

Quantum gravity in large dimensions

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Quantum gravity is investigated in the limit of a large number of space-time dimensions d , using as an ultraviolet regularization the simplicial lattice path integral formulation. In the weak field limit the appropriate expansion parameter is determined to be $1/d$. For the case of a simplicial lattice dual to a hypercube, the critical point is found at $k_c/\lambda = 1/d$ (with $k = 1/8\pi G$) separating a weak coupling from a strong coupling phase, and with $2d^2$ degenerate zero modes at k_c . The strong coupling, large G , phase is then investigated by analyzing the general structure of the strong coupling expansion in the large d limit. Dominant contributions to the curvature correlation functions are described by large closed random polygonal surfaces, for which excluded volume effects can be neglected at large d , and whose geometry we argue can be approximated by unconstrained random surfaces in this limit. In large dimensions the gravitational correlation length is then found to behave as $|\log(k_c - k)|^{1/2}$, implying for the universal gravitational critical exponent the value $\nu = 0$ at $d = \infty$.

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I. INTRODUCTION

The lack of perturbative renormalizability for quantum gravitation in physical dimensions [1–6] has brought to the forefront the need to develop field theoretic approximation schemes that do not rely on the assumption of weak gravitational fields, and which are sophisticated enough to deal with the rich physical structure of nonrenormalizable theories [7–11]. The hope is that more powerful covariant methods, better suited to the nonperturbative regime, will eventually shed some light on the elusive long-distance properties of quantum gravitation, which could ultimately have a bearing on a number of long-standing and fundamental issues, such as the short distance nature of space-time, the emergence of the semiclassical limit and the problem of large-scale quantum cosmology [12–15].

Approaches based on the simplicial lattice formulation for gravity [16–24], the $2 + \epsilon$ expansion [8,25–28] and approximate renormalization-group methods based on Wilson’s momentum slicing technique [29,30] have suggested the existence of a nontrivial ultraviolet fixed point in and around four dimensions, separating a weakly coupled (but physically unattractive) phase from a strongly coupled one, the latter phase being characterized by a finite invariant correlation length, and close to smooth geometries at large distances. Substantial uncertainties remain in each of the three approximation methods mentioned above, both about the results themselves and their relationship to each

other, but also regarding their ultimate physical significance and how they might relate to physical gravitational phenomena, and both early and late time cosmology. It would be clearly desirable if one could find a limit in the quantum gravity case where nonperturbative aspects of the theory could be fully explored by covariant analytical means. In the significantly simpler Yang-Mills case, the evidence so far is that the lattice is the only reliable nonperturbative method, capable of producing reasonably unambiguous quantitative results, within a controlled approximation based on the zero lattice spacing limit. It will be this method that will be therefore the focus of our work.

In this paper we study a set of approximation methods based on an expansion in the inverse number of dimensions. Increasing the number of space-time dimensions above four only worsens the renormalizability problem, which implies that the need for a nonperturbative approach, such as the lattice one, becomes even more acute. The so-called $1/d$ expansion was originally developed for statistical mechanics systems, and later extended to the study of quantum field theory, where it has since met with a number of considerable successes, including an understanding of triviality for scalar field theories above four dimensions (which, incidentally, are not perturbatively renormalizable for any $d > 4$). The above expansion is known to be intimately tied up with the mean field theory treatment of quantum mechanical systems, but not necessarily equivalent to it (as was already noted in the gauge theory case), and exploits the fact that in large dimensions each point is typically surrounded by many neighbors, whose action can then be either treated exactly, or included

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as some sort of local average. For classical spin systems at finite temperature, the $1/d$ expansion was originally developed in [31–33] by examining the structure of the high temperature expansion.

In many ways the $1/d$ expansion is similar to the very successful $1/N$ expansion for statistical mechanics systems [the $O(N)$ vector model being one thoroughly explored and well-understood example [34,35]] and $SU(N)$ gauge theories, where it leads to the planar diagram approximation [36,37] and the many phenomenological successes that follow from it. In the gravitational case it is less obvious how to attach color degrees (or any other internal degree) of freedom to the graviton, so this particular avenue seems unfruitful at the moment.

In this paper we will study large-dimensional pure gravitation, without any matter fields, which could then be added at a later stage. We recall here that for pure gravity in d dimensions there are $d(d+1)/2$ independent components of the metric, and the same number of algebraically independent components of the Ricci tensor. The contracted Bianchi identities then reduce the count by d , and so does general coordinate invariance, leaving $d(d+1)/2 - d - d = d(d-3)/2$ physical gravitational degrees of freedom in d dimensions. As a result, the number of physical degrees of freedom of the gravitational field grows rather rapidly (quadratically) with the number of dimensions.

The paper is organized as follows. In Sec. II we discuss the machinery of the $1/d$ expansion for the lattice theory of gravity based on Regge's simplicial construction. The action simplifies considerably in the large d limit, and we are able to exhibit the location of the critical point in the variable $k = 1/8\pi G$, at least in the weak field limit, as well as the nature of the excitation spectrum around it. In Sec. III we follow a complementary route to the large d limit, where we perform a simultaneous $1/d$ and strong coupling (small k) expansion. Since the strong coupling expansion for simplicial lattice gravity has not been discussed before in the literature, we will present here some general aspects of it. We then show how the relevant (in the long-distance limit) critical behavior can be extracted from the strong coupling expansion by analyzing the geometric structure of its dominant terms. In Sec. IV we provide some contact with results obtained in the continuum in and above $d = 4$, and compare and contrast with what has been found in the previous two sections from the simplicial lattice theory. Appendix A contains a brief summary of the large d limit for scalar lattice field theories, while Appendix B discusses some results relevant to non-Abelian gauge fields on the lattice.

II. EXPANSION IN INVERSE POWERS OF THE DIMENSION

Our first concern will be an approximate evaluation, in the large d limit, of an appropriately discretized form of the

continuum Euclidean functional integral for pure gravity without matter, which we write here as

$$Z_{cont} = \int [dg_{\mu\nu}] \exp\left(-\lambda \int d^d x \sqrt{g} + \frac{1}{16\pi G} \int d^d x \sqrt{g} R\right). \quad (2.1)$$

In the following we will therefore first address the key issue of precisely what type of terms in the discrete action, based on the simplicial lattice formulation [16], become dominant in this limit.

A. General formulas in d dimensions

We will consider here a general simplicial lattice in d dimensions, made out of a collection of flat d -simplices glued together at their common faces so as to constitute a triangulation of a smooth continuum manifold, such as the d -torus or the surface of a sphere. Each simplex is endowed with $d+1$ vertices, and its geometry is completely specified by assigning the lengths of its $d(d+1)/2$ edges. We will label the vertices by $1, 2, 3, \dots, d+1$ and denote the square edge lengths by $l_{12}^2 = l_{21}^2, \dots, l_{1,d+1}^2$. The vertices of the simplex can be specified by a set of vectors $\mathbf{v}_1 = 0, \mathbf{v}_2, \dots, \mathbf{v}_{d+1}$. The matrix

$$g_{ij} = \langle \mathbf{v}_{i+1} | \mathbf{v}_{j+1} \rangle, \quad (2.2)$$

with $1 \leq i, j \leq d$, is positive definite, and, in terms of the edge lengths $l_{ij} = |\mathbf{v}_i - \mathbf{v}_j|$, it is given by

$$g_{ij} = \frac{1}{2}(l_{1,i+1}^2 + l_{1,j+1}^2 - l_{i+1,j+1}^2). \quad (2.3)$$

The volume of a d -simplex is then given by the d -dimensional generalization of the well-known formula for a tetrahedron

$$V_d = \frac{1}{d!} \sqrt{\det g_{ij}}. \quad (2.4)$$

An equivalent form can be given in terms of a determinant of a $(d+2) \times (d+2)$ matrix,

$$V_d = \frac{(-1)^{(d+1)/2}}{d! 2^{d/2}} \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & l_{12}^2 & \dots & l_{1,d+1}^2 \\ 1 & l_{21}^2 & 0 & \dots & l_{2,d+1}^2 \\ 1 & l_{31}^2 & l_{32}^2 & \dots & l_{3,d+1}^2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & l_{d+1,1}^2 & l_{d+1,2}^2 & \dots & 0 \end{vmatrix}^{1/2}. \quad (2.5)$$

Then the dihedral angle in a d -dimensional simplex of volume V_d , between faces of volume V_{d-1} and V'_{d-1} , is obtained from

$$\sin \theta_d = \frac{d}{d-1} \frac{V_d V_{d-2}}{V_{d-1} V'_{d-1}}. \quad (2.6)$$

In the equilateral case we record here the particularly simple result for the volume of a simplex

$$V_d = \frac{1}{d!} \sqrt{\frac{d+1}{2^d}}, \quad (2.7)$$

and for the dihedral angle

$$\cos \theta_d = \frac{1}{d}. \quad (2.8)$$

The d -dimensional Euclidean lattice action, involving cosmological constant and scalar curvature terms, is then given by

$$I(l^2) = \lambda \sum V_d - k \sum \delta_d V_{d-2} \quad (2.9)$$

(where δ_d is the d -dimensional deficit angle, $\delta_d = 2\pi - \sum_{\text{simplices}} \theta_d$) and appears in the partition function as

$$Z(\lambda, k) = \int [dl^2] \exp(-I(l^2)). \quad (2.10)$$

B. Weak field expansion

The above formulas for volumes and angles are quite complicated in the general case, and therefore of limited use in large dimensions. The next step consists in expanding them out in terms of small edge length variations,

$$l_{ij}^2 = l_{ij}^{(0)2} + \delta l_{ij}^2. \quad (2.11)$$

We will set for convenience from now on $\delta l_{ij}^2 = \epsilon_{ij}$. Unless stated otherwise, we will be considering the expansion about the equilateral case, and set $l_{ij}^{(0)} = 1$ (we will later relax this last restriction). Furthermore one has the well-known expansion for determinants

$$\begin{aligned} \det(1 + M) &= e^{\text{tr} \ln(1+M)} \\ &= 1 + \text{tr} M + \frac{1}{2!} [(\text{tr} M)^2 - \text{tr} M^2] + \frac{1}{3!} \\ &\quad \times [(\text{tr} M)^3 - 3 \text{tr} M \text{tr} M^2 + 2 \text{tr} M^3] + \dots \end{aligned} \quad (2.12)$$

One can then rewrite the expression in Eq. (2.5) for the volume of a d -simplex as

$$V_d = \frac{(-1)^{(d+1)/2}}{d! 2^{d/2}} \sqrt{\det M_d}, \quad (2.13)$$

and expanding out to quadratic order one finds

$$\begin{aligned} \sqrt{-\det M_2} &= \sqrt{3} + \frac{1}{\sqrt{3}} \epsilon_{12} + \dots + \frac{2}{3\sqrt{3}} \epsilon_{12} \epsilon_{13} + \dots \\ &\quad - \frac{2}{3\sqrt{3}} \epsilon_{12}^2 + \dots, \end{aligned} \quad (2.14)$$

$$\begin{aligned} \sqrt{\det M_3} &= \sqrt{4} + \frac{1}{\sqrt{4}} \epsilon_{12} + \dots + \frac{3}{4\sqrt{4}} \epsilon_{12} \epsilon_{13} + \dots \\ &\quad - \frac{1}{4\sqrt{4}} \epsilon_{12} \epsilon_{34} + \dots - \frac{9}{2 \cdot 4\sqrt{4}} \epsilon_{12}^2 + \dots \end{aligned} \quad (2.15)$$

and for general d

$$\begin{aligned} \frac{1}{\sqrt{d+1}} \sqrt{\pm \det M_d} &= 1 + \frac{1}{d+1} \epsilon_{12} + \frac{d}{(d+1)^2} \epsilon_{12} \epsilon_{13} \\ &\quad - \frac{1}{(d+1)^2} \epsilon_{12} \epsilon_{34} - \frac{d^2}{2(d+1)^2} \epsilon_{12}^2 \\ &\quad + \dots + O(\epsilon^2). \end{aligned} \quad (2.16)$$

For large d the last expression simplifies to

$$\begin{aligned} \frac{1}{\sqrt{d+1}} \sqrt{\pm \det M_d} &= 1 + \frac{1}{d} (\epsilon_{12} + \dots) + \frac{1}{d} (\epsilon_{12} \epsilon_{13} + \dots) \\ &\quad - \frac{1}{d^2} (\epsilon_{12} \epsilon_{34} + \dots) - \frac{1}{2} (\epsilon_{12}^2 + \dots) \\ &\quad + O(\epsilon^3). \end{aligned} \quad (2.17)$$

Here the terms $\epsilon_{12} \epsilon_{13}$ refer to two edges sharing a common vertex, whereas the terms $\epsilon_{12} \epsilon_{34}$ denote terms with opposite edges, not sharing a common vertex.

As a result, the volume term appearing in the d -dimensional Euclidean lattice action of Eq. (2.9) becomes

$$\begin{aligned} V_d \underset{d \rightarrow \infty}{\sim} \frac{\sqrt{d}}{d! 2^{d/2}} \left\{ 1 + \frac{1}{d} (\epsilon_{12} + \dots) + \frac{1}{d} (\epsilon_{12} \epsilon_{13} + \dots) \right. \\ \left. - \frac{1}{d^2} (\epsilon_{12} \epsilon_{34} + \dots) - \frac{1}{2} (\epsilon_{12}^2 + \dots) + \dots \right\}, \end{aligned} \quad (2.18)$$

or, equivalently, ordering the terms in powers of $1/d$,

$$\begin{aligned} V_d \underset{d \rightarrow \infty}{\sim} \frac{\sqrt{d}}{d! 2^{d/2}} \left\{ 1 - \frac{1}{2} \epsilon_{12}^2 + \dots \right. \\ \left. + \frac{1}{d} (\epsilon_{12} + \dots + \epsilon_{12} \epsilon_{13} + \dots) + O\left(\frac{1}{d^2}\right) \right\}. \end{aligned} \quad (2.19)$$

To leading order, it involves a lattice sum over all squared edge length deviations. Note that the terms linear in ϵ (the so-called tadpole terms in the continuum), which would have required a shift in the ground state value of ϵ for a nonvanishing cosmological constant λ , vanish to leading order in $1/d$. The full volume term $\lambda \sum V_d$ appearing in the action can then be easily written down using the above expressions.

Next one needs to expand the dihedral angle. In the equilateral case one has for the dihedral angle

$$\theta_d = \arcsin \frac{\sqrt{d^2 - 1}}{d} \underset{d \rightarrow \infty}{\sim} \frac{\pi}{2} - \frac{1}{d} - \frac{1}{6d^3} + \dots, \quad (2.20)$$

which will require *four* simplices to meet on a hinge, to give a deficit angle of $2\pi - 4 \times \frac{\pi}{2} \approx 0$ in large dimen-

sions. One notes that in large dimensions the simplices look locally (i.e. at a vertex) more like hypercubes. Several d -dimensional simplices will meet on a $(d-2)$ -dimensional hinge, sharing a common face of dimension $d-1$ between adjacent simplices. Each simplex has $(d-2)(d-1)/2$ edges “on” the hinge, some more edges are then situated on the two “interfaces” between neighboring simplices meeting at the hinge, and finally one edge lies “opposite” to the hinge in question. In two dimensions, one finds for the dihedral angle at vertex 1, to

quadratic order,

$$\begin{aligned} \theta_2 = & \frac{\pi}{3} - \frac{1}{2\sqrt{3}}(\epsilon_{12} + \epsilon_{13}) + \frac{1}{\sqrt{3}}\epsilon_{23} + \frac{1}{12\sqrt{3}}(\epsilon_{12}^2 + \epsilon_{13}^2) \\ & + \frac{2}{3\sqrt{3}}\epsilon_{12}\epsilon_{13} - \frac{1}{3\sqrt{3}}(\epsilon_{12} + \epsilon_{13})\epsilon_{23} - \frac{1}{6\sqrt{3}}\epsilon_{23}^2, \end{aligned} \quad (2.21)$$

whereas in three dimensions one has for the dihedral angle at edge 12, to the same order,

$$\begin{aligned} \theta_3 = & \arcsin\frac{2\sqrt{2}}{3} + \frac{1}{3\sqrt{2}}\epsilon_{12} - \frac{1}{3\sqrt{2}}(\epsilon_{13} + \epsilon_{14} + \epsilon_{23} + \epsilon_{24}) + \frac{1}{\sqrt{2}}\epsilon_{34} + \frac{7}{72\sqrt{2}}\epsilon_{12}^2 - \frac{1}{72\sqrt{2}}(\epsilon_{13}^2 + \epsilon_{14}^2 + \epsilon_{23}^2 + \epsilon_{24}^2) \\ & - \frac{7}{36\sqrt{2}}\epsilon_{12}(\epsilon_{13} + \epsilon_{14} + \epsilon_{23} + \epsilon_{24}) - \frac{1}{4\sqrt{2}}(\epsilon_{13}\epsilon_{24} + \epsilon_{14}\epsilon_{23}) + \frac{3}{4\sqrt{2}}(\epsilon_{13}\epsilon_{14} + \epsilon_{23}\epsilon_{24}) + \frac{11}{36\sqrt{2}}(\epsilon_{13}\epsilon_{23} + \epsilon_{14}\epsilon_{24}) \\ & + \frac{1}{4\sqrt{2}}\epsilon_{12}\epsilon_{34} - \frac{1}{4\sqrt{2}}(\epsilon_{13} + \epsilon_{14} + \epsilon_{23} + \epsilon_{24})\epsilon_{34} - \frac{1}{8\sqrt{2}}\epsilon_{34}^2. \end{aligned} \quad (2.22)$$

In the general d -dimensional case the expansion coefficients for the dihedral angle at the hinge labeled by $1, 2, \dots, d-1$ are given by the following expressions (as well as their large d limit)

$$\begin{aligned} \frac{2}{d\sqrt{d^2-1}}\epsilon_{12} &\rightarrow \frac{2}{d^2}\epsilon_{12} & -\frac{d-1}{d\sqrt{d^2-1}}\epsilon_{1,d} &\rightarrow -\frac{1}{d}\epsilon_{1,d} & \frac{d-1}{\sqrt{d^2-1}}\epsilon_{d,d+1} &\rightarrow \epsilon_{d,d+1} \\ \frac{2(d^3-2d^2-d+1)}{d^2(d^2-1)^{3/2}}\epsilon_{12}^2 &\rightarrow \frac{2}{d^2}\epsilon_{12}^2 & -\frac{(d^2-2d-2)(d-1)^2}{2d^2(d^2-1)^{3/2}}\epsilon_{1,d}^2 &\rightarrow -\frac{1}{2d}\epsilon_{1,d}^2 \\ -\frac{(d-1)^2}{2(d^2-1)^{3/2}}\epsilon_{d,d+1}^2 &\rightarrow -\frac{1}{2d}\epsilon_{d,d+1}^2 & \frac{2(d^3-4d^2-d+2)}{d^2(d^2-1)^{3/2}}\epsilon_{12}\epsilon_{13} &\rightarrow \frac{2}{d^2}\epsilon_{12}\epsilon_{13} \\ -\frac{4(2d^2-1)}{d^2(d^2-1)^{3/2}}\epsilon_{12}\epsilon_{34} &\rightarrow -\frac{8}{d^3}\epsilon_{12}\epsilon_{34} & -\frac{(d-1)(d^3-2d^2+d+2)}{d^2(d^2-1)^{3/2}}\epsilon_{12}\epsilon_{1,d+1} &\rightarrow -\frac{1}{d}\epsilon_{12}\epsilon_{1,d+1} \\ \frac{2(d^2-d-1)}{d^2(d+1)\sqrt{d^2-1}}\epsilon_{34}\epsilon_{1,d+1} &\rightarrow \frac{2}{d^2}\epsilon_{34}\epsilon_{1,d+1} & \frac{2}{(d+1)\sqrt{d^2-1}}\epsilon_{12}\epsilon_{d,d+1} &\rightarrow \frac{2}{d^2}\epsilon_{12}\epsilon_{d,d+1} \\ \frac{d(d-1)}{(d+1)\sqrt{d^2-1}}\epsilon_{1,d}\epsilon_{1,d+1} &\rightarrow \epsilon_{1,d}\epsilon_{1,d+1} & \frac{(d-1)(3d+2)}{d^2(d+1)\sqrt{d^2-1}}\epsilon_{1,d}\epsilon_{3,d} &\rightarrow \frac{3}{d^2}\epsilon_{1,d}\epsilon_{3,d} \\ -\frac{(d-1)}{(d+1)\sqrt{d^2-1}}\epsilon_{1,d}\epsilon_{3,d+1} &\rightarrow -\frac{1}{d}\epsilon_{1,d}\epsilon_{3,d+1} & -\frac{(d-1)}{(d+1)\sqrt{d^2-1}}\epsilon_{1,d}\epsilon_{d,d+1} &\rightarrow -\frac{1}{d}\epsilon_{1,d}\epsilon_{d,d+1}. \end{aligned} \quad (2.23)$$

In the large d limit one then obtains, to leading order

$$\begin{aligned} \theta_d \underset{d \rightarrow \infty}{\sim} & \arcsin\frac{\sqrt{d^2-1}}{d} + \epsilon_{d,d+1} + \epsilon_{1,d}\epsilon_{1,d+1} + \dots \\ & + \frac{1}{d}\left(-\epsilon_{1,d} + \dots - \frac{1}{2}\epsilon_{1,d}^2 + \dots - \frac{1}{2}\epsilon_{d,d+1}^2\right. \\ & \left. - \epsilon_{12}\epsilon_{1,d+1} - \epsilon_{1,d}\epsilon_{3,d+1} - \epsilon_{1,d}\epsilon_{d,d+1} + \dots\right) \\ & + O\left(\frac{1}{d^2}\right). \end{aligned} \quad (2.24)$$

To evaluate the curvature term $-k\sum\delta_d V_{d-2}$ appearing in the gravitational lattice action, one needs the hinge volume V_{d-2} , which is easily obtained from Eq. (2.19), by reducing

$d \rightarrow d-2$,

$$\begin{aligned} V_{d-2} \underset{d \rightarrow \infty}{\sim} & \frac{2d^{3/2}(d-1)}{d!2^{d/2}} \left\{ 1 - \frac{1}{2}\epsilon_{12}^2 + \dots \right. \\ & \left. + \frac{1}{d}(\epsilon_{12} + \dots + \epsilon_{12}\epsilon_{13} + \dots) + O\left(\frac{1}{d^2}\right) \right\}, \end{aligned} \quad (2.25)$$

whereas the deficit angle δ is given by

$$\begin{aligned} \delta_d = & 2\pi - \sum_{\text{simplices}} \theta_d \\ = & 2\pi - \sum_{\text{simplices}} \left\{ \arcsin\frac{\sqrt{d^2-1}}{d} + \dots \right\}, \end{aligned} \quad (2.26)$$

with the expansion of the *arcsin* function given in Eq. (2.20).

C. Evaluation of the lattice action

We now specialize to the case where four simplices meet at a hinge. When expanded out in terms of the ϵ 's, one obtains for the deficit angle

$$\begin{aligned} \delta_d = & 2\pi - 4 \cdot \frac{\pi}{2} + \sum_{\text{simplices}} \frac{1}{d} - \epsilon_{d,d+1} + \cdots - \epsilon_{1,d}\epsilon_{1,d+1} \\ & + \cdots - \frac{1}{d} \left(-\epsilon_{1,d} - \frac{1}{2}\epsilon_{1,d}^2 - \frac{1}{2}\epsilon_{d,d+1}^2 - \epsilon_{12}\epsilon_{1,d+1} \right. \\ & \left. - \epsilon_{1,d}\epsilon_{3,d+1} - \epsilon_{1,d}\epsilon_{d,d+1} + \cdots \right) + O\left(\frac{1}{d^2}\right). \end{aligned} \quad (2.27)$$

The action contribution involving the deficit angle is then, for a single hinge,

$$\begin{aligned} -k\delta_d V_{d-2} = & (-k) \frac{2d^{3/2}(d-1)}{d!2^{d/2}} \left\{ 1 - \frac{1}{2}\epsilon_{12}^2 + \cdots \right\} \\ & \times \left\{ \frac{4}{d} + \cdots - \epsilon_{d,d+1} + \cdots - \epsilon_{1,d}\epsilon_{1,d+1} \right. \\ & \left. + \cdots \right\} \\ = & (-k) \frac{2d^{3/2}(d-1)}{d!2^{d/2}} (-\epsilon_{d,d+1} + \cdots \\ & - \epsilon_{1,d}\epsilon_{1,d+1} + \cdots). \end{aligned} \quad (2.28)$$

It involves two types of terms: one linear in the (single) edge opposite to the hinge, as well as a term involving a product of two distinct edges, connecting any hinge vertex to the two vertices opposite to the given hinge. Since there are four simplices meeting on one hinge, one will have 4 terms of the first type, and $4(d-1)$ terms of the second type. Combining the cosmological constant and the curvature contributions one then obtains

$$\begin{aligned} \frac{\sqrt{d}}{d!2^{d/2}} \left[\lambda \left(1 - \frac{1}{2}\epsilon_{12}^2 + \frac{\sigma}{4}\epsilon_{12}^4 + \cdots \right) \right. \\ \left. - k \cdot 2d(d-1)(-\epsilon_{d,d+1} + \cdots - \epsilon_{1,d}\epsilon_{1,d+1} + \cdots) \right]. \end{aligned} \quad (2.29)$$

The first term in the above expression refers to a single simplex, the second one to a single hinge. To obtain the total action, a sum over all simplices, respectively hinges, has still to be performed.

We have also added a term $\sigma\epsilon^4$ in order to impose a cutoff at large edge lengths $|\epsilon|$. The justification for this choice comes from the fact that numerical simulations show convincingly that very large, as well as very small, edge lengths are exponentially suppressed by the lattice gravitational measure, and, in particular, by a nontrivial

interplay between the λ term and the generalized triangle inequalities [20,21,24,38,39] (as such, σ is not really a parameter that one is allowed to vary, and should rather be fixed to some suitable numeric value). Dropping the irrelevant constant term and summing over edges, one obtains for the total action $\lambda \sum V_d - k \sum \delta_d V_{d-2}$ in the large d limit

$$\lambda \left(-\frac{1}{2} \sum \epsilon_{ij}^2 + \frac{\sigma}{4} \sum \epsilon_{ij}^4 \right) - 2kd^2 \left(-\sum \epsilon_{jk} - \sum \epsilon_{ij}\epsilon_{ik} \right), \quad (2.30)$$

up to an overall multiplicative factor $\sqrt{d}/d!2^{d/2}$, which will play no essential role in the following. The $\epsilon_{ij}\epsilon_{ik}$ coupling terms in the expression above can always of course be rewritten in terms of finite differences,

$$\epsilon_{ij}\epsilon_{ik} = -\frac{1}{2}(\epsilon_{ij} - \epsilon_{ik})^2 + \frac{1}{2}\epsilon_{ij}^2 + \frac{1}{2}\epsilon_{ik}^2, \quad (2.31)$$

and for smooth enough fields the first term on the right-hand side (r.h.s.) can be regarded as a discrete approximation to a derivative.

From the action in of Eq. (2.30), one notices that its form leads naturally to a first rough estimate for the critical point, defined as the point where the competing λ and curvature terms achieve comparable magnitudes, namely $k_c \sim \lambda/d^2$. This result will be further improved below when we perform an explicit calculation, which takes into account the actual number of neighbors for each point, given a specific choice of lattice and its associated coordination number [see Eq. (2.40)].

D. Action for the surface of the cross polytope

The next step involves the choice of a specific lattice on which the action is then evaluated. One possibility would be the hypercubic lattice, divided into simplices as originally discussed in [18]. This type of lattice has $2^d - 1$ edges emanating from each site in d dimensions.¹ Here we will evaluate the above action for the cross polytope β_{d+1} . The cross polytope β_n is the regular polytope in n dimensions corresponding to the convex hull of the points formed by permuting the coordinates $(\pm 1, 0, 0, \dots, 0)$, and has therefore $2n$ vertices. It is named so because its vertices are located equidistant from the origin, along the Cartesian axes in n -space. The cross polytope in n dimensions is bounded by 2^n $(n-1)$ -simplices, has $2n$ vertices and $2n(n-1)$ edges. In three dimensions, it represents the convex hull of the octahedron, while in four dimensions the cross polytope is the 16-cell [40]. In the general case it

¹This result should be compared to the $\sim d^2/2$ transverse-traceless degrees of freedom of the continuum gravitational field in d dimensions. The exponential growth for this particular lattice implies the existence of many redundant degrees of freedom in the large d limit. Amusingly, it is reminiscent of the Dirac spinor case, for which the number of degrees of freedom is also exponential, $\sim 2^{d/2}$ for large d .

is dual to a hypercube in n dimensions, with the “dual“ of a regular polytope being another regular polytope having one vertex in the center of each cell of the polytope one started with.

When we consider the surface of the cross polytope in $d + 1$ dimensions, we have an object of dimension $n - 1 = d$, which corresponds to a triangulated manifold with no boundary, homeomorphic to the sphere (as an example, see Fig. 1). From Eqs. (2.6), (2.26), and (2.27), the deficit angle is then given to leading order by

$$\delta_d = 0 + \frac{4}{d} - (\epsilon_{d,d+1} + 3 \text{ terms} + \epsilon_{1,d}\epsilon_{1,d+1} + \dots) + O(1/d^2, \epsilon/d, \epsilon^2/d) \quad (2.32)$$

and therefore close to flat in the large d limit (due to our choice of an equilateral starting configuration). Indeed if the choice of triangulation is such that the deficit angle is not close to zero, then the discrete model leads to an average curvature whose magnitude is comparable to the lattice spacing or ultraviolet cutoff, which from a physical point of view does not seem very attractive: one obtains a space-time with curvature radius comparable to the Planck length. In addition, the small fluctuation excitation spectrum for such strongly curved lattices looks disturbingly different from what one would expect in the continuum for transverse-traceless modes [21].

When evaluated on such a manifold the lattice action becomes

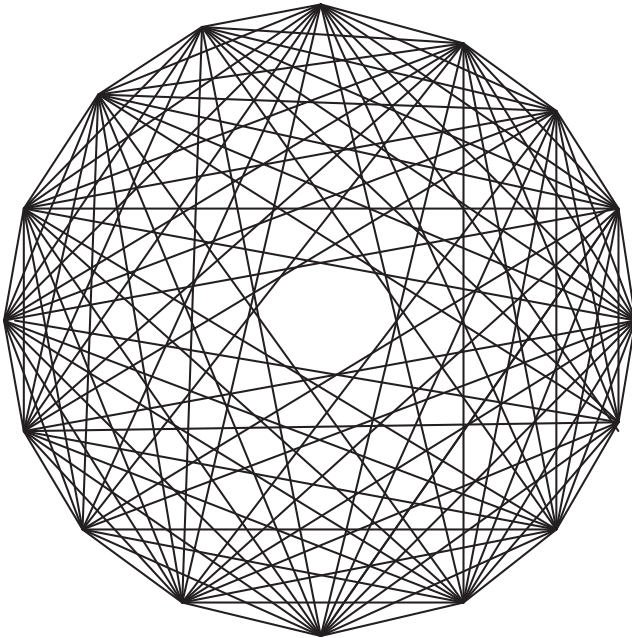


FIG. 1. Cross polytope β_n with $n = 8$ and $2n = 16$ vertices, whose surface can be used to define a simplicial manifold of dimension $d = n - 1 = 7$. For general d , the cross polytope β_{d+1} will have $2(d + 1)$ vertices, connected to each other by $2d(d + 1)$ edges.

$$\frac{\sqrt{d}2^{d/2}}{d!}2(\lambda - kd^3)\left[1 - \frac{1}{8}\sum\epsilon_{ij}^2 + \frac{1}{d}\left(\frac{1}{4}\sum\epsilon_{ij} + \frac{1}{8}\sum\epsilon_{ij}\epsilon_{ik}\right) + O(1/d^2)\right]. \quad (2.33)$$

Dropping the $1/d$ correction, one obtains therefore to leading order

$$\frac{\sqrt{d}2^{d/2}}{d!}2(\lambda - kd^3)\left(1 - \frac{1}{8}\sum\epsilon_{ij}^2 + \dots\right) \quad (2.34)$$

and, up to the irrelevant constant term and an overall multiplicative factor (which can be absorbed into a rescaling of the ϵ 's), the action reduces to the simple form

$$-\frac{1}{2}(\lambda - kd^3)\sum\epsilon_{ij}^2. \quad (2.35)$$

Since there are $2d(d + 1)$ edges in the cross polytope, one finds therefore that, at the critical point $kd^3 = \lambda$, the quadratic form in ϵ , defined by the above action, develops $2d(d + 1) \sim 2d^2$ zero eigenvalues.²

It is worth noting here that the competing curvature (k) and cosmological constant (λ) terms will have comparable magnitude when

$$k_c = \frac{\lambda l_0^4}{d^3 l_0^2}. \quad (2.36)$$

Here we have further allowed for the possibility that the average lattice spacing $l_0 = \langle l^2 \rangle^{1/2}$ is not equal to one (in other words, we have restored the appropriate overall scale for the average edge length, which is in fact largely determined by the value of λ). This then gives for $\lambda = 1$ [using the large- d expression for the average lattice spacing l_0 , obtained later in this section in Eq. (2.38)], the estimate $k_c = \sqrt{3}/(16 \times 5^{1/4}) = 0.0724$ in $d = 4$, to be compared with $k_c = 0.0636(11)$ obtained in [24] by direct numerical simulation in four dimensions. Even in $d = 3$ one finds for $\lambda = 1$, from Eqs. (2.36) and (2.38), $k_c = 2^{5/3}/27 = 0.118$, to be compared with $k_c = 0.112(5)$ obtained in [41] by direct numerical simulation. Again, the dependence of k_c on inverse powers of d is not surprising, as fluctuations, which are stronger in smaller dimensions, will require an increasingly larger value of the coupling k to make the transition happen in small dimensions.

The average lattice spacing l_0 is easily estimated from the following argument. The volume of a general equi-

²This result is quite close to the $d^2/2$ zero eigenvalues expected in the continuum for large d , with the factor of 4 discrepancy presumably attributed to an underlying intrinsic ambiguity that arises when trying to identify lattice points with points in the continuum. It would therefore suggest that, in this context, four lattice points should be identified with one continuum point. It is quite possible that the value of the numerical coefficient might depend on the specific lattice structure.

lateral simplex is given by Eq. (2.7), multiplied by an additional factor of l_0^d . In the limit of small k the average volume of a simplex is largely determined by the cosmological term, and can therefore be computed from

$$\langle V \rangle = -\frac{\partial}{\partial \lambda} \log \int [dl^2] e^{-\lambda V(l^2)}, \quad (2.37)$$

with $V(l^2) = (\sqrt{d+1}/d! 2^{d/2}) l^d \equiv c_d l^d$. Doing the single surviving integral over l^2 , $\int_0^\infty dl^2 \exp(-\lambda c_d l^d) = (c_d \lambda)^{-2/d} \Gamma((d+2)/d)$, gives $\langle V \rangle = 2/d \lambda = c_d l_0^d$. Solving this last expression for l_0^2 then gives the desired result

$$l_0^2 = \frac{1}{\lambda^{2/d}} \left[\frac{2}{d} \frac{d! 2^{d/2}}{\sqrt{d+1}} \right]^{2/d} \quad (2.38)$$

(which, for example, gives $l_0 = 2.153$ for $\lambda = 1$ in four dimensions, in reasonable agreement with the actual value $l_0 \approx 2.43$ found in [24] near the transition point). The result of Eq. (2.36), extended to d dimensions, should then read

$$k_c = \frac{\lambda l_0^d}{d^3 l_0^{d-2}} = \frac{\lambda l_0^2}{d^3}, \quad (2.39)$$

which is in fact the same result as before in $d = 4$. Using Eq. (2.38) inserted into Eq. (2.39) one then obtains in the large d limit for the naturally dimensionless combination $k/\lambda^{(d-2)/d}$

$$\frac{k_c}{\lambda^{(d-2)/d}} = \frac{2^{1+(2/d)}}{d^3} \left[\frac{\Gamma(d)}{\sqrt{d+1}} \right]^{2/d} \sim \frac{2}{e^2} \frac{1}{d}. \quad (2.40)$$

This result would then lead us to conclude that the above critical dimensionless ratio of couplings is given in the large- d limit by $k_c/\lambda \sim 1/d$. One should be careful though not to assign any deep physical significance to this result, which is only meant to help determine the critical values for the bare coupling constants.

In the following we will now revert back, for simplicity, to the case of an expansion about $l_0 = 1$. Returning to the partition function (and averages derived from it) associated with Eq. (2.34), we note that it can be formally computed via

$$Z = \int \prod_{i=1}^N d\epsilon_i e^{-\epsilon M \epsilon} = \frac{\pi^{N/2}}{\sqrt{\det M}} = \frac{\pi^{N/2}}{\sqrt{\prod_{i=1}^N \lambda_i}}, \quad (2.41)$$

with $N = 2d(d+1)$. Convergence of the Gaussian integral then requires $kd^3 > \lambda$. From

$$\ln Z = \frac{N}{2} \ln \pi - \frac{1}{2} \sum_{i=1}^N \ln \lambda_i \sim \frac{N}{2} \ln \pi - \frac{1}{2} \int_0^\infty ds \rho(s) \ln \lambda(s), \quad (2.42)$$

and using the fact that for the cross polytope to leading order in $1/d$ all eigenvalues are equal, one has

$$\log Z = \frac{\sqrt{d} 2^{(d/2)+1}}{d!} (kd^3 - \lambda) + d(d+1) \times \log \left[8\pi \frac{\sqrt{d} 2^{(d/2)+1}}{d!} (kd^3 - \lambda) \right], \quad (2.43)$$

with the first term arising from the constant term in the action, and the second term from the ϵ -field Gaussian integral. Therefore the general structure, to leading order in the weak field expansion at large d , is $\log Z = c_1(kd^3 - \lambda) - d(d+1) \log(kd^3 - \lambda) + c_2$ with c_1 and c_2 d -dependent constants, and therefore $\partial^2 \log Z / \partial k^2 \sim 1/(kd^3 - \lambda)^2$ with divergent curvature fluctuations in the vicinity of the critical point at $kd^3 = \lambda$.

E. Inclusion of higher order terms

It seems legitimate to ask what happens if the fluctuations in the ϵ 's are large enough so that the quadratic approximation is no longer adequate. Then one has from Eq. (2.34), to lowest order in $1/d$,

$$\frac{\sqrt{d} 2^{d/2}}{d!} 2 \left[(\lambda - kd^3) \left(1 - \frac{1}{8} \sum \epsilon_{ij}^2 + \dots \right) + \frac{\sigma \lambda}{16} \sum \epsilon_{ij}^4 \right], \quad (2.44)$$

where we have again included a cutoff term, proportional to σ , for each edge. Then, again up to the constant term and an overall multiplicative factor, the action reduces to

$$-\frac{1}{2} (\lambda - kd^3) \sum \epsilon_{ij}^2 + \frac{\sigma \lambda}{4} \sum \epsilon_{ij}^4. \quad (2.45)$$

At strong coupling $k \rightarrow 0$, the minimum lies at a non-vanishing value of the ϵ 's, namely $\epsilon_{ij} = \pm 1/\sqrt{\sigma}$. Since we started out with equilateral simplices with unit edges, this result is telling us that the edges have to be slightly extended (or shortened) to reach the minimum. As k is increased, the minimum eventually moves to the origin for $k = \lambda/d^3$. Neglecting the effects of fluctuations in the ϵ fields, $\langle \epsilon \cdot \epsilon \rangle - \langle \epsilon \rangle^2 = 0$, which is similar to the Landau treatment of ferromagnetic transitions, one then obtains

$$-\frac{1}{2} (\lambda - kd^3) \epsilon^2 + \frac{\sigma \lambda}{4} \epsilon^4. \quad (2.46)$$

For $kd^3 > \lambda$ the minimum is at the origin, whereas for $kd^3 < \lambda$ it moves away from it. For $\lambda > kd^3$ one has a shifted minimum at $\epsilon_0 = \pm(1 - kd^3/\lambda\sigma)^{1/2}$ and a total action $I(\epsilon_0) = -\lambda(1 - kd^3/\lambda)^2/4\sigma$. As a result ϵ_0 vanishes at $k = \lambda/d^3$, and so does $I(\epsilon_0)$.

If we apply the ideas of mean field theory, we need to keep the terms of order $1/d$ in Eq. (2.33). In the $\epsilon_{ij}\epsilon_{ik}$ term, we assume that the fluctuations are small and replace ϵ_{ik} by its average $\bar{\epsilon}$. Each ϵ_{ij} has $4d - 2$ neighbors (edges with one vertex in common with it); this has to be divided by 2

to avoid double counting in the sum, so the contribution is $(2d-1)\bar{\epsilon}$. Then to lowest order in $1/d$, the action is proportional to

$$(\lambda - kd^3) \left[1 - \frac{1}{8} \sum \epsilon_{ij}^2 + \frac{1}{4} \bar{\epsilon} \sum \epsilon_{ij} \right]. \quad (2.47)$$

This gives rise to the same partition function as obtained earlier, and using it to calculate the average value of ϵ_{ij} gives $\bar{\epsilon}$, as required for consistency.

To summarize, in this section we have developed an expansion in powers of $1/d$, which relies on a combined and simultaneous use of the weak field expansion. It can therefore be regarded as a double expansion in $1/d$ and ϵ , valid wherever the fields are smooth enough and the geometry is close to flat, which presumably is the case to some extent at large distances in the vicinity of the lattice critical point at k_c . In the next section we will develop a different and complementary $1/d$ expansion, which will not require weak fields, but will rely instead on the strong coupling (small $k = 1/8\pi G$, or large G) limit. As such it should now be considered as a double expansion in $1/d$ and k . Its validity will be in a regime where the fields are not smooth, and in fact will rely on considering lattice gravitational field configurations which are very far from smooth at short distances.

III. STRONG COUPLING EXPANSION IN LARGE DIMENSIONS

In this section we discuss the strong coupling [small $k = 1/(8\pi G)$] expansion of the lattice gravitational partition function, first in the general case, and subsequently for large d . The resulting series is expected to be useful up to some $k = k_c$, where k_c is the lattice critical point [as determined, for example, from Eq. (2.40)], at which the partition function develops a singularity. It appears that the phase $k > k_c$ is of limited physical interest, since in that phase space-time collapses into a two-dimensional manifold [20,22,23] (in fact, one of the first examples of compactification due to nonperturbative dynamics, as opposed to a specific choice of boundary conditions).

There will be two main aspects to the following discussion. The first aspect will be the development of a systematic expansion for the partition function and the correlation functions in powers of k , and a number of rather general considerations that follow from it. The second main aspect will be a detailed analysis and interpretation of the individual terms which appear order by order in the strong coupling expansion. This second part will then lead to a discussion of what happens for large d .

A. The measure

We will therefore first focus on the four-dimensional case, and then later exhibit its more or less immediate generalization to $d > 4$. The 4-dimensional Euclidean lat-

tice action [16,20,21] contains the usual cosmological constant and Regge scalar curvature terms

$$I_{latt} = \lambda \sum_h V_h(l^2) - k \sum_h \delta_h(l^2) A_h(l^2), \quad (3.1)$$

with $k = 1/(8\pi G)$, and possibly additional higher derivative terms as well. The action only couples edges which belong either to the same simplex or to a set of neighboring simplices, and can therefore be considered as *local*, just like the continuum action. It leads to a lattice partition function defined as

$$Z_{latt} = \int [dl^2] e^{-\lambda \sum_h V_h + k \sum_h \delta_h A_h}, \quad (3.2)$$

where, as customary, the lattice ultraviolet cutoff is set equal to one (i.e. all length scales are measured in units of the lattice cutoff). For definiteness the measure will be of the form [20,21,38]

$$\int [dl^2] = \int_0^\infty \prod_s (V_d(s))^\sigma \prod_{ij} dl_{ij}^2 \Theta[l_{ij}^2]. \quad (3.3)$$

The lattice partition function Z_{latt} should be compared to the continuum Euclidean Feynman path integral

$$Z_{cont} = \int [dg_{\mu\nu}] e^{-\lambda \int dx \sqrt{g} + [1/(16\pi G)] \int dx \sqrt{g} R}, \quad (3.4)$$

which involves a functional integration over all metrics, with functional measure [2,42]

$$\begin{aligned} \int [dg_{\mu\nu}] &= \int \prod_x (g(x))^{(d-4)(d+1)/8} \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \\ &\xrightarrow{d=4} \int \prod_x \prod_{\mu \geq \nu} dg_{\mu\nu}(x). \end{aligned} \quad (3.5)$$

Other popular measures would include an additional, local factor involving the determinant of the metric, such as $\prod_x g^{-5/2}(x)$ in $d = 4$ [43].

Since we will be doing an expansion in the kinetic term proportional to k , it will be convenient to include the λ -term in the measure. We will set therefore in this section

$$d\mu(l^2) \equiv [dl^2] e^{-\lambda \sum_h V_h}. \quad (3.6)$$

It should be clear that this last expression represents a fairly nontrivial quantity, both in view of the relative complexity of the expression for the volume of a simplex, Eq. (2.5), and because of the generalized triangle inequality constraints already implicit in $[dl^2]$. But, like the continuum functional measure, it is certainly *local*, to the extent that each edge length appears only in the expression for the volume of those simplices which explicitly contain it. Also, we note that in general the integral $\int d\mu$ can only be

evaluated numerically; nevertheless this can be done, at least in principle, to arbitrary precision. Furthermore, λ sets the overall scale and can therefore be set equal to one without any loss of generality [one can also conveniently normalize the integration measure, so that $Z_0 \equiv \int d\mu(l^2) = 1$, but this will not be necessary here].

To summarize, the effective strong coupling measure of Eq. (3.6) has the properties that (i) it is local in the lattice metric of Eq. (2.3), to the same extent that the continuum measure is ultralocal, (ii) it restricts all edge lengths to be positive, and (iii) it imposes a soft cutoff on large simplices due to the λ -term and the generalized triangle inequalities. Apart from these constraints, it does *not* significantly restrict the fluctuations in the lattice metric field at short distances. It will be the effect of the curvature term to restrict such fluctuation, by coupling the metric field between simplices, in the same way as the derivatives appearing in the continuum Einstein term couple the metric between infinitesimally close space-time points.

B. Expansion in powers of k

From now on we will discuss Z_{latt} only, and drop the subscript *latt*. As a next step, Z is expanded in powers of k ,

$$\begin{aligned} Z(k) &= \int d\mu(l^2) e^{k \sum_h \delta_h A_h} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} k^n \int d\mu(l^2) \left(\sum_h \delta_h A_h \right)^n. \end{aligned} \quad (3.7)$$

It is easy to show that $Z(k) = \sum_{n=0}^{\infty} a_n k^n$ is analytic at $k = 0$, so this expansion is well defined, up to the nearest singularity in the complex k plane. An estimate for the expected location of such a singularity in the large- d limit was given in Eq. (2.40) of the previous section. Beyond this singularity $Z(k)$ can sometimes be extended, for example, via Padé or differential approximants [44,45].³ The above expansion is of course analogous to the high temperature expansion in statistical mechanics systems, where the on-site terms are treated exactly and the kinetic or hopping term is treated as a perturbation. Singularities in the free energy or its derivatives can usually be pinned down with the knowledge of a large enough number of terms in the relevant expansion [44]. The often surprisingly rich structure of singularities in the complex coupling plane and their volume dependence has been explored in detail for

³It is well known that a first order transition cannot affect the singularity structure of $Z(k)$ as viewed from the strong coupling phase, as the free energy is C_∞ at a first order transition. $Z(k)$, as defined from the strong coupling phase, will be nonanalytic only at the second order, end point transition, modulo an exponentially small imaginary part appearing in the metastable phase, if one exists. Approaching the phase transition from the strong coupling side detects the physically relevant end point singularity, where the correlation length diverges and scale invariance is presumably recovered [24].

some simple exactly soluble models with a finite number of degrees of freedom [46].

Next consider a fixed, arbitrary hinge on the lattice, and call the corresponding curvature term in the action δA . Such a contribution will be denoted in the following, as is customary in lattice gauge theories, a *plaquette* contribution. For the average curvature on that hinge one has

$$\langle \delta A \rangle = \frac{\sum_{n=0}^{\infty} \frac{1}{n!} k^n \int d\mu(l^2) \delta A (\sum_h \delta_h A_h)^n}{\sum_{n=0}^{\infty} \frac{1}{n!} k^n \int d\mu(l^2) (\sum_h \delta_h A_h)^n}. \quad (3.8)$$

After expanding out in k the resulting expression, one obtains for the cumulants

$$\langle \delta A \rangle = \sum_{n=0}^{\infty} c_n k^n, \quad (3.9)$$

with

$$c_0 = \frac{\int d\mu(l^2) \delta A}{\int d\mu(l^2)}, \quad (3.10)$$

whereas to first order in k one has

$$\begin{aligned} c_1 &= \frac{\int d\mu(l^2) \delta A (\sum_h \delta_h A_h)}{\int d\mu(l^2)} \\ &\quad - \frac{\int d\mu(l^2) \delta A \cdot \int d\mu(l^2) \sum_h \delta_h A_h}{(\int d\mu(l^2))^2}. \end{aligned} \quad (3.11)$$

This last expression clearly represents a measure of the fluctuation in δA , namely $[\langle (\sum_h \delta_h A_h)^2 \rangle - \langle \sum_h \delta_h A_h \rangle^2]/N_h$, using the homogeneity properties of the lattice $\delta A \rightarrow \sum_h \delta_h A_h/N_h$. Equivalently, it can be written in an even more compact way as $N_h[\langle (\delta A)^2 \rangle - \langle \delta A \rangle^2]$. To second order in k one has

$$\begin{aligned} c_2 &= \frac{\int d\mu(l^2) \delta A (\sum_h \delta_h A_h)^2}{2 \int d\mu(l^2)} \\ &\quad - \frac{\int d\mu(l^2) \sum_h \delta_h A_h \cdot \int d\mu(l^2) \delta A \sum_h \delta_h A_h}{(\int d\mu(l^2))^2} \\ &\quad - \frac{\int d\mu(l^2) (\sum_h \delta_h A_h)^2 \int d\mu(l^2) \delta A}{2(\int d\mu(l^2))^2} \\ &\quad + \frac{\int d\mu(l^2) \delta A \cdot (\int d\mu(l^2) \sum_h \delta_h A_h)^2}{(\int d\mu(l^2))^3} \end{aligned} \quad (3.12)$$

which now corresponds to $c_2 = N_h^2[\langle (\delta A)^3 \rangle - 3\langle \delta A \rangle \langle (\delta A)^2 \rangle + 2\langle \delta A \rangle^3]/2$. At the next order one has $c_3 = N_h^3[\langle (\delta A)^4 \rangle - 4\langle \delta A \rangle \langle (\delta A)^3 \rangle - 3\langle (\delta A)^2 \rangle^2 + 12\langle \delta A \rangle^2 \langle (\delta A)^2 \rangle - 6\langle \delta A \rangle^4]/6$, and so on. Note that the expressions in square parentheses become rapidly quite small, $O(1/N_h^n)$ with increasing order n , as a result of large cancellations that must arise eventually between individual terms inside the square parentheses. In principle, a careful and systematic numerical evaluation of the above integrals (which is quite feasible in practice) would allow the determination of the expansion coefficients in k for the average curvature

$\langle \delta A \rangle$ to rather high order, but we shall not pursue this line of inquiry here.⁴

It is advantageous to isolate in the above expressions the *local* fluctuation term, from those terms that involve *correlations* between different hinges. To see this, one needs to go back, for example, to the first order expression in Eq. (3.11) and isolate in the sum \sum_h the contribution which contains the selected hinge with value δA , namely

$$\sum_h \delta_h A_h = \delta A + \sum'_h \delta_h A_h, \quad (3.15)$$

where the primed sum indicates that the term containing δA is *not* included. The result is

$$\begin{aligned} c_1 = & \frac{\int d\mu(l^2)(\delta A)^2}{\int d\mu(l^2)} - \frac{(\int d\mu(l^2)\delta A)^2}{(\int d\mu(l^2))^2} \\ & + \frac{\int d\mu(l^2)\delta A \sum'_h \delta_h A_h}{\int d\mu(l^2)} \\ & - \frac{(\int d\mu(l^2)\delta A)(\int d\mu(l^2)\sum'_h \delta_h A_h)}{(\int d\mu(l^2))^2}. \end{aligned} \quad (3.16)$$

One then observes the following: the first two terms describe the *local* fluctuation of δA on a given hinge; the third and fourth terms describe *correlations* between δA terms on *different* hinges. But because the action is *local*, the only nonvanishing contribution to the last two terms comes from edges and hinges which are in the immediate vicinity of the hinge in question. For hinges located further apart (indicated below by “*not nn*”) one has that their fluctuations remain uncorrelated, leading to a vanishing variance

⁴As an example, consider a nonanalyticity in the average scalar curvature

$$\mathcal{R}(k) = \frac{\langle \int dx \sqrt{g(x)} \mathcal{R}(x) \rangle}{\langle \int dx \sqrt{g(x)} \rangle}, \quad (3.13)$$

assumed to be of the form of an algebraic singularity at k_c , namely $\mathcal{R}(k) \sim_{k \rightarrow k_c} A_{\mathcal{R}}(k_c - k)^\delta$. It will lead to a behavior, for the general term in the series in k , of the type

$$(-1)^n A_{\mathcal{R}} \frac{(\delta - n + 1)(\delta - n + 2) \dots \delta}{n! k_c^{n-\delta}} k^n. \quad (3.14)$$

Given enough terms in the series, the singularity structure can then be investigated using a variety of increasingly sophisticated methods [44,47–49]. In Ref. [24] the curvature $\mathcal{R}(k)$ was computed numerically for various values of k , from which one can extract an approximate value for the coefficients, namely $\mathcal{R}(k) = -9.954 + 62.11k + 195.94k^2 - 1340.65k^3 + 40483.75k^4 + O(k^5)$. A better and much more accurate way would be a direct determination of the individual coefficients, via the edge length integrals of Eqs. (3.11) and (3.12).

$$\begin{aligned} & \frac{\int d\mu(l^2)\delta A \sum'_h \delta_h A_h}{\int d\mu(l^2)} \\ & - \frac{(\int d\mu(l^2)\delta A)(\int d\mu(l^2)\sum'_h \delta_h A_h)}{(\int d\mu(l^2))^2} = 0, \end{aligned} \quad (3.17)$$

since for uncorrelated random variables X_n 's, $\langle X_n X_m \rangle - \langle X_n \rangle \langle X_m \rangle = 0$. Therefore the only nonvanishing contributions in the last two terms in Eq. (3.16) come from hinges which are *close* to each other.

The above discussion makes it clear that a key quantity is the *correlation* between different plaquettes,

$$\langle (\delta A)_h (\delta A)_{h'} \rangle = \frac{\int d\mu(l^2)(\delta A)_h (\delta A)_{h'} e^{k \sum_h \delta_h A_h}}{\int d\mu(l^2) e^{k \sum_h \delta_h A_h}}, \quad (3.18)$$

or, better, its *connected* part (denoted here by $\langle \dots \rangle_C$)

$$\langle (\delta A)_h (\delta A)_{h'} \rangle_C \equiv \langle (\delta A)_h (\delta A)_{h'} \rangle - \langle (\delta A)_h \rangle \langle (\delta A)_{h'} \rangle, \quad (3.19)$$

which subtracts out the trivial part of the correlation. Here again the exponentials in the numerator and denominator can be expanded out in powers of k , as in Eq. (3.8). The lowest order term in k will involve the correlation

$$\int d\mu(l^2)(\delta A)_h (\delta A)_{h'}. \quad (3.20)$$

But unless the two hinges are close to each other, they will fluctuate in an uncorrelated manner, with $\langle (\delta A)_h (\delta A)_{h'} \rangle - \langle (\delta A)_h \rangle \langle (\delta A)_{h'} \rangle = 0$. In order to achieve a nontrivial correlation, the path between the two hinges h and h' needs to be tiled by at least as many terms from the product $(\sum_h \delta_h A_h)^n$ in

$$\int d\mu(l^2)(\delta A)_h (\delta A)_{h'} \left(\sum_h \delta_h A_h \right)^n \quad (3.21)$$

as are needed to cover the distance l between the two hinges. One then has

$$\langle (\delta A)_h (\delta A)_{h'} \rangle_C \sim k^l \sim e^{-l/\xi}, \quad (3.22)$$

with the correlation length $\xi = 1/|\log k| \rightarrow 0$ to lowest order as $k \rightarrow 0$ [here we have used the usual definition of the correlation length ξ , namely, that a generic correlation function is expected to decay as $\exp(-\text{distance}/\xi)$ for large separations].⁵ This last result is quite general, and holds, for example, irrespective of the boundary conditions (unless of course $\xi \sim L$, where L is the linear size of the

⁵This statement, taken literally, oversimplifies the situation a bit, as depending on the spin (or tensor structure) of the operator appearing in the correlation function, the large distance decay of the corresponding correlator is determined by the lightest excitation in that specific channel. But in the gravitational context one is mostly concerned with correlators involving spin two (transverse-traceless) objects, evaluated at fixed geodesic distance.

system, in which case a path can be found which wraps around the lattice).

But further thought reveals that the above result is in fact not completely correct, due to the fact that in order to achieve a nonvanishing correlation one needs, at least to lowest order, to connect the two hinges by a narrow tube. The previous result should then read correctly as

$$\langle (\delta A)_h (\delta A)_{h'} \rangle_C \sim (k^{n_d})^l, \quad (3.23)$$

where, as will be shown in more detail below, $n_d l$ represents (approximately) the minimal number of dual lattice polygons that are needed to form a closed surface connecting the hinges h and h' , with l the actual distance (in lattice units) between the two hinges.

C. Rotation matrices, Voronoi loops and closed surfaces

Up to this point our considerations have been quite general, and therefore do not take into account yet the detailed nature of the local interaction expressed in the action term $\sum_h \delta_h A_h$. It is well known that the deficit angle δ_h describes the rotation of a vector V^μ parallel transported around a closed loop encircling the hinge h . This full rotation is best described in terms of a (Lorentz) rotation matrix \mathbf{R} , an element of $SO(4)$ or $SO(3, 1)$, depending on the signature of the metric, and whose matrix elements will depend on the specific choice of coordinates at the point in question. In d dimensions the corresponding objects would be $SO(d)$ or $SO(d-1, 1)$ rotations, in the Riemannian and pseudo-Riemannian case, respectively.⁶

Just as in the continuum, where the affine connection and therefore the infinitesimal rotation matrix is determined by the metric and its first derivatives, on the lattice the elementary rotation matrix between simplices $\mathbf{R}_{s,s+1}$ is fixed by the difference between the g_{ij} 's of Eq. (2.3) within neighboring simplices. Consider therefore a closed path Γ encircling a hinge h and passing through each of the simplices that meet at that hinge. In particular, one may take Γ to be the boundary of the polyhedral dual (or Voronoi) area surrounding the hinge [20]. We recall that the Voronoi polyhedron dual to a vertex P is the set of all points on the lattice which are closer to P than any other vertex; the corresponding new vertices then represent the sites on the dual lattice. A unique closed parallel transport path can then be assigned to each hinge, by suitably connecting sites in the dual lattice.

⁶The preceding observations can in fact be developed into a consistent first order (Palatini) formulation of Regge gravity, with suitably chosen independent transformation matrices and metrics, related to each other by a set of appropriate lattice equations of motion [50]. One would expect the first and second order formulations to ultimately describe the same quantum theory, with common universal long-distance properties. How to consistently define finite rotations, frames and connections in Regge gravity was first discussed systematically in [51].

With each neighboring pair of simplices $s, s+1$ one associates a Lorentz transformation \mathbf{R}^α_β , which describes how a given vector V_μ transforms between the local coordinate systems in these two simplices,

$$V'^\alpha = [\mathbf{R}_{s,s+1}]^\alpha_\beta V^\beta. \quad (3.24)$$

The above Lorentz transformation is then directly related to the continuum path-ordered (P) exponential of the integral of the local affine connection $\Gamma^\lambda_{\mu\nu}$ via

$$\mathbf{R}^\alpha_\beta = [P e^{\int_{\text{between simplices}} \Gamma_\lambda dx^\lambda}]^\alpha_\beta. \quad (3.25)$$

Next consider moving a vector V once around a Voronoi loop, i.e. a loop formed by Voronoi edges surrounding a chosen hinge. The change in the vector V is given by

$$\delta V^\alpha = (\mathbf{R} - \mathbf{1})^\alpha_\beta V^\beta, \quad (3.26)$$

where $\mathbf{R} \equiv \prod_s \mathbf{R}_{s,s+1}$ is now the total rotation matrix associated with the given hinge. Since in the continuum δV is given by $\delta V^\alpha = \frac{1}{2} R^\alpha_{\beta\mu\nu} A_\Gamma^{\mu\nu} V^\beta$, where $A_\Gamma^{\mu\nu}$ is the antisymmetric bivector representing the loop area, one has the identification

$$\frac{1}{2} R^\alpha_{\beta\mu\nu} A_\Gamma^{\mu\nu} = (\mathbf{R} - \mathbf{1})^\alpha_\beta. \quad (3.27)$$

To first order in the deficit angle δ , one then recovers the well-known result

$$R^\alpha_{\beta\mu\nu} = \frac{\delta}{A_\Gamma} U^\alpha_\beta U_{\mu\nu}, \quad (3.28)$$

where $U_{\alpha\beta}$ represents the hinge bivector, $U_{\alpha\beta} = \frac{1}{2A_h} \epsilon_{\alpha\beta\mu\nu} l_1^\mu l_2^\nu$, with l_1 and l_2 the two hinge vectors and A_h the area of the hinge, and use has been made of the relationship between the original volumes and their dual counterparts, $A_\Gamma^\Gamma U^{\alpha\beta} = 2A_\Gamma$. As a result, one can relate the deficit angle directly to the effect of a complete rotation of a vector around a hinge,

$$\left[\prod_s \mathbf{R}_{s,s+1} \right]^\mu_\nu = [e^{\delta_h U^{(h)}}]^\mu_\nu. \quad (3.29)$$

In other words, the product of rotation matrices around the closed elementary loop describes a rotation in a plane perpendicular to the hinge, by an angle δ_h . Equivalently, this last expression can be rewritten in terms of a surface integral of the Riemann tensor, projected along the surface area element bivector $A_\Gamma^{\alpha\beta}$ associated with the loop,

$$\left[\prod_s \mathbf{R}_{s,s+1} \right]^\mu_\nu \approx [e^{(1/2) \int_s R^\cdot{}_{\alpha\beta} A_\Gamma^{\alpha\beta}}]^\mu_\nu. \quad (3.30)$$

Let us now return to the strong coupling expansion, and it will be advantageous now to focus on general properties of the parallel transport matrices \mathbf{R} .⁷ For smooth enough geometries, with small curvatures, the above rotation matrices can be chosen to be close to the identity. Small fluctuations in the geometry will then imply small deviations in the \mathbf{R} 's from the identity matrix. But for strong coupling ($k \rightarrow 0$) it was already emphasized before that the measure $\int d\mu(l^2)$ does not significantly restrict fluctuations in the lattice metric field. As a result these fields can be regarded in this regime as basically unconstrained random variables, only subject to the relatively mild constraints implicit in the measure $d\mu$. The geometry is generally far from smooth since there is no coupling term to enforce long range order (the coefficient of the lattice Einstein term is zero), and one has as a consequence large local fluctuations in the geometry. The matrices \mathbf{R} will therefore fluctuate with the local geometry, and average out to zero, or a value close to zero.⁸

This is quite similar of course to what happens in $SU(N)$ Yang-Mills theories, or even more simply in (compact) QED, where the analogs of the $SO(d)$ rotation matrices \mathbf{R} are phase factors $U_\mu(x) = e^{iaA_\mu(x)}$. One has there $\int \frac{dA_\mu}{2\pi} U_\mu(x) = 0$ and $\int \frac{dA_\mu}{2\pi} U_\mu(x) U_\mu^\dagger(x) = 1$. In addition, for two contiguous closed paths C_1 and C_2 sharing a common side, one has

$$e^{i \oint_{C_1} \mathbf{A} \cdot d\mathbf{l}} e^{i \oint_{C_2} \mathbf{A} \cdot d\mathbf{l}} = e^{i \oint_C \mathbf{A} \cdot d\mathbf{l}} = e^{i \int_S \mathbf{B} \cdot n dA}, \quad (3.31)$$

with C the slightly larger path encircling the two loops. For a closed surface tiled with many contiguous infinitesimal closed loops the last expression evaluates to 1, due to the divergence theorem. In the lattice gravity case the discrete analog of this last result is considerably more involved, and ultimately represents the (exact) lattice analog of the contracted Bianchi identities [53]. An example of a closed surface tiled with parallel transport polygons (here chosen for simplicity to be triangles) is shown in Fig. 2.

⁷The role of continuous rotation matrices in Regge gravity is brought out in a particularly clear way by the first order approach of Ref. [50].

⁸In the sense that, for example, the $SO(4)$ rotation

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

averages out to zero when integrated over θ . In general an element of $SO(n)$ is described by $n(n-1)/2$ independent parameters, which in the case at hand can be conveniently chosen as the six $SO(4)$ Euler angles. The uniform (Haar) measure over the group is then $d\mu_H(\mathbf{R}) = \frac{1}{32\pi^3} \times \int_0^{2\pi} d\theta_1 \int_0^\pi d\theta_2 \int_0^\pi d\theta_3 \int_0^\pi d\theta_4 \sin\theta_4 \int_0^\pi d\theta_5 \sin\theta_5 \int_0^\pi d\theta_6 \sin^2\theta_6$. This is just a special case of the general n result [52], which reads $d\mu_H(\mathbf{R}) = (\prod_{i=1}^n \Gamma(i/2) / 2^n \pi^{n(n+1)/2}) \times \prod_{i=1}^{n-1} \prod_{j=1}^i \sin^{j-1} \theta_j d\theta_j$ with $0 \leq \theta_k^1 < 2\pi$, $0 \leq \theta_k^j < \pi$.

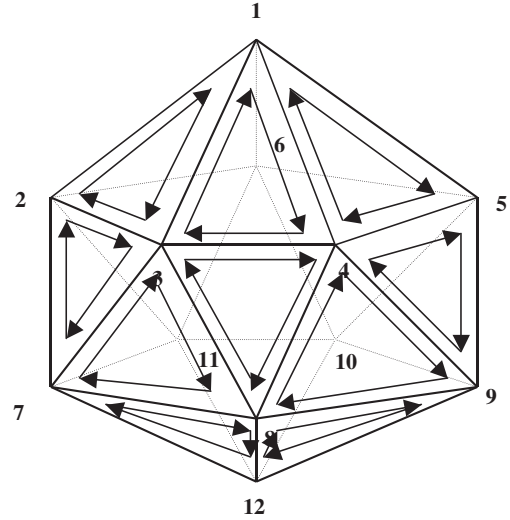


FIG. 2. Elementary closed surface tiled with parallel transport polygons. For each link of the dual lattice, the parallel transport matrices \mathbf{R} are represented by an arrow. In spite of the fact that the Lorentz matrices \mathbf{R} fluctuate with the local geometry, two contiguous, oppositely oriented arrows always give $\mathbf{R}\mathbf{R}^{-1} = 1$.

We can now reexamine the question, left open earlier in this section, of the value for the quantity n_d appearing in Eq. (3.23). This last quantity counts the number of polygons needed to obtain a closed surface around a hinge, in the framework of the strong coupling expansion for the curvature correlation function. For concreteness, we will consider a simplicial lattice built up of d -dimensional hypercubes divided up into simplices, as originally discussed in [18,20] in the four-dimensional case, although similar considerations should equally apply to other semi-regular d -dimensional lattices as well. Simply put the issue is then: how many polygons does it take to form the smallest closed surface attached to two hinges, separated from each other by l lattice steps?

First let us consider a slightly simpler case, namely, the smallest nontrivial closed surface made out of elementary parallel transport loops, and built around a *single* given hinge. In the four-dimensional hypercubic lattice the number of triangles per edge is either 14 (for the coordinate edges and the hyperdiagonal) or 8 (for the body and face diagonals). For a d -dimensional lattice, one needs the number of $(d-2)$ -simplices on each $(d-3)$ -simplex. This again is 14 for some $(d-3)$ -simplices, and somewhat less for others. For example, using the binary notation for the vertices as in [18], if the vertices of the $(d-3)$ -simplex are taken to be $(0, 0, 0, \dots)$, $(1, 0, 0, \dots)$, $(1, 1, 0, \dots)$, \dots up to the vertex with $(d-3)$ 1s followed by 3 0s, then the number of $(d-2)$ -simplices hinging on this, in the forward direction will be the same as the number of ways of inserting 1s in the 3 remaining places with 0s, which is 7. There will be the same number of $(d-2)$ -simplices in the backward direction. Thus for a typical $(d-3)$ -simplex, one needs 14 polygons to form a closed surface.

The next step then involves considering the minimal closed surface connecting *two* hinges separated by l lattice steps. If one is trying to connect two polygonal half-spheres with what resembles a closed tube, one needs to take a path through $(d - 2)$ -simplices connecting the $(d - 3)$ -simplices at the centers of the half-spheres. Suppose the path goes through l $(d - 2)$ -simplices, then the tube will consist of 26 (from the ends) plus $12(l - 1)$ polygons = $12l + 14$. One noteworthy aspect of this result is that it gives a large power of k , namely $n_d \sim 12$ in the notation of Eq. (3.23), but note that at the same time the power does not grow with d .

In the extreme strong coupling limit this then gives, from Eqs. (3.22) and (3.23),

$$\xi_{k \rightarrow 0} \sim \frac{l_0}{|\log k^{12}|} + \dots, \quad (3.32)$$

where the corrections (indicated here by the dots) arise from surfaces which are not minimal, i.e. deformations of the original minimal surface obtained by adding polygonal outgrowths to it, and therefore involving additional powers of k .

D. Random surfaces and the value of the universal exponent ν

In general for k not too small the random surface spanned by the parallel transport polygons will have a rather complex shape. The systematic counting of such surfaces is a rather challenging task, say compared to a regular hypercubic lattice, in view of the simplicial nature of the underlying lattice geometry. When discussing the average scalar curvature, given by the expectation value of $\delta_d V_{d-2}$, such a surface will be anchored on a given polyhedral loop, whereas when considering the correlation function of Eqs. (3.18) and (3.19) it will be anchored on two such parallel transport polygons, separated from each other by some fixed distance.⁹

⁹One might worry that the effects of large strong coupling fluctuations in the \mathbf{R} matrices might lead to a phenomenon similar to confinement in non-Abelian lattice gauge theories [54,55]. That this is most likely not the case can be seen from the fact that the analog of the Wilson loop $W(\Gamma)$ (defined here as a path-ordered exponential of the affine connection $\Gamma_{\mu\nu}^\lambda$ around a closed loop) does *not* give the static gravitational potential. The potential is instead determined from the correlation of (exponentials of) geodesic line segments, as in $\exp[-\mu_0 \int d\tau \sqrt{g_{\mu\nu}(x)} [(dx^\mu)/(d\tau)][(dx^\nu)/(d\tau)]]$, where μ_0 is the mass of the heavy source, as discussed already in some detail in [56,57]. The expected decay of near-planar Wilson loops with area A , $W(\Gamma) \sim \exp(\int_S R_{\cdot\mu\nu} A_{\Gamma}^{\mu\nu}) \sim \exp(-A/\xi^2)$ [58], where A is the minimal area spanned by the loop, gives instead the magnitude of the large-scale, averaged curvature, operationally determined by the process of parallel-transporting test vectors around very large loops, and which therefore is of order $R \sim 1/\xi^2$.

As one approaches the critical point, $k \rightarrow k_c$, one is interested in random surfaces which are of very large extent. Let n_p be the number of polygons in the surface, and set $n_p = T^2$ since after all one is describing a surface. The critical point then naturally corresponds to the appearance of surfaces of infinite extent,

$$n_p = T^2 \sim \frac{1}{k_c - k} \rightarrow \infty. \quad (3.33)$$

A legitimate parallel is to the simpler case of scalar field theories, where random walks of length T describing particle paths become of infinite extent at the critical point, situated where the inverse of the (renormalized) mass $\xi = m^{-1}$, expressed in units of the ultraviolet cutoff, diverges [59–63].

In the present case of polygonal random surfaces, one can provide the following concise argument in support of the identification in Eq. (3.33). First approximate the discrete sums over n , as they appear, for example, in the strong coupling expansion for the average curvature, Eq. (3.8) or its correlation, Eq. (3.18), by continuous integrals over areas

$$\sum_{n=0}^{\infty} c_n \left(\frac{k}{k_c}\right)^n \rightarrow \int_0^{\infty} dA A^{\gamma-1} \left(\frac{k}{k_c}\right)^A = \Gamma(\gamma) \left(\log \frac{k_c}{k}\right)^{-\gamma}, \quad (3.34)$$

where $A \equiv T^2$ is the area of a given surface. The $A^{\gamma-1}$ term can be regarded as counting the multiplicity of the surface (its entropy, in statistical mechanics terms). The exponent γ depends on the specific quantity one is looking at. For the average curvature one has $\gamma = -\delta$, while for its derivative, the curvature fluctuation (the curvature correlation function at zero momentum), one expects $\gamma = 1 - \delta$. The same type of singularity is of course obtained from the original series in Eq. (3.34), if one assumes for the coefficients c_n

$$c_n \sim \binom{-\gamma}{n} \underset{n \rightarrow \infty}{\sim} \frac{\Gamma(1 - \gamma) \sin \pi(n + \gamma)}{\pi} \times n^{\gamma-1} \left(1 - \frac{\gamma(1 - \gamma)}{2n} + \dots\right), \quad (3.35)$$

which in retrospect explains the appearance of the factor $A^{\gamma-1}$ in Eq. (3.34). In the last step we have used the well-known asymptotic expansion for the binomial coefficient $\binom{-\gamma}{n}$ for large n . Although we know its value exactly, the integral in Eq. (3.34) can also be evaluated by standard saddle point methods. The saddle point is located at

$$A = \frac{(\gamma - 1)}{\log \frac{k_c}{k}} \underset{k \rightarrow k_c}{\sim} \frac{(\gamma - 1)k_c}{k_c - k}. \quad (3.36)$$

Carried further, the saddle point method then leads to an approximation to the exact result for the quantity in Eq. (3.34), namely

$$e^{1-\gamma}(\gamma-1)^{\gamma-1}\sqrt{2\pi(\gamma-1)}\left(\log\frac{k_c}{k}\right)^{-\gamma}, \quad (3.37)$$

which agrees with the answer given above, up to an irrelevant overall multiplicative factor. From this discussion one then concludes that close to the critical point very large areas dominate, as claimed in Eq. (3.33).

Furthermore, one would expect that the universal geometric scaling properties of such a (closed) surface would not depend on its short distance details, such as whether it is constructed out of say triangles or more complex polygons. In general excluded volume effects at finite d will provide constraints on the detailed geometry of the surface, but as $d \rightarrow \infty$ these constraints can presumably be neglected and one is dealing then with a more or less unconstrained random surface. This should be regarded as a direct consequence of the fact that as $d \rightarrow \infty$ there are infinitely many dimensions for the random surface to twist and fold into, giving a negligible contribution from unallowed (by interactions) directions. Appendices A and B provide further specific examples for what is meant in this context, for the simpler cases of random walks and random polymers. In the following we will assume that this is indeed the case, and that no special pathologies arise, such as the collapse of the random surface into narrow tubelike, lower dimensional geometric configurations. Then in the large d limit the problem simplifies considerably.

Following [62], one can define the partition function for such an ensemble of unconstrained random surfaces as

$$Z_{RS} = \int \prod_{n,m=1}^T d^d X_{n,m} \exp\left[-\beta \sum_{\Delta} A_{\Delta}(\mathbf{X})\right], \quad (3.38)$$

where the integral is over d -component vectors $\mathbf{X}_{n,m}$ defined on two-dimensional triangular lattice sites, with sites labeled here by integers n and m . Up to a multiplicative constant, the term appearing in the exponent is just the total area of the surface, written as a sum of individual triangle areas. Introducing the induced two-dimensional metric tensor on each triangle allows one to recast the above partition function in the form of a two-dimensional massless field theory, which in a more compact continuum notation now reads

$$\begin{aligned} Z_{RS} = \text{const} \int [d\lambda][dg][d\mathbf{X}] \\ \times \exp\left[i \int d^2x \sqrt{g} \lambda^{ab} (g_{ab} - G_{ab}) - \beta \int d^2x \sqrt{g}\right], \end{aligned} \quad (3.39)$$

with $G_{ab} = \partial_a \mathbf{X} \cdot \partial_b \mathbf{X}$. The above action is now quadratic in the free massless \mathbf{X} -fields, whose propagator involves λ -dependent weights. We note that in the original gravitational context, the introduction of the coordinate vectors $\mathbf{X}(x)$ for describing the random surface spanned by poly-

gons, originally embedded in a fluctuating *curved* geometry, would seem plausible in view of the fact that as one approaches the critical point the expectation value of the scalar curvature does indeed go to zero [24].

As shown in [62], the overall *size* of the random surface, as embedded in the original d -dimensional space and suitably defined in the discrete case as

$$\langle \mathbf{X}^2 \rangle = \frac{1}{T^2} \sum_{n,m=1}^T \mathbf{X}_{n,m}^2 \quad (3.40)$$

is then immediately obtained from the free field infrared behavior of \mathbf{X} as $\langle \mathbf{X}^2 \rangle \sim \int_{1/T} d^2 p / p^2 \sim \log T$. Thus the mean square size of the surface increases logarithmically with the intrinsic area of the surface. This last result is usually interpreted as the statement that an unconstrained random surface has infinite fractal (or Hausdorff) dimension. Although made of very many triangles (or polygons), the random surface remains quite compact in overall size, as viewed from the original embedding space. In a sense, an unconstrained random surface is a much more compact object than an unconstrained random walk, for which $\langle \mathbf{X}^2 \rangle \sim T$. Identifying the size of the random surface with the gravitational correlation length ξ then gives

$$\xi \sim \sqrt{\log T} \underset{k \rightarrow k_c}{\sim} |\log(k_c - k)|^{1/2}. \quad (3.41)$$

From the definition of the exponent ν , namely $\xi \sim (k_c - k)^{-\nu}$, the above result then implies $\nu = 0$ (i.e. a weak logarithmic singularity).¹⁰ We note that the previous result for ξ in Eq. (3.32) only applied to the extreme strong coupling limit $k \rightarrow 0$.

Let us discuss next what the implications of this last result might be. As already outlined in Refs. [24,57,64], the exponent ν determines the universal renormalization-group evolution of the dimensionless coupling $\tilde{G} \equiv G\lambda^{(d-2)/d}$ in the vicinity of the ultraviolet fixed point. In particular, if one defines the dimensionless function $F(\tilde{G})$ via $m \equiv \xi^{-1} = \Lambda F(\tilde{G})$, where Λ is the ultraviolet cutoff (the inverse lattice spacing), then by differentiation of the renormalization-group invariant quantity m , $\Lambda \frac{d}{d\Lambda} m(\Lambda, \tilde{G}(\Lambda)) = 0$, one immediately obtains the Callan-Symanzik beta function $\beta(\tilde{G})$ [55]. From the definition

$$\Lambda \frac{d}{d\Lambda} \tilde{G}(\Lambda) = \beta(\tilde{G}(\Lambda)), \quad (3.42)$$

one gets an equivalent form for the beta function in terms of the function $F(\tilde{G})$ introduced above, namely

$$\beta(\tilde{G}) = -\frac{F(\tilde{G})}{\partial F(\tilde{G})/\partial \tilde{G}}. \quad (3.43)$$

¹⁰In four dimensions one finds for lattice quantum gravity $\nu \approx 1/3$ instead [24,64].

The generic beta function equation, determining the scale evolution of the coupling [obtained from Eq. (3.42), and identical in form to it],

$$\mu \frac{d}{d\mu} \tilde{G}(\mu) = \beta(\tilde{G}(\mu)), \quad (3.44)$$

can then be integrated in the vicinity of the fixed point. It then leads to a definite relationship between the relevant coupling \tilde{G} , the renormalization-group invariant (cutoff independent) quantity $m = 1/\xi$, and an arbitrary sliding scale $\mu = 1/r$. Up to scales of order ξ , it determines the universal running of \tilde{G} , which will give rise to macroscopic effects provided the nonperturbative scale ξ is very large. In [15,64] this scale was naturally identified with the scaled cosmological constant, which here would correspond to the ratio λ/G . The result of Eq. (3.41) then corresponds to the limiting case $\nu \rightarrow 0$. In the language of Refs. [15,64], it leads in the vicinity of the fixed point to an exponentially small (for $r/\xi \rightarrow 0$) renormalization-group running of $\tilde{G}(\mu)$ or $\tilde{G}(r)$, namely, from Eqs. (3.41) and (3.44)

$$\tilde{G}(r) - \tilde{G}_c \underset{\tilde{G} \rightarrow \tilde{G}_c}{\sim} e^{-c(\xi/r)^2}. \quad (3.45)$$

The quantum correction on the r.h.s is therefore quite small as long as $r \ll \xi$.

All of the above was in the limit of infinite dimension. In Ref. [64] it was suggested, based on a simple geometric argument, that $\nu = 1/(d-1)$ for large d . Moreover, for the lattice theory in finite dimensions one finds no phase transition in $d = 2$ [65], $\nu \approx 0.60$ in $d = 3$ [41] and $\nu \approx 0.33$ in $d = 4$ [24,64], which then leads to the (almost constant) sequence $(d-2)\nu = 1, 0.60$ and 0.66 in the three cases, respectively. After interpolating this last series of values with a quadratic polynomial in $1/d$, one obtains $\nu \approx 1.9/d$ for large d . On the other hand, in Ref. [29] the value $\nu = 1/2d$ was obtained in the same limit with a Wilson-type continuum renormalization-group approach, in which a momentum space slicing technique is combined with a truncation to the Einstein-Hilbert action and a cosmological term. It seems that in either case our analytical results for the large d limit are consistent with, and to some extent corroborate, these previous findings. For completeness let us mention here that in the extreme opposite case, namely, close to 2 dimensions, one has the by now well-established result $\nu = 1/(d-2) + O((d-2)^0)$ [25,26].

It is of interest to contrast the result $\nu \sim 0$ for gravity in large dimensions with what one finds for scalar [7,32] and gauge [66] fields, in the same limit $d = \infty$. Known results, and what we have found here so far, can be combined and summarized as follows:

$$\begin{aligned} \text{scalar field} & \quad \nu = \frac{1}{2} \\ \text{lattice gauge field} & \quad \nu = \frac{1}{4} \\ \text{lattice gravity} & \quad \nu = 0. \end{aligned} \quad (3.46)$$

The first rather well-known result is rederived in Appendix A. The second one, obtained for non-Abelian gauge theories at large d , is recalled in Appendix B. It should be regarded as encouraging that the new value obtained here, namely $\nu = 0$ for gravitation, appears to some extent to be consistent with the general trend observed for lower spin, at least at infinite dimension.

As far as $1/d$ corrections are concerned, the result obtained previously in this section hinge on the crucial assumption that the random surface is noninteracting, in other words that any self-intersection or folding of the surface does not carry additional statistical weights. This is similar to an unconstrained random walk, where the effects of path intersection and backtracking are neglected. While these assumptions seem legitimate at infinite d (since there are infinitely many orthogonal dimensions to move into), they are no longer valid at finite d . As a result, the problem becomes much more complex, and one expects that ν will then no longer be equal to zero. Indeed in four dimensions $\nu \approx 1/3$ [24]. In the much simpler random walk case, a systematic expansion can be developed, leading for n intersections to an effective $\phi^{2(n-1)}$ interaction for the scalar field associated with the random walk. Unfortunately in the gravitational case it is much less clear how to develop such a systematic expansion.

IV. THE CONTINUUM CASE

For quantum gravity formulated in dimensions greater than four there are a number of natural questions that come to mind. Are there any special dimensions for gravity? How do the Feynman rules depend on d ? What does continuum gravity look like in large dimensions? Before discussing the gravitational case, it might be useful to examine and contrast the somewhat simpler cases of scalar and vector (gauge) theories.

A. Special values of d in field theories

In *scalar* field theories the special role of dimension four is easily brought out by writing the action, simply using dimensional arguments, as

$$\begin{aligned} S = & \frac{1}{2} \int d^d x [(\partial_\mu \phi(x))^2 - m_0^2 \phi^2(x)] - \frac{\lambda_0}{\Lambda^{d-4}} \\ & \times \int d^d x \phi(x)^4, \end{aligned} \quad (4.1)$$

where Λ is the ultraviolet cutoff, λ_0 the bare self-coupling, m_0 the bare mass, and with the fields having canonical dimension $m^{(d-2)/2}$. The self-coupling is dimensionless only in dimension four, and above that the model is de-

scribed in the long-distance, infrared limit by a free field [63]. The interaction term is *relevant* for $d < 4$, and *irrelevant* above $d = 4$. In particular for any $d > 4$ one can prove that the correlation length exponent ν equals one-half, the free field value [7,32]. The long-distance, infrared behavior is the same as for a free field.

In the case of $SU(N)$ non-Abelian *gauge theories*, one has that the coupling is, again, dimensionless only in four dimensions, a well-known signature of perturbative renormalizability. Above four dimensions purely dimensional arguments indicate the appearance of a nontrivial ultraviolet fixed point [a zero of the Callan-Symanzik $\beta(g)$ function] close to the origin,

$$\beta(g) = (d-4)g - \beta_0 g^3 + \dots, \quad (4.2)$$

with a nontrivial fixed point at $g_c^2 = (d-4)/\beta_0 + O((d-4)^2)$ separating what is believed to be a Coulomb, non-confining phase, from the confining phase known to exist for sufficiently strong coupling [54]. Since the theory is not perturbatively renormalizable above four dimensions, the analysis of either phase is rather problematic in the continuum. The transition is characterized by nontrivial critical exponents, and the Green's functions in the scaling region correspond to an interacting theory, which can only be reconstructed in the Coulomb phase $g < g_c$ as an expansion in $\epsilon = d-4$ [9].

One might wonder if anything special happens in dimensions $d > 4$, beyond what has just been discussed. In $SU(N)$ Yang-Mills with (Euclidean) classical action

$$I_{cl} = \frac{1}{g^2 N} \int d^d x \frac{1}{4} \text{tr} F_{\mu\nu}^2, \quad (4.3)$$

one has to one loop for the divergent part of the effective action

$$\Gamma_{div}^{(1)} = \frac{1}{4-d} \frac{26-d}{3} \frac{g^2 N}{16\pi^2} I_{cl}, \quad (4.4)$$

which vanishes in $d = 26$, and to two loops [67]

$$\Gamma_{div}^{(2)} = \frac{1}{4-d} \frac{34}{3} \left(\frac{g^2 N}{16\pi^2} \right)^2 I_{cl}. \quad (4.5)$$

One would be hard pressed though to conclude that the above results suggest anything dramatic might happen at $d = 26$ in the Yang-Mills case, as the change of sign in the one-loop divergence is still counteracted by the two-(and higher)-loop terms for sufficiently large g^2 . It seems in general that the structure of the continuum theory at large d remains quite complicated and possibly still not amenable to a perturbative treatment.

On the lattice on the other hand the presence of a phase transition has been clearly established in the large d limit, in fact largely irrespective of the specific choice of continuous symmetry group [66]. For the group $SU(N)$ a critical point in g appears at $2d(2N/g^2)^4 = \text{const}$ (with the constant depending of the specific choice of N), and

with an exponent at the transition given by $\nu = 1/4$ [66]. But it seems that finding such a transition critically hinges on using nonperturbative methods, which allow one to explore the strong coupling regime, and, in particular, the existence of two physically distinct phases.

In the case of *gravity*, the expression analogous to Eq. (4.1) is

$$\begin{aligned} & \lambda \int d^d x \sqrt{g} - \frac{1}{16\pi G} \int d^d x \sqrt{g} R + \frac{\alpha_0}{\Lambda^{4-d}} \\ & \times \int d^d x \sqrt{g} R_{\mu\nu} R^{\mu\nu} + \frac{\beta_0}{\Lambda^{4-d}} \int d^d x \sqrt{g} R^2 + \dots, \end{aligned} \quad (4.6)$$

which shows the suppression of the curvature squared terms in the infrared region, by factors $O(1/\Lambda^2)$ when compared to the Einstein term, whose coefficient also involves a dimensionful quantity, namely $\Lambda^{d-2}/(16\pi G_0)$ (here α_0 and β_0 , as well as $G_0 \equiv \Lambda^{d-2}G$, are taken to be dimensionless couplings).¹¹ It then seems legitimate to ask if there are any special dimensions for gravity, in particular, above $d = 4$. As already mentioned in the introduction, one has $d(d+1)/2$ independent components of the metric in d dimensions, and the same number of algebraically independent components of the Ricci tensor appearing in the field equations. The contracted Bianchi identities reduce the count by d , and so does general coordinate invariance, leaving $d(d-3)/2$ physical gravitational degrees of freedom in d dimensions. As a result, the number of physical degrees of freedom of the gravitational field grows rather rapidly (quadratically) with the number of dimensions.

The first step is naturally to examine tree level gravity, where all loop (quantum) effects are neglected [1,69,70]. Then in the nonrelativistic, static limit gravitational interactions are described by

$$\begin{aligned} I_2[T] &= -\frac{\kappa^2}{2} \int d^d x [T_{\mu\nu} \square^{-1} T^{\mu\nu} - (d-2)^{-1} T_{\mu}{}^{\mu} \square^{-1} T_{\nu}{}^{\nu}] \\ &\rightarrow -\frac{d-3}{d-2} \frac{\kappa^2}{2} \int d^{d-1} x T^{00} \mathcal{G} T^{00}, \end{aligned} \quad (4.7)$$

where the Green's function \mathcal{G} is the static limit of $1/\square$, and $\kappa^2 = 16\pi G$. The above result then incorporates at least two well-known facts, namely, that there are no Newtonian forces in $d = 2 + 1$ dimensions, and that the Einstein tensor vanishes identically in $d = 1 + 1$ dimensions. But nothing particularly noteworthy seems to happen, at least at tree level, above $d = 3$. At the same time, four space-time dimensions is known to be the lowest dimension for

¹¹Adding curvature squared terms to the bare action cures the perturbative nonrenormalizability problem, but raises new issues related to unitarity [68]. Curvature squared terms are expected to play important roles at very short distances, comparable to the cutoff scale, where fluctuations in the curvature can become of order $\sim \Lambda^2/G_0$.

which Ricci flatness does not imply the vanishing of the gravitational field, $R_{\mu\nu\lambda\sigma} = 0$, and therefore the first dimension to allow for gravitational waves and their quantum counterparts, gravitons. The tree level *static* gravitational potential above $d > 3$ is simply obtained by Fourier transform using

$$\int d^d x \frac{1}{k^2} e^{ik \cdot x} = \frac{\Gamma(\frac{d-2}{2})}{4\pi^{d/2} (x^2)^{d/2-1}} \quad (4.8)$$

and therefore implies $\int d^{d-1} \mathbf{x} e^{i\mathbf{k} \cdot \mathbf{x}} / \mathbf{k}^2 \sim 1/r^{d-3}$.

When quantum loop effects are turned on [3,4], one finds that the one-loop divergence, proportional to curvature squared terms, vanishes on shell,

$$\Gamma_{div}^{(1)} = \frac{1}{4-d} \frac{\hbar}{16\pi^2} \int d^4 x \sqrt{g} \left(\frac{7}{20} R_{\mu\nu} R^{\mu\nu} + \frac{1}{120} R^2 \right), \quad (4.9)$$

using the well-known result $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = -R^2 + 4R_{\mu\nu} R^{\mu\nu} +$ total derivative to eliminate Riemann squared terms. The complete set of one-loop divergences, computed using the heat kernel expansion and zeta function regularization close to 4 dimensions, can be found in the comprehensive review cited in [12], and further references therein. At two loops it was shown some time ago [5,6] that there is a nonremovable on-shell two-loop R^3 -type divergence

$$\Gamma_{div}^{(2)} = \frac{1}{4-d} \frac{209}{2880} \frac{\hbar^2 G}{(16\pi^2)^2} \int d^4 x \sqrt{g} R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\kappa\lambda} R_{\kappa\lambda}{}^{\mu\nu}. \quad (4.10)$$

In the last quoted reference it is argued that in the above expression the 209 arises from 11×19 , with the factor of 11 coming from $(26-d)/2$, as expected from closed string theory [6]. Thus the latter divergence might vanish again at $d = 26$, but it is not expected that the same will happen at higher loops.

Recent two-loop results based on the $2 + \epsilon$ expansion for gravity with a cosmological constant [25], inspired by the $2 + \epsilon$ of other, simpler field theory models [8,71,72], show the appearance of a nontrivial ultraviolet fixed point in the G beta function above two dimensions,

$$\begin{aligned} \beta(G) &= (d-2)G - \frac{2}{3}(25 - n_f)G^2 \\ &\quad - \frac{20}{3}(25 - n_f)G^3 + \dots \end{aligned} \quad (4.11)$$

(for n_f massless real scalar fields minimally coupled to gravity). They could be possibly relevant as a first crude approximation to the four-dimensional theory (to the extent that they represent a manifestly gauge-invariant resummation of those diagrams which can be regarded as dominant close to 2 dimensions). But unfortunately they can hardly be thought as useful in the limit $d \rightarrow \infty$, especially in view of the fact that the Borel summability in $\epsilon =$

$d - 2$ [73,74] of such an expansion still remains a largely open question.

B. Feynman rules in d dimensions

A direct examination of the Feynman rules for continuum gravity at large d indeed reveals the occurrence of some degree of simplification. But first we should clarify our conventions and notation for this section, which are taken from [75], and where one expands around the flat Minkowski space-time metric, with signature given by $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1, \dots)$. The Einstein-Hilbert action in d dimensions is then given by

$$S_E = + \frac{1}{16\pi G} \int d^d x \sqrt{-g(x)} R(x), \quad (4.12)$$

with $g(x) = \det(g_{\mu\nu})$ and R the scalar curvature (it will also be assumed in the following that the bare cosmological constant is zero). Furthermore the coupling of gravity to scalar particles of mass m is described by the action

$$\begin{aligned} S_m &= \frac{1}{2} \int d^d x \sqrt{-g(x)} [g^{\mu\nu}(x) \partial_\mu \phi(x) \partial_\nu \phi(x) \\ &\quad - m^2 \phi^2(x)]. \end{aligned} \quad (4.13)$$

Usually in perturbation theory the metric $g_{\mu\nu}(x)$ is expanded around the flat metric $\eta_{\mu\nu}$ [3], by writing

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa \tilde{h}_{\mu\nu}(x), \quad (4.14)$$

with $\kappa^2 = 32\pi G$. In the de Donder gauge the graviton propagator is then given by

$$D_{\mu\nu\rho\sigma}(p) = \frac{i}{2} \frac{\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \frac{2}{d-2} \eta_{\mu\nu} \eta_{\rho\sigma}}{p^2 + i\epsilon}, \quad (4.15)$$

which suggests that the conformal mode contribution might go away as $d \rightarrow \infty$. But further thought reveals that this conclusion might perhaps be fallacious, as a different type of expansions seem to lead to slightly different conclusions.

If one follows the method of Refs. [76,77], then one defines the small fluctuation graviton field $h_{\mu\nu}(x)$ instead via

$$g^{\mu\nu}(x) \sqrt{-g(x)} = \eta^{\mu\nu} + \kappa h^{\mu\nu}(x). \quad (4.16)$$

One advantage of this expansion over the previous one is that it leads to considerably simpler Feynman rules, both for the graviton vertices and for the scalar-graviton vertices. A gauge fixing term can then be added [78,79], for example, of the form

$$\frac{1}{\kappa^2} (\partial_\mu \sqrt{-g(x)} g^{\mu\nu})^2, \quad (4.17)$$

as again used in [77]. The bare graviton propagator is then given simply by

$$D_{\mu\nu\rho\sigma}(p) = \frac{i}{2} \frac{\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}}{p^2 + i\epsilon}, \quad (4.18)$$

whose structure is now unaffected by the limit $d \rightarrow \infty$. Thus with the latter definition for the gravitational field, there are no factors of $1/(d-2)$ for the graviton propa-

gator in d dimensions; such factors appear instead in the expressions for the Feynman rules for the vertices. For the three-graviton and two ghost-graviton vertex the relevant expressions are quite complicated. The three-graviton vertex can be written as

$$\begin{aligned} U(q_1, q_2, q_3)_{\alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_3} = & -i\frac{\kappa}{2} \left[q_{(\alpha_1}^2 q_{\beta_1)}^3 \left(2\eta_{\alpha_2(\alpha_3} \eta_{\beta_3)\beta_2} - \frac{2}{d-2} \eta_{\alpha_2\beta_2} \eta_{\alpha_3\beta_3} \right) \right. \\ & + q_{(\alpha_2}^1 q_{\beta_2)}^3 \left(2\eta_{\alpha_1(\alpha_3} \eta_{\beta_3)\beta_1} - \frac{2}{d-2} \eta_{\alpha_1\beta_1} \eta_{\alpha_3\beta_3} \right) \\ & + q_{(\alpha_3}^1 q_{\beta_3)}^2 \left(2\eta_{\alpha_1(\alpha_2} \eta_{\beta_2)\beta_1} - \frac{2}{d-2} \eta_{\alpha_1\beta_1} \eta_{\alpha_2\beta_2} \right) + 2q_{(\alpha_2}^3 \eta_{\beta_2)(\alpha_1} \eta_{\beta_1)(\alpha_3} q_{\beta_3)}^2 \\ & + 2q_{(\alpha_3}^1 \eta_{\beta_3)(\alpha_2} \eta_{\beta_2)(\alpha_1} q_{\beta_1)}^3 + 2q_{(\alpha_1}^2 \eta_{\beta_1)(\alpha_3} \eta_{\beta_3)(\alpha_2} q_{\beta_2)}^1 \\ & + q^2 \cdot q^3 \left(\frac{2}{d-2} \eta_{\alpha_1(\alpha_2} \eta_{\beta_2)\beta_1} \eta_{\alpha_3\beta_3} + \frac{2}{d-2} \eta_{\alpha_1(\alpha_3} \eta_{\beta_3)\beta_1} \eta_{\alpha_2\beta_2} - 2\eta_{\alpha_1(\alpha_2} \eta_{\beta_2)(\alpha_3} \eta_{\beta_3)\beta_1} \right) \\ & + q^1 \cdot q^3 \left(\frac{2}{d-2} \eta_{\alpha_2(\alpha_1} \eta_{\beta_1)\beta_2} \eta_{\alpha_3\beta_3} + \frac{2}{d-2} \eta_{\alpha_2(\alpha_3} \eta_{\beta_3)\beta_2} \eta_{\alpha_1\beta_1} - 2\eta_{\alpha_2(\alpha_1} \eta_{\beta_1)(\alpha_3} \eta_{\beta_3)\beta_2} \right) \\ & \left. + q^1 \cdot q^2 \left(\frac{2}{d-2} \eta_{\alpha_3(\alpha_1} \eta_{\beta_1)\beta_3} \eta_{\alpha_2\beta_2} + \frac{2}{d-2} \eta_{\alpha_3(\alpha_2} \eta_{\beta_2)\beta_3} \eta_{\alpha_1\beta_1} - 2\eta_{\alpha_3(\alpha_1} \eta_{\beta_1)(\alpha_2} \eta_{\beta_2)\beta_3} \right) \right]. \end{aligned} \quad (4.19)$$

Again one notes that some terms become negligible as $d \rightarrow \infty$, but the remaining ones can have either sign, giving rise to nontrivial cancellations even for large d . The ghost-graviton vertex is given by

$$V(k_1, k_2, k_3)_{\alpha\beta, \lambda\mu} = i\kappa [-\eta_{\lambda(\alpha} k_{1\beta)} k_{2\mu} + \eta_{\lambda\mu} k_{2(\alpha} k_{3\beta)}], \quad (4.20)$$

and the two scalar-one graviton vertex is given by

$$\frac{i\kappa}{2} \left(p_{1\mu} p_{2\nu} + p_{1\nu} p_{2\mu} - \frac{2}{d-2} m^2 \eta_{\mu\nu} \right), \quad (4.21)$$

where the p_1, p_2 denote the four-momenta of the incoming and outgoing scalar field, respectively. Finally the two scalar-two graviton vertex is given by

$$\frac{i\kappa^2 m^2}{2(d-2)} \left(\eta_{\mu\lambda} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\lambda} - \frac{2}{d-2} \eta_{\mu\nu} \eta_{\lambda\sigma} \right), \quad (4.22)$$

where one pair of indices (μ, ν) is associated with one graviton line, and the other pair (λ, σ) is associated with the other graviton line. Again one notices some simplification in the limit $d \rightarrow \infty$. These rules follow readily from the expansion of the gravitational action to order $G^{3/2}$ (κ^3), and of the scalar field action to order G (κ^2).

The next step would involve a careful analysis of what the dominant diagrams are in the large d limit (still keeping in mind the serious shortcoming of assuming a vanishing bare cosmological constant), assuming that such a procedure remains reliable in this limit, in the sense that a

complete resummation can be performed, and that there are no large nonperturbative, nonanalytic contributions. But it seems so far that in the case of gravity there are conflicting claims in the literature [80–82] as to what exactly happens in the continuum as $d \rightarrow \infty$.

In Refs. [80,81] a gauge-invariant expansion in $1/d$ was developed for vanishing bare cosmological constant, considering both the case where the extra dimensions are noncompact and the case where they are highly compactified. The observation was made that there are order by order (in $1/d$) cancellations of large numbers of graphs, but the origin of such cancellation remained a puzzle. However, it was found that the leading term of any Green's function was given by a set of disjoint bubble graphs. It was then determined that the graviton propagator acquires a physical pole near the Planck mass, unfortunately in a region where the validity of the expansion appears questionable. Finally it was claimed that at $d = \infty$ phase-space factors suppress the Feynman integrations and the theory is therefore finite.

In the recent work of [82] it is also claimed that a consistent leading large- d limit exists for the Einstein theory without cosmological term, and that it can be constructed using a subclass of planar diagrams, which seems somewhat in disagreement with the class of diagrams identified in the previous references. It is then found that the large- d quantum gravity limit is well defined and renormalizable, provided the space-time integrations are not extended to the full d -dimensional space-time, in other words if the full space-time allows for compactified di-

mensions (the last result does not seem entirely surprising, as compactifying and shrinking extra dimensions leads to an effectively lower dimensional theory, with possibly convergent momentum integrations, depending on how the limits are taken).

But it seems difficult to reconcile the above quoted results with the fact that (a) the perturbative nonrenormalizability issue only gets worse in the continuum as one increases the dimension, and (b) that at least close to 2 dimensions an ultraviolet fixed point is *known* to exist, and somehow completely fails to show up in the large d diagrammatic treatment. The more likely scenario is perhaps that the theory remains perturbatively nonrenormalizable even in the large d case, and therefore just as intractable in the continuum as the equally difficult large- d Yang-Mills case. So far the continuum perturbative diagrammatic treatment has not led yet to any conclusive predictions about the behavior of physical gravitational observables (such as scale dependence and renormalization of couplings, nontrivial fixed points, anomalous scaling dimensions etc.), which makes it difficult to compare with recent nonperturbative lattice [24,64] and continuum [25,29] results in low dimensions, both of which project a rather different picture.

V. CONCLUSIONS

In this paper we have examined the lattice formulation of quantum gravity in the large d limit. Such a line of inquiry was stimulated by the fact that statistical systems based on local interactions generally tend to simplify considerably in this limit, where each point is found to be surrounded by a large number of neighbors, and mean field theory methods apply. Even when mean field theory does not apply, the hope was that the theory would simplify significantly, to a point where it could be solved exactly. In view of the general lack of analytical results, aside from perturbation theory and some other investigations restricted to low and somewhat unphysical dimensions, one would expect such results would help shed new light on the true nonperturbative ground state of quantum gravity in four dimensions.

While $d = \infty$ at first seems rather remote from the physical case $d = 4$, one can make the case that the well-known $1/N$ expansion of statistical mechanics system and $SU(N)$ gauge theories (the planar limit) has lead to some remarkable insights into the finite N structure of these theories, and in some cases even to quantitatively accurate answers for critical exponents (in the statistical mechanics context) and specific phenomenological predictions (for example in low energy QCD applications). Indeed more than once it has been argued that in the case of QCD, based on both theoretical and phenomenological arguments, that $1/N = 0$ is not too remote from the physical case $1/N = 1/3$. In the same spirit, $1/d = 0$ might not be as remote as

it seems at first from the real world case of $1/d = 1/4$ theory.

In pursuing the $1/d$ expansion for gravity we have followed two somewhat complementary approaches. In the first approach, various terms which appear in the lattice gravitational (Regge) action were expanded in powers of $1/d$. Since the resulting expressions are still rather cumbersome, we resorted to a combined weak field expansion, perturbing arbitrarily coordinated lattices built out of nearly equilateral simplices. The resulting expressions were then evaluated for the cross polytope, a triangulation of the d -dimensional sphere based on the dual of a d -dimensional hypercube. These were then shown to lead to a second order phase transition at a critical point $k_c \sim \lambda/d$, summarized in the result of Eq. (2.40). Near this critical point it was found that all $\sim d^2$ lattice degrees of freedom become massless (in the sense that all eigenvalues of the quadratic fluctuation matrix have the same sign and approach zero), suggesting a complete disappearance of the conformal mode instability in the Euclidean theory at $d = \infty$, in agreement with the naive conclusion from Eq. (4.15).

The second, and perhaps more ambitious, approach was based on a combined strong coupling (small $k = 1/8\pi G$) and large d expansion. First, in the strong coupling expansion, it was found that the relevant diagrams for the curvature correlation function to a given order in k can be identified with closed surfaces, built out of parallel transport polygons, with each polygon identified with the parallel transport of a test vector around an elementary loop residing within the dual lattice. We then argued that in the large d limit it should be possible to neglect surface self-intersections. One then finds that such surfaces, based on their equivalent description in terms of a two-dimensional massless field theory, naturally give rise to a logarithmic divergence of the correlation length at the critical point at k_c , leading in this limit to the exact (and presumably universal) result of Eq. (3.41).

The natural question then arises, and which is difficult to ignore, of whether these large d results have any relevance for a physical four-dimensional world. To the extent that the two cases are physically not too far apart, one would be tempted to conclude that the dependence of the correlation length ξ on the gravitational coupling, as expressed in Eq. (3.41), and, conversely, the dependence of the running gravitational coupling on ξ , as expressed in Eq. (3.45), would suggest, for large d , finite but *exponentially* small corrections to classical gravity, at least in a scaling regime where the relevant distances involved are much smaller than the macroscopic curvature scale, $r \ll \xi \sim 1/\sqrt{R}$, but still much larger than the Planck scale, $r \gg l_p \sim \sqrt{G}$. It is noteworthy that the quantum corrections computed here are nonanalytic at $r = 0$, in spite of the fact that at short distances they become rather small, and thus provide to some extent a justification for the semiclassical picture of

quantum gravity. In terms of the parameters relevant for vacuum structure, the above-mentioned nonperturbative curvature scale then corresponds to a graviton vacuum condensate of order $\xi^{-1} \sim 10^{-30}$ eV, extraordinarily tiny compared to the QCD color condensate ($\Lambda_{\text{QCD}} = 220$ MeV) and the electroweak Higgs condensate ($v = 250$ GeV). Furthermore, as has been stressed before, the quantum gravity theory, at least in its present framework, does not and cannot provide a value for the nonperturbative curvature scale ξ , which ultimately needs to be fixed by phenomenological input. But, to the extent that this curvature scale clearly does not coincide with the Planck scale (the cutoff scale), there is some room left for it to take a very large, even cosmological, value. The lattice gravity model in fact provides a clear case where the naive identification of the curvature scale with the Planck scale can be shown to be incorrect, due to the highly nontrivial renormalization effects of strongly fluctuating quantum gravitational fields, which cleverly arrange for the two scales to differ significantly in magnitude, the more so as one approaches the critical point.

Finally, in the last section, we have attempted to make contact with known results for the continuum theory above four dimensions, and, in particular, those which have some degree of relevance for the limit $d \rightarrow \infty$. Generally, and in analogy with the non-Abelian gauge theory case, it appears that the continuum theory does not seem to lead to the same level of simplification as the regularized lattice gravity model discussed in Secs. II and III (and this in spite of their purported, but so far proven only for $d = 3, 4$, equivalence in the lattice continuum limit). Indeed in either case (gravity and gauge), the issue of perturbative nonrenormalizability only gets worse with increasing dimension. Ultimately we would tend to ascribe this state of affairs to the fact that it appears quite challenging to perform the

needed resummation of the continuum theory with a bare cosmological constant (as done explicitly only close to 2 dimensions), perhaps an essential ingredient required to determine the true nonperturbative, long-distance behavior of quantum gravitation—even in infinitely many dimensions.

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APPENDIX A: SCALAR CASE AND RANDOM WALKS

The scalar field case is quite straightforward and is therefore worth reproducing here. It relies on the well-known equivalence between the $\lambda\phi^4$ scalar field theory and the Ising model, as far as their critical or long-distance behavior is concerned [83]. The Ising partition function is given in any dimension by

$$Z(\beta) = \sum_{S_i = \pm 1} \exp\left[\beta \sum_{\langle ij \rangle} S_i S_j\right], \quad (\text{A1})$$

where $\langle ij \rangle$ denotes a sum over nearest neighbors ($2d$ in d dimensions, for a simple cubic lattice). The corresponding scalar field theory is obtained by using a straightforward Gaussian integral representation for the Ising statistical weight, which reads

$$\sum_{S_i = \pm 1} e^{(\beta/2) \sum_{ij} S_i M_{ij} S_j} = (2\pi)^{-(N/2)} (\det \beta M)^{-(1/2)} \int \prod_i d\phi_i e^{-[1/(2\beta)] \sum_{ij} \phi_i M_{ij}^{-1} \phi_j + \sum_i \log(2 \cosh \phi_i)}, \quad (\text{A2})$$

and then expanding the exponent in powers of the field ϕ and its derivatives. In either case the critical point is located where the renormalized mass of the lowest excitation vanishes. Returning to the Ising case, the spin susceptibility is then given by

$$\chi(\beta) = \frac{1}{Z(\beta)} \sum_k \sum_{S_i = \pm 1} S_0 S_k \exp\left[\beta \sum_{\langle ij \rangle} S_i S_j\right], \quad (\text{A3})$$

and coincides with the spin correlation function $\langle S_0 S_k \rangle$, summed over sites k . Equivalently, it can be regarded as the Fourier transform of the spin correlation function, evaluated at zero momentum. It is convenient to rewrite the formula for the partition function as

$$Z(\beta) = (\cosh \beta)^N \sum_{S_i = \pm 1} \prod_{\langle ij \rangle} [1 + t S_i S_j], \quad (\text{A4})$$

with $t = \tanh \beta$, N the number of sites on the lattice, and the product ranging over all links on the lattice. The expansion in t has an obvious diagrammatic representation [84], consisting in the case of $\chi(\beta)$ of open paths linking the site 0 to any site k , with each link appearing at most once (but multiple times, if the expansion in β is used instead). Write $\chi = \sum_n \chi_n t^n$, where χ_n is now the number of open paths of length n with fixed origin. We obtain a path of length $n + 1$ by adding a link at its end, which can be done in $2d - 1$ ways, giving $\chi_{n+1} \sim 2d\chi_n$, and so for large d one has [32]

$$\chi(\beta) \propto \sum_{n=0}^{\infty} (2dt)^n = \frac{1}{1-2dt}. \quad (\text{A5})$$

Here use has been made of the fact that for large d excluded volume effects can be neglected, so that the factor $2d - 1$ can simply be replaced by $2d$. Then from $\chi \sim 1/(p^2 + m^2)|_{p=0} = \xi^2$ (the spin correlation function evaluated at zero momentum) one obtains $\xi \sim 1/(t_c - t)^\nu$ with $t_c = 1/2d$ and $\nu = 1/2$.

APPENDIX B: VECTOR CASE AND q -COORDINATED CAYLEY TREES

In the large d limit dominant diagrams in the strong coupling expansion of lattice gauge theories are represented not by surfaces, but by trees made out of three-dimensional cubes [66,85]. In the case of the plaquette-plaquette correlation function, these are all the trees which can be constructed such that they are anchored on the two given plaquettes.

The generating function for a q -coordinated Cayley tree [86] (a Bethe lattice with q links emanating from each vertex) is given by [66]

$$g(t) = \frac{u(1 - \frac{q}{2}u)}{(1-u)^2}, \quad (\text{B1})$$

with parameters u and t related by

$$t = u(1-u)^{q-2}; \quad (\text{B2})$$

$q = 3$ corresponds to a trivalent or binary tree. In the $SU(N)$ gauge case, one has $t = 2d\beta^4$, with $\beta \approx 2N/g^2$ at strong coupling, and then the above is essentially the same as the free energy of the gauge theory (up to various inessential constants). Also, in the gauge case $q = 6$ (since a cube has six faces) and a new cube can be attached to any of the six faces of the original cube (again ignoring excluded volume effects at large d), thus creating a continuous tree made out of cubes. The free energy is then equal to a sum over all possible trees of arbitrary length, giving rise to hydralike configurations as viewed from the diagrammatic perspective of the strong coupling expansion.

In particular, the plaquette-plaquette correlation function is obtained from the second derivative of the above generating function $g(t)$ with respect to the coupling β . It

can be represented as the sum of all trees of arbitrary shape (with coordination $q = 6$), but now with two fixed end points. Extending the analysis to general q , one can show that in fact the key result is in fact independent of q for $q > 2$. The relevant singularity in the second derivative of the free energy $g(t)$ corresponds to $u_c = 1/(q-1)$. Expanding Eq. (B2) in the vicinity of this point, one finds

$$t = t_c - \frac{1}{2}(q-1)^{4-q}(q-2)^{q-3} \left(u - \frac{1}{q-1}\right)^2 + \dots \quad (\text{B3})$$

i.e. the linear term vanishes. In the above expression t_c is the critical point,

$$t_c = \frac{(q-2)^{q-2}}{(q-1)^{q-1}}. \quad (\text{B4})$$

Thus $t_c - t \sim (u - u_c)^2$ for any $q > 2$. First and second derivatives of the free energy $g(t)$ with respect to t can then be calculated via

$$\begin{aligned} \frac{dg}{du} &= \frac{(q-1)u-1}{(u-1)^3} & \frac{dg}{dt} &= \frac{dg}{du} \frac{du}{dt} = \frac{1}{(1-u)^q} \\ \frac{d^2g}{dt^2} &= \frac{q(1-u)^{2-2q}}{1+(1-q)u}, \end{aligned} \quad (\text{B5})$$

which for any $q > 2$ behaves in the limit $t \rightarrow t_c$ as

$$\frac{d^2g}{dt^2} \sim \frac{q(q-2)^{(1-3q)/2}}{\sqrt{2}(q-1)^{(2-3q)/2}} \frac{1}{\sqrt{t_c-t}} \sim \xi^2. \quad (\text{B6})$$

Here use has been made of the fact that the second derivative of the free energy brings down two plaquette terms, giving the plaquette-plaquette correlation function, summed over both plaquette coordinates, and which is therefore equivalent to the Fourier transform of the plaquette-plaquette correlation at zero momentum. Thus one obtains the momentum space plaquette-plaquette correlation at zero momentum, or $1/(p^2 + m^2)|_{p=0}$, with $m = \xi^{-1}$, and this then gives $\xi \sim 1/(t_c - t)^{1/4}$ and thus $\nu = 1/4$ for any $q > 2$. It is further observed in [66] that the second order phase transition of the gauge theory described by $g(t)$ bears a striking similarity to the condensation of branched polymers, with the polymer chain built out of (trees of) three-dimensional cubes.

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