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# HIGHER DERIVATIVE QUANTUM GRAVITY ON A SIMPLICIAL LATTICE

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We generalize the action of Regge calculus to include the equivalent to both a cosmological constant term and a higher derivative term involving the integral of $R^2$. We compare our expression for these terms with the continuum values for the regular tessellations of 2-, 3- and 4-dimensional spheres, and describe how the formalism may be applied to calculations in quantum gravity.

## 1. Introduction

A major difficulty with conventional formulations of euclidean quantum gravity is the fact that the Einstein action $I_E$ can become arbitrarily negative [1]. This means that the path integral of $\exp(-I_E)$ does not converge. The problem persists for lattice formulations of gravity [2] and provides an obstacle to progress for calculations of anything other than the weak field limit [3].

A possible solution to the problem has been described by Hawking [1], who suggests performing the integration in a conformal gauge in which the Einstein action is non-negative, and then integrating over all conformal factors. A second possibility is to add to the Einstein action extra terms, including higher derivative ones [4] like $R^2$, in a carefully chosen combination which makes the total action non-negative.

This paper is based on the description of gravity known as Regge calculus [2, 5] in which the Einstein theory is expressed in terms of simplicial decompositions of space-time manifolds. Its use in quantum gravity is prompted by the desire to make use of techniques developed in lattice gauge theories, but with a lattice which reflects the structure of space-time rather than just providing a flat passive background. The difficulty of defining conformal transformations for the simplicial lattice leads us to explore the second of the two possible solutions mentioned above. In particular, the

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replacement of the Einstein action

$$I_{\mathrm{E}} = \frac{1}{16\pi G} \int \mathrm{d}^4x \sqrt{g}\, R\,, \tag{1.1}$$

where $R$ is the curvature scalar and $g$ the modulus of the determinant of the metric tensor, by

$$I = \int \mathrm{d}^4x \sqrt{g} \left[ a + \tfrac{1}{2}bR + \tfrac{1}{4}cR^2 \right] \tag{1.2}$$

leads to a non-negative action, provided the couplings $a > 0$, $b$ and $c > 0$ satisfy $4ac - b^2 \geqslant 0$. Of course the inclusion of a higher derivative term in the action has other effects, some desirable, some problematic. Stelle has shown [6] that the theory based on an action of the form (1.2) plus a term involving the square of the Weyl tensor, is renormalizable in four dimensions. Furthermore it has been demonstrated that the general fourth-order action leads to an asymptotically free theory [7]. Massless renormalizable asymptotically free theories appear to be at the present moment the only candidates for field theories in four dimensions that possess a non-trivial continuum limit. However in higher derivative gravity there appear to be difficulties with unitarity, at least in perturbation theory, and it is not clear at the present moment to what extent these problems arise because of the splitting of the weak field action into quadratic and non-quadratic parts, and if they persist in the full quantum theory. For reviews on the subject we refer the reader to refs. [8]. Other work on higher derivative gravity is described in refs. [9].

Most lattice formulations of gravity so far have been based on hypercubical lattices (see for example Das, Kaku and Townsend [10], Smolin [11], and Mannion and Taylor [12]). For such lattices, the formulation of $R^2$ terms in four dimensions involves constraints between the connections and the tetrads, which are difficult to handle. Also there is no simple way of writing down topological invariants, which are either related to the Einstein action (in two dimensions), or are candidates for extra terms to be included in the action [13]. Tomboulis [14] has written down an $R^2$ action which is reflection positive but has a very cumbersome form. We shall see how these difficulties need not be present on a simplicial lattice (except that it is not known how to write the Hirzebruch signature in lattice terms [15]).

It may be objected that since in Regge calculus where the curvature is restricted to the hinges which are subspaces of dimension 2 less than that of the space considered, then the curvature tensor involves $\delta$-functions with support on the hinges [3, 16], and so higher powers of the curvature tensor are not defined [17]. (This argument clearly does not apply to the Euler characteristic

$$\chi = \frac{1}{128\pi^2} \int \mathrm{d}^4x \sqrt{g}\, R_{abcd} R_{efgh} \varepsilon^{abef} \varepsilon^{cdgh}\,, \tag{1.3}$$

and the Hirzebruch signature

$$\tau = \frac{1}{96\pi^2} \int d^4x \sqrt{g} R_{abcd} R^{ab}{}_{ef} \varepsilon^{cdef}, \tag{1.4}$$

which are both integrals of 4-forms.) However it is a common procedure in lattice field theory to take powers of fields defined at the same point, and we see no reason why one should not consider similar terms in lattice gravity. Of course we would like our expressions to correspond to the continuum ones as the edge lengths of the simplicial lattice become smaller and smaller.

In sect. 2 we shall describe our formalism for writing down extra terms in the Regge calculus action and their values for regular tessellations of $S^2$, $S^3$ and $S^4$ are given in sect. 3. The possibility of writing down other higher derivative terms is explored in sect. 4, and in the final section we discuss some possible applications and numerical work in progress.

## 2. Formalism for $R^2$ on the simplicial lattice

In a $d$-dimensional Regge calculus space-time, the usual form of the action is

$$I_R = \sum_{\text{hinges h}} A_h \delta_h, \tag{2.1}$$

where the hinges are the $(d-2)$-dimensional subspaces on which the curvature is distributed, $A_h$ is the area of the hinge and $\delta_h$ is the deficit angle there, which is given by

$$\delta_h = 2\pi - \sum_{\substack{\text{blocks}\\ \text{meeting on h}}} \vartheta_d, \tag{2.2}$$

where $\vartheta_d$ is the dihedral angle. The action is the equivalent for a simplicial decomposition of the continuum expression $\frac{1}{2}\int d^dx \sqrt{g} R$, and indeed it has been shown [16–18] that $I_R$ tends to the continuum expression as the Regge block size (or the average edge length) tends to zero. Variation of $I_R$ with respect to the edge lengths of the blocks gives the simplicial analogue of Einstein's equations.

We now look at generalizations of the Regge calculus equivalent of the Einstein action. Firstly a cosmological constant term, which in the continuum theory takes the form $\Lambda \int d^dx \sqrt{g}$, can clearly be represented on the simplicial lattice by a term in the action of the form

$$I_\Lambda = \Lambda \times (\text{total volume}). \tag{2.3}$$

Secondly, we wish to find a term equivalent to the continuum expression $\frac{1}{4}\int d^dx \sqrt{g} R^2$, and the remainder of this section will be concerned with this problem.

Since the curvature is restricted to the hinges, it is natural that expressions for curvature integrals should involve sums over hinges as in (2.1). The curvature tensor, which involves second derivatives of the metric, is of dimension $L^{-2}$. Therefore $\frac{1}{4}\int d^d x \sqrt{g} R^n$ is of dimension $L^{d-2n}$. Thus if we postulate that an $R^2$ term will involve the square of $A_h \delta_h$, which is of dimension $L^{2(d-2)}$, then we shall need to divide by some $d$-dimensional volume to obtain the correct dimension for the extra term in the action. Now any hinge is surrounded by a number of $d$-dimensional simplices, so the procedure of dividing by a $d$-dimensional volume seems ambiguous. The crucial step is to realize that there is a unique $d$-dimensional volume associated with each hinge, and its construction will now be described.

There is a well-established procedure for constructing a dual lattice for any given lattice [19]. This involves constructing polyhedral cells, known in the literature as Voronoi polyhedra, around each vertex, in such a way that the cell around each particular vertex contains all points which are nearer to that vertex than to any other vertex. Thus the cell is made up from $(d-1)$-dimensional subspaces which are the perpendicular bisectors of the edges in the original lattice, $(d-2)$-dimensional subspaces which are orthogonal to the 2-dimensional subspaces of the original lattice, and so on.

It is not easy to visualize a dual lattice in an arbitrary dimension, so we shall now look specifically at the cases of interest to us, 2, 3 and 4 dimensions. Note that it is only in 2 dimensions that the hinges coincide with the vertices, and so the volume associated with the hinge is equal to the volume of the dual cell. In higher dimensions we shall need to define more carefully what we mean by the volume associated with each hinge.

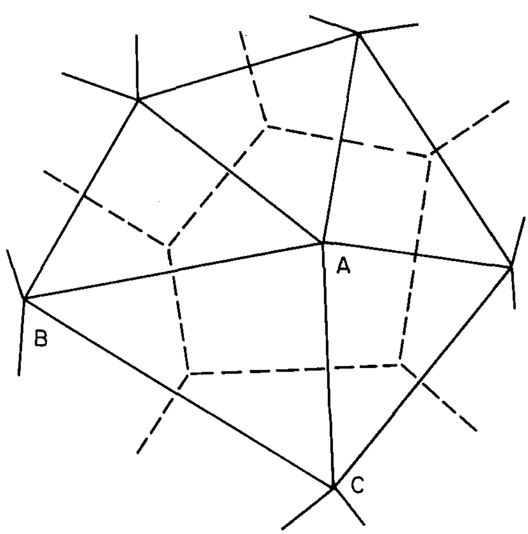


Fig. 1.

In two dimensions, we consider a network of triangles. (In Regge calculus it is conventional to use simplicial lattices, i.e. lattices based on triangles, tetrahedra, 4-simplices,..., since the edge lengths then determine uniquely the shape of the block.) The dual lattice has been sketched in dashed lines in fig. 1. We see that the "volume" (area here) of the dual cell around the vertex or hinge A, consists of the sum of area contributions from 5 different triangles. The contribution from one triangle, say ABC, is determined as follows. Suppose that the sides are $l_1$, $l_2$ and $l_3$ (see fig. 2), and that $F$ is equidistant from A, B, and C.

The area of AFB is $2A_1$ etc. Then the contribution of this triangle to the area of the polyhedron based on A is given by $A_1 + A_3$, with

$$A_1 = \frac{l_1^2 \Sigma_{213}}{32 A_{123}}, \tag{2.4}$$

where $A_{123}$ is the area of the triangle ABC, and throughout this paper we define $\Sigma_{\alpha\beta\gamma}$ by

$$\Sigma_{\alpha\beta\gamma} \equiv l_\alpha^2 - l_\beta^2 + l_\gamma^2 . \tag{2.5}$$

The other areas may be found by permuting the indices.

In 3 dimensions, the hinges are the edges of tetrahedra. Consider a number of tetrahedra meeting on an edge AB. There will be a polyhedral area in the dual lattice orthogonal to AB formed by joining the points at the centers of the tetrahedra. This is shown by the dashed line in fig. 3.

G is the point equidistant from the vertices of the tetrahedron ABCD. The volume associated with the hinge AB will be the volume of the object formed by joining the vertices of the polyhedron to the points A and B. This will be the sum of volume

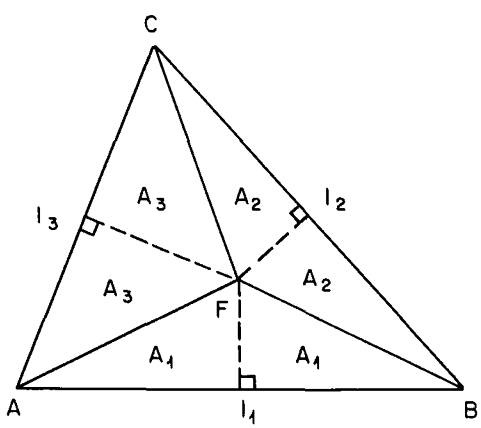


Fig. 2.

contributions from all the tetrahedra meeting on AB. For example the contribution from the tetrahedron ABCD is determined as follows. Suppose that F″ is the point in the triangle ABD equidistant from its vertices (see fig. 4). Then the volume in ABCD associated with AB is the sum of the volumes ABFG and ABF′G, which we denote by $V_{1(23)}$ and $V_{1(45)}$ respectively, with for example

$$V_{1(23)} = \frac{l_1^2 \Sigma_{213} \Sigma_{123(456)}}{4608 A_{123}^2 V_{123456}}, \tag{2.6}$$

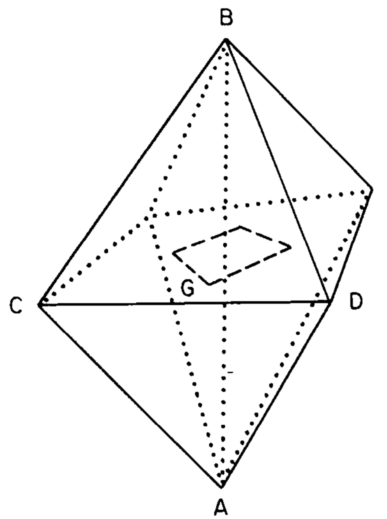


Fig. 3.

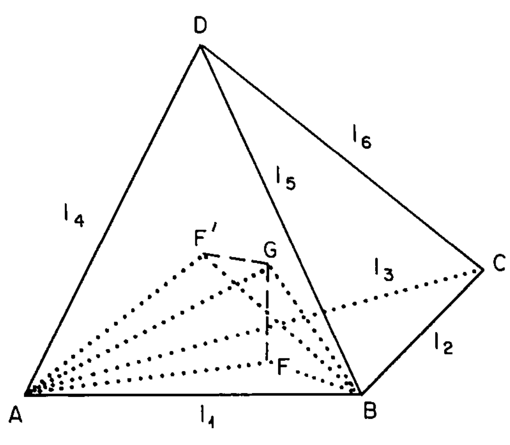


Fig. 4.

where $V_{123456}$ is the volume of ABCD, and we define $\Sigma_{\alpha\beta\gamma(\lambda\mu\nu)}$ by

$$\Sigma_{\alpha\beta\gamma(\lambda\mu\nu)} \equiv -2l_\alpha^2 l_\beta^2 l_\gamma^2 + l_\alpha^2 l_\nu^2 \Sigma_{\beta\alpha\gamma} + l_\beta^2 l_\lambda^2 \Sigma_{\alpha\beta\gamma} + l_\gamma^2 l_\mu^2 \Sigma_{\alpha\gamma\beta}. \tag{2.7}$$

In four dimensions, we consider a number of 4-simplices meeting on a triangular hinge. Again there will be a polyhedral cell in the dual lattice, orthogonal to the triangle; its vertices will be the points like H which are equidistant from the vertices of each 4-simplex like ABCDE (see fig. 5). The volume associated with the hinge ABC is that of the object formed by joining the vertices of the polyhedron to those of the hinge. The contribution from the 4-simplex ABCDE will be as follows. Suppose that G′ is the point in the tetrahedron ABCE equidistant from its vertices (see fig. 6).

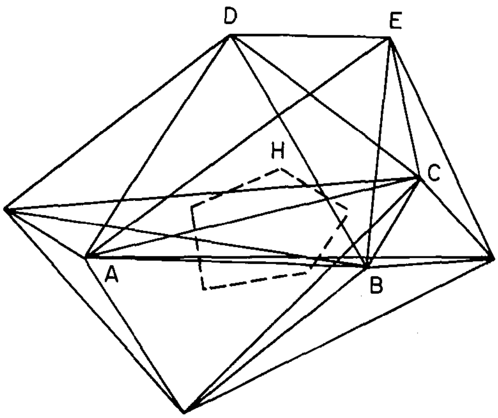


Fig. 5.

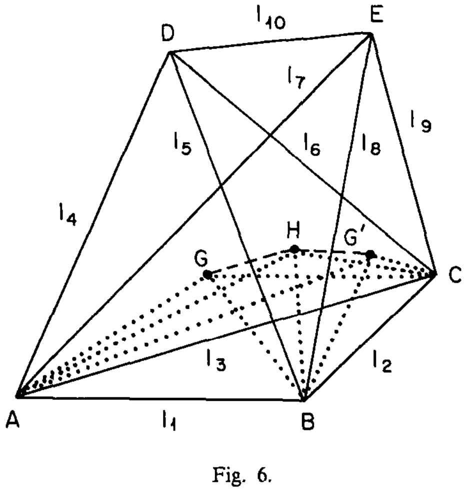


Fig. 6.

Then the volume in ABCDE associated with ABC is the sum of the volumes of ABCGH and ABCG′H which we denote by $V_{123(456)}$ and $V_{123(789)}$ respectively, with, for example

$$V_{123(456)} = \frac{\Sigma_{123(456)}}{1769472 A_{123}^2 V_{123456}^2 V} \times \Big\{ 12 V_{123456}^2 \Sigma_{123(789)} - \Sigma_{123(456)} \times \big[ V_{123456}^2 \big( \Sigma_{123(789)} + \sigma_{123(789)} \big) - 256 V^2 A_{123}^2 \big]^{1/2} \Big\}, \tag{2.8}$$

where $V$ is the volume of ABCDE, and we have defined $\sigma_{\alpha\beta\gamma(\lambda\mu\nu)}$ by

$$\sigma_{\alpha\beta\gamma(\lambda\mu\nu)} = l_\alpha^2 l_\beta^2 l_\gamma^2 - l_\alpha^2 l_\nu^4 - l_\beta^2 l_\lambda^4 - l_\gamma^2 l_\mu^4 + l_\lambda^2 l_\mu^2 \Sigma_{\beta\alpha\gamma} + l_\mu^2 l_\nu^2 \Sigma_{\alpha\beta\gamma} + l_\nu^2 l_\lambda^2 \Sigma_{\alpha\gamma\beta}. \tag{2.9}$$

For completeness we list the area $A_{123}$ and the volumes $V_{123456}$ and $V$ mentioned above:

$$A_{123} = \tfrac{1}{4}\big[ -l_1^4 - l_2^4 - l_3^4 + 2\big( l_1^2 l_2^2 + l_2^2 l_3^2 + l_3^2 l_1^2 \big) \big]^{1/2}, \tag{2.10}$$

$$V_{123456} = \tfrac{1}{12}\big[ l_1^2 l_6^2 \big( -l_1^2 - l_6^2 + l_2^2 + l_3^2 + l_4^2 + l_5^2 \big) + l_2^2 l_4^2 \big( -l_2^2 - l_4^2 + l_1^2 + l_3^2 + l_5^2 + l_6^2 \big) + l_3^2 l_5^2 \big( -l_3^2 - l_5^2 + l_1^2 + l_2^2 + l_4^2 + l_6^2 \big) - l_1^2 l_2^2 l_3^2 - l_1^2 l_4^2 l_5^2 - l_2^2 l_5^2 l_6^2 - l_3^2 l_4^2 l_6^2 \big]^{1/2}, \tag{2.11}$$

$$V = \tfrac{1}{48}\sqrt{\tfrac{1}{2}}\, \Big\{ 128\big[ A_{123}^2 A_{4710}^2 + A_{178}^2 A_{346}^2 + A_{145}^2 A_{379}^2 \big] - 16 l_1^2 l_3^2 l_4^2 l_7^2 + 2\big[ l_1^2 \Sigma_{364} \Sigma_{397} \Sigma_{4107} + l_3^2 \Sigma_{154} \Sigma_{187} \Sigma_{4107} + l_4^2 \Sigma_{123} \Sigma_{187} \Sigma_{397} + l_7^2 \Sigma_{123} \Sigma_{154} \Sigma_{364} \big] - \big[ \Sigma_{123} \Sigma_{154} \Sigma_{397} \Sigma_{4107} + \Sigma_{123} \Sigma_{187} \Sigma_{364} \Sigma_{4107} + \Sigma_{154} \Sigma_{187} \Sigma_{364} \Sigma_{397} \big]^{1/2} \Big\}. \tag{2.12}$$

Of course these may be written as determinants [5], but we have expanded them here to make the symmetries manifest in (2.10) and (2.11).

Before writing our explicit formula for $\frac{1}{4}\int d^d x R^2$, we should comment on one further point. If the integral of $R$ is represented by $\Sigma_h A_h \delta_h$, then there is some ambiguity in deciding how to represent $R^2$—should it involve $\Sigma_{hh'} A_h A_{h'} \delta_h \delta_{h'}$, or merely $\Sigma_h A_h^2 \delta_h^2$ without any cross terms? Our claim is that to represent the square of the *scalar* curvature, we do not need the cross terms. (This point will be discussed further in sect. 4.)

Our expression for the cosmological constant term involves the total volume of the simplicial complex. This may be written as

$$\text{total volume} = \sum_{d\text{-simplices}} (\text{volume of } d\text{-simplices}), \tag{2.13}$$

or equivalently as

$$\text{total volume} = \sum_{\text{hinges h}} V_{\text{h}}, \tag{2.14}$$

where $V_{\text{h}}$ is the volume associated with each hinge, as described above. Thus we may regard the invariant volume element $\sqrt{g}\,\mathrm{d}^d x$ as being represented by $V_{\text{h}}$ when one performs the sum over hinges

$$\int \mathrm{d}^d x \sqrt{g} \rightarrow \sum_{\text{hinges h}} V_{\text{h}}. \tag{2.15}$$

This means that we must regard the scalar curvature $R$ as being represented at each hinge by $2A_{\text{h}}\delta_{\text{h}}/V_{\text{h}}$, which is then consistent with eq. (2.1)

$$\tfrac{1}{2}\int \mathrm{d}^d x \sqrt{g}\, R \rightarrow \sum_{\text{hinges h}} V_{\text{h}} \frac{A_{\text{h}}\delta_{\text{h}}}{V_{\text{h}}} = \sum_{\text{hinges h}} A_{\text{h}}\delta_{\text{h}}. \tag{2.16}$$

It is then straightforward to see that

$$\frac{1}{2^n}\int \mathrm{d}^d x \sqrt{g}\, R^n \rightarrow \sum_{\text{hinges h}} V_{\text{h}} \left(\frac{A_{\text{h}}\delta_{\text{h}}}{V_{\text{h}}}\right)^n. \tag{2.17}$$

In particular, for $n = 2$, we have the equivalent form

$$\tfrac{1}{4}\int \mathrm{d}^d x \sqrt{g}\, R^2 \rightarrow \sum_{\text{hinges h}} \frac{A_{\text{h}}^2\delta_{\text{h}}^2}{V_{\text{h}}}, \tag{2.18}$$

which is the main formula in this paper. Note that our definition for $R$ is closely related to the conventional definition of sectional curvature [20].

At this stage it is useful to interpret our formulae in terms of the parallel transport of a test vector round a small loop. Consider a closed path $\Gamma$ encircling a hinge h and passing through each of the simplices that meet at that hinge. In particular we may take $\Gamma$ to be the boundary of the polyhedral area surrounding the hinge in figs. 1, 3 and 5.

For each neighboring pair of simplices $j$, $j+1$, we may write down a Lorentz transformation $L_\mu^{\nu}$, which describes how a given vector $\varphi_\mu$ transforms between the local coordinate systems in these two simplices [19]:

$$\varphi'_\mu = [L(j, j+1)]_\mu^{\ \nu} \varphi_\nu . \tag{2.19}$$

(Note that it is possible to choose coordinates so that $L_\mu^\nu$ is the unit matrix for one pair of simplices, but it will not then be unity for other pairs.) The Lorentz transformation is related to the path-ordered (P) exponential of the integral of the connection $(\Gamma_\lambda)_\mu^{\ \nu} = \Gamma^\nu_{\mu\lambda}$ by

$$L_\mu^{\ \nu} = \left[ \mathrm{P} \exp\left( \int_{\substack{\text{path} \\ \text{between simplices}}} \Gamma_\lambda \mathrm{d}x^\lambda \right) \right]_\mu^{\ \nu} . \tag{2.20}$$

The connection here has support only on the common interface between the two simplices. The product of these Lorentz transformations around a closed elementary Wilson loop $\Gamma$ is then given, for smooth enough manifolds, by

$$\left[ \prod_{\substack{\text{pairs of} \\ \text{simplices on } \Gamma}} L(j, j+1) \right]_\mu^{\ \nu} \approx \left[ \mathrm{e}^{R_{\cdot\rho\sigma} \Sigma^{\rho\sigma}} \right]_\mu^{\ \nu}, \tag{2.21}$$

where $(R_{\cdot\rho\sigma})_\mu^{\ \nu} = R^\nu_{\ \mu\rho\sigma}$ is the curvature tensor and $\Sigma^{\rho\sigma}$ is a bivector in the plane of $\Gamma$, with magnitude equal to $\sqrt{\frac{1}{2}}$ times the area of the loop $\Gamma$. (For a parallelogram with edges $a^\rho$ and $b^\rho$, $\Sigma^{\rho\sigma} = \frac{1}{2}(a^\sigma b^\rho - a^\rho b^\sigma)$.)

The total change in a vector $\varphi_\mu$ which undergoes parallel transport around $\Gamma$ is then given by

$$\varphi'_\mu = \varphi_\mu + \delta\varphi_\mu = \left[ \prod_{\substack{\text{pairs of} \\ \text{simplices on } \Gamma}} L(j, j+1) \right]_\mu^{\ \nu} \varphi_\nu , \tag{2.22}$$

which reproduces to lowest order the usual parallel transport formula

$$\delta\varphi_\mu = R^\nu_{\ \mu\rho\sigma} \Sigma^{\rho\sigma} \varphi_\nu . \tag{2.23}$$

On the Regge skeleton the effect of parallel transport around $\Gamma$ is described by

$$\left[ \prod_j L(j, j+1) \right]_{\mu\nu} = \left[ e^{\delta_\mathrm{h} U^{(\mathrm{h})}} \right]_{\mu\nu} , \tag{2.24}$$

where $U^{(\mathrm{h})}_{\mu\nu}$ is a bivector orthogonal to the hinge h, defined in 4 dimensions by

$$U^{(\mathrm{h})}_{\mu\nu} = \frac{1}{2A_\mathrm{h}} \varepsilon_{\mu\nu\rho\sigma} l^\rho_{(\mathrm{a})} l^\sigma_{(\mathrm{b})} , \tag{2.25}$$

and $l^{\rho}_{(a)}$ and $l^{\sigma}_{(b)}$ are the vectors forming two sides of the hinge h. It is these parallel transporters around closed elementary loops that satisfy the lattice analogues of the Bianchi identities [2, 3].

Comparison of eqs. (2.21) and (2.24) means that we may make the identification

$$R_{\mu\nu\rho\sigma}\Sigma^{\rho\sigma} \to \delta_{\mathrm{h}} U^{(\mathrm{h})}_{\mu\nu} . \tag{2.26}$$

It is important to notice here that this relation does not give complete information about the Riemann tensor, but only about its projection in the plane of the loop $\Gamma$ orthogonal to the hinge. Thus in going from (2.25) to Regge's expression for the Riemann tensor at the hinge,

$$R^{(\mathrm{h})}_{\mu\nu\rho\sigma} = \rho_{\mathrm{h}} \delta_{\mathrm{h}} U^{(\mathrm{h})}_{\mu\nu} U^{(\mathrm{h})}_{\rho\sigma} , \tag{2.27}$$

where $\rho_{\mathrm{h}}$ is the "density of hinges", we are possibly neglecting certain terms in $R_{\mu\nu\rho\sigma}$ which vanish when projected in the plane of $\Gamma$. (This is a reason why we meet problems if we try to use (2.27) as a formula giving full information about $R_{\mu\nu\rho\sigma}$ in Regge calculus. This point is further discussed in sect. 4.) Note however that (2.27) does have all the correct symmetries of the Riemann tensor.

Using eq. (2.27) in (2.26), we find that $\rho_{\mathrm{h}}$ must satisfy

$$\rho_{\mathrm{h}} U^{(\mathrm{h})}_{\rho\sigma} \Sigma^{\rho\sigma} = 1 . \tag{2.28}$$

Now $U^{(\mathrm{h})}_{\rho\sigma}$ and $\Sigma^{\rho\sigma}$ are both in the plane orthogonal to the hinge and their product is proportional to the area of the loop $\Gamma$. Thus the factor $\rho_{\mathrm{h}}$ must be equal, up to a numerical factor, to the inverse of the area of the loop dual to the hinge h. This last quantity in turn is proportional to the area of the hinge $A_{\mathrm{h}}$ divided by the volume $V_{\mathrm{h}}$ associated with it, which means that we may write (2.27) as

$$R^{(\mathrm{h})}_{\mu\nu\rho\sigma} = \frac{A_{\mathrm{h}}\delta_{\mathrm{h}}}{V_{\mathrm{h}}} U^{(\mathrm{h})}_{\mu\nu} U^{(\mathrm{h})}_{\rho\sigma} . \tag{2.29}$$

This leads to

$$R^{(\mathrm{h})} = \frac{2A_{\mathrm{h}}\delta_{\mathrm{h}}}{V_{\mathrm{h}}} , \tag{2.30}$$

which agrees with the form we have used for $R$ in eqs. (2.16), (2.17), and shows that we have chosen the numerical factor correctly in (2.29).

Before applying the expressions (2.15)–(2.18) to particular simplicial decompositions, we must discuss an important point. In theory it is possible for the volume $V_{\mathrm{h}}$ to become zero or even negative. To see this in two dimensions note firstly that the point equidistant from the vertices of a triangle need not be inside the triangle (see fig. 7).

In this case, the areas of the shaded triangles (each called $A_3$ in fig. 2) are actually negative. (This is essential to ensure that the total area $2(A_1 + A_2 + A_3)$ equals the area of the original triangle.) Now consider a network of triangles, with two such "elongated" triangles at a particular vertex, as shown in fig. 8. The polyhedron associated with vertex A has a part which "crosses over" itself. This triangular part will give a negative contribution to the area. One can imagine constructions (with the elongated triangles becoming narrower and narrower) where the negative contribution becomes larger than the positive one, and the $V_h$ becomes negative. Similar effects can occur in higher dimensions.

One might be tempted to deal with this situation by imposing the condition that $V_h$ should always be positive (in analogy with the requirement that the triangle

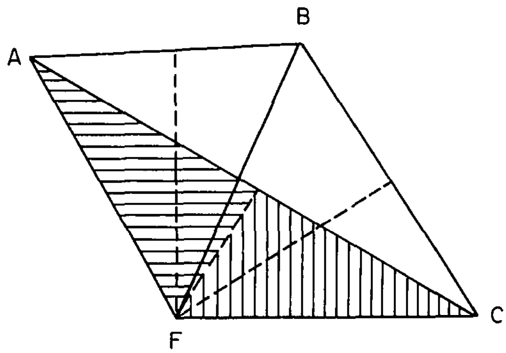


Fig. 7.

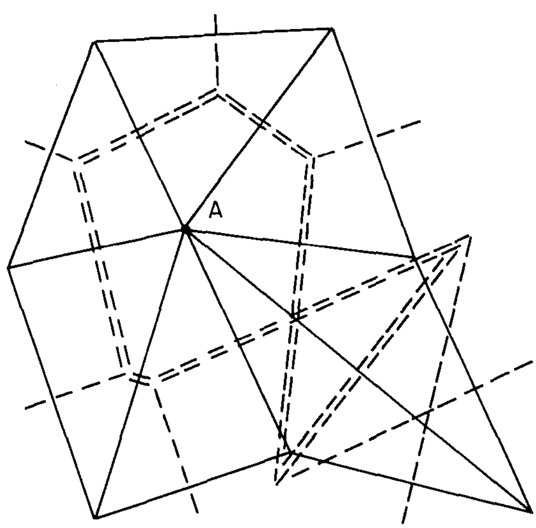


Fig. 8.

inequalities and the higher-dimensional analogues be satisfied for the original lattice). However it turns out that this is not necessary. If one allows the edge lengths to vary by large discrete amounts, the volumes $V_{\mathrm{h}}$ for some hinge could become negative, but if the edge lengths vary only smoothly, then as the volume $V_{\mathrm{h}}$ for some hinge tends toward zero (from above), the contribution from such a configuration to the path integral is exponentially suppressed by its small Boltzmann factor $\exp(-I_{\mathrm{R}})$. Thus the form of the $R^2$ term provides an infinite barrier to prevent any volume $V_{\mathrm{h}}$ from becoming negative.

## 3. Results for the regular tessellations of $S^2$, $S^3$ and $S^4$

We now study how the formulae (2.15), (2.16) and (2.18) compare with the continuum values for the regular tessellations of the two-sphere, the three-sphere and the four-sphere. (These correspond to the regular polyhedra in three, four and five dimensions [21].) Note that for regular tessellations, the volumes $V_{\mathrm{h}}$ take a very simple form since each $d$-dimensional simplex has its volume divided into $p$ equal parts, where $p$ is the number of hinges per simplex. If $q$ $d$-simplices meet at each hinge, then $V_{\mathrm{h}}$ is just the sum of $q$ of these contributions:

$$V_{\mathrm{h}} = \frac{q}{p} V, \tag{3.1}$$

where $V$ is the volume of the $d$-simplex. Then in 2 dimensions, $\{p, q\}$, with $p$ and $q$ as defined here, is just the Schläfli symbol. In 3 dimensions, the Schläfli symbol is $\{a, b, q\}$, where $\{a, b\}$ is the Schläfli symbol of the 2-dimensional simplex used to build the 3-dimensional ones. (Thus $\{a, b\}$ determines the value of $p$ as defined above.)

For regular tessellations, the dihedral angles $\vartheta_{\mathrm{d}}$ also take particularly simple forms. For example, the dihedral angle $\vartheta_{\mathrm{d}}$ at the $(d-2)$-dimensional hinge in a $d$-dimensional simplex satisfies

$$\cos \vartheta_{\mathrm{d}} = \frac{1}{d}. \tag{3.2}$$

(More generally such a dihedral angle may be worked out from the formula [22]

$$\sin \vartheta_{\mathrm{d}} = \frac{d}{d-1} \frac{V_d V_{d-2}}{V_{d-1} V'_{d-1}}, \tag{3.3}$$

where $V_{d-2}$ is the volume of the hinge, $V_d$ is the volume of the $d$-simplex, and $V_{d-1}$, $V'_{d-1}$ the volumes of the two $(d-1)$-dimensional faces that meet on the hinge.)

We now list the regular tessellations of $S^2$, $S^3$ and $S^4$ (note again that a tessellation of $S^n$ corresponds in Coxeter [21] to a regular polyhedron in $(n+1)$

dimensions). We give the Regge calculus formula for the total volume, $\frac{1}{2}\int d^dx\sqrt{g}R$ and $\frac{1}{4}\int d^dx\sqrt{g}R^2$ in each case, together with the numerical values, and compare them with the continuum value. We set the scale for each tessellation by requiring the edge lengths $l$ to give the same total volume as a sphere in that dimension, of radius $r$. (A list of the Regge calculus values for the Einstein action for the regular tessellations of $S^3$ is given by Warner [23]. We include our predictions of this quantity here both for the sake of completeness, and to correct some misprints in Warner's formulae.) The scalar curvature for $S^n$ is $n(n-1)/r^2$ [24].

Table 1 shows the simplicial lattice predictions for the various tessellations of $S^2$. In two dimensions the number of hinges is equal to the number of sites $N_0$. The second expression in the last column is the form taken by the first expression there when the length scale is set in the way described above. We see that for $S^2$, the Regge calculus equivalent of the Einstein action is exact, as indeed it must be by the Gauss-Bonnet theorem. Note that our Regge calculus expression for $\frac{1}{4}\int d^dx\sqrt{g}R^2$ is also exact in this case!

In table 2a we have listed the expressions for the regular tessellations of $S^3$. The length scale has been set in columns 4 and 5 by comparing volumes with those in the continuum case. The numerical values are shown in table 2b, and we see that as the number of vertices increases, the values of $\Sigma_h l_h\delta_h$ and $\Sigma_h l_h^2\delta_h^2/V_h$ tend clearly towards the continuum expression $\frac{1}{2}\int d^3x\sqrt{g}R$ and $\frac{1}{4}\int d^3x\sqrt{g}R^2$.

TABLE 1
Regular tessellations of $S^2$

| Tessellation | $N_0$ | Volume | $\sum_h \delta_h$ $\left(\equiv \frac{1}{2}\int d^2x\sqrt{g}R\right)$ | $\sum_h \frac{\delta_h^2}{V_h}$ $\left(\equiv \frac{1}{4}\int d^2x\sqrt{g}R^2\right)$ |
|---|---|---|---|---|
| tetrahedron $\alpha_3$ | 4 | $\sqrt{3}\,l^2$ | $4\pi$ | $\frac{16\pi^2}{\sqrt{3}\,l^2} \equiv \frac{4\pi}{r^2}$ |
| octahedron $\beta_3$ | 6 | $2\sqrt{3}\,l^2$ | $4\pi$ | $\frac{8\pi^2}{\sqrt{3}\,l^2} \equiv \frac{4\pi}{r^2}$ |
| cube $\gamma_3$ | 8 | $6l^2$ | $4\pi$ | $\frac{8\pi^2}{3l^2} \equiv \frac{4\pi}{r^2}$ |
| icosahedron | 12 | $5\sqrt{3}\,l^2$ | $4\pi$ | $\frac{16\pi^2}{5\sqrt{3}\,l^2} \equiv \frac{4\pi}{r^2}$ |
| dodecahedron | 20 | $\frac{3\cdot 5^{3/4}}{2^{3/2}}(1+\sqrt{5})^{3/2}l^2$ | $4\pi$ | $\frac{342\sqrt{2}\,\pi^2}{35^{3/4}(1+\sqrt{5})^{3/2}l^2} \equiv \frac{4\pi}{r^2}$ |
| continuum | | $4\pi r^2$ | $4\pi$ | $\frac{4\pi}{r^2}$ |

TABLE 2a
Regular tessellations of $S^3$

| Tessellation | $N_0$ | Volume | $\sum_h l_h \delta_h$ $\left(\equiv \frac{1}{2}\int d^3x\sqrt{g}\,R\right)$ | $\sum_h \frac{l_h^2 \delta_h^2}{V_h}$ $\left(\equiv \frac{1}{4}\int d^3x\sqrt{g}\,R^2\right)$ |
|---|---|---|---|---|
| 5-cell $\alpha_4$ | 5 | $\frac{5}{6}\sqrt{2}\,l^3$ | $10\sqrt{2}\left[\frac{6}{5}\pi^2\right]^{1/3} r \times(2\pi - 3\cos^{-1}\frac{1}{3})$ | $120\left[\frac{5}{6\pi^2}\right]^{1/3} r^{-1} \times(2\pi - 3\cos^{-1}\frac{1}{3})^2$ |
| 16-cell $\beta_4$ | 8 | $\frac{4}{3}\sqrt{2}\,l^3$ | $12\sqrt{2}\left[3\pi^2\right]^{1/3} r \times(2\pi - 4\cos^{-1}\frac{1}{3})$ | $\frac{432}{[3\pi^2]^{1/3}} r^{-1} \times(2\pi - 4\cos^{-1}\frac{1}{3})^2$ |
| tessaract $\gamma_4$ | 16 | $8l^3$ | $16\pi\left[\frac{1}{4}\pi^2\right]^{1/3} r$ | $32\pi^2\left[\frac{4}{\pi^2}\right]^{1/3} r^{-1}$ |
| 24-cell | 24 | $8\sqrt{2}\,l^3$ | $48\sqrt{2}\left[\frac{1}{2}\pi^2\right]^{1/3} r \times\left(2\pi - 6\cos^{-1}\sqrt{\frac{1}{3}}\right)$ | $1152\left[\frac{2}{\pi^2}\right]^{1/3} r^{-1} \times\left(2\pi - 6\cos^{-1}\sqrt{\frac{1}{3}}\right)^2$ |
| 600-cell | 120 | $50\sqrt{2}\,l^3$ | $720\sqrt{2}\left[\frac{1}{100}\pi^2\right]^{1/3} r \times(2\pi - 5\cos^{-1}\frac{1}{3})$ | $5184\left[\frac{100}{\pi^2}\right]^{1/3} r^{-1} \times(2\pi - 5\cos^{-1}\frac{1}{3})^2$ |
| 120-cell | 600 | $\frac{15}{4}\sqrt{5}\,(1+\sqrt{5})^4 l^3$ | $480\sqrt{5}\left[\frac{\pi^2}{3(1+\sqrt{5})^4}\right]^{1/3} r \times\left(2\pi - 6\cos^{-1} \times\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{1}{5}}\,1\right)}\right)$ | $192000\left[\frac{3(1+\sqrt{5})^4}{\pi^2}\right]^{1/3} r \times\left(2\pi - 6\cos^{-1} \times\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{1}{5}}\right)}\right)^2$ |
| continuum | | $2\pi^2 r^3$ | $6\pi^2 r$ | $\frac{18\pi^2}{r}$ |

TABLE 2b
Numerical results for the tessellations of $S^3$

| Tessellation | Sites $N_0$ | Hinges $N_1$ | $\sum_h l_h \delta_h$ $\left(\equiv \frac{1}{2}\int d^3x\sqrt{g}\,R\right)$ | $\sum_h \frac{l_h^2 \delta_h^2}{V_h}$ $\left(\equiv \frac{1}{4}\int d^3x\sqrt{g}\,R^2\right)$ |
|---|---|---|---|---|
| 5-cell | 5 | 10 | $8.461\pi^2 r$ | $35.789\pi^2 r^{-1}$ |
| 16-cell | 8 | 24 | $7.231\pi^2 r$ | $26.144\pi^2 r^{-1}$ |
| tessaract | 16 | 32 | $6.880\pi^2 r$ | $23.681\pi^2 r^{-1}$ |
| 24-cell | 24 | 96 | $6.455\pi^2 r$ | $20.836\pi^2 r^{-1}$ |
| 600-cell | 120 | 720 | $6.121\pi^2 r$ | $18.735\pi^2 r^{-1}$ |
| 120-cell | 600 | 1200 | $6.077\pi^2 r$ | $18.467\pi^2 r^{-1}$ |
| continuum | | | $6\pi^2 r$ | $18\pi^2 r^{-1}$ |

TABLE 3a
Regular tessellations of $S^4$

| Tessellation | $N_0$ | Volume | $\sum_h A_h\delta_h$ $\left(\equiv \frac{1}{2}\int d^4x\sqrt{g}\,R\right)$ | $\sum_h \frac{A_h^2\delta_h^2}{V_h}$ $\left(\equiv \frac{1}{4}\int d^4x\sqrt{g}\,R^2\right)$ |
|---|---|---|---|---|
| regular simplex $\alpha_5$ | 6 | $\frac{1}{16}\sqrt{5}\,l^4$ | $5^{3/4}8\sqrt{2}\,\pi r^2(2\pi - 3\cos^{-1}\frac{1}{4})$ | $240\sqrt{5}\,(2\pi - 3\cos^{-1}\frac{1}{4})^2$ |
| cross polytope $\beta_5$ | 10 | $\frac{1}{3}\sqrt{5}\,l^4$ | $5^{3/4}8\sqrt{6}\,\pi r^2(2\pi - 4\cos^{-1}\frac{1}{4})$ | $720\sqrt{5}\,(2\pi - 4\cos^{-1}\frac{1}{4})^2$ |
| measure polytope $\gamma_5$ | 32 | $10l^4$ | $16\sqrt{\frac{5}{3}}\,\pi^2r^2$ | $160\pi^2$ |
| continuum | | $\frac{8}{3}\pi^2r^4$ | $16\pi^2r^2$ | $96\pi^2$ |

TABLE 3b
Numerical results for the tessellations of $S^4$

| Tessellation | Sites $N_0$ | Hinges $N_2$ | $\sum_h A_h\delta_h$ $\left(\equiv \frac{1}{2}\int d^4x\sqrt{g}\,R\right)$ | $\sum_h \frac{A_h^2\delta_h^2}{V_h}$ $\left(\equiv \frac{1}{4}\int d^4x\sqrt{g}\,R^2\right)$ |
|---|---|---|---|---|
| $\alpha_5$ | 6 | 20 | $28.04\pi^2r^2$ | $294.9\pi^2$ |
| $\beta_5$ | 10 | 80 | $21.08\pi^2r^2$ | $166.6\pi^2$ |
| $\gamma_5$ | 32 | 80 | $20.7\pi^2r^2$ | $160\pi^2$ |
| continuum | | | $16\pi^2r^2$ | $96\pi^2$ |

We have also found that for both operators the approach to the continuum is very close to an $N_0^{-2/3}$ (or $N_1^{-2/3}$) behavior. These results are further evidence that our formula (2.18) is indeed the appropriate representation for an $R^2$ term in the lattice action.

The Regge calculus expressions for the regular tessellations of $S^4$ are shown in table 3a, again with the length scales set in column 4 by equating the volumes with those in the continuum. The convergence of the numerical results in table 3b is not as impressive as for $S^3$. The problem lies in the fact that there are no other regular tessellations of $S^4$, and ones with 6, 10 or even 32 vertices are certainly very crude approximations to the continuum. Hence we cannot expect in this case strong evidence from the regular tessellations on the convergence to the continuum of our simplicial lattice expressions.

## 4. Other higher derivative terms in the lattice gravity action

So far we have discussed the inclusion of only one of many possible higher derivative terms in the Regge calculus action. The others usually considered are

$\int d^dx\sqrt{g}R_{\mu\nu}R^{\mu\nu}$, $\int d^dx\sqrt{g}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ and $\int d^dx\sqrt{g}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor, and finally the Euler characteristic and the Hirzebruch signature (defined in eqs. (1.3) and (1.4)). Let us consider these last two quantities first.

The Euler characteristic $\chi$ for a simplicial decomposition may be obtained from a particular case of the general formula for the analogue of the Lipschitz-Killing curvatures of smooth riemannian manifolds for piecewise flat spaces [18]. In two dimensions, the formula of Cheeger, Müller and Schrader reduces of course to

$$\chi=\frac{1}{2\pi}\sum_{\mathrm{h}}\delta_{\mathrm{h}}, \tag{4.1}$$

which is the exact equivalent of the Gauss-Bonnet theorem

$$\chi=\frac{1}{4\pi}\int d^2x\sqrt{g}R. \tag{4.2}$$

In four dimensions the formula becomes

$$\chi=\sum_{\sigma^0}\left[1-\sum_{\sigma^2\supset\sigma^0}(0,2)-\sum_{\sigma^4\supset\sigma^0}(0,4)+\sum_{\sigma^4\supset\sigma^2\supset\sigma^0}(0,2)(2,4)\right], \tag{4.3}$$

where $\sigma^i$ denotes an $i$-dimensional simplex and $(i, j)$ denotes the (internal) dihedral angle at an $i$-dimensional face of a $j$-dimensional simplex. Thus, for example, (0,2) is the angle at the vertex of a triangle and (2,4) is the dihedral angle at a triangle in a 4-simplex (Cheeger, Müller and Schrader normalize the angles so that the volume of a sphere in any dimension is one; thus planar angles are divided by $2\pi$, 3-dimensional solid angles by $4\pi$ and so on).

Of course there is a much simpler formula for the Euler characteristic of a simplicial complex

$$\chi=N_0-N_1+N_2-N_3+N_4, \tag{4.4}$$

where $N_i$ is the number of simplices of dimension $i$; this is equivalent, by the Dehn-Sommerville equations [25], to

$$\chi=N_0-\tfrac{1}{2}N_2+N_4. \tag{4.5}$$

However, it may turn out to be useful in quantum gravity calculations to have a formula for $\chi$ in terms of the angles (and hence of the edge lengths) of the simplicial decomposition. In practice, an obstacle to the use of (4.3) is that, as far as we know, there is no simple formula for (0,4), the solid angle at the vertex of a general 4-simplex. (This is equivalent to the long-standing problem of the volume of a spherical tetrahedron [26].) (For a regular 4-simplex, it can be shown that (0,4) =

$-\frac{1}{5}+(1/2\pi)\cos^{-1}\frac{1}{4}$.) Furthermore, there seems to be no equivalent formula for the Hirzebruch signature for a simplicial decomposition.

Before leaving the Euler characteristic let us remark that formula (4.3) does not appear to be bilinear in the deficit angles, as one would have expected from our general arguments about $R^2$ type terms. However this may be due to the fact that the Euler characteristic is a total divergence, and so this formula is probably equivalent in some sense to evaluating the surface integral of the curvature two-form times the connection one-forms.

Let us now look in more detail at the other possible higher derivative terms listed at the beginning of this section. In 2 and 3 dimensions the Weyl tensor vanishes. In fact in 2 dimensions the Riemann and Ricci tensors satisfy

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}=\tfrac{1}{2}R_{\mu\nu}R^{\mu\nu}=R^2, \tag{4.6}$$

so the $R^2$ term, which we have already written down, is the only possible term of dimension four. In 3 dimensions one has

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}=4R_{\mu\nu}R^{\mu\nu}+3R^2, \tag{4.7}$$

and so we need also to find an expression for $\int d^3x\sqrt{g}\,R_{\mu\nu}R^{\mu\nu}$. In 4 dimensions the Riemann tensor satisfies

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}=C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}+2R_{\mu\nu}R^{\mu\nu}-\tfrac{1}{3}R^2, \tag{4.8}$$

which means that we need a lattice expression also for the integral of $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ and $R_{\mu\nu}R^{\mu\nu}$. Moreover the Euler characteristic may be written in the form [1]

$$\chi=\frac{1}{32\pi^2}\int d^4x\sqrt{g}\,C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}+\frac{1}{48\pi^2}\int d^4x\sqrt{g}\left(R^2-3R_{\mu\nu}R^{\mu\nu}\right). \tag{4.9}$$

Combined with (4.3) and (2.18), this means that we need to write a Regge calculus formula for only *one* of $\int d^4x\sqrt{g}\,C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ and $\int d^4x\sqrt{g}\,R_{\mu\nu}R^{\mu\nu}$. A term of this form involves information about the curvature in different directions, so we would expect there to be cross-terms in the lattice formula, involving contributions from different neighboring hinges (in analogy with the situation for $F_{\mu\nu}\tilde{F}^{\mu\nu}$ in lattice gauge theories where one needs contributions from orthogonal plaquettes). In practice we could write down expressions of the form

$$\sum_{\langle h,h'\rangle}\frac{\delta_h A_h}{V_h^{\frac{1}{2}}}\frac{\delta_{h'}A_{h'}}{V_{h'}^{\frac{1}{2}}} \tag{4.10}$$

where $\langle h,h'\rangle$ means a sum over hinges h and h′ such that the corresponding

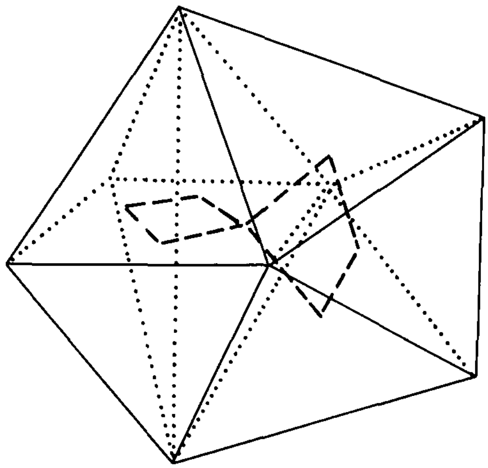

Fig. 9.

elementary parallel transport loops touch at one point on the dual lattice and do not have any edges in common (see fig. 9).

Another possibility is to construct suitable quantities based on Regge's approximate expression [2] in eq. (2.27) for the Riemann tensor at the hinge h, with the "density of hinges" $\rho_{\mathrm{h}}$ chosen suitably, as in eq. (2.29) say.

The naive use of (2.27) as an exact formula for the Riemann tensor can lead to problems. For example, if one uses it to evaluate the contribution to the Euler characteristic on each hinge one obtains zero, and it is therefore clear that one needs cross-terms involving contributions from different hinges. Even then it seems impossible to obtain the correct integer value for a particular simplicial decomposition by this method, and formula (4.3) has to be used. In spite of these difficulties, it may be possible to represent the $R_{\mu\nu}R^{\mu\nu}$ and $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ terms by expressions involving (2.27), with contributions from hinges meeting at a vertex but no edges in common. Note that since the value of (2.27) depends on the coordinate system used, we shall be able to consider, in this formulation, cross-terms only from those hinges which can be covered by the same coordinate system. We hope to return to this question in a future publication.

## 5. Prospects for calculations in quantum gravity

The formalism we have developed enables us to perform quantum gravity calculations based on the lattice lagrangian

$$I_{\mathrm{R}} = \sum_{\text{hinges h}} V_{\mathrm{h}} \left( a + b \frac{A_{\mathrm{h}} \delta_{\mathrm{h}}}{V_{\mathrm{h}}} + c \frac{A_{\mathrm{h}}^2 \delta_{\mathrm{h}}^2}{V_{\mathrm{h}}^2} \right), \tag{5.1}$$

which is equivalent, for smooth enough manifolds, to the continuum lagrangian

$$I = \int d^4x \sqrt{g}\left[a + \tfrac{1}{2}bR + \tfrac{1}{4}cR^2\right]. \tag{5.2}$$

We should remark that it is not clear to us at the present moment that the action of eq. (5.1) is *not* reflection positive [27] about some set of appropriate $(d-1)$-dimensional hyperplanes (appropriate in the sense that they have to contain the hyper-body-diagonals and satisfy other constraints). This in turn would lead to the possibility of constructing a positive self-adjoint transfer matrix of norm less than one (and hence a positive self-adjoint hamiltonian), and defining a unitary theory for finite lattice spacing.

Using a lattice which starts either as a regular tessellation of $S^n$ or as a hypercubical lattice, divided into simplices [3] by introducing the appropriate body and face diagonals in order to make the lattice rigid (see fig. 10 for the three-dimensional case), we allow the edge lengths to make small arbitrary variations. The new edge lengths are then accepted or rejected according to the probability distribution $\exp(-I_R)$. Such numerical work is in progress and the results will be presented in a forthcoming paper [28].

Some of the basic questions in higher derivative quantum gravity are the non-perturbative renormalization of the coupling constants such as $a, b, c\ldots,$ the structure of the phase diagram, the domain of attraction of the fixed points and the amount of necessary fine tuning of the bare couplings to get agreement with phenomenology. The technique known in the context of lattice gauge theories as block-spinning may be applied to study the renormalization of the different operators that arise on the lattice [29, 30]. The general idea is to look at the large scale behavior of the system by grouping together sets of neighboring simplices into larger block simplices. Expectation values of block quantities then provide information about the block correlations, and therefore, indirectly, about the block action, on a larger scale than the original action defined on the original lattice.

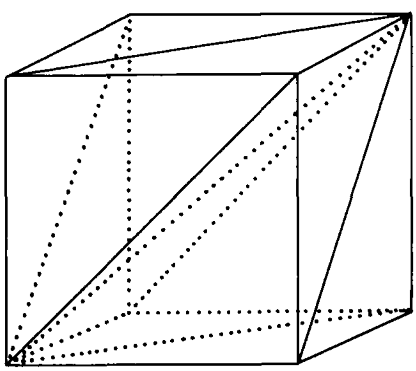

Fig. 10.

For a two-dimensional network of triangles topologically equivalent to a square lattice, with periodic boundary condition and divided up into triangles, we may replace groups of 4 triangles by the larger triangle shown in fig. 11. Among the possible block edge length definitions we have used, we mention the one that has the more direct physical interpretation as the geodesic distance

$$L_1 = l_1^2 + l_2^2 + 2 l_1 l_2 \cos\tfrac{1}{2}\delta , \tag{5.3}$$

where $\delta$ is the deficit angle at B. This is equivalent to replacing ABC by the geodesic distance between A and C. The construction of block spins in higher dimensions proceeds in analogous fashion.

In general one cannot have the luxury of restricting oneself to a small set of operators, and follow only the renormalization of those, since at each blocking new interaction can, at least temporarily, be generated. In principle one would have to add to the action terms that are more complicated than the ones we have written down (which always involve the parallel transport around an elementary closed Wilson loop) and contain information about the parallel transport around a loop entangling more than one hinge. (The simplest case is a loop in two dimensions encircling two sites and made out of ten dual links, for which again both a deficit angle and an area can be defined.)

The hope is that Wilson loops of arbitrary shape and increasing size will have rapidly increasing coefficients at the non-trivial fixed point(s) [29]. Technically one will also have to face the problem of reconstructing the block probability distributions from their moments (or cumulants), which are some appropriate functions of the block correlations. On the other hand some very accurate numerical results have been obtained for statistical mechanics systems using the real-space renormalization

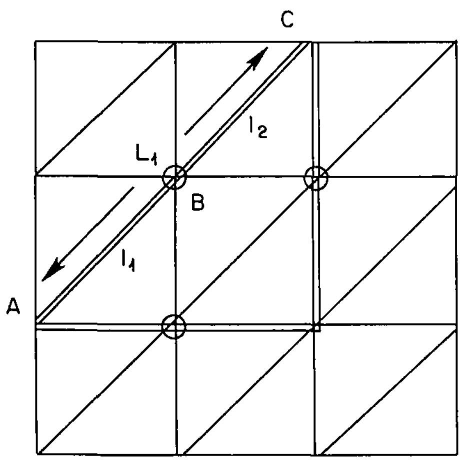


Fig. 11.

group approach with a reasonably small set of operators, and it is clear that the type of questions that one would like to ask in quantum gravity (the value of the renormalized cosmological constant, for example) are of a more general and basic nature.

We intend to use the methods described here to make predictions about the renormalization properties of the coupling constants in higher derivative gravity theories, as well as for the cosmological constant term.

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