1 is then given by the solution of

$$H_0(\Psi_0\Psi_1) + H_1\Psi_0 = 0. \tag{103}$$

Higher-order corrections can then be obtained in analogous fashion. It would seem natural to search for a solution (here specifically in 3 + 1 dimensions) of the form

$$\Psi \sim \exp\left\{-\alpha(\lambda, G)\sum_\sigma V_\sigma + \beta(\lambda, G)\sum_h \delta_h l_h + \dots\right\}, \tag{104}$$

with α and β given by power series

$$\alpha(\lambda, G) = \frac{\sqrt{\lambda}}{G}\sum_{n=0}^{\infty}\alpha_n(G\lambda)^n, \qquad \beta(\lambda, G) = \left(\frac{\sqrt{\lambda}}{G}\right)^{1/3}\sum_{n=0}^{\infty}\beta_n(G\lambda)^n. \tag{105}$$

The dots in Eq. (104) indicate possible higher-derivative terms in the exponent of the wave function.

VIII. VARIATIONAL METHOD FOR THE WAVE FUNCTION Ψ

In this section, we will describe some simple applications of the variational method for quantum gravity, based on the lattice Wheeler-DeWitt equation proposed earlier. The power of the variational method is well known and appreciated in nonrelativistic quantum mechanics, atomic physics, and many other physically relevant applications. Its success generally rests on the ability of finding a suitable, often physically motivated, wave function with the lowest possible energy, thereby providing an approximation to both the ground-state energy and the ground-state wave function. In practice, the wave function is often written as some sort of product of orbitals, dependent on a number of suitable parameters, which are later determined by minimization.

Here, we will write therefore an ansatz for the variational wave function, dependent on a number of free variational parameters

$$\Psi[l^2] = \Psi[l^2; \alpha, \beta, \gamma \dots], \tag{106}$$

and later require that the resulting wave function either satisfy the Wheeler-DeWitt equation, or that its energy functional

$$E(\alpha, \beta, \gamma\dots) = \frac{\langle\Psi[l^2]|\{-16\pi G \cdot G_{ij}(l^2)\frac{\partial^2}{\partial l_i^2 \partial l_j^2} - \frac{1}{16\pi G}\sqrt{g(l^2)}[{}^3R(l^2) - 2\lambda]\}|\Psi[l^2]\rangle}{\langle\Psi[l^2]|\Psi[l^2]\rangle} \tag{107}$$

be as close to zero as possible, $|E|^2 = \min$. This procedure should then provide a useful algebraic relation between the variational parameters, and thus allow their determination.[3]

Here, we will consider the following correlated product variational wave function (in general dimension):

$$\Psi[l^2] = Z^{-1/2} e^{-\alpha\sum_\sigma V_\sigma + \beta\sum_h \delta_h V_h + \dots} = Z^{-1/2}\prod_\sigma (e^{-\alpha V_\sigma})\prod_h (e^{\beta\delta_h V_h}) \times \dots, \quad (109)$$

with variational parameters $\alpha, \beta, \dots$ real or complex. Here, the $\sum_\sigma V_\sigma$ is the usual volume term in d dimensions, and $\sum_h \delta_h V_h$ the usual Regge curvature term, in the same number of dimensions. The dots indicate possible additional contributions, perhaps in the form of invariant curvature squared terms. In the atomic physics literature, these types of product wave functions are sometimes known as Jastrow-Slater wave functions [33,34]. Note that the above wave function is very different from the ones used in minisuperspace models, as it still depends on infinitely many lattice degrees of freedom in the thermodynamic limit (the limit in which the number of lattice sites is taken to infinity).

The wave-function normalization constant $Z(\alpha, \beta, \gamma \dots)$ is given by

$$Z = \int [dl^2] |\Psi[l^2; \alpha, \beta, \dots]|^2 = \int [dl^2] \exp\left\{-2\mathrm{Re}\alpha\sum_\sigma V_\sigma + 2\mathrm{Re}\beta\sum_h \delta_h V_h + \dots\right\} \quad (110)$$

and represents the partition function for Euclidean lattice quantum gravity, but *in one dimension less*. One would expect at least $\mathrm{Re}\alpha > 0$ to ensure convergence of the path integral; the trick we shall employ below is to obtain the relevant averages by analytic continuation in α and β of the corresponding averages in the Euclidean theory (for which α and β are real). Here, the expression $[dl^2]$ is the usual integration measure over the edge lengths [35], a lattice version of the DeWitt invariant functional measure over continuum metrics $[dg_{\mu\nu}]$. The definition of Z requires that the functional integral in Eq. (110) actually exists, which might or might not require some suitable regularization, for example, by the addition of curvature squared terms whose amplitude is sent to zero at the end of the calculation.

Next, one needs to compute the expectation value

$$\langle\Psi[l^2]|H|\Psi[l^2]\rangle, \quad (111)$$

with

$$H \equiv -16\pi G \cdot G_{ij}(l^2)\frac{\partial^2}{\partial l_i^2 \partial l_j^2} - \frac{1}{16\pi G}\sqrt{g(l^2)}[{}^3R(l^2) - 2\lambda], \quad (112)$$

which in turn is made up of three contributions, each of which can be evaluated separately. In terms of explicit lattice averages, one needs the three averages, or expectation values,

$$\langle\Psi[\alpha, \beta, \dots]\left|\left\{-\sum_\sigma \Delta_{l^2}(\sigma)\right\}\right|\Psi[\alpha, \beta, \dots]\rangle, \quad (113)$$

$$\langle\Psi[\alpha, \beta, \dots]\left|\left\{\sum_\sigma V_\sigma\right\}\right|\Psi[\alpha, \beta, \dots]\rangle, \quad (114)$$

$$\langle\Psi[\alpha, \beta, \dots]\left|\left\{2\sum_h \delta_h l_h\right\}\right|\Psi[\alpha, \beta, \dots]\rangle, \quad (115)$$

with

$$\Delta_{l^2}(\sigma) \equiv G_{ij}(l^2)\frac{\partial^2}{\partial l_i^2 \partial l_j^2}. \quad (116)$$

Note that we have summed over all lattice points by virtue of the assumed homogeneity of the lattice: the local *average* is expected to be the same as the average of the corresponding sum, divided by the overall number of simplices. Thus, for example, $\langle\Psi|\sum_\sigma V_\sigma|\Psi\rangle = N_\sigma\langle\Psi|V_\sigma|\Psi\rangle$, etc. At the same time, one has, by virtue of our choice of wave function,

$$\sum_\sigma V_\sigma|\Psi[\alpha, \beta, \gamma, \dots]\rangle = -\frac{\partial}{\partial\alpha}|\Psi[\alpha, \beta, \dots]\rangle, \quad (117)$$

$$\sum_h \delta_h l_h|\Psi[\alpha, \beta, \gamma, \dots]\rangle = \frac{\partial}{\partial\beta}|\Psi[\alpha, \beta, \dots]\rangle \quad (118)$$

and also for a given simplex labeled by σ,

$$-\Delta_{l^2}(\sigma)e^{-\alpha V_\sigma} = f(V_\sigma), \quad (119)$$

where f is some known function. More specifically, in $2+1$ dimensions one finds (here A_Δ is the area of the relevant triangle)

$$-\Delta_{l^2}(\sigma)A_\Delta^n = \tfrac{1}{4}n(n+1)A_\Delta^{n-1}, \quad (120)$$

[3]The continuum analog of the above expression would have the following general structure:

$$E = \frac{\int d^3\mathbf{x} \int[dg]\Psi^*[g]\cdot[-G\Delta_g - \frac{1}{G}\sqrt{g}(R - 2\lambda)]\cdot\Psi[g]}{\int[dg]\Psi^*[g]\cdot\Psi[g]}. \quad (108)$$

Similar energy functionals were considered some time ago by Feynman in his variational study of Yang-Mills theory in $2+1$ dimensions [32]. The main difference with gauge theories is that here the Hamiltonian contains two terms (kinetic and curvature terms) that enter with *opposite* signs, whereas in the gauge theory case both terms (the $\mathbf{E}^2$ term and the $(\nabla\times\mathbf{A})^2$ term) just add to each other. Feynman then argues that in the gauge theory the state of lowest energy corresponds necessarily to a minimum for both contributions.

$$-\Delta_{l^2}(\sigma)F(A_\Delta) = \frac{1}{2}\frac{dF}{dA_\Delta} + \frac{A_\Delta}{4}\frac{d^2F}{dA_\Delta^2} \tag{121}$$

and therefore

$$-\Delta_{l^2}(\sigma)e^{-\alpha A_\Delta} = \tfrac{1}{4}\alpha(\alpha A_\Delta - 2)e^{-\alpha A_\Delta}, \tag{122}$$

whereas in $3+1$ dimensions one has (here V_T is the volume of the relevant tetrahedron)

$$-\Delta_{l^2}(\sigma)V_T^n = \frac{1}{16}n(n+6)V_T^{n-1}, \tag{123}$$

$$-\Delta_{l^2}(\sigma)F(V_T) = \frac{7}{16}\frac{dF}{dV_T} + \frac{V_T}{16}\frac{d^2F}{dV_T^2} \tag{124}$$

and therefore

$$-\Delta_{l^2}(\sigma)e^{-\alpha V_T} = \frac{1}{16}\alpha(\alpha V_T - 7)e^{-\alpha V_T}. \tag{125}$$

In addition, in $3+1$ dimensions one needs to evaluate

$$-\Delta_{l^2}(\sigma)e^{\beta\sum_h l_h\delta_h}, \tag{126}$$

which is considerably more complicated. Nevertheless, in $2+1$ dimensions the corresponding result is zero, by the Gauss-Bonnet theorem. One identity can be put to use to relate one set of averages to another; it follows from the scaling properties of the lattice measure $[dl^2]$ in d dimensions with curvature coupling $k = 1/8\pi G$ and unscaled cosmological constant $\lambda_0 \equiv \lambda/8\pi G$ [31]. In three dimensions, it reads

$$2\lambda_0\left\langle\sum_T V_T\right\rangle - k\left\langle\sum_h \delta_h l_h\right\rangle - 7N_0 = 0, \tag{127}$$

where in the first term the sum is over all tetrahedra, and in the second term the sum is over all hinges (edges). The quantity N_0 is the number of sites on the lattice, the coefficient in front of it in general depends on the lattice coordination number, but for a cubic lattice subdivided into simplices it is equal to 7, since there are seven edges within each cube (three body principals, three face diagonals, and one body diagonal). The above sum rule can then be used by making the substitution $\lambda_0 \to 2\mathrm{Re}\alpha$ and $k \to 2\mathrm{Re}\beta$. In two dimensions, the analogous result reads

$$2\lambda_0\left\langle\sum_\Delta A_\Delta\right\rangle - k\left\langle\sum_h \delta_h\right\rangle - 3N_0 = 0, \tag{128}$$

with $2\sum_h \delta_h = \int\sqrt{g}R = 4\pi\chi = \mathrm{const.}$ by the Gauss-Bonnet theorem.

From now on, we will focus on the $2+1$ case exclusively. In this case, the curvature average of Eq. (115) is very simple

$$\left\langle\int\sqrt{g}R\right\rangle \to 4\pi\chi, \tag{129}$$

where χ is the Euler characteristic for the two-dimensional manifold. It will also be convenient to avoid a large number of factors of 16π's and make the replacement $16\pi G \to G$ for the rest of this section. Putting everything together, one then finds

$$\frac{E(\alpha)}{GN_T} = \frac{1}{4}\alpha(\alpha\bar{A}_\Delta - 2) + \frac{2\lambda}{G^2}\bar{A}_\Delta - \frac{1}{G^2}\frac{4\pi\chi}{N_T}. \tag{130}$$

One is not done yet, since what is needed next is an estimate for the average area of a triangle, $\bar{A}_\Delta$. This quantity is given, for a general measure over edges in two dimensions of the form $\prod dl^2 \cdot \prod_T (A_\Delta)^\sigma$, by

$$\langle A_\Delta\rangle = \frac{1+\frac{2}{3}\sigma}{2\alpha}, \tag{131}$$

again with the requirement $\mathrm{Re}\alpha > 0$ for the average to exist. It will be convenient to just set in the following $\bar{A}_\Delta = \langle A_\Delta\rangle = \sigma_0/\alpha$ with $\sigma_0 \equiv (1+\frac{2}{3}\sigma)/2$. One then obtains, finally, the relatively simple result

$$\frac{E(\alpha)}{GN_T} = \frac{\sigma_0 - 2}{4}\cdot\alpha + \frac{2\lambda\sigma_0}{G^2}\cdot\frac{1}{\alpha} - \frac{4\pi\chi}{G^2N_T}. \tag{132}$$

It would seem that, in order to avoid a potential instability, it might be safer to choose $\sigma_0 > 2$. The roots of this equation (corresponding to the requirement $\langle\Psi|H|\Psi\rangle = 0$) are given by

$$\alpha_0 = \frac{1}{G^2N_T(\sigma_0-2)}\{8\pi\chi \pm \sqrt{\Delta}\}, \tag{133}$$

with

$$\Delta \equiv 64\pi^2\chi^2 - 8G^2N_T^2\lambda\sigma_0(\sigma_0-2), \tag{134}$$

so that Δ is zero for

$$G = G_c = \pm\frac{2\sqrt{2}\pi\chi}{N_T\sqrt{\lambda\sigma_0(\sigma_0-2)}}. \tag{135}$$

Here, we select, on physical grounds, the positive root. When $\Delta = 0$ (or $G = G_c$), the two complex roots become real, or vice versa, with

$$\alpha_0(G_c) = \frac{N_T\lambda\sigma_0}{\pi\chi} > 0 \quad \text{if } \chi > 0. \tag{136}$$

Thus, for strong coupling (large $G > G_c$) α is almost purely imaginary

$$\alpha_0 = \pm\frac{i2\sqrt{2\lambda}}{G\sqrt{1-2/\sigma_0}} + \frac{8\pi\chi}{G^2N_T(\sigma_0-2)} + O(1/G^3), \tag{137}$$

whereas for weak coupling (small $G < G_c$) the two roots become

$$\alpha_0 \to \frac{N_T \lambda \sigma_0}{2\pi\chi} + O(G^2),$$
$$\alpha_0 \to \frac{16\pi\chi}{G^2 N_T(\sigma_0 - 2)} - \frac{N_T \lambda \sigma_0}{2\pi\chi} + O(G^2). \tag{138}$$

Note that an identical set of results would have been obtained if one had computed $|E(\alpha)|^2$ for complex alpha, and looked for minima. This is the quantity displayed in Figs. 6 and 7.

Next, we come to a brief discussion of the results. One interpretation is that the variational method, using the proposed correlated product wave function in $2+1$ dimensions, suggests the presence of a phase transition for pure gravity in G, located at the critical point $G = G_c$. This picture found here would then be in accordance with the result found in the *Euclidean* lattice theory in Ref. [36], which also gave a phase transition in three-dimensional gravity between a smooth phase (for $G > G_c$) and a branched polymer phase (for $G < G_c$). A similar transition was found on the lattice in four dimensions as well [23]. Finally, the presence of a phase transition is also inferred from continuum calculations for pure gravity in $\epsilon \equiv d-2 > 2$, although the latter does not give a clear indication on which phase is physical; nevertheless, simple renormalization group arguments suggest that the weak-coupling phase describes gravitational screening, while the strong-coupling phase implies gravitational antiscreening. This last expansion then gives a critical point for pure gravity in $2+1$ dimensions at $G_c = \frac{3}{25}(d-2) + \frac{45}{1250} \times (d-2)^2 + \ldots$, or $G_c \approx 0.024$ in units of the cutoff [37–39]. The Euclidean lattice calculation quoted earlier gives, in the same dimensions, $G_c \approx 0.355$. Note that the numerical magnitude of the critical point G in lattice units, contrary to the critical exponents, is not expected to be universal, and thus cannot be compared directly between formulations utilizing different ultraviolet regulators. We shall not enter here into some of the known peculiarities of three-dimensional gravity, including the absence of perturbative transverse-traceless radiation modes, and the absence of a sensible Newtonian limit; a recent discussion of these and related issues can be found, for example, in [25], and further references cited therein.

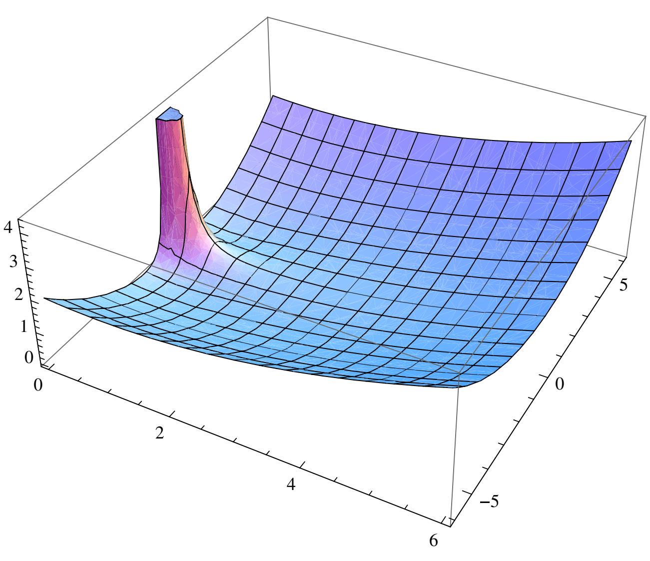

FIG. 6 (color online). Energy surface $|E(\alpha)|^2$ in $2+1$ dimensions at strong coupling, $G \gg G_c$ in the (Reα, Imα) plane. Note the presence of two almost purely imaginary, complex conjugate roots. The specific values used here are $\chi = 2$, $N_T = 10$, $\sigma_0 = 3$ and $\lambda = 1$.

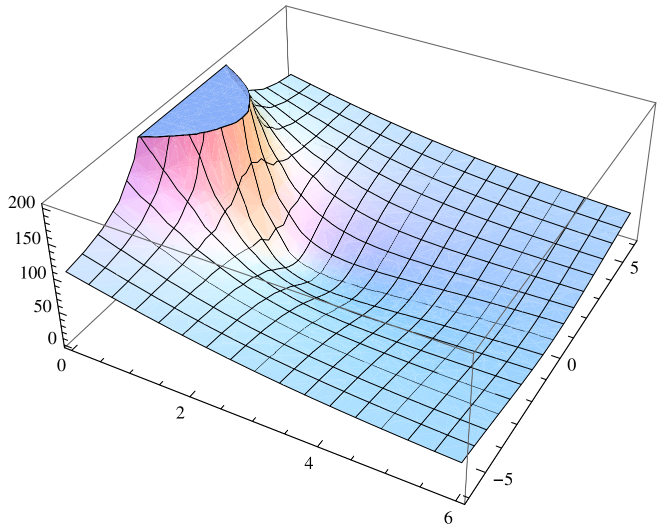

FIG. 7 (color online). Energy surface $|E(\alpha)|^2$ in $2+1$ dimensions for weak coupling, $G \ll G_c$. In this case both roots are along the real α axis.

In $3+1$ dimensions, the variational calculation is quite a bit more complex, since the integrated curvature term is no longer a constant. In the small curvature limit and for small variational parameter β, we have obtained the following expansion for the variational parameter α:

$$\alpha_0 = \pm i 4\sqrt{2}\frac{\sqrt{\lambda}}{G}\sqrt{\frac{\sigma_0}{\sigma_0 - 7}} - \frac{8c_0\beta}{\sigma_0 - 7} + O(\beta^2). \tag{139}$$

Here, c_0 is a real constant whose value we have not been able to determine yet. The two roots are found to become degenerate and real for

$$G = G_c \equiv \frac{\sqrt{\lambda\sigma_0(\sigma_0 - 7)}}{\sqrt{2}c_0\beta}, \tag{140}$$

which is again suggestive of a phase transition at G_c in $3+1$ dimensions, as found previously in the Euclidean theory in four dimensions [23,31]. More detailed calculations in the $3+1$ case are in progress, and will be presented elsewhere [40].

We conclude this section by observing that our results suggest a rather intriguing relationship between the ground-state wave functional of quantum gravity in $n+1$ dimensions, and averages computed within the Euclidean Feynman path integral formulation in n dimensions, i.e. in one dimension less. Moreover, since the variational calculations presented here rely on what could be regarded as an improved mean-field calculation, they are expected to become more accurate in higher dimensions, where the number of neighbors to each lattice point (or simplex) increases rapidly.

IX. WEAK-FIELD EXPANSION

In this section, we will discuss briefly the weak-field expansion for the proposed lattice Wheeler-DeWitt equation. The purpose here is to show how the weak-field expansion is performed, and how results analogous to the continuum ones are obtained for sufficiently smooth manifolds. Such results would be of relevance to the weak-coupling (small-G) expansion, and to an application of the WKB method on the lattice, for example. More generally, a clear connection to the continuum theory, and thus between lattice and continuum operators, is desirable, if not essential, in order to understand the meaning of physical gravitational averages, such as average curvature, etc. First, we note here that the lattice kinetic term (the one involving G_{ij}) has the correct continuum limit, essentially by construction. On the other hand, the curvature term appearing in the discrete Wheeler-DeWitt equation in $3+1$ dimensions is nothing but the integrand in the Regge expression for the Einstein-Hilbert action in three dimensions,

$$I_E = -k \sum_{\text{edges h}} l_h \delta_h. \tag{141}$$

The expansion of this action around flat space was already considered in some detail in Ref. [36], and shown to agree with the weak-field expansion in the continuum. Here, we provide a very short summary of the methods and results of this work. Following Ref. [20], one takes as background space a network of unit cubes divided into tetrahedra by drawing in parallel sets of face and body diagonals, as shown in Fig. 8. With this choice, there are $2^d - 1 = 7$ edges per lattice point emanating in the positive lattice directions: three body principals, three face diagonals, and one body diagonal, giving a total of seven components per lattice point.

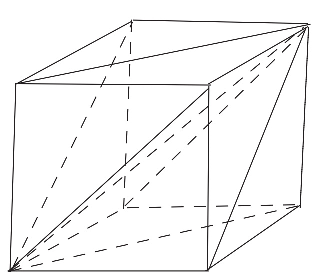

FIG. 8. A cube divided into simplices.

It is convenient to use a binary notation for edges, so that the edge index corresponds to the lattice direction of the edge, expressed as a binary number

$$\begin{aligned} &(0,0,1)\to 1, \qquad (0,1,1)\to 3, \qquad (1,1,1)\to 7,\\ &(0,1,0)\to 2, \qquad (1,0,1)\to 5, \qquad (1,0,0)\to 4,\\ &\qquad\qquad (1,1,0)\to 6. \end{aligned} \tag{142}$$

The edge lengths are then allowed to fluctuate around their flat space values $l_i = l_i^0(1+\epsilon_i)$, and the second variation of the action is expressed as a quadratic form in ϵ,

$$\delta^2 I = \sum_{mn} \epsilon^{(m)T} M^{(m,n)} \epsilon^{(n)}, \tag{143}$$

where n, m label the sites on the lattice, and M_{mn} is some Hermitian matrix. The general aim is then to show that the above quadratic form is equivalent to the expansion of the continuum Einstein-Hilbert action to quadratic order in the metric fluctuations. The infinite-dimensional matrix $M^{(m,n)}$ is best studied by going to momentum space; one assumes that the fluctuation ϵ_i at the point j steps from the origin in one coordinate direction, k steps in another coordinate direction, and l steps in the third coordinate direction, is related to the corresponding fluctuation ϵ_i at the origin by

$$\epsilon_i^{(j+k+l)} = \omega_1^j \omega_2^k \omega_4^l \epsilon_i^{(0)}, \tag{144}$$

with $\omega_i = e^{ik_i}$. In the smooth limit, $\omega_i = 1 + ik_i + O(k_i^2)$, the lattice action and the continuum action are then expected to agree. Note also that it is convenient here to set the lattice spacing in the three principal directions equal to one; it can always be restored at the end by using dimensional arguments.

It is desirable to express the lattice action in terms of variables which are closer to the continuum ones, such as $h_{\mu\nu}$ or $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{2}{3}\eta_{\mu\nu}h_{\lambda\lambda}$. Up to terms that involve derivatives of the metric (and which reflect the ambiguity of where precisely on the lattice the continuum metric should be defined), this relationship can be obtained by considering one tetrahedron, and using the expression for the invariant line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the diagonal flat metric. Inserting $l_i = l_i^0(1+\epsilon_i)$, with $l_i^0 = 1, \sqrt{2}, \sqrt{3}$ for the body principal ($i = 1, 2, 4$), face diagonal ($i = 3, 5, 6$), and body diagonal ($i = 7$), respectively, one obtains

$$
\begin{aligned}
(1+\epsilon_1)^2 &= 1 + h_{11},\\
(1+\epsilon_2)^2 &= 1 + h_{22},\\
(1+\epsilon_4)^2 &= 1 + h_{33},\\
(1+\epsilon_3)^2 &= 1 + \tfrac{1}{2}(h_{11}+h_{22}) + h_{12},\\
(1+\epsilon_5)^2 &= 1 + \tfrac{1}{2}(h_{11}+h_{33}) + h_{13},\\
(1+\epsilon_6)^2 &= 1 + \tfrac{1}{2}(h_{22}+h_{33}) + h_{23},\\
(1+\epsilon_7)^2 &= 1 + \tfrac{1}{3}(h_{11}+h_{22}+h_{33}) + \tfrac{2}{3}(h_{12}+h_{23}+h_{13})
\end{aligned} \tag{145}
$$

(note that we use the binary notation for edges, but maintain the usual index notation for the field $h_{\mu\nu}$). The above relationship can then be inverted to give the ϵ's in terms of the h's. Note that there are seven ϵ_i variables, but only six $h_{\mu\nu}$'s [in general in d dimensions we have $2^d - 1$ ϵ_i variables and $d(d+1)/2$ $h_{\mu\nu}$'s, which leads to a number of redundant lattice variables for $d > 2$].

Thus, to lowest order in $h_{\mu\nu}$, one can perform a field rotation on the lattice in order to go from the ϵ_i variables to the $h_{\mu\nu}$'s (or $\bar{h}_{\mu\nu}$'s),

$$\epsilon^T M_\omega \epsilon = (\epsilon^T V^{\dagger -1}) V^\dagger M_\omega V (V^{-1}\epsilon), \tag{146}$$

with

$$\epsilon = U_1 h, \qquad h = U_2 \bar{h}, \tag{147}$$

and so

$$\epsilon = V\bar{h}, \qquad V = U_1 U_2. \tag{148}$$

Here, V and U_1 are 7×6 matrices, while U_2 is a 6×6 matrix,

$$U_1 = \begin{pmatrix} \frac12 & 0 & 0 & 0 & 0 & 0\\ 0 & \frac12 & 0 & 0 & 0 & 0\\ 0 & 0 & \frac12 & 0 & 0 & 0\\ \frac14 & \frac14 & 0 & \frac12 & 0 & 0\\ \frac14 & 0 & \frac14 & 0 & \frac12 & 0\\ 0 & \frac14 & \frac14 & 0 & 0 & \frac12\\ \frac16 & \frac16 & \frac16 & \frac13 & \frac13 & \frac13 \end{pmatrix}, \tag{149}$$

$$U_2 = \begin{pmatrix} \frac13 & -\frac23 & -\frac23 & 0 & 0 & 0\\ -\frac23 & \frac13 & -\frac23 & 0 & 0 & 0\\ -\frac23 & -\frac23 & \frac13 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{150}$$

The above rotation is an essential step in transforming the lattice action into a form that looks like the continuum action, to quadratic order in the weak fields. For the Regge-Einstein term, the matrix M_ω describing the small fluctuations around flat space is given by

$$
\begin{aligned}
&(M_\omega)_{1,1} = -2, \quad (M_\omega)_{1,2} = -\omega_1\omega_4 - \bar\omega_2\bar\omega_4,\\
&(M_\omega)_{1,4} = 2 + 2\bar\omega_2, \quad (M_\omega)_{1,6} = 2\omega_1 + 2\bar\omega_2\bar\omega_4,\\
&(M_\omega)_{1,7} = -3\bar\omega_2 - 3\bar\omega_4, \quad (M_\omega)_{4,4} = -8,\\
&(M_\omega)_{4,5} = -4\omega_2 - 4\bar\omega_4, \quad (M_\omega)_{4,7} = 6 + 6\bar\omega_4,\\
&\qquad (M_\omega)_{7,7} = -18,
\end{aligned} \tag{151}
$$

with the remaining matrix elements obtained by permuting the appropriate indices. Because of its structure, which is of the form

$$M_\omega = \begin{pmatrix} A_6 & b\\ b^\dagger & -18 \end{pmatrix}, \tag{152}$$

where A_6 is a 6×6 matrix, a rotation can be done which completely decouples the fluctuations in ϵ_7,

$$M'_\omega = S^\dagger_\omega M_\omega S_\omega = \begin{pmatrix} A_6 + \frac{1}{18} b b^\dagger & 0\\ 0 & -18 \end{pmatrix}, \tag{153}$$

with

$$S_\omega = \begin{pmatrix} I_6 & 0\\ \frac{1}{18} b^\dagger & 1 \end{pmatrix}. \tag{154}$$

One then finds the first important result, namely, that the small fluctuation matrix M'_ω has three zero eigenvalues, corresponding to the translational zero modes for M_ω,

$$\begin{pmatrix} \epsilon_1\\ \epsilon_2\\ \epsilon_4\\ \epsilon_3\\ \epsilon_5\\ \epsilon_6\\ \epsilon_7 \end{pmatrix} = \begin{pmatrix} 1-\omega_1 & 0 & 0\\ 0 & 1-\omega_2 & 0\\ 0 & 0 & 1-\omega_4\\ \frac12(1-\omega_1\omega_2) & \frac12(1-\omega_1\omega_2) & 0\\ \frac12(1-\omega_1\omega_4) & 0 & \frac12(1-\omega_1\omega_4)\\ 0 & \frac12(1-\omega_2\omega_4) & \frac12(1-\omega_2\omega_4)\\ \frac13(1-\omega_1\omega_2\omega_4) & \frac13(1-\omega_1\omega_2\omega_4) & \frac13(1-\omega_1\omega_2\omega_4) \end{pmatrix} \times \begin{pmatrix} x_1\\ x_2\\ x_3 \end{pmatrix}, \tag{155}$$

where x_1, x_2, x_3 are three arbitrary parameters. The remaining eigenvalues are -18 (once) and $O(k^2)$ (3 times). Notice that one mode, corresponding to the fluctuations in the body diagonal ϵ_7, completely decouples. The next step is to transform the lattice weak-field action into a form similar (in fact identical) to the continuum form. One further rotation by the (6×6) matrix T_ω, defined by

$$T_\omega = \begin{pmatrix} \frac{\omega_1}{6} & -\frac{\omega_1}{3} & -\frac{\omega_1}{3} & 0 & 0 & 0 \\ -\frac{\omega_2}{3} & \frac{\omega_2}{6} & -\frac{\omega_2}{3} & 0 & 0 & 0 \\ -\frac{\omega_4}{3} & -\frac{\omega_4}{3} & \frac{\omega_4}{6} & 0 & 0 & 0 \\ -\frac{\omega_1\omega_2}{12} & -\frac{\omega_1\omega_2}{12} & -\frac{\omega_1\omega_2}{3} & \frac{1}{2} & 0 & 0 \\ -\frac{\omega_1\omega_4}{12} & -\frac{\omega_1\omega_4}{3} & -\frac{\omega_1\omega_4}{12} & 0 & \frac{1}{2} & 0 \\ -\frac{\omega_2\omega_4}{3} & -\frac{\omega_2\omega_4}{12} & -\frac{\omega_2\omega_4}{12} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (156)$$

gives the new small fluctuation matrix

$$L_\omega = T_\omega^\dagger M_\omega' T_\omega. \qquad (157)$$

This last transformation is equivalent to a change to trace-reversed metric variables. Finally, one defines the gauge-fixed matrix

$$\tilde{L}_\omega = L_\omega - \tfrac{1}{2} C_\omega^\dagger C_\omega, \qquad (158)$$

where C_ω is introduced in order to give the lattice analog of the harmonic gauge-fixing term, with

$$C_\omega = \frac{1}{6}\begin{pmatrix} 5(-1+\omega_1) & 1-\omega_1 & 1-\omega_1 & 6\left(1-\frac{1}{\omega_2}\right) & 6\left(1-\frac{1}{\omega_4}\right) & 0 \\ 1-\omega_2 & 5(-1+\omega_2) & 1-\omega_2 & 6\left(1-\frac{1}{\omega_1}\right) & 0 & 6\left(1-\frac{1}{\omega_4}\right) \\ 1-\omega_4 & 1-\omega_4 & 5(-1+\omega_4) & 0 & 6\left(1-\frac{1}{\omega_1}\right) & 6\left(1-\frac{1}{\omega_2}\right) \end{pmatrix}. \qquad (159)$$

Then, the form of $\tilde{L}_\omega$ is precisely equivalent to the corresponding continuum expression, in trace-reversed variables and in the harmonic gauge [36]. The seemingly complex combined S_ω and T_ω rotations just correspond to a rotation, from the original lattice edge fluctuation variables (ϵ) to the trace-reversed metric variables ($\bar{h}$).

Perhaps the most important aspect of the above proof of convergence of the lattice curvature term, and more generally of the whole lattice Wheeler-DeWitt equation, towards the corresponding continuum expression is in its relevance to the weak-field limit, to a perturbative expansion in G, and to a WKB expansion of the wave function. The latter are all issues that have already earned some consideration in the continuum formulation [4,6,7,10]. The present work suggests that most of those continuum results will remain valid, as long as they are derived in the context of the stated approximations. In particular, there is no reason to expect the *lattice* semiclassical wave function to have a different form (apart from the use of different variables, whose correspondence has been detailed in this section) compared to the continuum one [7].

X. CONCLUSIONS

In this paper, we have presented a lattice version of the Wheeler-DeWitt equation of quantum gravity. The present 3 + 1 approach is based on the canonical, and therefore Lorentzian, formulation of quantum gravity, and should therefore be regarded as complementary to the four-dimensional Euclidean lattice version of the same theory discussed earlier in other papers. The equations are explicit enough to allow a number of potentially useful practical calculations, such as the strong-coupling expansion, mean-field theory, and the variational method. In the preceding sections, we have outlined a number of specific calculations to illustrate the mechanics of the lattice theory, and the likely physical interpretation of the results. Because of its reliance on a different set of nonperturbative approximation methods, the formulation presented here should be useful in viewing the older Euclidean lattice results from a very different perspective; in a number of instances, we have been able to show the physical similarities between the two types of results.

Nevertheless, the phenomenal complexity of the original continuum theory, and of the Euclidean lattice approach, with all its issues of, for example, perturbative nonrenormalizability in four dimensions, has not gone away; it just got reformulated in a rather different language involving a Schrödinger-like equation, wave functionals, operators, and states. The hope is that the present approach will allow the use of a different set of approximation methods, and numerical algorithms, to explore what in some instances are largely known issues, but now from an entirely different perspective. Among the problems one might want to consider, we list: the description of invariant correlation functions [41,42], the behavior of the fundamental gravitational correlation length ξ as a function of the coupling G, the approach to the lattice continuum limit at G_c, estimates for the critical exponents in the vicinity of the fixed point, and the large-scale behavior of the gravitational Wilson loop [43]. Note that what is sometimes referred to as the problem of time does not necessarily affect the above issues, which in our opinion can be settled by looking exclusively at certain types of invariant correlations along the spatial directions only. These Green's functions should then provide adequate information about the nature of correlations in the full theory, without ever having to make reference to a time variable.

ACKNOWLEDGMENTS

The authors wish to thank Thibault Damour and the IHES in Bures-sur-Yvette for a very warm hospitality. Discussions with Reiko Toriumi are gratefully acknowledged. The work of H. W. H. was supported in part by the IHES and the University of California. The work of R. M. W. was supported in part by the UK Science and Technology Facilities Council.

APPENDIX A: THE MATRIX M IN 3 + 1 DIMENSIONS

The matrix of coefficients of the second partial derivative operators in 3 + 1 dimensions is given by $\frac{\mathbf{M}}{24V}$, where $\mathbf{M}$ is a symmetric 6×6 matrix with entries as follows:

$$
\begin{aligned}
M_{11} &= 2a^2, \\
M_{12} &= a^2 + b^2 + c^2 - 2ab - 2b, \\
M_{13} &= a^2 + d^2 + e^2 - 2ae - 2de, \\
M_{14} &= a^2 + b^2 + c^2 - 2ac - 2bc, \\
M_{15} &= a^2 + d^2 + e^2 - 2ad - 2de, \\
M_{16} &= b^2 + c^2 + d^2 + e^2 - 2af + 2bd + 2ce \\
&\quad - 2bc - 2be - 2cd - 2de, \\
M_{22} &= 2c^2, \\
M_{23} &= c^2 + d^2 + f^2 - 2cf - 2df, \\
M_{24} &= a^2 + b^2 + c^2 - 2ab - 2ac, \\
M_{25} &= a^2 + b^2 + d^2 + f^2 + 2af + 2bd - 2ce \\
&\quad - 2ab - 2ad - 2bf - 2df, \\
M_{26} &= c^2 + d^2 + f^2 - 2cd - 2df, \\
M_{33} &= 2d^2, \\
M_{34} &= a^2 + c^2 + e^+ f^2 + 2af - 2bd + 2ce \\
&\quad - 2ac - 2ae - 2cf - 2ef, \\
M_{35} &= a^2 + d^2 + e^2 - 2ad - 2ae, \\
M_{36} &= c^2 + d^2 + f^2 - 2cd - 2cf, \\
M_{44} &= 2b^2, \\
M_{45} &= b^2 + e^2 + f^2 - 2bf - 2ef, \\
M_{46} &= b^2 + e^2 + f^2 - 2be - 2ef, \\
M_{55} &= 2e^2, \\
M_{56} &= b^2 + e^2 + f^2 - -2be - 2bf, \\
M_{66} &= 2f^2.
\end{aligned}
\tag{A1}
$$

APPENDIX B: LATTICE HAMILTONIAN FOR GAUGE THEORIES

It is of interest to see how the Hamiltonian approach has fared for ordinary $SU(N)$ gauge theories, whose nontrivial infrared properties (confinement, chiral symmetry breaking) cannot be seen to any order in perturbation theory, and require therefore some sort of nonperturbative approach, based, for example, on the strong-coupling expansion. In the continuum, one starts from the Yang-Mills action

$$I = -\frac{1}{4g^2} \int d^4x F^a_{\mu\nu} F^{\mu\nu a}, \tag{B1}$$

with field strength

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu \tag{B2}$$

and gauge fields A^a_μ with $(a = 1 \dots N^2 - 1)$, where the quantities f_{abc} are the structure constants of the Lie group, such that the generators satisfy $[T_a, T_b] = i f_{abc} T_c$.

A lattice-regularized form of the gauge action in Eq. (B1) was given in Ref. [44], see also [45]. The theory is defined on a d-dimensional hypercubic lattice with lattice spacing a, vertices labeled by an index $\mathbf{n}$, and directions by μ. The group elements $U_{n\mu} = \exp(iagA^a_\mu T_a)$ are defined in the fundamental representation, and reside on the links of the lattice. The pure gauge Euclidean action involves a sum of traces of path-ordered products [with $U_{-\mu}(n+\nu) = U^\dagger_\mu(n)$] of unitary $U_\mu(n)$ matrices around an elementary square loop ("plaquettes," here denoted by $\square$),

$$I[U] = -\frac{a^{4-d}}{4g^2} \sum_{\square} \mathrm{tr}[UUU^\dagger U^\dagger + \mathrm{H.c.}]. \tag{B3}$$

The action is locally gauge invariant with respect to the change

$$U_\mu(n) \to V^\dagger(n) U_\mu(n) V(n+\nu), \tag{B4}$$

where V is an arbitrary $SU(N)$ matrix defined on the lattice sites.

The next step is to define the path integral as

$$Z(g^2) = \int [dU_H] \exp(-I[U]), \tag{B5}$$

where $[dU_H]$ is the Haar measure over the group $SU(N)$, one copy for each lattice link variable U. A lattice-regularized Hamiltonian can then be defined on a purely spatial lattice, by taking the zero-lattice-spacing limit in the time direction [46,47]. Local gauge invariance further allows one to set all the link variables in the time direction to unity, $U_{n0} = 1$, or $A^a_{n0} = 0$ in this lattice version of the temporal gauge. The $\dot{U}$ variables can now be eliminated by introducing generators of local rotations $E^a_i(\mathbf{n})$, defined on the links (with spatial directions labeled by i, $j = 1, 2, 3$) and satisfying the commutation relations

$$[E_i^a(\mathbf{n}), U_j(\mathbf{m})] = T^a U_i(\mathbf{n})\delta_{ij}\delta_{\mathbf{nm}}, \tag{B6}$$

along with the $SU(N)$ generator algebra commutation relation

$$[E_i^a(\mathbf{n}), E_j^b(\mathbf{m})] = if^{abc}E_i^c(\mathbf{n})\delta_{ij}\delta_{\mathbf{nm}}. \tag{B7}$$

This finally gives for the Hamiltonian of Wilson's lattice gauge theory [46]

$$H = \frac{g^2}{2a}\sum_{\text{links}} E^a E^a - \sum \frac{1}{4ag^2}\,\text{tr}[UUU^\dagger U^\dagger + \text{H.c.}]. \tag{B8}$$

The first term in Eq. (B8) is the lattice analog of the electric field term $\mathbf{E}^2$, while the second term is a lattice discretized version, involving lattice finite differences, of the magnetic field $(\nabla \times \mathbf{A})^2$ term. In this picture, the analog of Gauss's law is a constraint which needs to be enforced on physical states at each spatial site $\mathbf{n}$,

$$\sum_{i=1}^{6} E_i^a(\mathbf{n})|\Psi\rangle = 0. \tag{B9}$$

In general, and irrespective of the symmetry group chosen, the ground state in the strong-coupling $g^2 \to \infty$ limit has all the $SU(N)$ rotators in their ground state. In this limit, the Hamiltonian has the simple form

$$H_0 = \frac{g^2}{2a}\sum_{\text{links}} E_i^a E_i^a. \tag{B10}$$

Then, the vacuum is a state for which each link is in a color singlet state

$$E_i^a|0\rangle = 0. \tag{B11}$$

The lowest-order excitation of the vacuum is a state with one unit of chromoelectric field on each link of an elementary lattice square, and energy

$$E_\square = 4\cdot\frac{g^2}{2a}\frac{N^2-1}{2N}. \tag{B12}$$

Raleigh-Schrödinger perturbation theory can then be used to compute corrections to arbitrarily high order in $1/g^2$. But, ultimately one is interested in the limit $g^2 \to 0$, corresponding to the ultraviolet asymptotic freedom fixed point of the non-Abelian gauge theory, and thus to the lattice continuum limit $a \to 0$. Thus, in order to recover the original theory's continuum limit, one needs to examine a limit where the mass gap in units of the lattice spacing goes to zero, $am(g) \to 0$. This limit then corresponds to an infinite correlation length in lattice units; the zero-lattice-spacing limit so described is a crucial step in fully recovering desirable properties (rotational or Lorentz invariance, asymptotic freedom, massless perturbative gluon excitations, etc.) of the original continuum theory.

[1] P. A. M. Dirac, Proc. R. Soc. A **246**, 333 (1958); Phys. Rev. **114**, 924 (1959).

[2] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by Louis Witten (Wiley, New York, 1962), Chap. 7, p. 227; arXiv:gr-qc/0405109.

[3] B. S. DeWitt, in *Relativity, Groups and Topology*, edited by B. DeWitt and C. DeWitt, 1963 Les Houches Lectures (Gordon and Breach, New York, 1964).

[4] B. S. DeWitt, Phys. Rev. **160**, 1113 (1967); **162**, 1195 (1967); **162**, 1239 (1967).

[5] J. A. Wheeler, in *Batelle Rencontres: 1967 Lectures in Mathematics and Physics*, edited by C. DeWitt and J. A. Wheeler (Benjamin, New York, 1968), p. 242.

[6] H. Leutwyler, Phys. Rev. **134**, B1155 (1964).

[7] K. Kuchař, J. Math. Phys. (N.Y.) **17**, 777 (1976).

[8] K. Kuchař, in *Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics*, edited by G. Kunstatter, D. Vincent, and J. Williams (World Scientific, Singapore, 1992).

[9] C. Isham, in *Integrable Systems, Quantum Groups and Quantum Field Theories*, edited by L. A. Ibort and M. A. Rodriguez (Kluwer, Dordrecht, 1993) p. 157.

[10] A. O. Barvinsky and C. Kiefer, Nucl. Phys. **B526**, 509 (1998).

[11] R. P. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[12] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).

[13] W. F. Blyth and C. J. Isham, Phys. Rev. D **11**, 768 (1975).

[14] S. W. Hawking and J. C. Luttrell, Phys. Lett. **143B**, 83 (1984); Nucl. Phys. **B247**, 250 (1984).

[15] S. W. Hawking and Z. C. Wu, Phys. Lett. **151B**, 15 (1985).

[16] S. W. Hawking and D. N. Page, Nucl. Phys. **B264**, 185 (1986).

[17] D. N. Page, arXiv:hep-th/0610121.

[18] T. Regge, Nuovo Cimento, **19**, 558 (1961).

[19] J. A. Wheeler, in *Relativity, Groups and Topology*, edited by B. DeWitt and C. DeWitt, 1963 Les Houches Lectures (Gordon and Breach, New York, 1964).

[20] M. Roček and R. M. Williams, Phys. Lett. **104B**, 31 (1981); Z. Phys. C **21**, 371 (1984).

[21] J. Cheeger, W. Müller, and R. Schrader, in *Unified Theories Of Elementary Particles*, edited by P. Breitlohner and H. P. Dürr, Lecture Notes in Physics, (Springer, New York, 1982); Commun. Math. Phys. **92**, 405 (1984).

[22] C. Itzykson, in *Progress in Gauge Field Theory*, edited by G. 't Hooft *et al.*, NATO Advanced Study Institutes (Plenum Press, New York, 1984) p. 337.
[23] H. W. Hamber and R. M. Williams, Nucl. Phys. **B248**, 392 (1984); **B260**, 747 (1985); **B269**, 712 (1986); Phys. Lett. **157B**, 368 (1985).
[24] J. B. Hartle, J. Math. Phys. (N.Y.) **26**, 804 (1985); **27**, 287 (1986); **30**, 452 (1989).
[25] H. W. Hamber, *Quantum Gravitation*, Springer Tracts in Modern Physics (Springer, New York, 2009).
[26] F. Lund and T. Regge, ''Simplicial Approximation to some Homogeneous Cosmologies,'' (unpublished).
[27] J. B. Hartle, W. A. Miller, and R. M. Williams, Classical Quantum Gravity **14**, 2137 (1997).
[28] R. M. Williams, Classical Quantum Gravity **3**, 853 (1986).
[29] T. Piran and R. M. Williams, Phys. Rev. D **33**, 1622 (1986).
[30] R. M. Williams and P. A. Tuckey, Classical Quantum Gravity **7**, 2055 (1990).
[31] H. W. Hamber, Phys. Rev. D **45**, 507 (1992); Nucl. Phys. **B400**, 347 (1993); Phys. Rev. D **61**, 124008 (2000); Gen. Relativ. Gravit. **41**, 817 (2009).
[32] R. P. Feynman, Nucl. Phys. **B188**, 479 (1981).
[33] J. Cardy and H. Hamber, Nucl. Phys. **B170**, 79 (1980).
[34] P. Surany, Nucl. Phys. **B210**, 519 (1982).
[35] H. W. Hamber and R. M. Williams, Phys. Rev. D **59**, 064014 (1999).
[36] H. W. Hamber and R. M. Williams, Phys. Rev. D **47**, 510 (1993).
[37] T. Aida and Y. Kitazawa, Nucl. Phys. **B491**, 427 (1997).
[38] H. Kawai and M. Ninomiya, Nucl. Phys. **B336**, 115 (1990); H. Kawai, Y. Kitazawa, and M. Ninomiya, Nucl. Phys. **B393**, 280 (1993); **B404**, 684 (1993); Y. Kitazawa and M. Ninomiya, Phys. Rev. D **55**, 2076 (1997).
[39] M. Reuter, Phys. Rev. D **57**, 971 (1998); M. Reuter and H. Weyer, Gen. Relativ. Gravit. **41**, 983 (2009); E. Manrique, M. Reuter, and F. Saueressig, Ann. Phys. (N.Y.) **326**, 463 (2011).
[40] H. W. Hamber, R. Toriumi, and R. M. Williams (unpublished).
[41] H. W. Hamber, Phys. Rev. D **50**, 3932 (1994).
[42] S. B. Giddings, D. Marolf, and J. B. Hartle, Phys. Rev. D **74**, 064018 (2006).
[43] H. W. Hamber and R. M. Williams, Phys. Rev. D **76**, 084008 (2007); **81**, 084048 (2010).
[44] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
[45] R. Balian, J. M. Drouffe, and C. Itzykson, Phys. Rev. D **11**, 2104 (1975); **19**, 2514(E) (1979).
[46] J. B. Kogut and L. Susskind, Phys. Rev. D **11**, 395 (1975).
[47] J. B. Kogut, Rev. Mod. Phys. **55**, 775 (1983).