January 1987

QCD on the Lattice and Light Hadron Spectroscopy *

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ABSTRACT

Results in lattice Quantum Chromodynamics with fermions are reviewed. Estimates for the hadron spectrum on large lattices without and with light fermion polarization loops are discussed, and the results for different fermion regularizations are compared. The effects of the loops are found to be significant in lattice units, but to a greater part they can be re-absorbed in a redefinition of the overall scale. New estimates for the light quark masses with Wilson fermions are also presented. They differ substantially from previous estimates obtained by other methods in the continuum, and are much smaller. Finally it is argued that staggered fermions can lead to some anomalous results, which arise because of the doubling problem.

* Talk presented at the Irvine Conference on *Nonperturbative Methods in Field Theory*, January 7-9, 1987, to appear in Nuclear Physics B.

I - INTRODUCTION

Considerable progress has recently been made in understanding nonperturbative effects in QCD using numerical methods for the lattice formulation [1]. Among the quantities of physical interest that have been estimated in the pure gauge theory one can mention the string tension, the glueball mass and the finite temperature deconfinement transition. Of great interest in QCD is the hadron spectrum, because a wealth of experimental data has been available for quite some time. The values of the quark masses are also of central importance, and they can only be inferred indirectly from the observed hadron masses. Any grand unified theory that attempts to go beyond QCD and claims to explain the observed lepton and quark spectrum will have to be consistent with the observed values for the latter.

Since most of the lattice QCD computations have been done neglecting the fermion vacuum polarization contributions [2], one of the outstanding problems that has remained is the development of a set of efficient algorithms to reliably include these effects in a realistic study of the full low energy spectrum.

The direct introduction of fermions as anticommuting variables appears impractical in systems with dimension greater than two, and several approaches have been suggested in the literature to deal with fermions in an approximate but controllable way, and which are such that in four dimensions the requirements in computer time are kept at a reasonable level. It appears at the present moment that the pseudofermion method [3,4] is among the more promising ones, and it is the purpose here to describe some preliminary results obtained when the effects of the fermion loops are included this way. For other methods the reader should consult ref. [6] and [7].

In section II notation is introduced, and the features of the Monte Carlo and Langevin pseudofermion methods are reviewed. The sources of statistical and systematic errors associated with the approximations are also pointed out. For most of the following studies the Wilson fermion action [1,7] was preferred, because of the absence of the troublesome species doubling problem, and because of the clear advantages it presents as far as the identification of particles with definite spin and parity is concerned.

Section III contains a brief discussion of the methodology used in extracting the mass of the low-lying hadron states and other quantities of physical interest by Monte Carlo methods, and the strategy that should be followed to control statistical and systematic errors in the calculations.

In section IV results for hadron and quark masses without fermion polarization loops are discussed, using recent computations done on a $10 \times 10 \times 10 \times 20$ lattice at $g^2 = 1$. General agreement with the observed light hadron spectrum is found, but some significant lattice regularization dependence of the light quark masses is observed. The discrepancies in the different estimates can be explained by large anomalous dimension factor corrections. When the corrections are taken into account the quark masses agree reasonably well with recent QCD sum rule estimates in the continuum.

In section V a discussion follows on the effect of including light quark polarization effects on a comparably sized lattice. It is found that for three light flavors a significant readjustment of the cutoff takes place at fixed coupling constant. But when physical ratios are computed, the results for the light hadron spectrum are very similar to the case of no loops. Some of the preliminary results are amusing, if not unexpected. Surprisingly the invariant quark masses are now much smaller, by almost a factor of two compared to the results with no loops.

Finally section VI describes a computation of the η' mass which yields encouraging results, improving on a previous computation done on a much smaller lattice, and section VII contains the conclusions.

II - LATTICE QUANTUM CHROMODYNAMICS WITH FERMIONS

Here the notation will be introduced. The total action consists of a gauge contribution

$$S_G = -\frac{\beta}{6} \sum_{n,\mu<\nu} \mathrm{Tr}\, U_{n,\mu} U_{n+\mu,\nu} U^\dagger_{n+\nu,\mu} U^\dagger_{n,\nu} + c.c. \tag{2.1}$$

and a fermion contribution [1,7]

$$S_F = \; k \sum_f \sum_{n,\mu} [\bar\psi^{(f)}_n (r+\gamma_\mu) U_{n\mu} \psi^{(f)}_{n+\mu} + \bar\psi^{(f)}_{n+\mu} (r-\gamma_\mu) U^\dagger_{n\mu} \psi^{(f)}_n] - \sum_f \sum_n \bar\psi^{(f)}_n \psi^{(f)}_n \tag{2.2}$$

The U's are 3×3 complex matrices elements of the group $SU(3)$ and the γ's are euclidean Gamma matrices, which obey $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ and are hermitean $\gamma^\dagger_\mu = \gamma$. The gauge coupling g is related to β by $\beta = 6/g^2$, for simplicity the lattice spacing a is set equal to one in most of the following, and the lattice is taken to be periodic of size $L \times L \times L \times T$. Also most of the time only the cases $r=1$ (Wilson fermions) and $r=0$ (naive or Kogut-Susskind fermions) will be considered, and the number of fermion flavors n_f will be taken to be between 0 and 3, a choice which appears reasonable if one wants to investigate the properties of light hadrons. Eq. (2.2) is also somewhat restrictive in the sense that all flavors are assumed to have equal mass. If different flavors are given different masses, then the k_f's will differ in their values. Notation can be somewhat simplified by defining the matrix $\Delta[U]$

$$\Delta^{ab}_{\alpha\beta}(n,n') = -k[(r+\gamma_\mu)_{\alpha\beta} U^{ab}_{n-\mu,\mu} \delta_{n,n'-\mu} + (r-\gamma_\mu)_{\alpha\beta} U^{\dagger ab}_{n,\mu} \delta_{n,n'+\mu}] + \delta_{\alpha\beta}\delta^{ab}\delta_{nn'} \tag{2.3}$$

Then the total action becomes

$$S[U,\psi,\bar\psi] = S_G[U] - \sum_{ij} \bar\psi_i \Delta_{ij}[U] \psi_j \tag{2.4}$$

and the partition function is given by

$$Z = \int [dU][d\bar\psi][d\psi] \exp(-S_G - S_F) \tag{2.5}$$

where $[dU] = \prod_{n,\mu} dU_{n,\mu}$ is the invariant measure on SU(3), and the fermion integration means $[d\bar{\psi}][d\psi] = \prod_{n,\alpha,a,f} d\bar{\psi}^{a(f)}_{\alpha n} \prod_{m,\beta,b,f'} d\psi^{b(f')}_{\beta m}$.

Since the action is quadratic in the fermions, it is straightforward to integrate them out using the Matthews-Salam formula, which gives

$$S_{eff}[U] = S_G[U] - n_f \ \ln \det \Delta[U] \tag{2.6}$$

and the determinant now induces a nonlocal interaction among the gauge fields. Because of the symmetry $\gamma_5 \Delta[U] \gamma_5 = \Delta^\dagger[U]$ one has that the determinant of $\Delta[U]$ is real, and positive definite for $k < 1/8r$, and one can write

$$\Big\{ \det \Delta[U] \Big\}^{n_f} = \Big\{ \det \Delta[U] \Delta^\dagger[U] \Big\}^{n_f/2} \tag{2.7}$$

giving a real, positive definite euclidean measure. In the infinite volume limit one can argue that the determinant is real and positive also for $1/8r < k < k_c$, where k_c is the critical value of k at which the pseudoscalar meson mass vanishes [8].

The non-locality the fermion determinant poses special problems in a numerical simulation since it induces a long-range interaction between the gauge variables. On the other hand in the actual updating done during a simulation one is only interested in the change of the weight factor when one link is locally changed

$$U_{n\mu} \to U_{n\mu} + \delta U_{n\mu} \qquad \Delta[U] \to \Delta[U] + \delta\Delta[U] \tag{2.8}$$

Then one has the well known identity

$$\begin{aligned} \frac{\det(\Delta[U] + \delta\Delta[U])}{\det(\Delta[U])} &= \det(1 + \Delta^{-1}[U]\delta\Delta[U]) = \exp\big(\mathrm{Tr} \ln(1 + \Delta^{-1}[U]\delta\Delta[U])\big) \\ &\approx \exp\big(\mathrm{Tr}(\Delta^{-1}[U]\delta\Delta[U])\big) \end{aligned} \tag{2.9}$$

where the last step assumes a small δU. Note that for a change of a single link $\delta\Delta[U]$ has only a few non-vanishing matrix elements, since $\Delta[U]$ is sparse and the determinant is computed for a small matrix (12×12 for three colors and four spins). Thus the problem of evaluating a determinant of a large matrix has been reduced to the problem of evaluating a few matrix elements of the inverse.

In the pseudofermion method [3,4] the change in the effective action δS_{eff} is expanded to lowest order in δU, and terms of higher order in δU are neglected. One then obtains using eq. (2.7)

$$\begin{aligned} \delta S_{eff} &= \delta S_G - n_f \mathrm{Tr}\Big[\Delta^{-1}[U] \frac{\delta\Delta[U]}{\delta U_{n\mu}} \delta U_{n\mu}\Big] + O(\delta U^2) \\ &= \delta S_G - \frac{n_f}{2} \mathrm{Tr}\Big[\Delta^{-1}[U] \frac{\delta\Delta[U]}{\delta U_{n\mu}} \delta U_{n\mu} + \Delta^{\dagger -1}[U] \frac{\delta\Delta^\dagger[U]}{\delta U_{n\mu}} \delta U_{n\mu}\Big] + O(\delta U^2) \end{aligned} \tag{2.10}$$

When written out explicitly this expression becomes

$$\delta S_{eff} = \delta S_G - n_f \sum_{n,\mu} [< \bar{\psi}^a_n (r+\gamma_\mu) \psi^b_{n+\mu} > \delta U^{ab}_{n\mu} + < \bar{\psi}^a_{n+\mu} (r-\gamma_\mu) \psi^b_n > \delta U^{\dagger ab}_{n\mu}] + O(\delta U^2) \tag{2.11}$$

where one has defined the fermion 'current'

$$J^{ab}_{n\mu} \equiv < \bar{\psi}^a_n (r+\gamma_\mu) \psi^b_{n+\mu} > = \frac{1}{2} \mathrm{Tr} \big[(r+\gamma_\mu) (\Delta^{-1}[U])^{ab}_{n+\mu,n} + (r-\gamma_\mu) (\Delta^{\dagger -1}[U])^{ab}_{n+\mu,n} \big] \tag{2.12}$$

and the angular brackets ($<>$) here denote averaging over the fermion fields only.

In other words, the procedure replaces the currents in a given gauge field configuration by their averages, which can be computed by a second, separate stochastic (Monte Carlo or Langevin) process. The power of the method lies in the fact that for small enough δU the currents need not be re-computed when a single link is changed, but can be evaluated *once* before each sweep through the gauge configuration. Consider first the case in which the fermion currents are evaluated by an auxiliary Monte Carlo simulation. In order to evaluate the currents, introduce a set of complex boson fields ('pseudofermions') $\phi^a_{n\alpha}$ having the quantum numbers of the fermions, and define also for convenience a set of fields $\chi^a_{n\alpha}$ with $\chi_n = \sum_m (\Delta[U])_{nm} \phi_m$. A suitable positive action for these fields is then [9]

$$S_{pf}[U] = \sum_{nm} \phi^*_n (\Delta^\dagger[U] \Delta[U])_{nm} \phi_m = \sum_n \chi^*_n \chi_n \tag{2.13}$$

which is positive definite (all space-time, spin, color and flavor indices have been denoted here for brevity by m,n), and one obtains

$$\Big(\Delta[U] \Delta^\dagger[U] \Big)^{-1}_{nm} = \ll \phi^*_m \phi_n \gg = \frac{\int [d\phi^* d\phi] \; \phi^*_m \phi_n \; \exp(-S_{pf}[U])}{\int [d\phi^* d\phi] \; \exp(-S_{pf}[U])} \tag{2.14}$$

where the symbol $\ll \gg$ stands for an average over the pseudofermion distribution, in a given fixed background gauge field configuration. It then follows immediately that the average currents are given by formulae of the type

$$J^{ab}_{n\mu} \equiv < \bar{\psi}_n (r+\gamma_\mu) \psi_{n+\mu} > = \frac{1}{2} \big[\ll \chi^*_n (r+\gamma_\mu) \phi_{n+\mu} \gg + \ll \phi^*_n (r-\gamma_\mu) \chi_{n+\mu} \gg \big] \tag{2.15}$$

In practice it is sufficient, for the range of quark masses accessible in present studies (which in turn is limited by the size of the lattice), to compute the currents using a few (say in the range of twenty to one hundred) pseudofermionic sweeps, which will be denoted by n_{pf}. Since the change in the U's is small, it is sufficient to compute *all* the local currents between boson sweeps, instead of updating them after *each* local change. This is due to the fact that, to lowest order in δU, the product of determinant ratios (see eq.(2.9)) which gives the total acceptance probability for a whole sweep, is the same whether one does or does not continuously update the currents after each new link update. This makes the method rather efficient, and in the limit in which δU and $1/n_{pf}$ go to zero the method becomes exact [4]. When the actual computation is done δU is kept fixed at a small number, and n_{pf} is increased until stability in the results is obtained. The error is then expected to be linear in δU (see discussion below). Therefore one can either limit oneself to a very small value for δU, or study different larger δU's and extrapolate later to zero step size.

Alternatively the fermion currents can be computed by a Langevin stochastic process [10,11]. For a boson field $\phi(x)$ with generic action $S(\phi)$, one considers the solution of the stochastic differential equation with an auxiliary 'fifth' time t

$$\frac{\partial \phi(x,t)}{\partial t} = -\frac{\delta S(\phi(x,t))}{\delta \phi(x,t)} + \eta(x,t) \tag{2.16}$$

and η a white noise with zero mean and variance two. After averaging over the noise, the solution of the above equation provides in the long time limit all the Green's function of the theory [10]. When using the Langevin equation for computing the fermionic currents the procedure is somewhat simplified by the fact that one does not need to take the square of the Dirac operator, because the the action need not be positive definite for the Langevin process to converge (in fact the action can even be complex [12], which is useful if one wants to consider a theory in Minkowski space). One starts again by introducing the fictitious 'fifth' time t and writes down the two equations

$$\begin{aligned} \frac{\partial \phi_1(x,t)}{\partial t} &= -\,\Delta\,[U]\;\phi_1(x,t) + \eta(x,t) \\ \frac{\partial \phi_2(x,t)}{\partial t} &= -\,\Delta^{\dagger}[U]\;\phi_2(x,t) + \eta(x,t) \end{aligned} \tag{2.17}$$

with η a white noise (the same in the two equations)

$$< \eta(x,t)\eta(x',t') > = 2\;\delta(x-x')\delta(t-t') \tag{2.18}$$

and the angular brackets denote an average over the noise. Two (instead of one) equations are needed because $\Delta[U]$ is not a hermitean operator; in fact it is purely antihermitean in the massless case. The formal solution is easily found

$$\begin{aligned} \phi_1(x,t) &= \int_0^t d\tau \int dy\;\big(\exp(-\Delta\tau)\big)_{xy}\eta(y,t-\tau) \\ \phi_2(x,t) &= \int_0^t d\tau \int dy\;\big(\exp(-\Delta^{\dagger}\tau)\big)_{xy}\eta(y,t-\tau) \\ &= \int_0^t d\tau \int dy\;\eta(y,t-\tau)\big(\exp(-\Delta^{*}\tau)\big)_{xy} \end{aligned} \tag{2.19}$$

and exists if all the eigenvalues of the matrix $\Delta[U]$ have positive real part. This corresponds in general to the case of a fermion with positive real mass ($\Delta \sim \not{D} + m$). In the limit of large times one then gets

$$\lim_{t\to\infty} < \phi_1(x,t)\phi_2^{*}(x',t) > = \big(\Delta^{-1}[U]\big)_{x,x'} \tag{2.20}$$

From the solution of the Langevin equations one finds that for the fermions the relaxation times go as $1/m_q$, whereas they go as $1/m_q^2$ in the case of the Monte Carlo, because of the use of the square of the Dirac operator ($\sim -\not{D}^2 + m^2$) in the pseudofermion action. For a generic observable $\tilde{A}$ one then has

$$< \tilde{A}(t) > = < \tilde{A}(\infty) > + O(e^{-t/\tau}) \tag{2.21}$$

where τ is the relaxation time. For a free fermion the first order Langevin equation gives $\tau^{-1} = 2m$, and for a general interacting theory $\tau \sim \xi^z$ where ξ is the correlation length, and z is a dynamical critical exponent which is usually of order one in scalar field theories [13]. Close to a critical point the divergence of τ leads to critical slowing down. While the Langevin process converges faster to equilibrium, the price one pays is that it also has a stronger short wavelength noise, at least in the case $r = 0$. For large times, the size of the statistical fluctuations in turn is given by

$$\Delta\tilde{A} = \frac{\text{const}}{\sqrt{t/\tau}} \tag{2.22}$$

As stated above, the relation between τ and ξ depends on the algorithm, even for a given action. In the limit of small step size, the Monte Carlo and the Langevin process become equivalent (for the same action), as a consequence of a theorem due to Kolmogoroff [4]. An important difference between the two processes lies in the fact that the Monte Carlo method is constructed so as to give, in the large time limit, the exact equilibrium probability distribution also for arbitrary step size. The decision on which algorithm to choose thus depends on the problem, and the specific observable one is trying to compute.

In order to solve the above set of equations numerically, one needs to discretize the 'fifth' time t. With a time step ϵ and $t = \epsilon k$ eq. (2.16) becomes [11]

$$\phi^{(k+1)}(x) = \phi^{(k)}(x) - \epsilon \, \frac{\delta S(\phi^k(x))}{\delta \phi^k(x)} + \sqrt{2\epsilon} \, \eta^{(k)}(x) \tag{2.23}$$

with η a white noise with zero mean and unit variance

$$< \eta^{(k)}(x) > = 0 \qquad < \eta^{(k)}(x) \, \eta^{(k')}(x') > = \delta(x - x')\delta^{kk'} \tag{2.24}$$

By this procedure an error of order ϵ is introduced in the averages, which can in principle be reduced by going to small enough ϵ. The effect of the error is easily seen for example for a quadratic action $\frac{1}{2}\int \phi M \phi$, where the propagator (which should be M^{-1}) is altered by terms of order ϵ, when it is computed by solving eq. (2.23)

$$< \phi(x)\phi(x') > = \left(\frac{1}{M(1 - \frac{\epsilon}{2}M)} \right)_{x'x} \tag{2.25}$$

Algorithms have been used based on second-order Runge-Kutta integrators for which the error is reduced to order ϵ^2 [14]. As stated above, the Monte Carlo method itself reduces to the Langevin equation in the limit of small field changes, of order $\sqrt{\epsilon}$.

In practice when solving eq. (2.23) on the computer, the average over the noise is replaced by the 'fifth' time average

$$< \phi(x)\phi(x') > = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} dt' \; \phi(x,t')\phi(x',t') \tag{2.26}$$

A Langevin equation on the group manifold has been written for the $SU(3)$ variables also [4,14,15]. The discretization of the continuous time evolution equation is less obvious here, because of the necessity of maintaining at each time step the unitarity property of the $U_{n\mu}$'s. It is easy to show that the correct equilibrium distribution for the gauge fields $U_{n\mu}$ is recovered, up to terms of order ϵ, if they evolve according to the stochastic difference equation

$$U_{n\mu}^{(k+1)} = \exp\Big(i\sqrt{2\epsilon}\, \omega_{n\mu}^{\alpha} \frac{\lambda^{\alpha}}{2} - \epsilon P_{AT} \big[\frac{\delta S}{\delta U_{n\mu}^{(k)}} U_{n\mu}^{(k)} \big] \Big) U_{n\mu}^{(k)} \tag{2.27}$$

Here the λ^{a}'s are the $SU(3)$ Gell-Mann matrices, and the $\omega_{n\mu}^{a}$'s are random real numbers with zero mean and unit variance

$$< \omega_{n\mu}^{\alpha} > = 0 \qquad < \omega_{n\mu}^{\alpha}\, \omega_{n'\mu'}^{\beta} > = \delta_{nn'} \delta_{\mu\mu'} \delta^{\alpha\beta} \tag{2.28}$$

The operator P_{AT} projects out the antihermitian traceless part of the expression in parentheses, and its effect is to constrain the new element to lie still within the group manifold. The great advantage of the Langevin formulation (and any other process in which the evolution is given by fixed equations) over the Monte Carlo lies in the fact that the first obeys deterministic equations once the noise is fixed. It allows therefore to set up correlated processes in which the statistical error is substantially reduced by a coherent cancellation effect. The method can be used to compute quantities which otherwise would be too small to be measured because of the signal to noise problem.

Returning to the problem of errors in the pseudofermion algorithm, it is important to note the dependence of physical observables on the various parameters (ϵ, n_{pf}). The random change in the gauge matrices on a given link can be parametrized as

$$\delta U_{n\mu} = U'_{n\mu} - U_{n\mu} = \big[\exp(i\sqrt{2\epsilon}\, \omega_{n\mu}^{\alpha} \frac{\lambda_{\alpha}}{2}) - 1 \big] U_{n\mu} \tag{2.29}$$

where the $\omega_{n\mu}^{\alpha}$'s are random real numbers with zero mean and unit variance. Then it can be shown that the error E introduced for small δU and n_f, and for large n_{pf}, has the general form [4]

$$E \sim \frac{\epsilon}{n_{pf}} (A n_f + B n_f^2) + O(\frac{1}{n_{pf}^2}) \tag{2.30}$$

with A and B depending on the quantity measured, and $\sqrt{\epsilon}$ parametrizing the smallness of the boson step size δU. The fact that the error is linear in $(\delta U)^2$ (or ϵ) is a consequence of the approximations done in evaluating the variation in the fermion determinant. The fact that the term proportional to n_f^2 goes like $1/n_{pf}$ is a consequence of the statistical fluctuations in the currents, which decrease like $1/\sqrt{n_{pf}}$ for large samples. The term linear in n_f accounts for the systematic error due to the finite relaxation times of the pseudofermions.

III - COMPUTATION OF THE LIGHT HADRON MASSES: GENERALITIES

A quantity of interest is the condensate wavefunction [8,16]

$$< \bar{\psi}(x)\psi(x) > = Z^{-1} \int [dU] \; \Delta^{-1}[U]_{x,x} \exp(-S_G[U]) \tag{3.1}$$

It is only meaningful to calculate this quantity with the $r = 0$ action, since otherwise the chiral symmetry is explicitly broken and the above quantity is not a useful order parameter. The gauge coupling constant is usually chosen to lie inside the narrow (in the coupling constant) 'crossover' region ($5.8 \leq \beta \leq 6.2$), where at least approximate scaling behavior for the static quark-antiquark potential and other physical quantities starts to set in.

The way to proceed is to calculate $< \bar{\psi}\psi >$ in a given gauge configuration at fixed m_q and g^2, using the pseudofermion Monte Carlo method or the Langevin equation, average the results over many configurations and finally change m_q and extrapolate (using for example a polynomial) to $m_q = 0$ for each value of g^2. Because of long relaxation times for small quark masses and the finite box size, the range of quark masses typically useful and accessible in the crossover region is roughly between $m_q a = 0.3$ and $m_q a = 0.02$, corresponding to a physical quark mass which can be at most reduced to about 40 MeV, and which is still above the physical m_u and m_d quark masses. The dependence of the results on the box size can be studied by comparing results on lattices of different linear dimension. For $< \bar{\psi}\psi >$ there is in general a tendency to become smaller when the lattice size is increased. For weak enough coupling one expects $< \bar{\psi}\psi >$ to scale like Λ^3, the lattice scaling parameter, times an anomalous dimension factor

$$< \bar{\psi}\psi > = \; R \left(2\beta_0 g^2\right)^{-4/11} \Lambda^3_{latt} \tag{3.2}$$

On an $8 \times 8 \times 8 \times 8$ lattice and for $n_f = 0$ (no internal fermion loops) one finds $R = (110 \pm 10)^3$. The fact that $< \bar{\psi}\psi >$ is non-vanishing provides clear evidence for chiral symmetry breaking in QCD. A spectral representation for $< \bar{\psi}\psi >$ can be written

$$< \bar{\psi}\psi > = \frac{m_q}{V} \int_0^\infty d\lambda \; \frac{\rho(\lambda)}{\lambda^2 + m_q^2} \tag{3.3}$$

and chiral symmetry breaking then implies a non-vanishing spectral density at low frequency

$$\lim_{m_q \to 0} < \bar{\psi}\psi > = \frac{\pi}{V}\rho(0) \neq 0 \tag{3.4}$$

These results have also been extended to the case $n_f = 2$ and $n_f = 4$ [9,17], with $r = 0$ fermions also for the loops.

Unfortunately in the Wilson fermion case, because of the presence of terms in the fermion action that break chiral symmetry explicitly, the quantity $<\bar{\psi}\psi>$ is not a useful order parameter: the evaluation of the non-perturbative part of the condensate $<\bar{\psi}\psi>$ requires the subtraction of a divergent perturbative tail. For any finite r (Wilson fermions) a non-vanishing $<\bar{\psi}\psi>$ for $m \to 0$ does not imply the existence of a Goldstone particle with the quantum numbers of the pion, as can be seen from the identity, valid for the lattice Dirac operator $\not{D}+m$ that defines the fermion action in eq.(2.2), and any value of r

$$\frac{1}{\not{D}+m}+\gamma_5\,\frac{1}{\not{D}+m}\,\gamma_5=\frac{\not{D}+\not{D}^\dagger}{(\not{D}+m)(\not{D}^\dagger+m)}+\frac{2m}{(\not{D}+m)(\not{D}^\dagger+m)} \tag{3.5}$$

One has $\not{D}^\dagger = -\not{D}$ only for r=0. Therefore the first term on the right-hand side also vanishes only for $r=0$.

Of even greater interest is the hadron spectrum, since it is known very well experimentally. The masses of the lighter hadrons are obtained by computing the appropriate correlation functions of composite operators, and then by looking at their long distance behavior [8,16,18]. In the cases of interest here, they are given by the formulae

$$\begin{aligned}
<\bar{\psi}\psi(x)\;\bar{\psi}\psi(0)>_{I\neq 0} &= \int d\mu(U)\;\mathrm{Tr}[G(x,0|U)G(0,x|U)] \\
<\bar{\psi}\psi(x)\;\bar{\psi}\psi(0)>_{I=0} &= \int d\mu(U)\;\mathrm{Tr}[G(x,0|U)G(0,x|U)] \\
&\quad - n_f\int d\mu(U)\;\mathrm{Tr}[G(x,x|U)G(0,0|U)] \\
<\bar{\psi}\bar{\psi}\bar{\psi}(x)\;\psi\psi\psi(0)> &= \int d\mu(U)\;\mathrm{Tr}[G(x,0|U)G(x,0|U)G(x,0|U)]
\end{aligned} \tag{3.6}$$

where color and spinor indices have been suppressed for simplicity, and

$$G(x,0|U) = <x|\;\Delta^{-1}[U]\;|0> \qquad G(0,x|U)=\gamma_5\;G(x,0|U)^\dagger\;\gamma_5 \tag{3.7}$$

$I=0$ denotes here an isospin singlet, and it is assumed that the flavor symmetry is unbroken. The measure is given as usual by

$$d\mu(U)=Z^{-1}\prod_{n\mu}dU_{n\mu}\;\exp(-S_G+n_f\mathrm{Tr}\ln\Delta[U]) \tag{3.8}$$

The best methods available for computing $G(x,0|U)$ for large separations rely on an exact numerical iterative inversion of the matrix $\Delta[U]$, one column at a time [19]. One would like to solve for a fixed point source (say at position 0) the equation

$$X_{m0}=\Delta[U]^{-1}_{ml}\;\delta_{l0} \tag{3.9}$$

so that X is given by a column vector. In the simplest case of Jacobi relaxation one solves iteratively the equation (suppressing indices)

$$X^{(k+1)}=\lambda\delta+(1-\lambda\Delta[U])\,X^{(k)} \tag{3.10}$$

with λ a parameter that acts like a time step. Then for $k \to \infty$ one obtains

$$X = \frac{1}{1-(1-\lambda\Delta[U])}\,\lambda\delta = \frac{1}{\Delta}\,\delta \tag{3.11}$$

A necessary condition for convergence is that the matrix $\Delta[U]$ has no zero eigenvalues. Also for fermions $\Delta[U]$ is not positive definite (for $r = 0$ and $m = 0$ it is purely antihermitian). But if the largest eigenvalue of $1 - \lambda\Delta[U]$ is less than one, then the procedure will converge. For finite quark mass this can always be realized by adjusting the value of λ. The convergence in general will not be monotonic (since $\Delta[U]$ is a matrix with random entries) and will be slower for smaller quark masses. In general it appears that convergence can be attained in the case $r = 0$ for m_q different from zero, while in the Wilson case on a finite lattice convergence is guaranteed only for $k < 1/8r$. For $1/8r \leq k \leq k_c$ there is always an extremely small but finite probability on a finite lattice (of order $\exp(-N)$, where N is the number of sites on the lattice) of encountering a configuration sufficiently close to a pure gauge such that the inversion will not converge. Often convergence can be accelerated by using Runge-Kutta and over-relaxation methods. Conjugate gradient solvers and Fourier acceleration techniques, which are useful in solving large sparse system in other applications, have also been used in this context [20]. Needless to say, often this part of the computation is the most time consuming.

In most cases only meson states are considered which can be created from the vacuum by the local currents

$$\begin{array}{llll} \bar\psi^a(x)\gamma_5\psi^a(x) & J^{PC}=0^{-+} & \bar\psi^a(x)\psi^a(x) & J^{PC}=0^{++} \\ \bar\psi^a(x)\gamma_\mu\psi^a(x) & J^{PC}=1^{--} & \bar\psi^a(x)\gamma_5\gamma_\mu\psi^a(x) & J^{PC}=1^{++} \\ \bar\psi^a(x)\sigma_{\mu\nu}\psi^a(x) & J^{PC}=1^{-+} & & \end{array} \tag{3.12}$$

Extended lattice operators of the generic form $\bar\psi_{n+\mu}\Gamma U_{n\mu}\psi_n$ can also be used. For states with the quantum numbers of the proton $(1/2^+)$ and the Δ $(3/2^+)$ resonance a set of local operators is [21]

$$\begin{aligned} N_\alpha(x) &= \epsilon^{abc}\left[\psi^{Ta(u)}(x)C\gamma_5\psi^{b(d)}(x)\right]\psi_\alpha^{c(u)}(x) \\ \Delta^{++}_{\mu\alpha}(x) &= \epsilon^{abc}\left[\psi^{Ta(u)}(x)C\gamma_\mu\psi^{b(u)}(x)\right]\psi_\alpha^{c(u)}(x) \end{aligned} \tag{3.13}$$

with $C = i\gamma_2\gamma_0$ the charge conjugation matrix. An alternative set of operators for the baryons is given by the static flavor-spin $SU(6)$ wavefunctions used in ref. [6]. Choices of different operators lead to different propagator amplitudes, but are expected to give the same estimate for the mass of the physical state, which is determined by the lowest eigenvalue of the transfer matrix. In general any (local or nonlocal) gauge invariant operator $\bar\psi\Gamma\psi$ with a sizeable projection, in the small quark mass limit, on the physical state $|\Gamma>$ $(<0|\bar\psi\Gamma\psi|\Gamma>\neq 0)$ will be adequate. For some states (like the 2^{++}) no local operator exists with this property.

It is useful to denote by $G_M(x)$ and $G_B(x)$ the generic connected meson and baryon propagator, respectively, averaged over the gauge configurations

$$\begin{aligned} G_M(x) &= <\bar\psi\psi(x)\;\bar\psi\psi(0)> \\ G_B(x) &= <\bar\psi\bar\psi\bar\psi(x)\;\psi\psi\psi(0)> \end{aligned} \tag{3.14}$$

From the large distance behavior of the propagators the hadron masses are then computed by a judicious fit to the appropriate correlation function, summed over spatial hyperplanes to extract the zero spatial momentum state. Since periodic boundary conditions are used one then gets for the mesons

$$\tilde{G}_M(t) = \sum_{\vec{x}} G_M(x) \underset{t \gg a}{\sim} 2A_M \cosh[m_M(\frac{T}{2} - t)] + 2A_{M^*} \cosh[m_{M^*}(\frac{T}{2} - t)] + \cdots \quad (3.15)$$

where m_{M^*} is the mass of the first radial recurrence, which can be resolved if $m_{M^*} - m_M \gg T^{-1}$, where T is the temporal extent of the box. The neglected contributions are due to higher radial excitations of increasing mass. For the baryons it is known that a state with opposite intrinsic parity contributes and propagates backward in time

$$\begin{aligned} \tilde{G}_B(t) = \sum_{\vec{x}} G_B(x) \underset{t \gg a}{\sim} & \frac{1}{2}(1+\gamma_0)\big[A_{B^+} e^{-m_{B^+} t} + A_{B^-} e^{-m_{B^-}(T-t)}\big] \\ & + \frac{1}{2}(1-\gamma_0)\big[A_{B^+} e^{-m_{B^+}(T-t)} + A_{B^-} e^{-m_{B^-} t}\big] + \cdots \end{aligned} \quad (3.16)$$

where m_{B^+} and m_{B^-} refer to the baryons with opposite intrinsic parity.

In order to estimate the importance of finite lattice effects it is useful to consider alternative lattice fermion actions, which are nevertheless expected to lead to the same continuum limit. In the Kogut-Susskind formulation of lattice fermions [22] the action can be chosen as [23]

$$S_F = \frac{1}{2} \sum_f \sum_{n,\mu} \eta_{n,\mu} \big[\bar{\chi}^{(f)}_{n+\mu} U^\dagger_{n,\mu}\, \chi^{(f)}_n - \bar{\chi}^{(f)}_n U_{n,\mu}\, \chi^{(f)}_{n+\mu}\big] - \sum_f \sum_n m_f\, \bar{\chi}^{(f)}_n \chi^{(f)}_n \quad (3.17)$$

with $\eta_{n,\mu} = (-1)^{n_1 + \cdots n_{\mu-1}}$. The above action contains only single component Fermi fields but unfortunately describes four Dirac fermions in the continuum. It corresponds to the case $r = 0$ in eq. (2.2). A definite advantage of this formulation is that it has a residual continuous $[U(1)]^{n_f}$ chiral symmetry in the massless case, which prevents the appearance of fermion mass counterterms. In particular no parameters need to be fine tuned to obtain a massless pion in the limit of zero quark mass, while in the Wilson case k has to be tuned to its critical value k_c.

On the other hand in the Kogut-Susskind action only the *even* sites should be identified with the points in the physical euclidean space, and the four Dirac fields should be thought of as constructed out of the sixteen χ fields in the unit cell of size $2a$ [24]. This in particular implies that *a lattice of double the linear size* used for the Wilson fermions should be employed in order to make a meaningful comparison. In addition, the fermion doubling problem implies that there are sixteen pseudoscalar mesons, only *one* of which becomes massless in the zero quark mass limit, for finite lattice spacing. In ref. [25] this fact was explicitly demonstrated, showing that there is a range of couplings for which the remaining pseudoscalars, instead of becoming massless, have a mass comparable to the vector meson. A similar situation arises in the framework of Hamiltonian Kogut-Susskind fermions, where the massless pion is completely lost at strong coupling [26]. Thus restricting oneself to just this state, while convenient, is not really justified, and the chiral properties one predicts for continuum QCD do depend at this time still a lot on which state is chosen. The perturbative corrections to the quark masses are also much larger in the Kogut-Susskind case than in the Wilson case, as will be discussed below, and other anomalies are present in the numerical results for $g^2 \approx 1$, as will be discussed in the next section.

The spin-parity assignments in the case of the Kogut-Susskind fermions are less straightforward, because states of different spin-parity can contribute to the same correlation function [8,16]. In order to extract the masses one first considers the generic meson propagator

$$G_M(x) = \int d\mu(U) \sum_{ab} \mid G(x,0|U)_{ab} \mid^2 \tag{3.18}$$

and the baryon propagator

$$G_B(x) = \int d\mu(U) \sum_{abc} \sum_{a'b'c'} \epsilon^{abc} \epsilon^{a'b'c'} G_{aa'}(x,0|U) G_{bb'}(x,0|U) G_{cc'}(x,0|U) \tag{3.19}$$

In order to separate the states with different spin-parity it is convenient to set the external momentum to zero and consider the four kinds of corners of the spatial Brillouin zone. One defines [8,16]

$$\begin{aligned}
G_M^0(t) &= \sum_{\vec{x}} G_M(x) \\
G_M^1(t) &= \sum_{\vec{x}} [(-1)^x + (-1)^y + (-1)^z] G_M(x) \\
G_M^2(t) &= \sum_{\vec{x}} [(-1)^{x+y} + (-1)^{y+z} + (-1)^{z+x}] G_M(x) \\
G_M^3(t) &= \sum_{\vec{x}} (-1)^{x+y+z} G_M(x)
\end{aligned} \tag{3.20}$$

For large $t \gg a$ the above propagators behave like

$$\begin{aligned}
G_M^0(t) &\underset{t\gg a}{\sim} A_P e^{-m_P t} + (t \to T-t) + \cdots \\
G_M^1(t) &\underset{t\gg a}{\sim} A_V e^{-m_V t} + (-1)^t A_T e^{-m_T t} + (t \to T-t) + \cdots \\
G_M^2(t) &\underset{t\gg a}{\sim} A_V e^{-m_V t} + (-1)^t A_A e^{-m_A t} + (t \to T-t) + \cdots \\
G_M^3(t) &\underset{t\gg a}{\sim} A_P e^{-m_P t} + (-1)^t A_S e^{-m_S t} + (t \to T-t) + \cdots
\end{aligned} \tag{3.21}$$

where the subscripts P, V, S, A, and T as usual refer to pseudoscalar, vector, scalar, axial vector and tensor mesons respectively. It is clear that in the case of the Kogut-Susskind fermions the analysis of the correlation functions is made more difficult by the presence of particles with different spin. In the case of the baryons similar projections on the Brillouin boundary can be done, with the lowest state always being the spin 1/2 baryon.

Unfortunately, because of the doubling, it can be shown though that there are states with the quantum numbers of the pion whose mass does not vanish in the chiral limit for $g^2 \approx 1$. Thus instead of having the full four-flavor chiral symmetry in the massless limit, one has for finite lattice spacing only a residual continuous chiral symmetry

$$U(4)_V \otimes U(4)_A \to U(1)_V \otimes U(1)_A \tag{3.22}$$

which is reflected in only one out of sixteen pions becoming massless for $m_q = 0$. In the Wilson case one has the unattractive feature of having to fine-tune k in order to obtain a massless pion, but at least one knows in the chiral limit how many massless pions there are.

When the effects of internal fermion loops are taken into account, it is unclear what number of continuum flavors should be assigned in the staggered case. The Kogut-Susskind fermions at first appear to be particularly convenient here, since the evaluation of the fermion determinant with single-component spinors is so much simpler. If all pions can be made massless, then clearly $n_f = 4$. But since all but one pion are close to the rho mass in the chiral limit, except for the weakest couplings, the effective number of flavors will be much less than four! This point seems to have been completely missed in some contemporary literature. It is therefore not surprising that the earliest simulations with the loops [41] gave results that were to a certain extent inconsistent with the Wilson fermion results. This issue will not be resolved until all the pion masses, with loop effects included, are determined in the 'scaling' region.

Turning to the subject of errors, it is known that statistical and systematic errors tend to affect the computation of the masses. Fluctuations in the gauge field configurations contribute to the statistical errors in the masses. The higher the relative mass of the state, the higher in general the fluctuation. An example are the P-wave mesons, whose higher mass with respect to the pseudoscalar and vector meson arises in part because of cancellations between different spin components of the fermion propagator. The masses of the hadrons are obtained in practice by fitting the correlation functions to the expected exponential or hyperbolic cosine (or sum thereof). At short distances ($t \ll m^{-1}$) the correlation functions follow in general power law behavior, while at larger separation a contamination from higher excitations (the radial recurrences) will be present.

In order to extract the hadron masses and amplitudes the correlation functions are fitted (at least in the Wilson fermion case) to a sum of two hyperbolic cosines for the mesons and to two decaying exponential for the baryons. In the baryon case the contribution of the opposite parity state has to be taken into account in the fit. One notices in general that for smaller quark masses the expected asymptotic behavior sets in more slowly as a function of distance. By making the lattices longer in the imaginary 'time' direction these effects can be reduced. Of course the higher the separation, the larger the statistical errors will be, and more statistical accuracy is needed. In order to avoid biases coming from the short distance contribution the fits are often restricted to some $t > t_{min} \approx 5$. Fits with single hyperbolic cosines (or two opposite sign exponentials for the baryons) can also done for some even greater t. In the case of the Kogut-Susskind fermions, because of the presence of more than one particle, the radial recurrences are often not included in the fit and a cut is set at $t > t_{min}$, except for G_M^0 where two-exponential fits are usually still reliable. The fits turn out in general to be rather satisfactory, but one should keep in mind when estimating the error that the data for the correlation functions at different values of t are strongly correlated.

These fits are then finally compared with the distance-dependent mass estimates $m(t)$ obtained by solving the nonlinear equation

$$\frac{G(t)}{G(t-1)} = \frac{\cosh\left[m(t)(L/2-t)\right]}{\cosh\left[m(t)(L/2-t+1)\right]} \tag{3.23}$$

for the mesons and the analogous equation $G(t)/G(t-1) = \exp[-m(t)t]/\exp[-m(t)\,(t-1)]$ for the baryons. In figs. 1a and 1b graphs of $m(t)$ for the pseudoscalar meson and the nucleon are shown for different values of k (at $g^2 = 1.111$ and for $n_f = 3$), together with the estimate for $m(\infty)$.

Possible correlations between the configurations are studied by analyzing subsets of consecutive gauge configurations, and also by plotting for example the distance-dependent pion 'mass' $m_P(t)$ and baryon 'mass' $m_B(t)$ as a function of configuration label. The uncertainty in the mass resulting from the fits (or equivalently from extrapolating to large t) and the statistical fluctuations in the correlation functions leads then to the errors quoted in the following. Due to the large size of the lattice, the high accuracy in the matrix inversion and the substantial number of configurations analyzed the statistical and systematic effects are quite small, although an even longer lattice in the time direction at the smallest value of the quark mass (and for even smaller quark masses) would sometimes be desirable as a further check on the systematic error. Still it is difficult with presently available machines to decrease the error much below five percent. By the time the data is extrapolated to the chiral limit the error then increase to about ten percent. Note that since all hadron masses are extracted from the same set of fermion propagators which are evaluated exactly on the same gauge configuration, the errors quoted are correlated and the spin splittings are determined more accurately than one might have expected from the uncertainty in the masses themselves [9].

The physical region of very small quark masses has to be, at least with the presently available lattice sizes, obtained by extrapolation using the values of the hadron masses computed for larger quark masses. The smallest quark mass used in the present study corresponds roughly to the strange quark mass. Results for lighter quarks (up, down) are then obtained by extrapolation. Still one should keep in mind that the behavior for small quark mass is presumably *known* from current algebra, just as the behavior for small g^2 is known from perturbation theory, and one only needs to establish the necessary *matching*. For the hadron masses in lattice units, the simplest assumption is that they are given by expressions of the type [27]

$$M^2 a^2 = \left[A(g^2)\right]^2 + B(g^2)\, m + C(g^2)\, m^2 \tag{3.24}$$

where m is the average quark mass, and one expects that the coefficients A and B (up to anomalous dimension factors) will scale like Λ in the continuum limit, since they both have dimension of mass. If the quark masses differ from each other appreciably, then B

Figure 1: Pseudoscalar meson mass estimates $\mathbf{m(t)}$ as a function of imaginary time separation $\mathbf{t}$ for different values of $\mathbf{k}$, assuming a single exponential decay of the euclidean correlation function. The statistical errors are comparable to the size of the dots. Here $\beta = \mathbf{5.4}$ and $\mathbf{n_f = 3}$.

should depend on the quark type, e.g. for the physical proton mass strictly speaking one should have

$$M_p^2 = A^2 + B_u m_u + B_d m_d + B_s m_s + \cdots \qquad (3.25)$$

with $B_u =< p|\bar{u}u|p >$ expected to be about twice $B_d =< p|\bar{d}d|p >$, and $B_s =< p|\bar{s}s|p >$

Figure 2: As in fig. 1, but for the particle with the quantum number of the nucleon.

negligible. The B coefficients are related to the pion-nucleon σ term

$$\sigma = \frac{m_u + m_d}{4M_p} < p|\bar{u}u + \bar{d}d|p > \approx \frac{B(g^2)a^{-1}}{4M_p}(m_u + m_d) \tag{3.26}$$

which in turn is determined experimentally from pion-nucleon scattering to be about $60 \pm 10\ MeV$ in the chiral limit [27]. By measuring the nucleon mass slope on the lattice, the σ term can be estimated. As will be discussed below, present lattice values are a factor of two smaller than the experimental number, but also have large uncertainties.

If all quarks are taken with the same mass, then for very large m the constant C is expected to be equal to 4 for the mesons and 9 for the baryons, since in the limit of very large quark masses the bound state mass tends to the sum of the quark masses, but for intermediate m the situation might be more complicated. Actually when fermion polarization effects are included, it is known from chiral perturbation theory that pion emission and reabsorption gives rise to non-analytic corrections of the type

$$\sim n_f \, m^4 \, \log m^2 \quad \text{and} \quad \sim n_f \, m^{3/2} \tag{3.27}$$

with the second term absent for the pseudoscalar mesons, as a consequence of their pseudo-Goldstone nature [27]. The non-analytic corrections should be small for the mesons but can be significant for the baryons. Chiral perturbation theory allows one to estimate their size (more precisely, the coefficients of (3.27) are known in terms of A, which is not very well known though, since the experimental hadron masses are given for finite quark mass), and thus one should be careful and consider their effect when analyzing the data for the masses.

The finite spatial extent of the lattice poses a limit on the smallest pion mass that can be reached: the inverse pion mass in lattice units should be at most about one half the spatial size of the lattice. In general one should adopt a conservative cut $m_\pi L > 3$. This is equivalent to the requirement that the pion wavefunction should fit inside the space-time box. As an example, the lattice spacing at $g^2 = 1$ and $n_f = 0$ is about 0.1 Fermis, and thus for a lattice of side 10 the spatial box size is about 1 Fermi. Another way of phrasing these requirements is that the masses in lattice units should lie between the ultraviolet and the infrared lattice cutoff

$$\frac{\pi}{La} \ll M \ll \frac{\pi}{a} \tag{3.28}$$

and the hadron mass M can be made small only if L, the spatial extent of the box, is made large. Also, for pion masses greater than about 0.2-0.1 in lattice units the cutoff dependent corrections to the fermion propagator are significant, as can be seen comparing the hadron mass values for $r = 0$ and $r = 1$ using the same quark mass as input. (An improved fermion action as discussed in ref. [28,29] might make the extrapolation problem to small quark masses less acute). Finally, when fermion loops are included threshold effects also become important for small quark mass.

Finite size effects coming from the gluons are expected to be contained (but have still not been checked thoroughly), as shown for example by the small size dependence of the average plaquette. For periodic boundary conditions the finite size corrections are exponentially in the linear spatial box size, with a decay length given by the inverse mass gap in the relevant channel (e.g. the spin zero glueball for the plaquette). For the hadron masses the lowest state that contributes to the finite size correction is the pion, but the correction can still remain small in the chiral limit, since the pion become weakly interacting (assuming chiral symmetry is actually fully restored on the lattice; this is certainly *not* the case at strong coupling) [23]. As pointed out in the second of ref. [8] for the pion mass one finds, for a large box ($M_\infty^{-1} \ll L$),

$$M_L = M_\infty + A \frac{M_\infty^2}{f_\pi} (M_\infty L)^{-3/2} \exp(-M_\infty L) + \cdots \tag{3.29}$$

where the constant A is known: $A = 3/[2(2\pi)^{3/2} n_f]$ [30]. Note that the correction is given basically by a spatial free-field correlation function. In other words, on a finite lattice both the quarks and the gluons can wind around the periodic lattice if they are light enough, giving rise to unphysical results, whose magnitude can nevertheless be estimated. Recent estimates for the lowest (scalar) glueball place its mass in lattice units at $g^2 = 1$ at about $0.65(7)a^{-1}$, which in turn gives a glueball correlation length only slightly larger than one lattice spacing [31]. It is possible however that the hadron masses have larger finite size corrections. Work on this point is in progress.

Finally the physical masses of the hadrons are obtained, by extrapolating the results obtained at finite coupling to the limit of zero coupling using the renormalization group. One has to check that the results at finite coupling scale in accordance with the predictions of asymptotic freedom. The glueball correlation length (the inverse glueball mass) grows very rapidly in this region, and one cannot make the coupling too small because of boundary effects. Because of machine time limitation it is often possible to study only a few (1-3) values of the gauge coupling constant in the 'crossover' region. One then hopes that in this region one is close enough to the asymptotic behavior, since then for the masses the corrections are of order $\Lambda^2 a^2$. More precisely, for finite g^2 one expects from asymptotic freedom that for ratios of physical masses

$$\frac{M_a}{M_b} = C_{ab}\left[1 + O(\Lambda^2 a^2)\right] \tag{3.30}$$

and thus the lattice corrections are *exponentially* small in g^2 since $\Lambda a \sim \exp(-1/2\beta_0 g^2)$. For Symanzik's perturbatively improved gauge action [28] the contribution of dimension six operators is cancelled out order by order, and the claim is that the correction can be reduced to $O(\Lambda^4 a^4)$. The program can be extended to include fermions [29].

On the other hand there are quantities for which perturbative corrections proportional to g^2 are in general present. While the hadron mass ratios have no perturbative corrections (the corrections for small a are of order $a^2\Lambda^2$) and lead also in the explicit computation to comparable values, this is not true of the invariant quark masses, the fermion condensate and the decay constants, for which the corrections are known to be of order $1/\ln(a\Lambda)$. For the pion decay constants with Wilson fermions, obtained using the local operator of equation (3.12) on finds for example [32]

$$f_\pi^{cont} = \left[1 - 0.14\, g^2 + O(g^4)\,\right] f_\pi^{latt} \tag{3.31}$$

In the case of the light quark masses a perturbative evaluation of the finite quark mass renormalization on the lattice to one loop gives the relationship between the lattice quark mass $m(a)$ and the continuum invariant quark mass in the $\overline{MS}$ scheme. Since for the light quarks (u,d,s) this quantity is of great interest and is not known very well, it will be discussed in more detail here. The invariant quark mass is defined as

$$\begin{aligned} \tilde{m}_{\overline{MS}} &= m(\mu)\Big(\log\frac{\mu}{\Lambda_{\overline{MS}}}\Big)^{12/(33-2n_f)} \\ &= m(a)\Big(\log\frac{C_m a^{-1}}{\Lambda_{\overline{MS}}}\Big)^{12/(33-2n_f)} \end{aligned} \tag{3.32}$$

where the constant C_m relates the scale parameter on the lattice to the scale parameter in the continuum, as obtained by looking at the fermion mass. Since

$$\log \frac{1}{a\Lambda_{\overline{MS}}} = \frac{1}{2\beta_0 g_{latt}^2} + \log \frac{\Lambda_{latt}}{\Lambda_{\overline{MS}}} \tag{3.33}$$

one obtains

$$\tilde{m}_{\overline{MS}} = m(a)\Big(\frac{1}{2\beta_0 g_{latt}^2} + \log \frac{\Lambda_{latt}}{\Lambda_{\overline{MS}}} + \log C_m\Big)^{12/(33-2n_f)} \tag{3.34}$$

For the group $SU(3)$ with n_f flavors one has for the scaling violation parameter

$$\Lambda_{\overline{MS}} = \frac{\pi}{a}\Big(\frac{8\pi^2}{33-2n_f}\,\tilde{\beta}\Big)^{(459-57n_f)/(33-2n_f)^2} \exp\Big(-\frac{4\pi^2}{33-2n_f}\tilde{\beta}\Big)\ \Big[1+O(\frac{1}{\beta})\Big] \tag{3.35}$$

In the presence of fermions and for $r=1$ the relationship between $\beta_{latt}=6/g^2$ and $\tilde{\beta}$ is [33]

$$\tilde{\beta} = \beta_{latt} - 1.852 - 0.020\ n_f \tag{3.36}$$

while for $r=0$ the constant multiplying n_f is -0.042 instead. Therefore one finds $\log(\Lambda_{latt}/\Lambda_{\overline{MS}}) = -3.22$ for $n_f=0$ and $g^2=1$, and -3.72 for $n_f=3$ and $g^2=1.111$ (and also $r=1$ in this second case).

Also, by comparing the fermion self-energy on a lattice to the fermion self-energy in the continuum, one finds $\log C_m = 2.15$ and 6.53 for $r=1$ and $r=0$, respectively [34]. Collecting these results one has for example (these values will be used later on)

$$\begin{aligned}
\tilde{m} &= m(a)(2\beta_0 g_{latt}^2)^{-4/11}(1-0.15\ g_{latt}^2)^{4/11} \\
&= m(a)\big(7.17-3.22+2.15\big)^{4/11} = 1.93\ m(a) \quad (r=1) \\
\tilde{m} &= m(a)(2\beta_0 g_{latt}^2)^{-4/11}(1+0.46\ g_{latt}^2)^{4/11} \\
&= m(a)\big(7.17-3.22+6.53\big)^{4/11} = 2.35\ m(a) \quad (r=0)
\end{aligned} \tag{3.37}$$

for $n_f=0$ and $g^2=1$, while for $n_f=3$ and $g^2=1.111$ one finds

$$\begin{aligned}
\tilde{m} &= m(a)(2\beta_0 g_{latt}^2)^{-4/9}(1-0.18\ g_{latt}^2)^{4/9} \\
&= m(a)\big(7.89-3.72+2.15\big)^{4/9} = 2.27\ m(a) \quad (r=1)
\end{aligned} \tag{3.38}$$

The lattice quark mass correction $\log C_m$ for the Wilson fermions ($r=1$) is quite small (15-19%), while it is disturbingly large for the Kogut-Susskind case ($r=1$) (46%). This last result is a consequence of the fermion doubling problem, which is known to lead to significant perturbative lattice corrections. It is clear that a two-loop computation would be quite useful. Finally for the tree-level improved fermion action of ref. [29] one finds $\tilde{m} = 1.90\ m(a)$ for $r=1$, $n_f=0$ and $g^2=1$, which is quite close to the ordinary (un-improved) Wilson result.

Sometimes the 'running' quark masses at a scale of say $\mu = 1\ GeV$ are quoted. These are related to the invariant quark masses by

$$m(\mu) = \tilde{m} \left(\log \frac{\mu}{\Lambda_{\overline{MS}}} \right)^{-12/(33-2n_f)} \tag{3.39}$$

and $\tilde{m}$ from eq. (3.37), (3.38) includes all the lattice corrections.

IV - RESULTS WITHOUT FERMION POLARIZATION EFFECTS

The evaluation of the determinant factor in the statistical weight by the methods described in the previous sections tends to slow down the upgrading of the link variables by a significant amount, and introduces additional statistical and systematic effects that are absent in the pure gauge computation. The first results on hadron masses and decay constants were obtained using an approximation in which the determinant is set equal to one [8,16,18] (for a review of recent results and a more complete list of references see ref. [2]). This would seem at first like a rather drastic approximation, since it is equivalent to neglecting fermion vacuum polarization effects. The approximation was in fact first used by Wilson [1] to compute the strong coupling mass spectrum.

Part of its usefulness stems from the fact that, besides the approximation being of interest in itself, it allows to test many of the techniques discussed previously for extracting the hadron spectrum, and determine the significance of statistical and systematic effects. Furthermore, there are theoretical and phenomenological arguments that suggest that the neglected effects, at least for some physical quantities, are small. Among these one can mention the fact that the approximation is exact for $N_c \to \infty$ since there fermion loop effects are suppressed by factors of n_f/N_c; that the Zweig rule is exact and hadrons are made only of valence quarks and gluons, which is not very different from the real world, which one knows is reasonably well described by the non-relativistic quark model; and finally the ratio of widths $\Gamma_\omega/\Gamma_\rho \approx 1/16$ and the ρ-ω mass splitting $(m_\omega - m_\rho)/m_\rho \approx 1/57$ can be taken as an indication for the size of loop effects. Of course the approximation has also some serious shortcomings such as the absence of the anomaly and an η' meson degenerate with the π, and zero widths for all hadrons. As will be shown later, numerical simulations of QCD with fermion loops included tend to indicate that most of the effect goes into a readjustment of the scale a^{-1}, as predicted by asymptotic freedom. A possible attitude is to consider the $n_f = 0$ theory as a starting point in a systematic expansion. Write for any observable O

$$O(n_f) = O(0) + O'(0)\, n_f + \frac{1}{2} O''(0)\, n_f^2 + \cdots \tag{4.1}$$

with $O'(0)$, $O''(0)$, ... calculable (at least in principle) in the pure gauge theory.

Having set aside the problem of the inclusion of the determinant in the statistical factor, one computes quantum averages using in the weight factor only the pure gauge action. In the following results will be presented for light hadron masses in lattice QCD at

$g^2 = 1$, obtained on a $10 \times 10 \times 10 \times 20$ lattice. Virtual quark loop effects will be included in a more complete study, described in the next section. To estimate the importance of finite cutoff effects introduced by the lattice regularization of the fermion action, results will be compared for two different fermion formulations, $r = 1$ (Wilson fermions) and $r = 0$ (Kogut-Susskind fermions).

Details on the actual numerical computation will now be given [35]. By Monte Carlo methods one generates a set of gauge configurations $C_1 ... C_N$ in equilibrium with respect to the measure $d\mu(U) = Z^{-1} \exp(-S_G)[dU]$ where S_G is the pure gauge action. Also the lattice is taken to be periodic of size $L \times L \times L \times T$, and for simplicity the lattice spacing a is set equal to one in the following. After 3000 sweeps for thermalization, 21 gauge field configurations were generated on the $10 \times 10 \times 10 \times 10$ lattice at $\beta = 6$, separated by 500 full lattice sweeps with 10 Metropolis hits per link. The average plaquette was estimated at

$$E_p\,(\beta = 6.0) = 0.40596(4) \tag{4.2}$$

The lattice was then duplicated in the time direction and the fermion propagators were subsequently evaluated (for different quark masses) to high accuracy by the relaxation method with 400 iterations, using a second order Runge-Kutta algorithm for the Kogut-Susskind fermions [8]. Periodic boundary conditions were adopted, as in previous studies. Several tests were performed to ensure that the fermion propagators at the largest separation ($t = 10$) had converged to better than one percent. Meson and baryon correlation functions were then computed and finally averaged over the gauge configurations. In the case of the Kogut-Susskind fermions the first 10 configurations were used in the propagator evaluation.

The results for hadron masses are as follows. In the **Wilson case** (with $r = 1$) the data is presented in Table I and in fig. 3. The pseudoscalar meson (pion) mass tends to zero at $k_c = 0.1564(2)$ and the quark mass can be defined [8] by

$$m = \ln\big[1 + \frac{1}{2}(\frac{1}{k} - \frac{1}{k_c})\big] \underset{k \to k_c}{\sim} \frac{1}{2}(\frac{1}{k} - \frac{1}{k_c}) \tag{4.3}$$

extending to the interacting case the definition of the quark mass in the free case. The pseudoscalar mass squared is well fitted by a linear function of $1/k$ and one finds

$$m_P^2 = A_P\; m + O(m^2) \tag{4.4}$$

where the slope is determined to be $A_P = 2.6(2) a^{-1}$. The other hadron masses squared are also reasonably well fitted, within the quoted errors, by linear functions of $1/k$. The lattice spacing can be determined for example from the physical rho mass ($m_\rho = 769\ MeV$). At k_c one finds

$$0.39(2) \quad 0.49(7) \quad 0.61(8) \quad 0.48(6) \quad 0.57(6) \tag{4.5}$$

for the mass of the state with the quantum numbers of the ρ, nucleon, Δ, δ and A_1 in lattice units (the B meson is found to be almost degenerate with the A_1). The error in the rho mass determination at k_c can be estimated by extrapolating to the chiral limit either

k	0.148	0.150	0.152	0.154	0.1564(2)
m	0.168	0.128	0.089	0.049	0.0
m_P	0.69(2)	0.59(3)	0.50(2)	0.37(2)	0.00
m_V	0.75(2)	0.66(3)	0.59(2)	0.51(2)	0.39(2)
$(m_V^2 \ - \ m_P^2)^{1/2}$	0.27(3)	0.30(3)	0.33(3)	0.36(4)	0.39(2)
m_N	1.24(2)	1.09(3)	0.94(5)	0.77(8)	0.49(7)
$(m_N^2 - 2.8\ m_P^2)^{1/2}$	0.45(5)	0.46(6)	0.44(5)	0.46(6)	0.46(6)
m_Δ	1.31(2)	1.14(4)	1.00(5)	0.85(8)	0.61(8)
$(m_\Delta^2 \ - \ m_N^2)^{1/2}$	0.35(2)	0.35(3)	0.35(2)	0.36(4)	0.35(4)
m_S	0.88(2)	0.77(3)	0.68(3)	0.59(5)	0.48(2)
m_A	0.98(2)	0.86(3)	0.77(4)	0.69(4)	0.57(6)
m_T	0.99(2)	0.87(3)	0.77(4)	0.70(4)	0.57(6)
f_P	0.13(1)	0.11(1)	0.092(7)	0.075(8)	0.057(7)

Table I
Hadron masses in lattice units at $g^2 = 1$ for Wilson fermions

m_V^2 or $(m_V^2 - m_P^2)^{1/2}$. One concludes that the lattice spacing in the Wilson fermion case is

$$a_W^{-1}(\beta = 6.0\ n_f = 0) = 1950(120)\ MeV \quad (r = 1) \tag{4.6}$$

The relatively large error in the proton mass in lattice units is mostly due to the fact that the baryon mass decreases quite rapidly as the quark mass is taken to zero. As for the rho and pion mass, one finds that while the baryon mass appears not to be linear in $1/k$, the mass squared shows only small deviations from linearity, and leads to the above extrapolated value. For finite quark mass the results for the vector meson are consistent with

$$m_V^2 = [0.39(2)a^{-1}]^2 + 2.2(2)a^{-1}\ m + O(m^2) \tag{4.7}$$

and for the nucleon with

$$m_N^2 = [0.49(7)a^{-1}]^2 + 7.3(9)a^{-1}\ m + O(m^2) \tag{4.8}$$

No substantial deviation from linearity is seen in the range of quark masses explored, indicating that the coefficient of the m^2 term is not too big (for large m it is expected to be 4 and 9 for the mesons and baryons, respectively). The dimensionless ratio between the nucleon mass slope in m and the pion mass slope is determined to be about 2.8, while the pion and rho slopes are comparable in magnitude. The physical mass of the nucleon is then estimated at

$$m_{nucleon} = 960(140)\ MeV \tag{4.9}$$

without taking into account the error due to the uncertainty in the lattice spacing, which would add another 6% error. The mass of the physical Δ resonance is then 1190(160) MeV and for the δ and A_1 one obtains 940(120) MeV and 1110(120) MeV, respectively. The mass of the nucleon appears to be slightly larger than the experimental value, while the nucleon-delta splitting is perhaps smaller.

Figure 3: Hadron masses in lattice units as a function of the parameter k (related to the bare quark mass) at $\mathbf{g^2 = 1}$ for Wilson fermions (data from Table I).

From eq. (4.4) or from fig. 3 one obtains for the sum of up and down quark masses on the lattice

$$m_u(a) + m_d(a) = 7.8(4)\ MeV \quad (r = 1) \tag{4.10}$$

The pion decay constant on the lattice is $f_\pi^{latt} = 0.057(7)a^{-1} = 110(14)\ MeV$. The lattice value should then be corrected by the perturbative renormalization factor relating the currents on the lattice to the currents in the continuum (see previous section, eq. (3.31))

$$f_\pi^{cont} = [1 - 0.14\, g^2 + O(g^4)]\ f_\pi^{latt} = 95(10)\ MeV \tag{4.11}$$

leading to a final estimate in good agreement with the experimental value. It is difficult in the Wilson case to compute the chiral condensate $<\bar{\psi}\psi>$ directly, but its value can be estimated from the well-known PCAC formula that relates it to the pseudoscalar meson mass; for the pion one has

$$m^2(\pi^+) = m^2(\pi^0) = \frac{2}{f_\pi^2}\left[m_u < \bar{u}u > + m_d < \bar{d}d >\right] \tag{4.12}$$

with similar formulae for the kaons. The data also provides information on how the decay constants vary with the meson mass. For both fermion formulations one notices that the pion decay constant increases with the quark mass. By interpolating the data in Table I, or alternatively from the graph in fig. 4, one can also obtain an estimate for the kaon decay constant $f_K/f_\pi = 0.068(5)/0.057(7) = 1.2(1)$. The above value is in reasonable agreement with the experimental ratio of 1.16 obtained from the baryon decays.

The strange quark mass can be estimated either by using the (nonexistent) $\bar{s}s$ pseudoscalar meson, whose 'experimental' mass is estimated using flavor $SU(3)$ at $686\ MeV$, or the $\phi(1020)$ vector meson mass or finally the (nonexistent) sss nucleon, whose experimental mass is estimated at about $1464(11)\ MeV$ using a simple hyperfine interaction mass formula. The three independent strange quark mass estimates are then 93(8), 109(10) and 107(15) MeV respectively, in reasonable agreement with each other. The final estimate for the strange quark mass with Wilson fermions at $g^2 = 1$ is then

$$m_s(a) = 103(10)\ MeV \quad (r = 1) \tag{4.13}$$

Turning to the **Kogut-Susskind fermion case** (see Table II and fig. 4) one finds that the pion mass squared is strikingly linear in the quark mass for $m < 0.25$, leading to the estimate $A_P = 5.7(4)a^{-1}$. (Unfortunately the measured slope is very close to the value at $\beta = 5.6$, $A_P = 6.3(3)$, [36] and thus shows no indication of even approximate scaling behavior). The masses of the rho and the nucleon show clearly some curvature as a function of the quark mass, while the masses squared are again remarkably linear. Extrapolating to $m = 0$ one finds

$$0.41(1) \quad 0.52(5) \tag{4.14}$$

for the mass of the state with the quantum numbers of the ρ and nucleon, respectively, in lattice units. The results for the other mesons (P-waves) were found to be quite unstable

m	0.3	0.25	0.20	0.15	0.10	0.05	0.025	0.0
m_P	1.31(1)	1.20(1)	1.08(1)	0.93(2)	0.76(2)	0.54(2)	0.38(3)	0.00
m_V	1.49(1)	1.37(1)	1.23(1)	1.09(3)	0.92(5)	0.71(3)	0.58(4)	0.41(1)
$(m_V^2 - m_P^2)^{1/2}$	0.71(3)	0.66(3)	0.61(2)	0.55(2)	0.49(2)	0.45(3)	0.43(2)	0.41(1)
m_N	2.44(12)	2.24(9)	2.01(12)	1.75(13)	1.47(8)	1.10(9)	0.86(15)	0.52(5)
$(m_N^2 - 2.8\, m_P^2)^{1/2}$	1.07(9)	0.99(9)	0.88(8)	0.80(7)	0.74(7)	0.63(5)	0.57(6)	0.52(5)
f_P	0.31(2)	0.30(2)	0.27(2)	0.22(2)	0.18(2)	0.12(2)	0.09(1)	0.06(1)
$< \bar{\psi}\psi >$	0.456(2)	0.421(4)	0.371(4)	0.309(3)	0.227(3)	0.125(4)	0.069(5)	0.018(2)

Table II
Hadron masses in lattice units at $g^2 = 1$ for Kogut-Susskind fermions

(due to the necessity, in this case, of an eight-parameter fit to the appropriate correlation functions). One concludes that the lattice spacing in the Kogut-Susskind fermion case is

$$a_{KS}^{-1}(\beta = 6.0\; n_f = 0) = 1850(60)\; MeV \quad (r = 0) \tag{4.15}$$

For finite quark mass the results for the vector meson are consistent with

$$m_V^2 = [0.41(2)a^{-1}]^2 + 6.3(5)a^{-1}\; m + O(m^2) \tag{4.16}$$

and for the nucleon with

$$m_N^2 = [0.52(5)a^{-1}]^2 + 16.2(1.9)a^{-1}\; m + O(m^2) \tag{4.17}$$

It is quite clear from the data that the nucleon mass itself is not linear in the quark mass, except very close to the chiral limit. Here, as in the Wilson case, the ratio between the nucleon and pion slopes is about 2.8. From the value for the lattice spacing the physical mass of the nucleon is estimated at

$$m_{nucleon} = 960(90)\; MeV \tag{4.18}$$

again without taking into account the uncertainty in the lattice spacing, which would add another 4% error. It is encouraging that this estimate is in good agreement with the results obtained before with Wilson fermions.

On the other hand from eq. (4.4) or from fig. 4 one obtains for the sum of up and down quark masses on the lattice

$$m_u(a) + m_d(a) = 3.66(14)\; MeV \quad (r = 0) \tag{4.19}$$

Figure 4: Hadron masses in lattice units as a function of the bare quark mass m at $\mathbf{g^2 = 1}$ for Kogut-Susskind fermions (data from Table II).

which is *substantially* smaller than in the Wilson fermion case. The fermion condensate at $m = 0$ is $< \bar{\psi}\psi > (a) = 0.018(2)a^{-3} = [485(30)\ MeV]^3$ and the quoted value refers to two flavors. It is of interest to compare the estimated value of the condensate at the up quark mass to the corresponding value at the strange quark mass (the latter is obtained by interpolation from the values in Table II)

$$< \bar{u}u > (a) = [385(30)\ MeV]^3 \qquad < \bar{s}s > (a) = [590(30)\ MeV]^3$$

$$\frac{<\bar{s}s>}{<\bar{u}u>} = 3.6(1.2) \tag{4.20}$$

Thus the condensates are strongly flavor- (or, more precisely, mass-) dependent. Note in particular that in this last expression for the ratio, anomalous dimension factors cancel out, and therefore lattice corrections might be smaller. These last results are completely consistent with current algebra, and state that it is really the quantity

$$\frac{<\bar{u}u + \bar{d}d>}{f_\pi^2} = \frac{2\, m_\pi^2}{m_u + m_d} \equiv A_P\, a^{-1} \tag{4.21}$$

that stays constant as a function of the light quark mass. The pion decay constant is given by $f_\pi = 0.060(10)a^{-1} = 110(15)\ MeV$, in reasonable agreement with the Wilson fermion estimate. By interpolating the data in Table II or from the graph in fig. 4 one also obtains the estimate $f_K/f_\pi = 0.073/0.060 = 1.2(1)$.

As in the Wilson fermion case, the strange quark mass can be deduced from the pseudoscalar or vector meson, or from the baryon. The three estimates are 44(4), 37(6) and 42(6) MeV respectively, again in good agreement with each other, leading to the final estimate for the strange quark for Kogut-Susskind fermions at $g^2 = 1$

$$m_s(a) = 41(5)\ MeV \quad (r = 0) \tag{4.22}$$

This number is again significantly smaller than the corresponding result for Wilson fermions. On the other hand one notices that the invariant ratio

$$\frac{2\, m_s(a)}{m_u(a) + m_d(a)} \tag{4.23}$$

is quite similar for the two fermion actions. The ratio is about 26(4) for Wilson fermions, and about 22(4) for Kogut-Susskind fermions. Similar numerical values (25.1(8)) also follow from lowest order current algebra identities for the pseudoscalar mesons masses [16]. Finally one should mention that the scaling violation parameter $\Lambda_{\overline{MS}}$ can also be estimated using the value for the lattice spacing computed above. One finds $\Lambda_{\overline{MS}} = 0.058a^{-1} = 110(10)\ MeV$ where the error quoted is only statistical.

The pion-nucleon σ term can be estimated from the nucleon slope (eqs. (4.8) and (4.17)), and one obtains

$$\sigma = 29(7)\ MeV \quad (r = 1) \qquad \sigma = 29(6)\ MeV \quad (r = 0) \tag{4.24}$$

with good agreement between the two determinations. To compute the asymmetries in the distribution one would have to consider quarks of unequal mass as was done on a small lattice in ref. [37]. Also note that σ is a renormalization group invariant as it stands, and thus has no perturbative corrections. Still the above value is smaller by about a factor of two than the experimental number (in the next section only a slight increase will be found when fermion loops effects are included). Reducing the quark mass even further in order to approach the chiral limit more closely would clearly help in reducing the error.

Generally speaking one finds that the fluctuations in the hadron masses are more contained in the Kogut-Susskind than in the Wilson case, a situation which was ascribed earlier to the quadratic mass renormalization effects which lead to the observed fluctuations in k_c in the latter case. The results for both Wilson and Kogut-Susskind fermions are also in reasonable agreement with older lower statistics studies on the same size lattice [38], and with the more recent results (for Kogut-Susskind fermions only) of ref. [39].

The slope of the pion mass squared versus the quark mass A_P, as well as the quark mass $m(a)$ itself, are not renormalization group invariant quantities. As discussed in the previous section, an invariant mass can be obtained by including the appropriate anomalous dimension factor of eq. (3.32)

$$\tilde{m}_q = m(\mu)\Big(\log \frac{\mu}{\Lambda_{\overline{MS}}}\Big)^{4/11} \tag{4.25}$$

Taking into account the additional perturbative lattice corrections discussed before in sec. III (see eq. (3.35)) one finds

$$\begin{aligned} &\tilde{m}_u + \tilde{m}_d = 15.1(8)\ MeV \quad \tilde{m}_s = 199(20)\ MeV \quad (r=1) \\ &\tilde{m}_u + \tilde{m}_d = 8.6(4)\ MeV \quad\ \ \tilde{m}_s = 96(12)\ MeV \quad\ \ (r=0) \end{aligned} \tag{4.26}$$

in the Wilson and Kogut-Susskind case, respectively. The difference between the two quark mass estimates cannot be accounted for by the statistical and systematic errors, and is presumably due to the different continuum approach of the the two fermion actions. The anomalous dimension lattice correction factors are too small to compensate for this difference.

The discrepancy in the quark mass estimates is not the only difference in the observed results for the two fermion actions. If one compares the estimates for the rho mass at different values of β, for the Wilson action ($r = 1$) one finds [2,40]

$$\begin{aligned} \frac{m_\rho}{\Lambda_{\overline{MS}}}(\beta = 5.4) &= \frac{0.77(5)a^{-1}}{0.116a^{-1}} \simeq 6.6 \\ \frac{m_\rho}{\Lambda_{\overline{MS}}}(\beta = 5.7) &= \frac{0.51(5)a^{-1}}{0.083a^{-1}} \simeq 6.1 \\ \frac{m_\rho}{\Lambda_{\overline{MS}}}(\beta = 6.0) &= \frac{0.39(2)a^{-1}}{0.060a^{-1}} \simeq 6.5 \end{aligned} \tag{4.27}$$

which is consistent with at least *approximate* scaling behavior. Similar behavior is observed for the slope A_P. On the other hand for the Kogut-Susskind case ($r = 0$) one finds [36,40]

$$\begin{aligned} \frac{m_\rho}{\Lambda_{\overline{MS}}}(\beta = 5.7) &= \frac{0.80(5)a^{-1}}{0.083a^{-1}} \simeq 9.6 \\ \frac{m_\rho}{\Lambda_{\overline{MS}}}(\beta = 6.0) &= \frac{0.41(2)a^{-1}}{0.060a^{-1}} \simeq 6.8 \end{aligned} \tag{4.28}$$

which shows *no* scaling behavior yet, even though the numbers at $\beta = 6.0$ are comparable (all these results are of course still without fermion loops). A similar result is found for the pion slope A_P: for $\beta = 5.7$ it is $6.3a^{-1}$ while for $\beta = 6.0$ it is $5.8a^{-1}$, a result at this point again inconsistent with scaling. The inclusion of fermion loops effect does not ameliorate the problem: for $r = 0$ one finds that while the lattice spacing decreases substantially, the pion mass squared slope A_P in lattice units is almost the same with (for $n_f = 2$) and without fermion loops at $\beta = 5.6$ [41]. For comparison the pion mass slope was also evaluated for the tree-level improved fermion action of ref. [29], and was found to be $A_P = 3.22(48)a^{-1}$ for $k_c = 0.187(2)$ and $a_{IA}^{-1} = 1900(200)\ MeV$. Therefore in this case $m_u + m_d(a) = 6.4(1.4)\ MeV$ and $\tilde{m}_u + \tilde{m}_d = 12.2(2.8)\ MeV$, which is between the previous two results, but differs from the Wilson fermion answer by $21\% \pm 22\%$.

The continuum answer for the light quark masses must lie between the Wilson and Kogut-Susskind fermion results (the lattice corrections have opposite signs in the two cases). The perturbative lattice correction factors appear to be too small though to compensate for the discrepancy. One can try to estimate the size of the higher order corrections by requiring that the invariant masses in the end be the same

$$\begin{aligned} \tilde{m}_W \equiv & (2\beta_0 g_{latt}^2)^{-4/11}(1 - 0.15x\ g_{latt}^2)^{4/11}\ m_W(a) = (2\beta_0 g_{latt}^2)^{-4/11} Z_m^W\ m_W(a) \\ = \tilde{m}_{KS} \equiv & (2\beta_0 g_{latt}^2)^{-4/11}(1 + 0.46x\ g_{latt}^2)^{4/11}\ m_{KS}(a) = (2\beta_0 g_{latt}^2)^{-4/11} Z_m^{KS}\ m_{KS}(a) \end{aligned} \tag{4.29}$$

with a ratio of coefficients given by the lowest order formula, and x a correction factor to be determined. From the above values of the lattice quark masses one finds for $g^2 = 1$ that $x \approx 4.2$, indeed a large number. In other words, at this value of $g^2 = 1$ the perturbative correction has to be about four times as large in order to compensate for the large lattice quark mass ratio. The purely lattice correction is thus estimated at $Z_m^W \approx 0.70$ and $Z_m^{KS} \approx 1.48$. In the continuum one then obtains the estimates

$$\tilde{m}_u + \tilde{m}_d \approx 11.0\ MeV \quad \tilde{m}_s \approx 136\ MeV \ \ (n_f = 0) \tag{4.30}$$

Since the ratio $m_d/m_u = 1.80$ is known from current algebra, one obtains the final estimate for the running quark masses at a scale of 1 GeV

$$m_u(1GeV) \approx 3.0\ MeV \ \ m_d(1GeV) \approx 5.3\ MeV \ \ m_s(1GeV) \approx 102\ MeV \ \ (n_f = 0) \tag{4.31}$$

To summarize, the results presented for light hadron spectroscopy at $g^2 = 1$ appear to be in reasonable agreement with phenomenology, even though a simplified model of QCD has been adopted in which virtual fermion loop effects are omitted. Some significant discrepancy is observed in the estimates for the light quark masses, and can be ascribed to problems associated with the fermion doubling and finite lattice corrections due to the additional light modes present for $r = 0$. One still will have to check that the results for the mass ratios at $g^2 = 1$ remain unchanged as the gauge coupling is decreased further to approach the continuum limit, and that the agreement found in this study is not purely accidental. Finally the effects of dynamical fermion loops have to be taken into account, which leads to the topic of the next section.

V - INCLUSION OF FERMION POLARIZATION EFFECTS

Before computing physical quantities like hadron masses with internal loops by the pseudofermion method it is important to test whether the method works for simpler observables [42]. A quantity of interest which can be measured rather easily is the plaquette energy

$$E_p = < 1 - \frac{1}{N} \text{ Tr } U_p > \tag{5.1}$$

where U_p is the ordered product of four link matrices around an elementary square. Its change when the dynamical fermion loops are introduced should give a first indication of the effects and, to a certain extent, of the reliability of the numerical algorithms used. It can be computed exactly in weak coupling lattice perturbation theory to one loop order, including fermionic effects [43]. For $SU(3)$ in four dimensions with n_f light fermion flavors the result is for $r = 1$

$$< 1 - \frac{1}{N} \text{ Tr } U_p > = \frac{2}{\beta} + (1.225 - 0.059\ n_f)\frac{1}{\beta^2} + O(\frac{1}{\beta^3}) \tag{5.2}$$

With the conventions chosen here, one would therefore expect that the fermion polarization would reduce the average plaquette and thus *increase* the order in the gauge fields.

One reaches a similar conclusion if one considers the limit of small k (large quark mass). By expanding the fermion determinant to lowest order in k one can easily convince oneself that in this limit the effects of the fermions can be re-absorbed by a shift in the gauge coupling β, with an effective coupling given by [44]

$$\beta_{eff} = \beta + 16\ N\ n_f\ k^4 + O(k^6) \tag{5.3}$$

Other quantities of interest are the average 'current'

$$< J \cdot U > = \frac{1}{8V} \sum_{n\mu} [< \bar{\psi}_n (r + \gamma_\mu) U_{n\mu} \psi_{n+\mu} > + < \bar{\psi}_{n+\mu} (r - \gamma_\mu) U^\dagger_{n\mu} \psi_n >] \tag{5.4}$$

and the fermion condensate

$$< \bar{\psi}\psi > = \frac{1}{V} \sum_n < \bar{\psi}_n \psi_n > \tag{5.5}$$

It is easy to see that the average 'current' $< J \cdot U >$ should *increase* when the loop effects are included as a consequence of the ordering effect of the determinant, while $< \bar{\psi}\psi >$ should *decrease*. In fact for the latter quantity in the continuum, on a given background gauge configuration, one can write the spectral representation

$$< \bar{\psi}\psi >_U = m \sum_{\lambda > 0} \frac{\rho_U(\lambda)}{\lambda^2 + m^2} \tag{5.6}$$

where $\rho_U(\lambda)$ is the density of eigenvalues of the operator $-\not{D}^2[U]$. It follows then that the average over field configurations is given by

$$< \bar{\psi}\psi > = \frac{1}{Z} \int [dU] \; e^{-S_G} \; [\prod_{\sigma > 0} \rho_U(\sigma)(\sigma^2 + m^2)]^{n_f/2} \; m \sum_{\lambda > 0} \frac{\rho_U(\lambda)}{\lambda^2 + m^2} \tag{5.7}$$

which shows the suppression of low energy modes by the fermion determinant, and also implies the inequality, valid for any *fixed* g^2, for the condensate in lattice units

$$< \bar{\psi}\psi >_{n_f = 0} \; > \; < \bar{\psi}\psi >_{n_f > 0} \tag{5.8}$$

Note that this does not necessarily imply that the condensate in physical units decreases for $n_f > 0$. Furthermore, the fermion condensate allows one to estimate the fermion contribution to the energy density through the identity

$$< k \; \frac{\partial}{\partial k} (\; n_f \; \mathrm{Tr} \ln \Delta[U]) > = \; n_f (NC - < \bar{\psi}\psi >) \tag{5.9}$$

where C is the number of components of a Dirac spinor (C=4 in d=4). It is valid in an arbitrary background gauge field configuration, even before averaging over the gauge field configurations.

Of course one of the outstanding problems in the numerical study of QCD is the inclusion of fermion vacuum polarization effects in a realistic study of the mass spectrum on a large lattice. While several algorithms have been proposed, it is unclear at this point how well some of these methods will perform in the realm a full calculation on a large lattice. A study has been done to extend a previous calculation on a $4 \times 4 \times 4 \times 16$ lattice [42], and has confirmed some of the qualitative features of the effects of fermion loops described there. The aim is on the one hand to provide accurate data at one value of the coupling constant for the full theory on a larger lattice, and at the same time gain some qualitative insight into the effects of the fermion loops on the hadron spectrum, the light quark masses and the QCD scaling parameter, among other things. Even though some of the results are quite preliminary and will have to be checked with higher accuracy on larger lattices, it appears that some new effects are quite visible, especially on the quark mass estimates, as will be discussed below. To what extent some of these results are contaminated by lattice artifacts is not entirely clear at this point.

The effects of virtual quark loops were included by employing the pseudofermion method described earlier. The results use the Wilson form ($r = 1$) of the fermion action. Part of the computation completely parallels the analysis in the case with no internal fermion loops as described in the earlier sections.

Details on the actual numerical computation will now be given [45]. Two completely independent sets of configurations were generated on a $10 \times 10 \times 10 \times 10$ lattice, one starting from a pure gauge configuration thermalized at $\beta = 6/g^2 = 5.6$ (hot start) with 2000 iterations, and another set from a pure gauge configuration thermalized at $\beta = 6.0$ (cold start), again thermalized with 2000 iterations. Then another 800 iterations were performed on both pure gauge configurations with the effects of the fermion determinant included for $n_f = 3$ and $\beta = 5.4$, at five values of the hopping parameter $k = 0.156$, 0.158, 0.160, 0.162 and 0.163. Let us recall that when the loop effects are taken into account, two new parameters need to be introduced (apart from the number of flavors n_f and k or the fermion mass m): the step size used in the gauge updating ϵ_B and the number of pseudofermion iterations needed to compute the fermion currents n_{pf}. As pointed out before, in the limit that ϵ_B goes to zero and n_{pf} goes to infinity the algorithm becomes exact [4].

k	0.1560	0.1580	0.1600	0.1620	0.1630
$< E_P >$	0.477(2)	0.475(2)	0.472(2)	0.467(3)	0.456(5)
$< J \cdot U >$	0.527(2)	0.535(3)	0.542(2)	0.550(2)	0.558(4)
$< \bar{\psi}\psi >$	0.922(2)	0.915(3)	0.908(3)	0.901(3)	0.896(3)

Table III

Average plaquette, average current and chiral condensate in lattice units at g^2=1.111 for Wilson fermions.

For the first set of five configurations (hot start) the propagators were then computed on 20 configurations separated by 20 gauge iterations (with 10 hits per link). For the second set of five configurations (cold start) the propagators were evaluated on 25 configurations separated by 20 gauge iterations. The pseudofermions needed in the gauge field updating were thermalized each time with 20+100 iterations (with 5 hits per site) and the step size for the gauge field updating was chosen to be $\epsilon_B = 0.01$. Thus the configurations at different values of k can be taken as almost statistically independent, with two independent sets for each k. Correlations between consecutive configurations at the same value of k were monitored by looking at local observables (plaquette energy, average fermion current and fermion condensate) and at the propagators, which were binned in groups of 5 and 10 to estimate the errors in the masses and amplitudes. The correlations were found to be small after 100 gauge iterations. Good agreement (to within statistical errors) was found between 'heating' and 'cooling' runs at the same value of k, indicating adequate thermalization. Extensive runs were done also for $\epsilon = 0.005$ and no noticeable change in the hadron masses was found, within statistical errors. Comparison with earlier results on a $4 \times 4 \times 4 \times 16$ lattice at the same value of β with $n_f = 3$ and $\epsilon_B = 0.005$ [42] suggests that at this value of β and for the chosen values of k finite size effects are contained (see discussion below). Results for the local observables at different values of k are shown in Table III.

The lattices were triplicated in the time direction to give a $10 \times 10 \times 10 \times 30$ lattice and the fermion propagators were subsequently evaluated (for different values of k or the quark mass) to high accuracy by the relaxation method with 200 iterations. Periodic boundary conditions were adopted for both the gauge fields and the fermions, as in previous studies. Several tests were performed to ensure that the fermion propagators at the largest separation ($t = 15$) had converged to better than one percent. Meson and baryon correlation functions were then computed and finally averaged over the gauge configurations. The lattice was sufficiently long in the time direction that the mass of the lowest state could be estimated unambiguously (as a check the correlation functions were also fitted to sums of exponentials).

The uncertainty in the mass resulting from the extrapolation to large t and the statistical fluctuations in the correlation functions leads then to the errors quoted in Tables IV and V. Due to the large size of the lattice, the high accuracy in the matrix inversion and the substantial number of configurations analyzed, the statistical and systematic effects are reasonably under control. At the same time more efficient fermion algorithm are also being developed and are expected to speed up the simulation significantly. It should also be emphasized that since all hadron masses are extracted from the *same* set of fermion propagators, which are evaluated exactly on the same gauge configuration, the errors quoted are correlated and the spin splittings are determined more accurately than one might have expected from the uncertainty in the masses themselves [8]. Also it should be pointed out that the fact that ϵ_B is finite is another source of error on the masses, which is difficult to measure unless the calculation is completely redone for another value of ϵ_B, say 0.005. This eventually has to be done. Since one expects the error generically to be $O(\epsilon_B)$ this means for example that the error quoted below for k_c, the critical value for the hopping parameter at which the pseudoscalar meson mass vanishes, refers only to this chosen value of ϵ_B.

As usual the physical region of very small quark masses has to be, at least with the presently available lattice sizes, obtained by extrapolation using the values of the hadron masses computed for larger quark mass. The extrapolation problem is eased though by the knowledge of how some of the masses (notably the pion) behave for small quark masses in the continuum. The smallest quark mass used in the present study corresponds roughly to the strange quark mass. Results for lighter quarks (up, down) are then obtained by smooth extrapolation. A second motivation for the specific lattice size used was that it allows a comparison with the data obtained without fermion polarization effects [35] (see previous section) on a lattice of comparable *physical* size. The finite spatial extent of the lattice poses a limit on the smallest pion mass that can be reached; the inverse pion mass in lattice units should be at most about one half the spatial size of the lattice. In the present study, as well as in the calculation of ref. [35], a conservative cut $m_P L > 3$ was adopted. Since the lattice spacing at $\beta = 5.6$ is about 0.08 Fermis (to be compared with 0.1 Fermis in the previous section and in ref. [35), the spatial box size is about 0.8 Fermis.

The results for hadron masses will now be discussed. Tables IV and V list the results for the hadron masses in lattice units. The symbols P, V, S, A, T refer to the pseudoscalar, vector, scalar, axial and tensor mesons respectively. N and Δ (or D in the figures) refer to the $s = 1/2$ and $s = 3/2$ baryon. The same results are then displayed in figs. 2a, 2b and 2c (the lines in the these figures are only intended as a guide to the eye). In all columns, except the second one, the mass of the 'sea' quarks generating the vacuum polarization effects is taken to be the same as the mass of the 'valence' quarks that enters into the external propagators. The second column (labelled with an asterisk) shows the results of a calculation where the 'sea' quarks have $k = 0.163$ and the 'valence' quarks have $k = 0.156$. In other words the 'valence' quarks are much heavier than the 'sea' quarks. As can be seen from the table the vacuum polarization contribution of the light quarks significantly decreases the mass of the heavier quark bound states. For light enough quarks the decays $\rho \to \pi\pi$ and $\Delta \to N\pi$ will take place, but this will require $m_P < 0.17a^{-1}$ and $m_P < 0.12a^{-1}$ respectively, which is smaller than the mass values considered here ($m_P \geq 0.39a^{-1}$).

Furthermore it appears that finite size effects for the values of k considered here are likely to be quite small (less than the statistical errors). For example on the $4 \times 4 \times 4 \times 16$ lattice at the same value of β and for $k = 0.160$ one finds [42] $m_P = 0.55(8)$, $(m_V^2 - m_P^2)^{1/2} = 0.35(9)$ and $m_N = 1.38(30)$ which agrees to 27%, 6% and 6% respectively with the values quoted in Table IV. One should keep in mind that finite size effects decrease exponentially with the spatial linear size of the box (see eq. (3.28)). Similar results were obtained later also with the hopping parameter expansion (see the second of ref. [2], and references therein).

From the data in the tables one notices that the hyperfine splittings ($P - V$, $N - \Delta$ and $S - A$) are almost independent of k (or of the quark mass) when expressed in terms of mass squared differences. This phenomenon is observed to a certain extent also in the absence of vacuum polarization effects, but while there the $P - V$ splitting decreases for larger quark mass, here it is found to increase, in agreement with the observation that the spin splitting in masses squared increases when one goes from the ρ to the J/Ψ. It is amusing to see that this effect is due to fermion polarization loops.

As in the case $n_f = 0$ [35] the vector meson mass and the baryon mass are not well fitted for small quark masses by a linear function of $1/k$ or the quark mass m, whereas the quality of the fit improves when the masses squared are used (as in fact is clearly the case for the pion). The same situation is observed when the mass squared differences are computed. Remarkably also the nucleon mass squared minus about 2.8 the pseudoscalar mass squared is roughly constant within the (large) errors (again, this difference is computed more accurately than one might expect from the error in the individual quantities themselves, because of the correlation in the errors coming from the exact evaluation of the propagator in a given background gauge field configuration). Thus all the extrapolated values are quoted assuming linear behavior in $1/k$ for all masses squared and mass squared differences.

The pseudoscalar meson mass tends to zero at $k_c = 0.1643(3)$ (to be compared with the estimate $k_c = 0.163(7)$ on the smaller lattice of ref. [42], which differs by less than one percent but also has a larger error). The quark mass can be defined again by $m = \ln[1 + \frac{1}{2}(\frac{1}{k} - \frac{1}{k_c})] \sim_{k \to k_c} \frac{1}{2}(\frac{1}{k} - \frac{1}{k_c})$. The pseudoscalar meson mass squared is well fitted by a linear function of $1/k$ and one finds for small m

$$m_P^2 = 6.4(5)a^{-1}\ m + O(m^2) \tag{5.10}$$

The lattice spacing can be determined for example from the rho mass. At k_c one finds

$$0.29(2) \quad 0.36(5) \quad 0.48(5) \quad 0.36(4) \quad 0.46(6) \quad 0.43(4) \tag{5.11}$$

k	0.1560	0.1560*	0.1580	0.1600	0.1620	0.1630	0.1643(3)
m	0.150	0.150	0.114	0.079	0.042	0.024	0.00
m_P	1.05(5)	0.73(3)	0.92(3)	0.75(2)	0.54(3)	0.39(4)	0.00
$m_{P'}$	1.52(5)	1.18(2)	1.39(3)	1.22(2)	1.01(2)	0.85(3)	0.46(5)
m_V	1.11(5)	0.79(3)	0.98(3)	0.82(2)	0.62(2)	0.49(3)	0.29(2)
$(m_V^2 - m_P^2)^{1/2}$	0.37(3)	0.29(4)	0.35(3)	0.33(3)	0.31(4)	0.30(3)	0.29(2)
m_N	1.78(6)	1.31(4)	1.57(7)	1.30(5)	0.97(7)	0.74(5)	0.36(5)
$(m_N^2 - 2.8m_P^2)^{1/2}$	0.27(5)	0.29(6)	0.31(6)	0.33(5)	0.34(5)	0.35(7)	0.36(5)
m_Δ	1.81(7)	1.33(4)	1.61(7)	1.34(5)	1.02(8)	0.81(6)	0.48(5)
$(m_\Delta^2 - m_N^2)^{1/2}$	0.32(5)	0.31(8)	0.35(4)	0.33(5)	0.33(4)	0.33(6)	0.33(4)
f_P	0.15(2)	0.07(2)	0.12(2)	0.09(1)	0.06(1)	0.05(1)	0.03(1)

Table IV
Hadron masses (S-waves) in lattice units at $\mathbf{g^2 = 1.111}$ for Wilson fermions. In the second column (*) the sea quarks have $\mathbf{k = 0.163}$.

for the mass of the state with the quantum numbers of the ρ, nucleon, Δ, δ, A_1 and B in lattice units. The error in the rho mass determination at k_c can be estimated by extrapolating either m_P^2 or $(m_V^2 - m_P^2)^{1/2}$. One concludes that the lattice spacing in the Wilson fermion case at $\beta = 5.4$ for $n_f = 3$ is much smaller than for $n_f = 0$

$$a_W^{-1}\ (\beta = 5.4,\ n_f = 3) = 2600(200)\ MeV$$
$$a_W^{-1}\ (\beta = 5.4,\ n_f = 0) = 980(100)\ MeV \tag{5.12}$$

which should be compared to the value 1800(700) MeV (for $n_f = 3$) obtained on the smaller lattice [42]. As in the case with no loops, the relatively large error in the proton mass in lattice units is mostly due to the fact that the baryon mass decreases quite rapidly as the quark mass is reduced to zero. For finite quark mass the results for the vector meson are consistent with

$$m_V^2 = [0.29(2)a^{-1}]^2 + 6.7(5)a^{-1}\ m + O(m^2) \tag{5.13}$$

and for the nucleon with

$$m_N^2 = [0.36(5)a^{-1}]^2 + 18.0(3.2)a^{-1}\ m + O(m^2) \tag{5.14}$$

No substantial deviation from linearity is seen in the range of quark masses explored, indicating that the coefficient of the m^2 term is not too big (for large m it is expected to

k	0.1560	0.1560*	0.1580	0.1600	0.1620	0.1630	0.1643(3)
m	0.150	0.150	0.114	0.079	0.042	0.024	0.00
m_S	1.47(9)	0.90(6)	1.23(8)	0.99(5)	0.74(5)	0.60(4)	0.36(4)
m_A	1.50(9)	0.93(6)	1.27(8)	1.03(5)	0.79(5)	0.67(4)	0.46(4)
m_T	1.49(9)	0.92(6)	1.26(8)	1.02(5)	0.77(5)	0.65(4)	0.43(4)
$(m_S^2 - m_V^2)^{1/2}$	0.95(5)	0.43(5)	0.74(5)	0.56(6)	0.40(4)	0.31(3)	0.20(4)
$(m_A^2 - m_S^2)^{1/2}$	0.32(4)	0.25(5)	0.31(4)	0.29(3)	0.30(4)	0.28(3)	0.29(4)
$(m_A^2 - m_T^2)^{1/2}$	0.16(4)	0.13(5)	0.17(4)	0.15(3)	0.16(4)	0.14(3)	0.15(4)

Table V
Hadron masses (P-waves) in lattice units at $\mathbf{g^2 = 1.111}$ for Wilson fermions. In the second column (*) the sea quarks have $\mathbf{k = 0.163}$.

be 4 and 9 for the mesons and baryons, respectively). The physical mass of the nucleon is then estimated at

$$m_{nucleon} = 970(130)\ MeV \tag{5.15}$$

without taking into account the error due to the uncertainty in the lattice spacing, which would add another 7% error. The difference in mass between the physical nucleon and the mass of the nucleon in the chiral limit ($m = 0$) can the be estimated at about 33 MeV. The mass of the physical Δ is then 1250(150) MeV and for the δ, A_1 and B one obtains about 940(100) MeV, 1200(100) MeV and 1120(100) MeV respectively. Note that the B meson is clearly lighter than the A_1. These masses are in qualitative agreement with the experimental values, but the errors are unfortunately still quite large, and the extrapolation to small quark mass is still to a certain extent a question of subjective judgement. In any case no gross inconsistency with the observed spectrum is found.

Next the quark masses and the pion decay constant will be considered. From eq. (5.10) or from fig. 5 one obtains for the sum of up and down quark masses on the lattice

$$m_u(a) + m_d(a) = 2.4(3)\ MeV \tag{5.16}$$

The pion decay constant on the lattice is $f_\pi^{latt} = 0.03(1)a^{-1} = 80(25)\ MeV$. The lattice value should then be corrected by the perturbative renormalization factor relating the currents on the lattice to the currents in the continuum [32] (as mentioned before, this finite factor arises because of the explicit chiral symmetry breaking inherent in the Wilson formulation)

$$f_\pi^{cont} = [1 - 0.14\, g^2 + O(g^4)]\ f_\pi^{latt} = 70(25)\ MeV \tag{5.17}$$

Figure 5: S-wave meson masses in lattice units as a function of the parameter k (related to the bare quark mass) at $\mathbf{g^2 = 1.111}$ for Wilson fermions (data from Table IV). P stands for pseudoscalar meson, V for vector meson. P/V denotes the spin splitting $(\mathbf{m_V^2 - m_P^2})^{1/2}$. f is the pseudoscalar decay constant.

leading to a final estimate in reasonable agreement with the experimental value. The data also provides information on how the decay constants vary with the meson mass. One notices that the pseudoscalar meson decay constant increases with the quark mass. By interpolating the data in Table IV, or alternatively from the graph in fig. 5, one finds $f_P = 0.03a^{-1} + 0.73(15)m$ and thus an estimate for the kaon decay constant can be

Figure 6: Baryon masses in lattice units as a function of the parameter k (related to the bare quark mass) at $\mathbf{g^2 = 1.111}$ for Wilson fermions (data from Table IV). N stands for the ($\mathbf{s = 1/2}$) nucleon, D for the ($\mathbf{s = 3/2}$) $\boldsymbol{\Delta}$ resonance. N/D denotes the spin splitting $\mathbf{(m_\Delta^2 - m_N^2)^{1/2}}$ and N/P the quantity $\mathbf{(m_N - 2.8\ m_P^2)^{1/2}}$.

obtained $f_K/f_\pi = 1 + 0.12(5) = 1.12(5)$. The above value is in reasonable agreement with the experimental ratio of 1.16 deduced from the baryon decays.

The strange quark mass can be estimated, as in the case $n_f = 0$, either by using the (nonexistent) $\bar{s}s$ pseudoscalar meson, the $\phi(1020)$ vector meson or finally the (nonexistent) sss nucleon. The three independent strange quark mass estimates are then

Figure 7: Same as in figs. 5 and 6, but for the scalar (S), axial (A) and tensor (T) P-wave mesons. The data is from Table V.

28(4), 27(4) and 28(5) MeV respectively, in reasonable agreement with each other. The final estimate for the lattice strange quark mass with Wilson fermions and $n_f = 3$ is then

$$m_s(a) = 28(5)\ MeV \tag{5.18}$$

which is substantially smaller than the value found (in the Wilson fermion case) for $n_f = 0$: there $m_u(a) + m_d(a) = 7.8(8)\ MeV$ and $m_s(a) = 103(15)\ MeV$. On the other hand one notices that the ratio $2m_s(a)/(m_u(a) + m_d(a))$ is unchanged: it is about 24(5) for $n_f = 3$ and about 26(5) for $n_f = 0$.

The pion-nucleon σ term again can be estimated from the nucleon slope in eq. (5.14), and one obtains

$$\sigma = 30(8)\ MeV \quad (r=1,\ n_f=3) \tag{5.19}$$

which is only slightly larger than the value obtained before for $n_f = 0$ (eq. (4.23)). The above value is still smaller by about a factor of two than the experimental number ($60 \pm 10\ MeV$), but has a large uncertainty because of the substantial extrapolation done to reach the chiral limit.

As usual the slope of the pseudoscalar meson mass squared versus the quark mass, as well as the quark mass $m(a)$ itself, are not renormalization group invariant quantities. An invariant mass can be obtained by including the appropriate anomalous dimension factor

$$\tilde{m}_q = m_q(a)\big(\log \frac{1}{a\Lambda_{\overline{MS}}}\big)^{4/9} \tag{5.20}$$

Taking into account the additional perturbative lattice corrections discussed in section III, one finds the invariant masses in the $\overline{MS}$ scheme (by multiplying by a factor of 2.27), which in turn can then be related to the quark masses at any scale by the equation

$$m_q(\mu) = \tilde{m}_q\ \big(\log \frac{\mu}{\Lambda_{\overline{MS}}}\big)^{-4/9} \tag{5.21}$$

For $\mu = 1\ GeV$ one then finds (for $r = 1$)

$$\begin{array}{lll} m_u + m_d(1GeV) = 4.0(5)\ MeV & m_s(1GeV) = 46(8)\ MeV & n_f = 3 \\ m_u + m_d(1GeV) = 11.3(6)\ MeV & m_s(1GeV) = 149(15)\ MeV & n_f = 0 \end{array} \tag{5.22}$$

It appears that the values for $n_f = 3$ are *unusually* low. If one argues that for very small quark masses ($m_q \ll \Lambda_{\overline{MS}}$) the number of light flavors should be chosen to be two and not three, one can increase the answer by 33% of the difference between the answer for $n_f = 3$ and $n_f = 0$. This operation would then give about $6.4(5)\ MeV$ for the sum of up and down quark masses and about $80(10)\ MeV$ for the strange quark, which is closer to the QCD sum rule estimates of 8 and 90 MeV respectively [27]. It seems though that such small quark masses would in turn justify the use of $n_f = 3$, since for all three light quarks one in fact has $m_q \ll \overline{MS}$. In any case it is surprising that such a strong flavor dependence goes into the estimates for the quark masses.

In the charmed quark mass region it seems reasonable to assume that one has three *light* flavors. From the experimental D meson mass one can then try to extract the quark mass (see the starred entry in table IV). One finds that roughly $m_c(a) = 0.31(5)a^{-1} = 810(120)\ MeV$. At the scale $\mu = 1GeV$ one then finds

$$m_c(1GeV) = 1320(200)\ MeV \tag{5.23}$$

which is slightly larger than the QCD sum rule estimate of about 1200 MeV for the charmed quark mass [27]. From the quoted numbers an invariant prediction would be of course $m_c/m_s = 29(7)$ for $n_f = 3$, but a more accurate estimate for the charmed quark mass would be welcome to better pin down the overall scale.

By comparing the above results with the quark masses obtained for $n_f = 0$, and with the relative size of the lattice perturbative corrections in the $r = 1$ and $r = 0$ case, one can argue that the actual physical quark mass should be smaller than the Wilson fermion estimate by a factor of $Z_m^W \approx 0.44$ (or, equivalently, that the ansatz (3.37) for the quark mass leads to an over-estimate; see eq. (4.29) in the previous section). (A comparison with $r = 0$ fermions, as was done before in the case of no fermion loops, is not easy to do here, because of the difficulties associated with determining what the actual effective number of light fermion flavors really is in this formulation, as discussed before, see section III). In other words, the assumption is that the correct anomalous dimension factor is only weakly n_f-dependent for a given g^2. Since again the ratio $m_d/m_u = 1.80$ is known from current algebra, one obtains for the running quark masses in the continuum at a scale of 1 GeV the final, best estimate

$$\begin{array}{ll} m_u(1GeV) \approx 0.6\ MeV & m_d(1GeV) \approx 1.1\ MeV \\ m_s(1GeV) \approx 20\ MeV & m_c(1GeV) \approx 580\ MeV \quad (n_f = 3) \end{array} \tag{5.24}$$

which are still *surprisingly small values*, even allowing for substantial lattice errors.

The scaling violation parameter $\Lambda_{\overline{MS}}$, for three light fermion flavors, can be estimated using the value for the lattice spacing, determined before from the physical rho meson mass. One finds for $n_f = 3$

$$\Lambda_{\overline{MS}} = 0.049a^{-1} = 125(10)\ MeV \tag{5.25}$$

(where the error quoted is only statistical), which is remarkably close to the value obtained for $n_f = 0$: $\Lambda_{\overline{MS}} = 0.049a^{-1} = 110(10)\ MeV$. For weak enough g^2 one expects that the rho mass should become proportional to the lambda parameter $\Lambda_{\overline{MS}}$, if scaling holds. From the rho mass on the one hand one finds

$$\beta = 5.4 \qquad \frac{am_\rho(n_f = 0)}{am_\rho(n_f = 3)} \approx \frac{0.75(2)}{0.29(2)} = 2.6(3) \tag{5.26}$$

From the ratio of the $\Lambda_{\overline{MS}}$ parameter (which is the natural scale in the continuum) one gets on the other hand

$$\beta = 5.4 \qquad \frac{\Lambda_{\overline{MS}}\,(n_f = 0)}{\Lambda_{\overline{MS}}\,(n_f = 3)} = \frac{0.111}{0.048} = 2.3 \tag{5.27}$$

The results therefore suggest [42,45] that when virtual quark loop effects are included, the rho mass in lattice units has changed by an amount approximately consistent with the ratio of scale parameters. (For completeness one should add that if one had chosen Λ_{latt} as the relevant scale parameter, one would have concluded that the ratio of rho masses in lattice units was too small by about a factor of two, since the ratio of Λ_{latt}'s is 4.14 at $\beta = 5.4$). In other words, in the region of β studied most of the fermion polarization effects can be accounted for by a shift in β (by approximately 0.5), and is consistent with the observation that with three light flavors the 'crossover' region has shifted to $\beta \approx 5.0$.

In conclusion the results presented here for light hadron spectroscopy at $\beta = 5.4$ appear to be in reasonable agreement with phenomenology, although the errors are still quite large. Surprisingly the quark masses have turned out to be quite small, and future calculations will have to check whether these results persist for even smaller quark masses and weaker coupling. Also one will have to check that the results for the hadron mass ratios remain unchanged as the gauge coupling is decreased further to approach the continuum limit. It is expected that a study of this type will require even larger lattice sizes.

VI - THE MASS OF THE ETA PRIME MESON

The classical action for massless QCD is known to have an $U(n_f)_{vector} \otimes U(n_f)_{axial}$ symmetry, which one believes is broken down dynamically to

$$SU(n_f)_{vector} \otimes SU(n_f)_{axial} \otimes U(1)_{baryon\ no.} \tag{6.1}$$

by the triangle anomaly in the isoscalar axial current. The solution to the $U(1)_A$ problem was suggested some years ago on the basis of the $1/N_c$ expansion and topological arguments. A consequence of the non-conservation of the flavor singlet axial current is that the mass of the corresponding pseudoscalar meson (the η') should be higher than the mass of pions and kaons.

A possible approach to the problem of computing the mass of the η' is through Witten's large N formula, relating the mass to the topological susceptibility

$$m_\eta^2 + m_{\eta'}^2 - 2m_K^2 = \frac{4n_f}{N_c^2 f_\pi^2} \, g^2 \, \chi_t \tag{6.2}$$

with

$$\chi_t = \frac{1}{V} < Q^2 > \qquad Q = \frac{1}{16\pi^2} \int_V d^4x \ \mathrm{Tr}[F\tilde{F}] \tag{6.3}$$

It is known though that the lattice transcription of $F\tilde{F}$ is not straightforward.

A different approach was pursued in the second of ref. [8], where it was suggested that the leading term of the η' mass in an expansion in positive powers of n_f could be computed in the pure gauge theory with no internal quark loops. To compute the isospin mass splittings one writes for a generic meson (not necessarily a pseudoscalar)

$$m_\Gamma^2 = m_{\Gamma 0}^2 + n_f \, m_{\Gamma 1}^2 + O(n_f^2) \tag{6.4}$$

and tries to determine $m_{\Gamma 1}$, assuming one knows already $m_{\Gamma 0}$ from a previous calculation. One hopes that $m_{\Gamma 1}$ will give a reasonable estimate of the mixing between flavors. The propagator in momentum space, and for small momenta, can be written as

$$\frac{A_\Gamma}{p^2 + m_\Gamma^2} \tag{6.5}$$

and one assumes that one can set

$$A_\Gamma = A_{\Gamma 0} + n_f A_{\Gamma 1} + O(n_f^2) \tag{6.6}$$

Then one has, to lowest order in n_f,

$$m_{\Gamma 1}^2 = \chi_\Gamma \; A_{\Gamma 0}^{-1} \; m_{\Gamma 0}^4 \tag{6.7}$$

and the quantity χ_Γ is given by

$$\chi_\Gamma = V(< M_\Gamma^2 >_U - < M_\Gamma >_U^2) \tag{6.8}$$

with

$$M_\Gamma = \frac{1}{V} \sum_x \; \mathrm{Tr}\, \Gamma \; \Delta^{-1}(x, x|U) \tag{6.9}$$

χ_Γ measures the fluctuation of $< \bar{\psi}\Gamma\psi >$ from (gauge) configuration to configuration. A systematic study is underway to estimate this quantity for the five local operators of eq. (3.12). The most direct computation of the η' mass from the iso-singlet propagator with internal fermion loops is much harder because of the bad signal to noise ratio [4].

k	0.150	0.152	0.154	0.1564(2)
m	0.128	0.089	0.049	0.0
m_P	0.59(3)	0.50(2)	0.37(2)	0.0
$m_{P\,1}$	0.34(8)	0.36(3)	0.46(9)	0.39(9)
$\sqrt{< Q^2 >}$	5.2(5)	5.3(5)	5.8(7)	5.4(8)

Table VI

Quantities (in lattice units) relevant for determining the $\eta\prime$ mass at $\mathbf{g^2 = 1}$ with Wilson fermions, as described in the text.

It is expected that the fluctuation is largest in the pseudoscalar channel, and indeed this is what one finds. Using 10 pure gauge configurations from the calculations of sect. IV, and computing on each of them all local matrix elements $< \bar{\psi}\Gamma\psi >$ using 1000+5000 pseudofermion Langevin iterations (the matrix elements are all quite small $\sim O(10^{-4})$, except for the scalar one, and thus large statistics are needed to measure them accurately) one obtains the numbers quoted in table V. For the pseudoscalar one finds, using the value for the lattice spacing $a^{-1} = 1950(120)\ MeV$ (eq. (4.6)),

$$m_{P\,1} = 760 \pm 180\ MeV \qquad m_{\eta'} = 1370 \pm 310\ MeV \tag{6.10}$$

The η' mass estimate uses the pion and kaon mass as input, and it appears to be somewhat larger than (but still consistent with) the experimental number. The above result is encouraging, since it shows that for Wilson fermions loop effects are not suppressed after all for small quark mass and intermediate coupling. In fact $m_{P\,1}$ itself is only slowly decreasing with increasing quark mass.

Turning to the case of the vector meson, one finds that $m_{V\,1}$ is more difficult to measure since the matrix element $< \bar{\psi}\gamma_\mu\psi >$ is much smaller in this case. At present the statistical accuracy is not sufficient to quote a number, except that the mixing is smaller by at least a factor of two than in the pseudoscalar case. For the P-wave mesons the mixing is also much smaller than the pseudoscalar case.

VII - CONCLUSIONS

For the first time it seems that a computation of the hadron spectrum from first principles is feasible, and the results appear encouraging and in qualitative agreement with phenomenology. In certain instances (like for the quark masses) it appears like some genuine predictions can be produced. An understanding of statistical and systematic effects is emerging, making more accurate calculations feasible and within reach of presently available computing resources. Computations on even larger lattices are in progress. It is hoped that the limitations imposed at the present moment by the finite extent of the lattice are in part going to be alleviated by the use of even larger lattices, improved actions and finite size scaling methods, and eventually renormalization group techniques.

Acknowledgements

The author would like to thank H. Rubinstein for useful discussions on the QCD sum rule estimates of the light quark masses. The computations presented in this paper were performed on the Cray X-MP 4/8 and the SCS-40 at the San Diego Supercomputer Center, operated by GA Technologies for the National Science Foundation. The author would like to thank Sid Karin, director of the Center, for making a substantial amount of time available on the SCS-40. This work was supported by an NSF grant NSF-PHY-8605552.

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