Wheeler-DeWitt equation in 2 + 1 dimensions

Herbert W. Hamber* and Reiko Toriumi†

Department of Physics and Astronomy, University of California, Irvine, California 92697-4575, USA

Ruth M. Williams‡

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom and Girton College, Cambridge CB3 0JG, United Kingdom

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The infrared structure of quantum gravity is explored by solving a lattice version of the Wheeler-DeWitt equations. In the present paper only the case of $2+1$ dimensions is considered. The nature of the wave function solutions is such that a finite correlation length emerges and naturally cuts off any infrared divergences. Properties of the lattice vacuum are consistent with the existence of an ultraviolet fixed point in G located at the origin, thus precluding the existence of a weak coupling perturbative phase. The correlation length exponent is determined exactly and found to be $\nu = 6/11$. The results obtained so far lend support to the claim that the Lorentzian and Euclidean formulations belong to the same field-theoretic universality class.

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I. INTRODUCTION

It is possible that the well-known ultraviolet divergences affecting the perturbative treatment of quantum gravity in four dimensions point to a fundamental vacuum instability of the full theory. If this is the case, then the correct identification of the true ground state for gravitation necessarily requires the introduction of a consistent nonperturbative cutoff. To this day the only known way to do this reliably in quantum field theory is via the lattice formulation. Nevertheless, previous work on lattice quantum gravity has dealt almost exclusively with the Euclidean formulation in d dimensions, treated via the manifestly covariant Feynman path integral method. Indeed, the latter is very well suited for numerical integration, and many analytical and numerical results have been obtained over the years within this framework. However, the issue of their relationship to the Lorentzian theory has remained largely open, at least from the point of view of a rigorous treatment. The main supporting arguments for the Euclidean approach come from the fact that the above equivalence holds true for other field theories (no exceptions are known) and from the fact that in gravity itself it is rigorously true to all orders in the weak field expansion.

In this paper we will focus on the Hamiltonian approach to gravity, which assumes from the beginning a metric with Lorentzian signature. In order to obtain useful insights regarding the nonperturbative ground state, a Hamiltonian lattice formulation was introduced based on the Wheeler-DeWitt equation, where the quantum gravity Hamiltonian is written down in the position-space representation. In a previous paper [1] a general discrete Wheeler-DeWitt equation was given for pure gravity, based on the simplicial lattice formulation originally developed by Regge and Wheeler. On the lattice the infinite-dimensional manifold of continuum geometries is replaced by a finite manifold of piecewise linear spaces, with solutions to the lattice equations then providing a suitable approximation to the continuum gravitational wave functional. The lattice equations were found to be explicit enough to allow the development of potentially useful practical solutions. As a result, a number of sample quantum gravity calculations were carried out in $2+1$ and $3+1$ dimensions. These were based mainly on the strong coupling expansion and on the Rayleigh-Ritz variational method, the latter implemented using a set of correlated product (Slater-Jastrow) wave functions.

Here, we extend the work initiated in Ref. [1] and show how exact solutions to the lattice Wheeler-DeWitt equations can be obtained in $2+1$ dimensions for arbitrary values of Newton's constant G. The procedure we follow is to solve the lattice equations exactly for several finite regular triangulations of the sphere and then extend the result to an arbitrarily large number of triangles. One finds that for large enough areas the exact lattice wave functional depends on geometric quantities only, such as the total area and the total integrated curvature (which in $2+1$ dimensions is just proportional to the Euler characteristic). The regularity condition on the solutions of the wave equation at small areas is shown to play an essential role in constraining the form of the wave functional, which we eventually find to be expressible in closed form as a confluent hypergeometric function of the first kind. Later it will be shown that the resulting wave function allows an exact evaluation of a number of useful (and manifestly diffeomorphism-invariant) averages, such as the average area of the manifold and its fluctuation.

From these results a number of suggestive physical results can be obtained, the first one of which is that the correlation length in units of the lattice spacing is found to be finite for all $G > 0$ and diverges at $G = 0$. Such a result

*HHamber@uci.edu
†RToriumi@uci.edu
‡R.M.Williams@damtp.cam.ac.uk

can be viewed as consistent with the existence of an ultraviolet fixed point (or a phase transition in statistical field theory language) in G located at the origin, thus entirely precluding the existence of a weak coupling phase for gravity in $2+1$ dimensions. Simple renormalization group arguments would then suggest that gravitational screening is not physically possible in $2+1$ dimensions and that gravitational antiscreening is the only physically realized option in this model. A second result that follows from our analysis is an exact determination of the critical correlation length exponent for gravity in $2+1$ dimensions, which is found to be $\nu = 6/11$. It is known that the latter determines, through standard renormalization group arguments, the scale dependence of the gravitational coupling in the vicinity of the ultraviolet fixed point.

A short outline of the paper is as follows. In Sec. II, as a general background to the rest of the paper, we briefly describe the formalism of classical canonical gravity, as originally formulated by Arnowitt, Deser and Misner. The continuum Wheeler-DeWitt equation and its invariance properties are introduced as well at this stage. In Sec. III we introduce the lattice Wheeler-DeWitt equation derived in a previous paper [1], and later Sec. IV makes more explicit various quantities appearing in it. This last section also discusses briefly the role of continuous lattice diffeomorphism invariance in the Regge framework as it applies to the present case of 2 + 1—dimensional gravity. Section V focuses on the scaling properties of the lattice equations and various sensible choices for the lattice coupling constants, with the aim of giving eventually a more transparent form to the wave function results. Section VI gives a detailed outline of the general method of solution for the lattice equations and then gives the explicit solution for a number of regular triangulations of the sphere. Later, a general form of the wave function is given that covers all the previous discrete cases and allows a subsequent study of the infinite volume limit. Section VII focuses on one of the simplest diffeomorphism-invariant averages that can be computed from the wave function, namely the average total area. A brief discussion follows on how the latter quantity relates to the corresponding averages computed in the Euclidean theory. Section VIII extends the calculation to the area fluctuation and shows how the critical exponents (anomalous dimensions) of the 2 + 1—gravity theory can be obtained from the exact wave function solution, using some rather straightforward scaling arguments. Section IX discusses some simple physical implications that can be inferred from the values of the exact exponents and the fact that quantum gravity in $2+1$ dimensions does not seemingly possess, in either the Euclidean or Lorentzian formulation, a weak coupling phase. Section X contains a summary of the results obtained so far.

II. CONTINUUM WHEELER-DEWITT EQUATION

Since this paper involves the canonical quantization of gravity, we begin here with a very brief summary of the classical canonical formalism [2] as derived by Arnowitt *et al.* [3]. While many of the results presented in this section are rather well known, it will be useful, in view of later applications, to recall the main results and formulas and provide suitable references for expressions used later in the paper.

The first step in developing a canonical formulation for gravity is to introduce a time-slicing of space-time, by introducing a sequence of spacelike hypersurfaces labeled by a continuous time coordinate t. The invariant distance is then written as

$$\begin{aligned} ds^2 &\equiv -d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \\ &= g_{ij} dx^i dx^j + 2 g_{ij} N^i dx^j dt - (N^2 - g_{ij} N^i N^j) dt^2, \end{aligned} \tag{1}$$

where x^i $(i = 1, 2, 3)$ are coordinates on a three-dimensional manifold and τ is the proper time, in units with $c = 1$.

Indices are raised and lowered with $g_{ij}(\mathbf{x})$ $(i, j = 1, 2, 3)$, which denotes the three-metric on the given spacelike hypersurface, and $N(\mathbf{x})$ and $N^i(\mathbf{x})$ are the lapse and shift functions, respectively. It is customary to mark four-dimensional quantities by the prefix ^{4}, so that all unmarked quantities will refer to three dimensions (and are occasionally marked explicitly by a^3). In terms of the original four-dimensional metric $^4 g_{\mu\nu}$, one has

$$\begin{pmatrix} ^4 g_{00} & ^4 g_{0j} \\ ^4 g_{i0} & ^4 g_{ij} \end{pmatrix} = \begin{pmatrix} N_k N^k - N^2 & N_j \\ N_i & g_{ij} \end{pmatrix}, \tag{2}$$

which then gives for the spatial metric and the lapse and shift functions,

$$g_{ij} = {}^4 g_{ij} \qquad N = (-{}^4 g^{00})^{-1/2} \qquad N_i = {}^4 g_{0i}. \tag{3}$$

For the volume element one has

$$\sqrt{-{}^4 g} = N \sqrt{g}, \tag{4}$$

where the latter involves the determinant of the three-metric, $g \equiv \det g_{ij}$. As usual g^{ij} denotes the inverse of the matrix g_{ij}.

A transition from the classical to the quantum description of gravity is obtained by promoting the metric g_{ij}, the conjugate momenta π^{ij}, the Hamiltonian density H and the momentum density H_i to quantum operators, with $\hat{g}_{ij}$ and $\hat{\pi}^{ij}$ satisfying canonical commutation relations. In particular, the classical constraints now select a physical vacuum state $|\Psi\rangle$, such that in the source-free case

$$\hat{H} |\Psi\rangle = 0 \qquad \hat{H}_i |\Psi\rangle = 0 \tag{5}$$

and in the presence of sources more generally

$$\hat{T} |\Psi\rangle = 0 \qquad \hat{T}_i |\Psi\rangle = 0, \tag{6}$$

where $\hat{T}$ and $\hat{T}_i$ now include matter contributions that should be added to $\hat{H}$ and $\hat{H}_i$. The momentum constraint involving $\hat{H}_i$, or more generally $\hat{T}_i$, ensures that the state functional does not change under a transformation of coordinates x^i, so that Ψ depends only on the intrinsic geometry of the 3-space. The Hamiltonian constraint is then the only remaining condition that the state functional must satisfy.

As in ordinary nonrelativistic quantum mechanics, one can choose different representations for the canonically conjugate operators $\hat{g}_{ij}$ and $\hat{\pi}^{ij}$. In the functional position representation, one sets

$$\hat{g}_{ij}(\mathbf{x}) \to g_{ij}(\mathbf{x}) \quad \hat{\pi}^{ij}(\mathbf{x}) \to -i\hbar \cdot 16\pi G \cdot \frac{\delta}{\delta g_{ij}(\mathbf{x})}. \tag{7}$$

In this picture, quantum states become wave functionals of the three-metric $g_{ij}(\mathbf{x})$,

$$|\Psi\rangle \to \Psi[g_{ij}(\mathbf{x})]. \tag{8}$$

The two quantum-constraint equations in Eq. (6) then become the Wheeler-DeWitt equation [4–6]

$$\left\{-16\pi G \cdot G_{ij,kl}\frac{\delta^2}{\delta g_{ij}\delta g_{kl}} - \frac{1}{16\pi G}\sqrt{g}({}^3R - 2\lambda) + \hat{H}^\phi\right\}\Psi[g_{ij}(\mathbf{x})] = 0 \tag{9}$$

and the momentum constraint listed below. Here, $G_{ij,kl}$ is the inverse of the DeWitt supermetric, given by

$$G_{ij,kl} = \frac{1}{2}g^{-1/2}(g_{ik}g_{jl} + g_{il}g_{jk} + \alpha g_{ij}g_{kl}), \tag{10}$$

with parameter $\alpha = -1$. The three-dimensional version of the DeWitt supermetric itself, $G^{ij,kl}(x)$, is given by

$$G^{ij,kl} = \frac{1}{2}\sqrt{g}(g^{ik}g^{jl} + g^{il}g^{jk} + \bar{\alpha}g^{ij}g^{kl}), \tag{11}$$

with parameter α in Eq. (10) related to $\bar{\alpha}$ in Eq. (11) by $\bar{\alpha} = -2\alpha/(2 + 3\alpha)$, so that $\alpha = -1$ gives $\bar{\alpha} = -2$ (note that this is dimension dependent). In the position representation, the diffeomorphism (or momentum) constraint reads

$$\left\{2ig_{ij}\nabla_k\frac{\delta}{\delta g_{jk}} + \hat{H}_i^\phi\right\}\Psi[g_{ij}(\mathbf{x})] = 0, \tag{12}$$

where $\hat{H}^\phi$ and $\hat{H}_i^\phi$ are possible matter contributions. In the following, we shall set both of these to zero, as we will focus here almost exclusively on the pure gravitational case.

A number of basic issues need to be addressed before one can gain a full and consistent understanding of the dynamical content of the theory (see, for example, Refs. [7–11] as a small set of representative references). These include possible problems of operator ordering and the specification of a suitable Hilbert space, which entails at some point a choice for the inner product of wave functionals, for example, in the Schrödinger form

$$\langle\Psi|\Phi\rangle = \int d\mu[g]\Psi^*[g_{ij}]\Phi[g_{ij}], \tag{13}$$

where $d\mu[g]$ is some appropriate measure over the three-metric g. Note also that the continuum Wheeler-DeWitt equation contains, in the kinetic term, products of functional differential operators which are evaluated at the same spatial point $\mathbf{x}$. One would expect that such terms could produce $\delta^{(3)}(0)$-type singularities when acting on the wave functional, which would then have to be regularized in some way. The lattice cutoff discussed below is one way to provide such an explicit ultraviolet regularization.

A peculiar property of the Wheeler-DeWitt equation, which distinguishes it from the usual Schrödinger equation $H\Psi = i\hbar\partial_t\Psi$, is the absence of an explicit time coordinate. As a result, the rhs term of the Schrödinger equation is here entirely absent. The reason is of course diffeomorphism invariance of the underlying theory, which expresses now the fundamental quantum equations in terms of fields g_{ij} and not coordinates.

III. LATTICE HAMILTONIAN FOR QUANTUM GRAVITY

In constructing a discrete Hamiltonian for gravity, one has to decide first what degrees of freedom one should retain on the lattice. One possibility, which is the one we choose to pursue here, is to use the more economical (and geometric) Regge-Wheeler lattice discretization for gravity [12,13], with edge lengths suitably defined on a random lattice as the primary dynamical variables. Even in this specific case, several avenues for discretization are possible. One could discretize the theory from the very beginning, while it is still formulated in terms of an action, and introduce for it a lapse and a shift function, extrinsic and intrinsic discrete curvatures, etc. Alternatively, one could try to discretize the continuum Wheeler-DeWitt equation directly, a procedure that makes sense in the lattice formulation, as these equations are still given in terms of geometric objects, for which the Regge theory is very well suited. It is the latter approach which we will proceed to outline here.

The starting point for the following discussion is therefore the Wheeler-DeWitt equation for pure gravity in the absence of matter, Eq. (9),

$$\left\{-(16\pi G)^2 G_{ij,kl}(\mathbf{x})\frac{\delta^2}{\delta g_{ij}(\mathbf{x})\delta g_{kl}(\mathbf{x})} - \sqrt{g(\mathbf{x})}({}^3R(\mathbf{x}) - 2\lambda)\right\}\Psi[g_{ij}(\mathbf{x})] = 0 \tag{14}$$

and the diffeomorphism constraint of Eq. (12),

$$\left\{2ig_{ij}(\mathbf{x})\nabla_k(\mathbf{x})\frac{\delta}{\delta g_{jk}(\mathbf{x})}\right\}\Psi[g_{ij}(\mathbf{x})] = 0. \tag{15}$$

Note that these equations express a constraint on the state $|\Psi\rangle$ at *every* $\mathbf{x}$, each of the form $\hat{H}(\mathbf{x})|\Psi\rangle = 0$ and $\hat{H}_i(\mathbf{x})|\Psi\rangle = 0$.

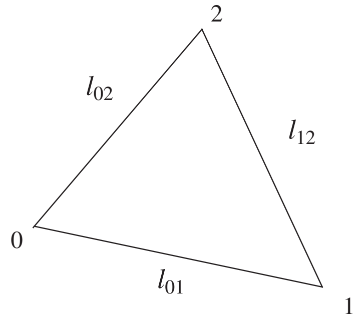

FIG. 1. A triangle with labels.

On a simplicial lattice [14–18] (see for example Ref. [19], and references therein, for a more complete discussion of the lattice formulation for gravity), one knows that deformations of the squared edge lengths are linearly related to deformations of the induced metric. In a given simplex σ, take coordinates based at a vertex 0, with axes along the edges from 0. The other vertices are each at unit coordinate distance from 0 (see Figs. 1–3 as an example of this labeling for a triangle). In terms of these coordinates, the metric within the simplex is given by

$$g_{ij}(\sigma) = \frac{1}{2}(l_{0i}^2 + l_{0j}^2 - l_{ij}^2). \tag{16}$$

Note that in the following discussion only edges and volumes along the spatial direction are involved. It follows that one can introduce in a natural way a lattice analog of the DeWitt supermetric of Eq. (11) by adhering to the following procedure [20,21]. First, one writes for the supermetric in edge length space

$$\|\delta l^2\|^2 = \sum_{ij} G^{ij}(l^2)\delta l_i^2 \delta l_j^2, \tag{17}$$

with the quantity $G^{ij}(l^2)$ suitably defined on the space of squared edge lengths. By a straightforward exercise of

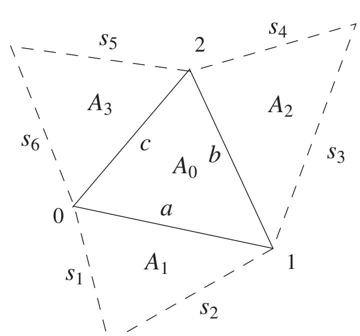

FIG. 2. Neighbors of a given triangle. The picture illustrates the fact that the Laplacian $\mathbf{\Delta}(l^2)$ appearing in the kinetic term of the lattice Wheeler-DeWitt equation (here in 2 + 1 dimensions) contains edges a, b, c that belong both to the triangle in question as well as to several neighboring triangles (here, three of them) with squared edges denoted sequentially by $s_1 = l_1^2 \ldots s_6 = l_6^2$.

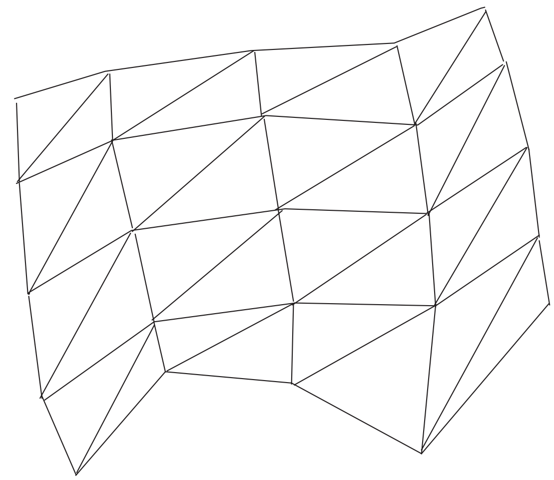

FIG. 3. A small section of a suitable dynamical spatial lattice for quantum gravity in 2 + 1 dimensions.

varying the squared volume of a given simplex σ in d dimensions

$$V^2(\sigma) = \left(\frac{1}{d!}\right)^2 \det g_{ij}(l^2(\sigma)) \tag{18}$$

to quadratic order in the metric (on the rhs), or in the squared edge lengths belonging to that simplex (on the lhs), one is led to the identification

$$G^{ij}(l^2) = -d! \sum_\sigma \frac{1}{V(\sigma)} \frac{\partial^2 V^2(\sigma)}{\partial l_i^2 \partial l_j^2}. \tag{19}$$

It should be noted that in spite of the appearance of a sum over simplices σ, $G^{ij}(l^2)$ is local, since the sum over σ only extends over those simplices which contain either the i or the j edge.

At this point one is finally ready to write a lattice analog of the Wheeler-DeWitt equation for pure gravity, which reads

$$\left\{-(16\pi G)^2 G_{ij}(l^2) \frac{\partial^2}{\partial l_i^2 \partial l_j^2} - \sqrt{g(l^2)}[{}^3R(l^2) - 2\lambda]\right\}\Psi[l^2] = 0, \tag{20}$$

with $G_{ij}(l^2)$ the inverse of the matrix $G^{ij}(l^2)$ given above. The range of the summation over i and j and the appropriate expression for the scalar curvature, in this equation, are discussed below and made explicit in Eq. (21).

Equations (9) or (20) express a constraint equation at each *point* in space. Here, we will elaborate a bit more on this point. On the lattice, points in space are replaced by a set of edge labels i, with a few edges clustered around each vertex in a way that depends on the dimensionality and the local lattice coordination number. To be more specific, the first term in Eq. (20) contains derivatives with respect to edges i and j connected by a matrix element G_{ij} which is nonzero only if i and j are close to each other, essentially nearest neighbor. One would therefore expect that the first term could be represented by just a sum of edge contributions, all from within one $(d-1)$-simplex σ (a tetrahedron in three dimensions). The second term containing ${}^3R(l^2)$ in Eq. (20) is also local in the edge lengths: it only involves a handful of edge lengths, which enter into the definition of areas, volumes and angles around the point $\mathbf{x}$, and follows from the fact that the local curvature at the original point $\mathbf{x}$ is completely determined by the values of the edge lengths clustered around i and j. Apart from some geometric factors, it describes, through a deficit angle δ_h, the parallel transport of a vector around an elementary dual lattice loop. It should, therefore, be adequate to represent this second term by a sum over contributions over all $(d-3)$-dimensional hinges (edges in 3 + 1 dimensions) h attached to the simplex σ, giving, therefore, in three dimensions

$$\Big\{-(16\pi G)^2 \sum_{i,j\subset\sigma} G_{ij}(\sigma)\frac{\partial^2}{\partial l_i^2 \partial l_j^2} - 2n_{\sigma h}\sum_{h\subset\sigma} l_h\delta_h + 2\lambda V_\sigma\Big\}\Psi[l^2] = 0. \tag{21}$$

Here δ_h is the deficit angle at the hinge h, l_h the corresponding edge length, and $V_\sigma = \sqrt{g(\sigma)}$ the volume of the simplex (tetrahedron in three spatial dimensions) labeled by σ. $G_{ij}(\sigma)$ is obtained either from Eq. (19) or from the lattice transcription of Eq. (10)

$$G_{ij,kl}(\sigma) = \frac{1}{2}g^{-1/2}(\sigma)[g_{ik}(\sigma)g_{jl}(\sigma) + g_{il}(\sigma)g_{jk}(\sigma) - g_{ij}(\sigma)g_{kl}(\sigma)], \tag{22}$$

with the induced metric $g_{ij}(\sigma)$ within a simplex σ given in Eq. (16). The combinatorial factor $n_{\sigma h}$ ensures the correct normalization for the curvature term, since the latter has to give the lattice version of $\int\sqrt{g}\,{}^3R = 2\sum_h \delta_h l_h 3$ (in three spatial dimensions) when summed over all simplices σ. The inverse of $n_{\sigma h}$ counts, therefore, the number of times the same hinge appears in various neighboring simplices and consequently depends on the specific choice of underlying lattice structure; for a flat lattice of equilateral triangles in two dimensions, $n_{\sigma h} = 1/6$.[1] The lattice Wheeler-DeWitt equation given in Eq. (21) was the main result of a previous paper [1].

IV. EXPLICIT SETUP FOR THE LATTICE WHEELER-DEWITT EQUATION

In this section, we shall establish our notation and derive the relevant terms in the discrete Wheeler-DeWitt equation for a simplex. From now on we shall focus almost exclusively on the case of 2 + 1 dimensions. The basic simplex in this case is, of course, a triangle, with vertices and squared edge lengths labeled as in Fig. 1. We set $l_{01}^2 = a$, $l_{12}^2 = b$, $l_{02}^2 = c$. The components of the metric for coordinates based at vertex 0, with axes along the 01 and 02 edges, are

$$g_{11} = a, \qquad g_{12} = \frac{1}{2}(a+c-b), \qquad g_{22} = c. \tag{23}$$

The area A of the triangle is given by

$$A^2 = \frac{1}{16}[2(ab+bc+ca) - a^2 - b^2 - c^2], \tag{24}$$

so the supermetric G^{ij}, according to Eq. (19), is

$$G^{ij} = \frac{1}{4A}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}. \tag{25}$$

Thus, for the triangle we have

$$G_{ij}\frac{\partial^2}{\partial s_i \partial s_j} = -4A\Big(\frac{\partial^2}{\partial a\partial b} + \frac{\partial^2}{\partial b\partial c} + \frac{\partial^2}{\partial c\partial a}\Big), \tag{26}$$

and the Wheeler-DeWitt equation is

$$\Big\{(16\pi G)^2 4A\Big(\frac{\partial^2}{\partial a\partial b} + \frac{\partial^2}{\partial b\partial c} + \frac{\partial^2}{\partial c\partial a}\Big) - 2n_{\sigma h}\sum_h \delta_h + 2\lambda A\Big\}\Psi[s] = 0, \tag{27}$$

where the sum is over the three vertices h of the triangle.

In the following sections we will be concerned at some point with various discrete, but generally regular, triangulations of the two-sphere, such as the tetrahedron, the octahedron and the icosahedron. These were already studied in some detail in Refs. [22,23]. A key aspect of the Regge theory is the presence of a continuous, local lattice diffeomorphism invariance, whose main aspects in regard to their relevance for the 3 + 1 formulation of gravity were already addressed in some detail in Ref. [1] in the context of the lattice weak field expansion. Here we will add some remarks about how this local invariance manifests itself in the 2 + 1 formulation and, in particular, for the discrete triangulations of the sphere studied later on in this paper. Of some relevance is the presence of exact zero modes of the gravitational lattice action, reflecting a local lattice diffeomorphism invariance, present already on a finite lattice. Since the Einstein action is a topological invariant in two dimensions, the relevant action in this case has to be a curvature-squared action supplemented by a cosmological constant term. Specifically, part of the results in Refs. [22,24] can be summarized as follows. For a given lattice, one finds for the counting of zero modes

[1]Instead of the combinatorial factor $n_{\sigma h}$, one could insert a ratio of volumes $V_{\sigma h}/V_h$ (where V_h is the volume per hinge [17] and $V_{\sigma h}$ is the amount of that volume in the simplex σ), but the above form is simpler.

$$\begin{aligned} &\text{Tetrahedron } (N_0 = 4)\text{: 2 zero modes} \\ &\text{Octahedron } (N_0 = 6)\text{: 6 zero modes} \\ &\text{Icosahedron } (N_0 = 12)\text{: 18 zero modes.} \end{aligned} \tag{28}$$

Thus, if the number of zero modes for each regular triangulation of the sphere is denoted by $N_{\mathrm{z.m.}}$, then the results can be reexpressed as

$$N_{\mathrm{z.m.}} = 2N_0 - 6, \tag{29}$$

which agrees with the expectation that, in the continuum limit, $N_0 \to \infty$, $N_{\mathrm{z.m.}}/N_0$ should approach the constant value d in d space-time dimensions, the expected number of local parameters for a diffeomorphism. Similar estimates were obtained when looking at deformations of a flat lattice in various dimensions [22]. The case of near-flat space is obviously the simplest: by moving the location of the vertices around in flat space, one can find a different assignment of edge lengths that represents the same flat geometry. This then leads to the $d \cdot N_0$-parameter family of transformations for the edge lengths in flat space.

In general, lattice diffeomorphisms correspond to local deformations of the edge lengths about a vertex, which leave the local geometry physically unchanged, the latter being described by the values of local lattice operators corresponding to local volumes, curvatures, etc. The lesson is that the correct count of continuum zero modes will, in general, only be recovered asymptotically for large triangulations, where N_0 is significantly larger than the number of neighbors to a point in d dimensions. With these observations in mind, we can now turn to a discussion of the solution method for the lattice Wheeler-DeWitt equation in $2+1$ dimensions.

One item that needs to be discussed at this point is the proper normalization of various terms (kinetic, cosmological and curvature) appearing in the lattice equation of Eq. (21). For the lattice gravity action in d dimensions, one has generally the following correspondence

$$\int d^d x \sqrt{g} \leftrightarrow \sum_\sigma V_\sigma, \tag{30}$$

where V_σ is the volume of a simplex; in two dimensions it is simply the area of a triangle. The curvature term involves deficit angles in the discrete case,

$$\frac{1}{2}\int d^d x \sqrt{g} R \leftrightarrow \sum_h V_h \delta_h, \tag{31}$$

where δ_h is the deficit angle at the hinge h, and V_h the associated "volume of the hinge" [12]. In four dimensions, the latter is the area of a triangle (usually denoted by A_h), whereas in three dimensions it is simply given by the length l_h of the edge labeled by h. In two dimensions, $V_h = 1$. In this work we will focus almost exclusively on the case of $2+1$ dimensions; consequently, the relevant formulas will be Eqs. (30) and (31) for dimension $d = 2$.

The continuum Wheeler-DeWitt equation is local, as can be seen from Eq. (14). One can integrate the Wheeler-DeWitt operator over all space and obtain

$$\left\{-(16\pi G)^2 \int d^2x \boldsymbol{\Delta}(g) + 2\lambda \int d^2 x\sqrt{g} - \int d^2x\sqrt{g}R\right\}\Psi = 0 \tag{32}$$

with the super-Laplacian on metrics defined as

$$\boldsymbol{\Delta}(g) \equiv G_{ij,kl}(\mathbf{x}) \frac{\delta^2}{\delta g_{ij}(\mathbf{x})\delta g_{kl}(\mathbf{x})}. \tag{33}$$

In the discrete case, one has one local Wheeler-DeWitt equation for *each* triangle [see Eqs. (20) and (21)], which therefore takes the form

$$\left\{-(16\pi G)^2 \boldsymbol{\Delta}(l^2) - \kappa \sum_{i\subset\Delta} \delta_i + 2\lambda A_\Delta\right\}\Psi = 0, \tag{34}$$

where $\boldsymbol{\Delta}(l^2)$ is the lattice version of the super-Laplacian, and we have set for convenience $\kappa = 2n_{\sigma h}$. As we shall see below, for a lattice of fixed coordination number, κ is a constant and does not depend on the location on the lattice. In the above expression, $\boldsymbol{\Delta}(l^2)$ is a discretized form of the covariant super-Laplacian, acting locally on the space of $s = l^2$ variables. From Eqs. (26) and (34), one has explicitly

$$\boldsymbol{\Delta}(l^2) = -4A_\Delta\left(\frac{\partial^2}{\partial a\partial b} + \frac{\partial^2}{\partial b\partial c} + \frac{\partial^2}{\partial c\partial a}\right). \tag{35}$$

Note that the curvature term involves three deficit angles δ_i, associated with the three vertices of a triangle. Now, Eq. (34) applies to a single given triangle, with one equation to be satisfied at each triangle on the lattice. One can also construct the total Hamiltonian by simply summing over all triangles, which leads to

$$\left\{-(16\pi G)^2 \sum_\Delta \boldsymbol{\Delta}(l^2) + 2\lambda \sum_\Delta A_\Delta - \kappa \sum_\Delta \sum_{i\subset\Delta} \delta_i\right\}\Psi = 0. \tag{36}$$

Summing over all triangles (Δ) is different from summing over all lattice sites (i), and the above equation is equivalent to

$$\left\{-(16\pi G)^2 \sum_\Delta \boldsymbol{\Delta}(l^2) + 2\lambda \sum_\Delta A_\Delta - \kappa q \sum_i \delta_i\right\}\Psi = 0, \tag{37}$$

where q is the lattice coordination number and is determined by how the lattice is put together (which vertices are neighbors to each other or, equivalently, by the so-called incidence matrix). Here, q is the number of neighboring simplexes that share a given hinge (vertex). For a flat triangular lattice $q = 6$, whereas for a tetrahedron, octahedron, and icosahedron, one has $q = 3, 4, 5$, respectively. For proper normalization in Eq. (36), one requires

$$\int d^2x\sqrt{g} \leftrightarrow \sum_\Delta A_\Delta \tag{38}$$

as well as

$$\frac{1}{2}\int d^2x\sqrt{g}R \leftrightarrow \sum_i \delta_i. \tag{39}$$

This last correspondence allows one to fix the overall normalization of the curvature term

$$\kappa \equiv 2n_{\sigma h} = \frac{2}{q}, \tag{40}$$

which then determines the relative weight of the local volume and curvature terms.

V. CHOICE OF COUPLING CONSTANTS

As in the Euclidean lattice theory of gravity, we will find it convenient here to factor out an overall irrelevant length scale from the problem and set the (unscaled) cosmological constant equal to one as was done in Ref. [17]. Indeed, recall that the Euclidean path integral weight always contains a factor $P(V) \propto \exp(-\lambda_0 V)$, where $V = \int\sqrt{g}$ is the total volume on the lattice, and λ_0 is the unscaled cosmological constant. The choice $\lambda_0 = 1$ then fixes this overall scale once and for all. Since $\lambda_0 = 2\lambda/16\pi G$, one then has $\lambda = 8\pi G$ in this system of units. In the following we will also find it convenient to introduce a scaled coupling $\tilde{\lambda}$ defined as

$$\tilde{\lambda} \equiv \frac{\lambda}{2}\left(\frac{1}{16\pi G}\right)^2 \tag{41}$$

so that for $\lambda_0 = 1$ (in units of the UV cutoff or, equivalently, in units of the fundamental lattice spacing), one has $\tilde{\lambda} = 1/64\pi G$. One can now rewrite the Wheeler-DeWitt equation so that the kinetic term (the term involving the Laplacian) has a unit coefficient and write Eq. (14) as

$$\left\{-\mathbf{\Delta} + \frac{2\lambda}{(16\pi G)^2}\sqrt{g} - \frac{1}{(16\pi G)^2}\sqrt{g}R\right\}\Psi = 0. \tag{42}$$

Note that in the extreme strong coupling limit ($G \to \infty$), the kinetic term is the dominant one, followed by the volume (cosmological constant) term (using the facts about $\tilde{\lambda}$ given above) and, finally, by the curvature term. Consequently, at least in a first approximation, the curvature R term can be neglected compared to the other two terms in this limit.

Two further notational simplifications will be done in the following. The first one is introduced in order to avoid lots of factors of 16π in many of the subsequent formulas. Consequently, from now on we shall write G as a shorthand for $16\pi G$,

$$16\pi G \to G. \tag{43}$$

In this notation one then has $\lambda = G/2$ and $\tilde{\lambda} = 1/4G$. The above notational choices then lead to a much more streamlined representation of the Wheeler-DeWitt equation,

$$\left\{-\mathbf{\Delta} + \frac{1}{G}\sqrt{g} - \frac{1}{G^2}\sqrt{g}R\right\}\Psi = 0. \tag{44}$$

A second notational choice will be dictated later on by the structure of the wave function solutions, which will commonly involve factors of $\sqrt{G}$. For this reason we will now define the new coupling g as

$$g \equiv \sqrt{G}, \tag{45}$$

so that $\tilde{\lambda} = 4/g^2$ (the latter g should not be confused with the square root of the determinant of the metric).

Later on it will be convenient to define a parameter β for the triangulations of the sphere, defined as

$$\beta \equiv \frac{2\pi}{\sqrt{\tilde{\lambda}}G^2}. \tag{46}$$

Factors of 2π arise here because we are looking at various triangulations of the two-sphere. More generally, for a two-dimensional closed manifold with arbitrary topology, one has by the Gauss-Bonnet theorem

$$\int d^2x\sqrt{g}R = 4\pi\chi \tag{47}$$

with χ as the Euler characteristic of the manifold. The latter is related to the genus g (the number of handles) via $\chi = 2 - 2g$ (note that for a discrete manifold in two dimensions, one has the equivalent form due to Euler $\chi = N_0 - N_1 + N_2$, where N_i denotes the number of simplices of dimension i). Thus for a general two-dimensional manifold, we will define

$$\beta = \frac{\pi\chi}{\sqrt{\tilde{\lambda}}G^2}. \tag{48}$$

Equivalently, using

$$\sqrt{\tilde{\lambda}}G^2 = \frac{1}{2\sqrt{G}}\cdot G^2 = \frac{1}{2}G^{3/2} \tag{49}$$

and then making use of the coupling g, one has simply

$$\beta = \frac{4\pi}{g^3} \tag{50}$$

for the sphere, and in the more general case

$$\beta = \frac{2\pi\chi}{g^3}. \tag{51}$$

VI. OUTLINE OF THE GENERAL METHOD OF SOLUTION

It should be clear from the previous discussion that in the strong coupling limit (large G), one can, at least at first, neglect the curvature term, which can then be included at a later stage. This simplifies the problem quite a bit, as it is the curvature term that introduces complicated interactions between neighboring simplices (this is evident from the lattice Wheeler-DeWitt equation of Eq. (21), where the deficit angles enter the curvature term only).

The general procedure for finding a solution will be as follows. First, a solution will be found for *equilateral* edge lengths s. Later, this solution will be extended to determine whether it is consistent to higher order in the weak field expansion. Consequently, we shall write for the squared edge lengths

$$l_{ij}^2 = s(1 + \epsilon h_{ij}), \tag{52}$$

with ϵ a small expansion parameter. Therefore, for example, in Eq. (35) one has $a = s(1+\epsilon h_a)$, $b = s(1+\epsilon h_b)$ and $c = s(1 + \epsilon h_c)$. The resulting solution for the wave function will then be given by a suitable power series in the h variables. Nevertheless, in some rare cases (such as the single-triangle case described below or the single tetrahedron in 3 + 1 dimensions [1]), one is lucky enough to find immediately an exact solution, without having to rely in any way on the weak field expansion.

To lowest order in h, a solution to the Wheeler-DeWitt equation is readily found using the standard power series (or Frobenius) method, appropriate for the study of quantum mechanical wave equations. In this method one first obtains the correct asymptotic behavior of the solution for small and large arguments and later constructs a full solution by writing the remainder as a power series or polynomial in the relevant variable. Of some importance in the following is the correct determination of the wave functional Ψ for small and large areas (small and large s) and to what extent the resulting wave function can be expressed in terms of invariants, such as areas and curvatures, or powers thereof.

In the following we will see that the natural variable for displaying results is the scaled total area x, defined as

$$x \equiv 2\sqrt{\tilde{\lambda}} A_{\mathrm{tot}} = 2\sqrt{\tilde{\lambda}} \sum_{\Delta} A_{\Delta}. \tag{53}$$

We will look at a variety of two-dimensional lattices, including the regular triangulations of the two- sphere given by the tetrahedron, octahedron and icosahedron, as well as the case of a triangulated torus with coordination number six. In the equilateral case the natural variable for displaying the results is then

$$x = 2\sqrt{\tilde{\lambda}} A_{\mathrm{tot}} = 2N_{\Delta}\sqrt{\tilde{\lambda}} A_{\Delta}. \tag{54}$$

Later on we will be interested in taking the infinite volume limit, defined in the usual way as

$$N_{\Delta} \to \infty, \qquad A_{\mathrm{tot}} \to \infty, \qquad \frac{A_{\mathrm{tot}}}{N_{\Delta}} \to \mathrm{const.} \tag{55}$$

It follows that this last ratio can be used to define a fundamental lattice spacing l_0, for example via $A_{\mathrm{tot}}/N_{\Delta} = A_{\Delta} = \sqrt{3} l_0^2/4$.

The full solution of the quantum mechanical problem will, in general, require that the wave functions be properly normalized, as in Eq. (13). This will introduce at some stage wave function normalization factors $\mathcal{N}$ and $\tilde{\mathcal{N}}$, which will be fixed by the standard rules of quantum mechanics. If the wave function depends on the total area only, then the relevant requirement becomes

$$\int_0^\infty dA_{\mathrm{tot}} |\Psi(A_{\mathrm{tot}})|^2 \equiv \frac{1}{2\sqrt{\tilde{\lambda}}} \int_0^\infty dx |\Psi(x)|^2 = 1. \tag{56}$$

As in nonrelativistic quantum mechanics, two solutions are expected, only one of which will be regular as the origin and thus satisfy the wave function normalizability requirement.

At this point it will be necessary to discuss each lattice separately in some detail. For each lattice geometry, we will break down the presentation into four separate items:

(a) *Equilateral case in the strong coupling limit* ($\epsilon = 0$). This subsection will find a solution in the extreme strong coupling limit (large G), without curvature term in the Wheeler-DeWitt equation. The solution will not rely on the weak field expansion, and the results will be exact to zeroth order in the weak field expansion of Eq. (52). In this case the simplices are all taken to be equilateral, and the lattice edge lengths fluctuate together.

(b) *Equilateral case with curvature term* ($\epsilon = 0$). Next, the curvature term is included. The solution again will not rely on the weak field expansion, and all the triangles will be taken to be equilateral. The resulting solution will, therefore, be valid again (and exact) to zeroth order in the ϵ expansion parameter of Eq. (52).

(c) *Large area in the strong coupling limit* ($\epsilon \neq 0$). In this case we will look at nonzero local fluctuations in Eq. (52). The method of solution will now rely on the weak field expansion for large areas (large s), but nevertheless it will turn out that an exact solution can be found in this case. The resulting answer will be found to be correct to arbitrarily large order $\mathcal{O}(\epsilon^n)$, with n a positive integer.

(d) *Small area in the strong coupling limit* ($\epsilon \neq 0$). Finally, we will look at the case of nonzero fluctuations [$\epsilon \neq 0$ in Eq. (52)] in the limit of small areas (small s). In this limit we will find that, in general, the solution can be written entirely in terms of invariants involving total areas and curvatures only up to order $\mathcal{O}(\epsilon)$ or $\mathcal{O}(\epsilon^2)$, depending on whether a further symmetrization of the problem is performed or not.

If the reader is not interested in the details for each lattice, he can skip the next few subsections and go directly to the summary presented in Sec. VI F.

A. Single triangle case

From Eq. (34) the Wheeler-DeWitt equation for a single triangle reads

$$\left\{(16\pi G)^2 4A_\Delta\left(\frac{\partial^2}{\partial a\partial b}+\frac{\partial^2}{\partial b\partial c}+\frac{\partial^2}{\partial c\partial a}\right)+2\lambda A_\Delta\right\}\Psi(a,b,c) = 0, \tag{57}$$

where a, b, c are the three squared edge lengths for the given triangle, and A_Δ is the area of the same triangle. Note that for a single triangle there can be no curvature term. Equivalently, one needs to solve

$$\left\{\frac{\partial^2}{\partial a\partial b}+\frac{\partial^2}{\partial b\partial c}+\frac{\partial^2}{\partial c\partial a}+\tilde{\lambda}\right\}\Psi(a,b,c)=0. \tag{58}$$

If one sets

$$\Psi[a,b,c]=\Phi[A_\Delta], \tag{59}$$

then one finds the following equivalent differential equation

$$A_\Delta\frac{d^2\Phi}{dA_\Delta^2}+2\frac{d\Phi}{dA_\Delta}+16\tilde{\lambda}A_\Delta\Phi=0. \tag{60}$$

For a single triangle the total area equals the area of the single triangle, $A_{\rm tot}=A_\Delta$. Here it will be convenient to define

$$x=4\sqrt{\tilde{\lambda}}A_{\rm tot}\equiv 4\sqrt{\tilde{\lambda}}A_\Delta \tag{61}$$

so that the solution will be a function of this variable only. Note that in this case, and in this case only, we will deviate from the general definition of the variable x given in Eq. (53). One can then write the solution to Eq. (60) in the form

$$\Psi(x)=\mathcal{N}\frac{J_n(x)}{x^n} \tag{62}$$

with

$$n=\frac{1}{2} \tag{63}$$

so that

$$\Psi(x)=\mathcal{N}\frac{J_{1/2}(4\sqrt{\tilde{\lambda}}A_{\rm tot})}{(4\sqrt{\tilde{\lambda}}A_{\rm tot})^{1/2}}. \tag{64}$$

The wave function normalization constant is given here by

$$\mathcal{N}=2\tilde{\lambda}^{1/4}. \tag{65}$$

Note that the above solution is exact and did not require, in any way, the weak field expansion. Two alternate forms of the wave function are

$$\Psi(A_{\rm tot})=\mathcal{N}\frac{\sin(4\sqrt{\tilde{\lambda}}A_{\rm tot})}{2\sqrt{2\pi}\sqrt{\tilde{\lambda}}A_{\rm tot}} = \mathcal{N}\sqrt{\frac{2}{\pi}}\exp(-4i\sqrt{\tilde{\lambda}}A_{\rm tot})\,{}_1F_1(1,2,8i\sqrt{\tilde{\lambda}}A_{\rm tot}). \tag{66}$$

Here, ${}_1F_1(a,b,z)$ is the confluent hypergeometric functions of the first kind. The usefulness of the latter representation will become clearer later, when other lattices are considered and the curvature term is included. Expanding the solution for small area, one obtains

$$\Psi(x)=\mathcal{N}\sqrt{\frac{2}{\pi}}\left[1-\frac{x^2}{6}+\frac{x^4}{120}+\mathcal{O}(x^6)\right], \tag{67}$$

which shows that it is indeed nonsingular and, thus, normalizable.

In the limit of large areas, a solution to the original differential equation is given either by the asymptotic behavior of the above Bessel (here, sine) function (J), the same limiting behavior for the corresponding Bessel function Y, or by the two corresponding Hankel functions (H).

$$\Psi\sim_{x\to\infty}\frac{1}{x}\exp(\pm ix)\sim\frac{1}{A_{\rm tot}}\exp(\pm 4i\sqrt{\tilde{\lambda}}A_{\rm tot}). \tag{68}$$

Nevertheless, among those four solutions, only one is regular and, therefore, physically acceptable.

The calculation for a single triangle can be regarded as a useful starting point, and a basic stepping stone, for the strong coupling expansion in $1/G$. It shows the physical characteristics of the wave function solution deep in the strong coupling regime: for $G\to\infty$ the coupling term between different simplices, which is caused mainly by the curvature term, disappears and one ends up with a completely decoupled problem, where the edge lengths in nonadjacent simplices fluctuate independently.

B. Tetrahedron

In the case of the tetrahedron, one has 4 triangles, 6 edges, 4 vertices, and 3 neighboring triangles for each vertex. Let us discuss again, here, the various cases individually.

(a) *Equilateral case in the strong coupling limit* ($\epsilon=0$) We first look at the case when $\epsilon=0$ in Eq. (52), deep in the strong coupling region and without the curvature term.
Following Eq. (53), we define the scaled area variable as

$$x=2\sqrt{\tilde{\lambda}}A_{\rm tot}=4\times 2\sqrt{\tilde{\lambda}}A_\Delta \tag{69}$$

and the solution will be found later to be a function of this variable only. For equilateral triangles, the wave function Ψ needs to satisfy

$$\Psi''+\frac{2}{x}\Psi'+\Psi=0. \tag{70}$$

The correct solution can be written in the form

$$\Psi(x)=\mathcal{N}\frac{J_n(x)}{x^n} \tag{71}$$

with

$$n=\frac{1}{2} \tag{72}$$

so that

$$\Psi(x) = \mathcal{N} \frac{J_{1/2}(2\sqrt{\tilde{\lambda}}A_{\rm tot})}{(2\sqrt{\tilde{\lambda}}A_{\rm tot})^{1/2}}. \quad (73)$$

The wave function normalization constant is given by

$$\mathcal{N} = \sqrt{2}\tilde{\lambda}^{\frac{1}{4}}. \quad (74)$$

Below are two equivalent forms of the same wave function

$$\Psi(A_{\rm tot}) = \mathcal{N} \frac{\sin(2\sqrt{\tilde{\lambda}}A_{\rm tot})}{\sqrt{2\pi}\sqrt{\tilde{\lambda}}A_{\rm tot}} = \mathcal{N}\sqrt{\frac{2}{\pi}}\exp(-2i\sqrt{\tilde{\lambda}}A_{\rm tot})\,{}_1F_1(1,2,4i\sqrt{\tilde{\lambda}}A_{\rm tot}) \quad (75)$$

for the equilateral case. In the limit of small area, one obtains

$$\Psi = \mathcal{N}\sqrt{\frac{2}{\pi}}\left[1 - \frac{x^2}{6} + \frac{x^4}{120} + \mathcal{O}(x^6)\right], \quad (76)$$

which again confirms that the wave function is regular at the origin. Since one is solving a second- order linear differential equation, one expects two solutions; here, one is singular and the other one is not, as is often the case in quantum mechanics. For the geometry of the tetrahedron, one solution can be written in terms of Bessel functions of the first kind (J)

$$\frac{J_{1/2}(x)}{\sqrt{x}} = \sqrt{\frac{2}{\pi}}\frac{\sin x}{x}. \quad (77)$$

The Bessel function of the second kind (Y) also satisfies the same differential equation, but since

$$\frac{Y_{1/2}(x)}{\sqrt{x}} = -\sqrt{\frac{2}{\pi}}\frac{\cos x}{x} \quad (78)$$

this second solution is not normalizable, it is therefore discarded on physical grounds. We shall see below that the same behavior at small x holds also for the nonzero curvature term. Note that both of the above solutions are real.[2]

(b) *Equilateral case with curvature term* ($\epsilon = 0$)
Next, we include the effects of the curvature term. To zeroth order in weak field expansions, when all edges fluctuate in unison, one now needs to solve the ordinary differential equation

$$\Psi'' + \frac{2}{x}\Psi' - \frac{2\beta}{x}\Psi + \Psi = 0, \quad (79)$$

with $\beta = 2\pi/\sqrt{\tilde{\lambda}}G^2$ as in Eq. (46). Since the deficit angle $\delta = \pi$ at each vertex, the curvature contribution for each triangle is $\kappa \cdot \pi \cdot 3$. In this case one has, therefore,

$$\kappa_{\rm tetra} = 2 \cdot \frac{1}{3} \quad (80)$$

and, therefore, the solution is given by

$$\Psi \simeq \exp(-2i\sqrt{\tilde{\lambda}}A_{\rm tot})\,{}_1F_1\left(1 - i\frac{3\pi\kappa_{\rm tetra}}{G^2\sqrt{\tilde{\lambda}}}, 2, 4i\sqrt{\tilde{\lambda}}A_{\rm tot}\right) = \exp(-2i\sqrt{\tilde{\lambda}}A_{\rm tot})\,{}_1F_1\left(1 - i\frac{2\pi}{G^2\sqrt{\tilde{\lambda}}}, 2, 4i\sqrt{\tilde{\lambda}}A_{\rm tot}\right) \quad (81)$$

in the equilateral case, up to an overall normalization factor. Note that in this case one had to include a factor of $A_{\rm tot}/(4A_\Delta)$ (which in the tetrahedron case equals one) in the imaginary part of the first argument of ${}_1F_1$.

(c) *Large area in the strong coupling limit* ($\epsilon \neq 0$)
Next, we look at the case $\epsilon \neq 0$ in Eq. (52). In the limit of large areas, one finds that the two independent solutions reduce to

$$\Psi \sim_{x\to\infty} \exp(\pm ix) \sim \exp(\pm 2i\sqrt{\tilde{\lambda}}A_{\rm tot}) \quad (82)$$

to all orders in ϵ. To show this, one sets $\Psi = e^{\alpha A_{\rm tot}}$, where $A_{\rm tot}$ is a sum of the four triangle areas that make up the tetrahedron, and then expands the edge lengths in the usual way according to Eq. (52), by setting $a = s(1 + \epsilon h_a)$ etc. Here we are interested specifically in the limit when s is large and ϵ is small. One then finds that the rhs of the lattice Wheeler-DeWitt equation is given to $\mathcal{O}(\epsilon^n)$ by

$$\frac{e^{\alpha\sqrt{3}s}}{4}\frac{1}{2^n\sqrt{3}n!n}\alpha^n(\alpha^2 + 4\tilde{\lambda})\epsilon^n s^n\left(\sum h\right)^n + \cdots. \quad (83)$$

One concludes that in this limit, it is sufficient to have

$$\alpha^2 + 4\tilde{\lambda} = 0, \quad (84)$$

or $\alpha = \pm 2i\sqrt{\tilde{\lambda}}$, to obtain an exact solution in the limit $n \to \infty$. Note that in the strong coupling limit, the two independent wave function solutions in Eq. (82) completely factorize as a product of single-triangle contributions.

(d) *Small area in the strong coupling limit* ($\epsilon \neq 0$)
In the limit of small area, we have shown before that the solution reduces to a constant in the equilateral case $[\mathcal{O}(\epsilon^0)]$ for small x or small areas. Beyond the equilateral case, one can write a general ansatz for the wave function in terms of geometric invariants

[2]There are also linear combinations of Bessel functions which give complex Hankel (H) functions. These satisfy the Wheeler-DeWitt equation as well; however, they are not physically acceptable since both are singular at the origin.

$$\Psi = \left(\prod_\Delta A_\Delta\right)^{\gamma_0}\left[1 + \gamma_2\left(\sum_\Delta A_\Delta\right)^2 + \gamma_4\left(\sum_\Delta A_\Delta\right)^4 + \cdots\right] \tag{85}$$

and then expand the solution in ϵ for small s. To zeroth order in ϵ, we had the solution $\Psi \sim J_n(x)/x^n$ with $x = 2\sqrt{\tilde{\lambda}}A_{\rm tot}$ and $n = 1/2$. This gives in Eq. (85) $\gamma_0 = 0$, $\gamma_2 = -\frac{2}{3}\tilde{\lambda}$ and $\gamma_4 = \frac{2}{15}\tilde{\lambda}^2$. To linear order $[\mathcal{O}(\epsilon)]$ one finds, though, that terms appear which cannot be expressed in the form of Eq. (85). But one also finds that, while these terms are nonzero if one uses the Hamiltonian density (the Hamiltonian contribution from just a single triangle), if one uses the sum of such triangle Hamiltonians, then the resulting solution is symmetrized, and the corrections to Eq. (85) are found to be of order $\mathcal{O}(\epsilon^2)$. Then, the wave function for small area is of the form

$$\Psi \sim 1 - \frac{2}{3}\tilde{\lambda}A_{\rm tot}^2 + \frac{2}{15}\tilde{\lambda}^2A_{\rm tot}^4 + \ldots \tag{86}$$

up to terms $\mathcal{O}(\epsilon^2)$.

C. Octahedron

The discussion of the octahedron proceeds in a way that is similar to what was done before for the tetrahedron. In the case of the octahedron, one has 8 triangles, 12 edges and 6 vertices, with 4 neighboring triangles per vertex. Again we will now discuss the various cases individually.

(a) *Equilateral case in the strong coupling limit* ($\epsilon = 0$)
Again, we look first at the case $\epsilon = 0$ in Eq. (52), deep in the strong coupling region and without the curvature term. Following Eq. (53) we define the scaled area variable as

$$x = 2\sqrt{\tilde{\lambda}}A_{\rm tot} = 8 \times 2\sqrt{\tilde{\lambda}}A_\Delta \tag{87}$$

and it is found that the solution is a function of this variable only. For equilateral triangles the wave function Ψ needs to satisfy

$$\Psi'' + \frac{4}{x}\Psi' + \Psi = 0. \tag{88}$$

The correct solution can be written in the form

$$\Psi(x) = \mathcal{N}\frac{J_n(x)}{x^n} \tag{89}$$

with

$$n = \frac{3}{2} \tag{90}$$

so that

$$\Psi(x) = \mathcal{N}\frac{J_{3/2}(2\sqrt{\tilde{\lambda}}A_{\rm tot})}{(2\sqrt{\tilde{\lambda}}A_{\rm tot})^{3/2}}. \tag{91}$$

The wave function normalization factor is given by

$$\mathcal{N} = \sqrt{15}\tilde{\lambda}^{1/4}. \tag{92}$$

Equivalent forms of the above wave function are

$$\begin{aligned}\Psi(A_{\rm tot}) &= \mathcal{N}\frac{1}{2^{3/2}\Gamma(\frac{5}{2})} \times \exp(-2i\sqrt{\tilde{\lambda}}A_{\rm tot})\,{}_1F_1(2, 4, 4i\sqrt{\tilde{\lambda}}A_{\rm tot})\\ &= \mathcal{N}\left[-\frac{\cos(2\sqrt{\tilde{\lambda}}A_{\rm tot})}{2\sqrt{2\pi}\tilde{\lambda}A_{\rm tot}^2} + \frac{\sin(2\sqrt{\tilde{\lambda}}A_{\rm tot})}{4\sqrt{2\pi}\tilde{\lambda}^{3/2}A_{\rm tot}^3}\right].\end{aligned} \tag{93}$$

These can be expanded for small $A_{\rm tot}$ or small x to give

$$\Psi = \mathcal{N}\frac{\sqrt{2}}{3\sqrt{\pi}}\left[1 - \frac{x^2}{10} + \frac{x^4}{280} + \mathcal{O}(x^6)\right]. \tag{94}$$

We note here again that both Bessel functions of the first (J) and second (Y) kind, in principle, give solutions for this case, as well as the two corresponding Hankel (H) functions. Nevertheless, only the solution associated with the Bessel J function is regular near the origin.

(b) *Equilateral case with curvature term* ($\epsilon = 0$)
Next, we include the effects of the curvature term. Since here the deficit angle $\delta = 2\pi/3$ at each vertex, the curvature contribution for each equilateral triangle is $\kappa \cdot \frac{2\pi}{3} \cdot 3 = 2\pi\kappa$. For the octahedron, one has in Eq. (40)

$$\kappa_{\rm octa} = 2 \cdot \frac{1}{4}. \tag{95}$$

With the curvature term, one finds

$$\begin{aligned}\Psi(A_{\rm tot}) &\simeq \exp(-2i\sqrt{\tilde{\lambda}}A_{\rm tot})\,{}_1F_1 \times \left(2 - i\frac{4\pi\kappa_{\rm octa}}{\sqrt{\tilde{\lambda}}G^2}, 4, 4i\sqrt{\tilde{\lambda}}A_{\rm tot}\right)\\ &= \exp(-2i\sqrt{\tilde{\lambda}}A_{\rm tot})\,{}_1F_1 \times \left(2 - i\frac{2\pi}{\sqrt{\tilde{\lambda}}G^2}, 4, 4i\sqrt{\tilde{\lambda}}A_{\rm tot}\right).\end{aligned} \tag{96}$$

Note that in this case one had to include a factor $A_{\rm tot}/(4A_\Delta)$, which in the octahedron case equals two.

(c) *Large area in the strong coupling limit* ($\epsilon \neq 0$)
In the limit of large areas, the two independent solutions reduce to

$$\Psi \sim_{x\to\infty} \exp(\pm ix) \sim \exp(\pm 2i\sqrt{\tilde{\lambda}}A_{\rm tot}) \tag{97}$$

to all orders in ϵ. In other words, to $\mathcal{O}(\epsilon^n)$ with $n \to \infty$, as for the tetrahedron case. Note also that in the strong coupling limit, the two independent wave function solutions again completely factorize as a product of single-triangle contributions.

(d) *Small area in the strong coupling limit* ($\epsilon \neq 0$)
In the limit of small area, the solution approaches a constant in the equilateral case. To go beyond the equilateral case, one can write again a general ansatz for the wave function, written in terms of geometric invariants as in Eq. (85). Then the solution can be expanded in ϵ for small s. To zeroth order in ϵ, the solution is $\Psi \sim J_n(x)/x^n$ with $n = 3/2$. This gives in Eq. (85) $\gamma_0 = 0$, $\gamma_2 = -\frac{2}{5}\tilde{\lambda}$ and $\gamma_4 = \frac{2}{35}\tilde{\lambda}^2$. However, to linear order [$\mathcal{O}(\epsilon)$], one finds again that linear terms in h appear which cannot be expressed in the form of Eq. (85). But one also finds that while these terms are nonzero if one uses the Hamiltonian density (the Hamiltonian contribution from just a single triangle), if one uses the sum of such triangle Hamiltonians, then the resulting solution is symmetrized, and the corrections to Eq. (85) are found to be of order $\mathcal{O}(\epsilon^2)$. Then the wave function for small area is of the form

$$\Psi \simeq 1 - \frac{2}{5}\tilde{\lambda}A_{\rm tot}^2 + \frac{2}{35}\tilde{\lambda}^2 A_{\rm tot}^4 + \ldots \tag{98}$$

up to terms of $\mathcal{O}(\epsilon)$.

D. Icosahedron

The discussion of the icosahedron proceeds in a way that is similar to what was done before for the other regular triangulations. Here one has 20 triangles, 30 edges and 12 vertices, with five neighboring triangles per vertex. Let us again discuss the various cases individually.

(a) *Equilateral case in the strong coupling limit* ($\epsilon = 0$)
Again, we look first at the case $\epsilon = 0$ in Eq. (52), deep in the strong coupling region and without curvature term. Following Eq. (53), we define the scaled area variable as

$$x = 2\sqrt{\tilde{\lambda}}A_{\rm tot} \equiv 20 \times 2\sqrt{\tilde{\lambda}}A_{\Delta} \tag{99}$$

and a solution is found which is a function of this variable only. For equilateral triangles, the wave function Ψ needs to satisfy

$$\Psi'' + \frac{10}{x}\Psi' + \Psi = 0. \tag{100}$$

A solution can then be found of the form

$$\Psi(x) = \mathcal{N}\frac{J_n(x)}{x^n} \tag{101}$$

with

$$n = \frac{9}{2} \tag{102}$$

so that

$$\Psi(x) = \mathcal{N}\frac{J_{9/2}(2\sqrt{\tilde{\lambda}}A_{\rm tot})}{(2\sqrt{\tilde{\lambda}}A_{\rm tot})^{9/2}}. \tag{103}$$

The wave function normalization factor is given by

$$\mathcal{N} = 9\sqrt{12155}\tilde{\lambda}^{1/4}. \tag{104}$$

Below is an equivalent form of the same solution

$$\Psi(A_{\rm tot}) = \mathcal{N}\frac{1}{2^{9/2}\Gamma(\frac{11}{2})} \times \exp(-2i\sqrt{\tilde{\lambda}}A_{\rm tot})\,{}_1F_1(5, 10, 4i\sqrt{\tilde{\lambda}}A_{\rm tot}). \tag{105}$$

For small area $A_{\rm tot}$ or small x, one obtains

$$\Psi = \mathcal{N}\frac{1}{2^{9/2}\Gamma(\frac{11}{2})}\left[1 - \frac{x^2}{22} + \frac{x^4}{1144} + \mathcal{O}(x^6)\right], \tag{106}$$

which shows that the above solution is regular at the origin and normalizable.

(b) *Equilateral case with curvature term* ($\epsilon = 0$)
Next, we include again the effects of the curvature term. Since now the deficit angle $\delta = \pi/3$ at each vertex, the curvature contribution for each triangle is $\kappa \cdot \frac{\pi}{3} \cdot 3 = \pi\kappa$. For the icosahedron, one has in Eq. (40)

$$\kappa_{\rm icosa} = 2 \cdot \frac{1}{5}. \tag{107}$$

Then, with the curvature term included for equilateral triangles, one obtains for equilateral triangles [$\mathcal{O}(\epsilon^0)$]

$$\begin{aligned}\Psi(A_{\rm tot}) &\simeq \exp(-2i\sqrt{\tilde{\lambda}}A_{\rm tot})\,{}_1F_1 \\ &\quad\times\left(5 - i\frac{5\pi\kappa_{\rm icosa}}{\sqrt{\tilde{\lambda}}G^2}, 10, 4i\sqrt{\tilde{\lambda}}A_{\rm tot}\right) \\ &= \exp(-2i\sqrt{\tilde{\lambda}}A_{\rm tot})\,{}_1F_1 \\ &\quad\times\left(5 - i\frac{2\pi}{\sqrt{\tilde{\lambda}}G^2}, 10, 4i\sqrt{\tilde{\lambda}}A_{\rm tot}\right),\end{aligned} \tag{108}$$

up to an overall wave function normalization constant. Note that in this case one had to include a factor $A_{\rm tot}/4A_{\Delta}$, which in the dodecahedron case equals five.

(c) *Large area in the strong coupling limit* ($\epsilon \neq 0$)
In the limit of large areas, the two independent solutions reduce to

$$\Psi \sim_{x\to\infty} \exp(\pm ix) \sim \exp(\pm 2i\sqrt{\tilde{\lambda}}A_{\rm tot}) \tag{109}$$

to all orders in the weak field expansion parameter ϵ, as for the tetrahedron and octahedron case. Note also that in the strong coupling limit, the two independent wave function solutions again completely factorize as a product of single-triangle contributions.

(d) *Small area in the strong coupling limit* ($\epsilon \neq 0$)
In the limit of small area, the solution approaches a constant in the equilateral case. To go beyond the equilateral case, one can write again a general ansatz for the wave function, written in terms of geometric invariants as in Eq. (85). Then the solution in ϵ for small s. To zeroth order in ϵ the solution is $\Psi \sim J_n(x)/x^n$ with $n = 9/2$. This gives in Eq. (85) $\gamma_0 = 0$, $\gamma_2 = -\frac{2}{11}\tilde{\lambda}$ and $\gamma_4 = \frac{2}{143}\tilde{\lambda}^2$. But to linear order $[\mathcal{O}(\epsilon)]$, one finds again that linear terms in h appear which cannot be expressed in the form of Eq. (85). But one also finds that, while these terms are nonzero if one uses the Hamiltonian density (the Hamiltonian contribution from just a single triangle), if one uses the sum of such triangle Hamiltonians, then the resulting solution is symmetrized, and the corrections to Eq. (85) are found to be of order $\mathcal{O}(\epsilon^2)$. Then the wave function for small area is of the form

$$\Psi \simeq 1 - \frac{2}{11}\tilde{\lambda}A_{\text{tot}}^2 + \frac{2}{143}\tilde{\lambda}^2 A_{\text{tot}}^4 + \ldots, \tag{110}$$

up to terms of $\mathcal{O}(\epsilon)$.

E. Torus

Finally, we will consider a regularly triangulated torus, which will consist here of an infinite lattice built out of triangles, with each triangle having 12 neighboring triangles. The torus topology is equivalent to requiring periodic boundary conditions in the two spatial directions. Of course, one could consider the same type of lattice but with some other sort of boundary condition, but we shall not pursue that aspect here.

Due to the local structure of the lattice Wheeler-DeWitt equation in Eq. (34), it will not be necessary to include in the wave function triangles that are arbitrarily far apart. Instead, it will be sufficient, in order to determine the overall structure of the solution, to include only those triangles that are affected in a nontrivial way by the interaction terms in the Wheeler-DeWitt equation. In the present case, this requires the consideration of one given triangle plus its 12 neighbors, giving a total of 13 triangles. Here, we will also set as before $x \equiv 2\sqrt{\tilde{\lambda}}A_{\text{tot}}$.

(a) *Equilateral case in the strong coupling limit* ($\epsilon = 0$)
For this case the relevant equation and its solution are largely in line with what was obtained for the previous cases. For equilateral triangles, the wave function Ψ has to satisfy

$$\Psi'' + \frac{13}{2x}\Psi' + \Psi = 0. \tag{111}$$

The wave function can now be written as

$$\Psi(x) = \mathcal{N}\frac{J_n(x)}{x^n} \tag{112}$$

with, here, (due to our specific choice of sublattice)

$$n = \frac{11}{4} \tag{113}$$

so that

$$\Psi(x) = \mathcal{N}\frac{J_{11/4}(2\sqrt{\tilde{\lambda}}A_{\text{tot}})}{(2\sqrt{\tilde{\lambda}}A_{\text{tot}})^{11/4}}. \tag{114}$$

The wave function normalization constant is given in this case by

$$\mathcal{N} = 4\sqrt{\frac{30\Gamma(\frac{13}{4})}{\Gamma(\frac{11}{4})}}\tilde{\lambda}^{1/4}. \tag{115}$$

For the above wave function, an equivalent form is

$$\Psi(A_{\text{tot}}) = \mathcal{N}\frac{1}{2^{11/4}\Gamma(\frac{15}{4})} \times \exp(-2i\sqrt{\tilde{\lambda}}A_{\text{tot}})\,{}_1F_1\left(\frac{13}{4}, \frac{13}{2}, 4i\sqrt{\tilde{\lambda}}A_{\text{tot}}\right). \tag{116}$$

Expanding the above solution for small area, one obtains

$$\Psi = \mathcal{N}\frac{1}{2^{11/4}\Gamma(\frac{15}{4})}\left[1 - \frac{x^2}{15} + \frac{x^4}{570} + \mathcal{O}(x^6)\right], \tag{117}$$

which shows the above solution is indeed regular at the origin.

(b) *Equilateral case with curvature term* ($\epsilon = 0$)
In the case of the torus, the curvature term is zero ($\chi = 0$), so there are no changes to the preceding discussion.

(c) *Large area in the strong coupling limit* ($\epsilon \neq 0$)
In the limit of large areas, the two independent solutions reduce to

$$\Psi \sim_{x\to\infty} \exp(\pm ix) \sim \exp(\pm i2\sqrt{\tilde{\lambda}}A_{\text{tot}}) \tag{118}$$

to all orders in ϵ. This is similar to what was found earlier for the other lattices. In particular, the two independent solutions again completely factorize as a product of single-triangle contributions.

(d) *Small area in the strong coupling limit* ($\epsilon \neq 0$)
In the limit of small area, the regular solution approaches a constant and the discussion, and solution, is rather similar to the previous cases. Here one finds

$$\Psi \simeq 1 - \frac{4}{15}\tilde{\lambda}A_{\text{tot}}^2 + \frac{8}{285}\tilde{\lambda}^2 A_{\text{tot}}^4 + \ldots, \tag{119}$$

up to terms of $\mathcal{O}(\epsilon^2)$.

F. Summary of results

In this section we will summarize the results obtained so far for the various finite lattices considered (tetrahedron, octahedron, icosahedron, and regularly triangulated torus).

(a) *Equilateral case in the strong coupling limit* ($\epsilon = 0$) It is rather remarkable that all of the previous cases (except the trivial case of a single triangle, which has no curvature) can be described by one single set of interpolating wave functions, where the interpolating variable is simply related to the overall lattice size (specifically, the number of triangles).
Indeed, for equilateral triangles and in the absence of curvature, the wave function $\Psi(x)$ for all previous cases is a solution to the following equation

$$\Psi'' + \frac{2n+1}{x}\Psi' + \Psi = 0, \tag{120}$$

with parameter n given by

$$n = \frac{1}{4}(N_\Delta - 2), \tag{121}$$

where $N_\Delta \equiv N_2$ is the total number of triangles on the lattice. Thus,

$$N_\Delta = 4\left(n + \frac{1}{2}\right) \tag{122}$$

and, consequently,

$$\begin{aligned} n_{\text{tetrahedron}} &= \frac{1}{4}(4-2) = \frac{1}{2}, \\ n_{\text{octahedron}} &= \frac{1}{4}(8-2) = \frac{3}{2}, \\ n_{\text{icosahedron}} &= \frac{1}{4}(20-2) = \frac{9}{2}, \\ n_{\text{torus}} &= \frac{1}{4}(13-2) = \frac{11}{4}. \end{aligned} \tag{123}$$

Note that for a single triangle, one has $n = \frac{1}{2}$ as well, but the definition of the scaled area is different in that case.
Furthermore, the differential equation in Eq. (120) describes, in spherical coordinates and with suitable choice of constants, the radial wave function for a free quantum particle in $D = 2n + 2$ dimensions. Indeed, recall that in D dimensions the Laplace operator in spherical coordinates has the form

$$\Delta\Psi = \frac{\partial^2\Psi}{\partial r^2} + \frac{D-1}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\Delta_{S^{D-1}}\Psi, \tag{124}$$

where $\Delta_{S^{D-1}}$ is the Laplace-Beltrami operator on the $(D-1)$ sphere. In our case, the wave function does not, to this order, depend on angles and therefore the last (angular variable) term does not contribute. The role of the angles is played, in our case, by the h variables, which to this order do not fluctuate.

A nonsingular, normalizable solution to Eq. (120) is then given by

$$\begin{aligned} \Psi(x) &= \mathcal{N}\frac{J_n(x)}{x^n} \\ &= \tilde{\mathcal{N}} e^{-ix}{}_1F_1\left(n + \frac{1}{2}, 2n+1, 2ix\right), \end{aligned} \tag{125}$$

where $\mathcal{N}$ is the wave function normalization constant

$$\mathcal{N} \equiv 2\left[\frac{\Gamma(n+\frac{1}{2})\Gamma(2n+\frac{1}{2})}{\Gamma(n)}\right]^{1/2}\tilde{\lambda}^{1/4} \tag{126}$$

and

$$\tilde{\mathcal{N}} \equiv \frac{1}{2^n\Gamma(n+1)}\mathcal{N}. \tag{127}$$

Here and in Eq. (125), ${}_1F_1(a, b; z)$ denotes the confluent hypergeometric function of the first kind, sometimes denoted also by $M(a, b, ; z)$. In either form, the above wave function is real, in spite of appearances. The general asymptotic behavior of the solution $\Psi(x)$ is found from Eq. (120). For small x one has

$$\Psi(x) \sim x^\alpha \tag{128}$$

with index $\alpha = 0, -2n$. The latter solution is singular and will be discarded. For large x one finds immediately

$$\Psi(x) \sim \frac{1}{x^{n+\frac{1}{2}}}\exp(\pm ix), \tag{129}$$

which is of course consistent with all the previous results. Indeed, the other possible independent solution of Eq. (120) would be

$$\Psi(x) \simeq \frac{Y_n(x)}{x^n}, \tag{130}$$

where $Y_n(x)$ is a Bessel function of the second kind (or Neumann function). However, the latter leads to a wave function Ψ which is singular as $x \to 0$,

$$\Psi(x) \sim -\frac{1}{\pi}\Gamma(n)2^n x^{-2n}, \tag{131}$$

and gives, therefore, a solution that is not normalizable. For completeness we record here the small x (small area) behavior of the normalized wave function in Eq. (125)

$$\Psi(x) \sim \mathcal{N}\frac{1}{2^n\Gamma(n+1)}, \tag{132}$$

and the corresponding large x (large area) behavior

$$\Psi(x) \sim \mathcal{N}\sqrt{\frac{2}{\pi}}\frac{1}{x^{n+\frac{1}{2}}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right), \tag{133}$$

both of which reflect well-known properties of the Bessel functions $J_n(x)$.

(b) *Equilateral case with curvature term* ($\epsilon = 0$)
When the curvature term is included in the Wheeler-DeWitt equation, and still in the limit of equilateral triangles, one obtains the following interpolating differential equation

$$\Psi'' + \frac{2n+1}{x}\Psi' - \frac{2\beta}{x}\Psi + \Psi = 0, \qquad (134)$$

which now describes the radial wave function for a quantum particle in $D = 2n + 2$ dimensions, with a repulsive Coulomb potential proportional to 2β. The nonsingular, normalizable solution is now given by

$$\Psi(x) \simeq e^{-ix}{}_1F_1\left(n + \frac{1}{2} - i\beta, 2n + 1, 2ix\right), \qquad (135)$$

up to an overall wave function normalization constant $\tilde{\mathcal{N}}(n, \beta)$. The normalization constant can be evaluated analytically but has a rather unwieldy form and will not be recorded here. Note that the imaginary part (β) of the first argument in the confluent hypergeometric function of Eq. (135) depends on the topology but does not depend on the number of triangles. In view of the previous discussion, the parameter n increases gradually as more triangles are included in the simplicial geometry. For the regular triangulations of the sphere, the total deficit angle (the sum of the deficit angles in a given simplicial geometry) is always 4π, so even if one writes for the wave functional $\Psi[A_{\rm tot}, \delta_{\rm tot}]$, the curvature contribution $\sum_h \delta_h$ is a constant and does not contribute in any significant way. Note also that, in spite of appearances, the above wave function is still real for nonzero β. That $\Psi(x)$ in Eq. (135) is a real function can be seen, for example, from its definition via the power series expansion

$$\Psi(x) \simeq 1 + \frac{2\beta}{2n+1}x - \frac{1 + 2n - 4\beta^2}{4 + 12n + 8n^2}x^2 - \frac{\beta(5 + 6n - 4\beta^2)}{6(3 + 11n + 12n^2 + 4n^3)}x^3 + \mathcal{O}(x^4) \qquad (136)$$

and again up to an overall normalization factor $\mathcal{N}(n, \beta)$.

The general asymptotic behavior of the solution $\Psi(x)$ is again easily determined from Eq. (134). For small x one has

$$\Psi(x) \sim x^{\alpha} \qquad (137)$$

with again $\alpha = 0, -2n$ and, therefore, independent of the curvature contribution involving β. The second solution is singular and will be discarded as before. For large x one finds immediately

$$\Psi(x) \sim \frac{1}{x^{n+\frac{1}{2}}}\exp\{\pm i(x - \beta \ln x)\}, \qquad (138)$$

which is of course consistent with all previous results. It also shows that the convergence properties of the wave function at large x are not affected by the β term. A second independent solution to Eq. (134) is given by

$$\Psi(x) \simeq e^{-ix}U\left(n + \frac{1}{2} - i\beta, 2n + 1, 2ix\right), \qquad (139)$$

where $U(a, b, ; z)$ is the confluent hypergeometric function of the second kind (sometimes referred to as Tricomi's function). This second solution is singular at the origin, leading to a wave function that is not normalizable and will not be considered further here.

The asymptotic behavior of the regular solution for large argument z (discussed in standard quantum mechanics textbooks such as Refs. [25,26] and whose notation we will follow here) can be obtained from the asymptotic form of the confluent hypergeometric function ${}_1F_1$, defined originally, for small z, by the series

$${}_1F_1(a, b, z) = 1 + \frac{az}{b1!} + \frac{a(a+1)z^2}{b(b+1)2!} + \cdots. \qquad (140)$$

It is common procedure to then write ${}_1F_1(a, b, z) = W_1(a, b, z) + W_2(a, b, z)$, where W_1 and W_2 are separately solutions of the confluent hypergeometric equation

$$z\frac{d^2F}{dz^2} + (b - z)\frac{dF}{dz} - aF = 0. \qquad (141)$$

Then an asymptotic expansion for ${}_1F_1$ (or M) is obtained from the following relations:

$$W_1(a, b, z) = \frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a}w(a, a-b+1, -z)$$
$$W_2(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)}e^z z^{a-b}w(1-a, b-a, z), \qquad (142)$$

where

$$w(\alpha, \beta, z) \sim_{z\to\infty} 1 + \frac{\alpha\beta}{z1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{z^2 2!} + \cdots, \qquad (143)$$

with the irregular (at the origin) solution given instead by the combination $G(a, b, z) = iW_1(a, b, z) - iW_2(a, b, z)$. One immediate and useful consequence of the above result is that, as anticipated before, the behavior of the regular solution close to the origin is not affected by the presence of the β (curvature) term. In other words, the wave function solution

$\Psi(x)$ in Eq. (135) is always well behaved for small areas and, therefore, leads to a perfectly acceptable, normalizable solution.

Furthermore, the combination and properties of arguments in the confluent hypergeometric function in Eq. (135) allow one to write it equivalently as a Coulomb wave function with (Sommerfeld) parameter η

$$C_l(\eta)\rho^{l+1}\cdot e^{-i\rho}{}_1F_1(l+1-i\eta,2l+2,2i\rho) = F_l(\eta,\rho), \tag{144}$$

where $F_l(\eta,\rho)$ denotes the *regular* Coulomb wave function that arises in the solution of the quantum mechanical three-dimensional Coulomb problem in spherical coordinates [25,26]. The latter is a solution of the radial differential equation

$$\frac{d^2F_l}{d\rho^2}+\left[1-\frac{2\eta}{\rho}-\frac{l(l+1)}{\rho^2}\right]F_l=0, \tag{145}$$

with the actual radial wave function then given by $R_l(r)=F_l(kr)/r$. After comparing the above equation with Eq. (135), one then identifies $\rho=x$, $l=n-\frac{1}{2}$ and $\eta=\beta$. Thus $l=N_\Delta/4-1$, where N_Δ is the number of triangles on the lattice. The proportionality constant C_l in Eq. (144) is given by the (Gamow) parameter

$$C_l(\eta)\equiv\frac{2^le^{-\frac{\pi\eta}{2}}|\Gamma(l+1+i\eta)|}{\Gamma(2l+2)}. \tag{146}$$

One then has immediately, from Eq. (135), an equivalent representation for the regular wave function as

$$\Psi(x)\simeq[C_{n-\frac{1}{2}}(\beta)]^{-1}\frac{1}{x^{n+\frac{1}{2}}}F_l(\beta,x), \tag{147}$$

again up to an overall wave function normalization constant $\tilde{\mathcal{N}}(n,\beta)$. Again, we note here that, on the other hand, the *irregular* Coulomb wave function [usually denoted by $G_l(\eta,\rho)$] is singular for small r and will, therefore, not be considered here. Further relevant properties of the Coulomb wave function can be found in Refs. [25–29].

The known asymptotics of Coulomb wave function [27–29] allow one to derive the following result for the wave function Ψ for large x

$$\Psi(x)\simeq\tilde{\mathcal{N}}\frac{1}{C_{n-\frac{1}{2}}(\beta)\cdot x^{n+\frac{1}{2}}}\sin\left[x-\beta\ln2x-\frac{(2n-1)\pi}{4}+\sigma_n\right] \tag{148}$$

with (Coulomb) phase shift

$$\sigma_n(\beta)=\arg\Gamma\left(n+\frac{1}{2}+i\beta\right). \tag{149}$$

Also, from Eq. (146),

$$C_{n-\frac{1}{2}}(\beta)\equiv\frac{2^{n-\frac{1}{2}}e^{-\frac{\pi\beta}{2}}|\Gamma(n+\frac{1}{2}+i\beta)|}{\Gamma(2n+1)}. \tag{150}$$

It is easy to check that the above result correctly reduces to the asymptotic expression given earlier for Ψ in Eq. (133) in the limit $\beta=0$. The structure of the wave function in Eq. (148) implies that the norm is still finite for $\beta\neq0$, since the convergence properties of the wave function are not affected by the curvature term.

(c) *Large area in the strong coupling limit* ($\epsilon\neq0$)
In the limit of large areas the two independent solutions reduce to

$$\Psi\underset{x\to\infty}{\sim}\exp(\pm ix), \tag{151}$$

where $x\propto A_{\rm tot}$. This is true without assuming the weak field expansion, as was already the case before (see, in particular, the section discussing the tetrahedron case).
Consequently, in the strong coupling limit, the two wave function solutions in Eq. (151) completely factorize as a product of single-triangle contributions,

$$\Psi\simeq\prod_\Delta\exp(\pm2i\sqrt{\lambda}A_\Delta), \tag{152}$$

again up to an overall normalization constant. The above result, anticipated in Ref. [1], was the basis for the variational treatment using correlated product wave functions given in our previous work. Note also, in view of the result of Eq. (133), that the correct solution, satisfying the required regularity condition for small areas, is actually a linear combination of the above factorized solutions.

(d) *Small area in the strong coupling limit* ($\epsilon\neq0$)
In the limit of small area, we have shown before in all cases that the solution reduces to a constant in the equilateral case [$\mathcal{O}(\epsilon^0)$] for small x or small areas. To linear order [$\mathcal{O}(\epsilon)$], the general result is still that linear terms in h appear which cannot be expressed in the form of Eq. (85). But one also finds that, while these terms are nonzero if one uses the Hamiltonian density (the Hamiltonian contribution from just a single triangle), if one uses the sum of such triangle Hamiltonians, then the resulting solution is symmetrized, and the corrections to Eq. (85) are found to be of order $\mathcal{O}(\epsilon^2)$. In other words, it seems that some residual lattice artifacts that survive at very short distances can be partially removed by a suitable coarse-graining procedure on the Hamiltonian density. One might wonder what lattices correspond to values

of n greater that $9/2$, which is the highest value attained for a regular triangulation of the sphere, corresponding to the icosahedron. For each of the three regular triangulations with N_0 sites, one has for the number of edges $N_1 = \frac{q}{2} N_0$ and for the number of triangles $N_2 = (\frac{q}{2} - 1)N_0 + 2$, where q is the number of edges meeting at a vertex (the local coordination number). In the three cases examined before, q was between three and five, with six corresponding to the regularly triangulated torus. Note that for a sphere $N_0 - N_1 + N_2 = 2$ always. The interpretation of other, even noninteger, values of q is then clear. Additional triangulations of the sphere can be constructed by considering irregular triangulations, where now the parameter q is interpreted as an *average* coordination number. Of course, the simplest example is a semiregular lattice with N_a vertices with coordination number q_a and N_b vertices with coordination number q_b, such that $N_a + N_b = N_0$. Various irregular and random lattices were considered in detail some time ago in Ref. [16], and we refer the reader to this work for a clear exposition of the properties of these lattices.

We conclude this section by briefly summarizing the key properties of the gravitational wave function given in Eqs. (135) and (147), which from now on will be used as the basis for additional calculations. First we note that the above wave function is a function of the total area and total curvature only and, as such, is manifestly diffeomorphism invariant and in accord with the spatial diffeomorphism constraint. While it was derived by looking at the discrete triangulations of the sphere, it contains a parameter n, related to the total number of triangles on the lattice by Eq. (121), that will allow us to go beyond the case of a finite lattice and investigate the physically meaningful, and presumably universal, infinite volume limit $n \to \infty$ [see Eq. (55)]. We have also shown that the above wave function is, in all cases, an exact solution of the full lattice Wheeler-DeWitt equation of Eq. (21) in the limit of large areas and to all orders in the weak field expansion. Again, this last case is most relevant for taking the infinite volume limit, defined previously in Eq. (55). Furthermore, the small area behavior of the wave function plays a crucial role in uniquely constraining, through the regularity condition, the correct choice of solution. In this last limit one also finds that the various individual lattice solutions agree with the universal form of Eqs. (135) and (147) only to a low order in the weak field expansion, which is expected given the different short distance lattice artifacts of the regular triangulation solutions. Nevertheless, knowledge of their behavior is completely adequate for extracting the most important physically relevant piece of information, namely the constraint on the wave function based on the stated regularity condition at small areas, which comes down to a simple integrability or power counting argument.

VII. AVERAGE AREA

In this section we will look at a natural quantum mechanical expectation value, the average total physical area of the lattice simplicial geometry. It is one of many quantities that can be calculated within the lattice quantum gravity formalism and is clearly both manifestly geometric and diffeomorphism invariant. Here, we will use the wave functions given in Eqs. (135) and (147), originally obtained for the tetrahedron, octahedron and icosahedron and later extended to any number of triangles N_Δ

$$\Psi(A_{\mathrm{tot}}) \simeq e^{-i\frac{A_{\mathrm{tot}}}{g}} {}_1F_1\left(n + \frac{1}{2} - i\beta, 2n + 1, 2i\frac{A_{\mathrm{tot}}}{g}\right), \tag{153}$$

with $n \equiv \frac{1}{4}(N_\Delta - 2)$, $\beta \equiv 4\pi/g^3$ and $g \equiv \sqrt{G}$, and again valid up to an overall wave function normalization constant. Due to the structure of the wave function, the resulting probability distribution for the area is rather nontrivial, having many peaks associated with the infinitely many minima and maxima of the hypergeometric function. Clearly, the most interesting limit is one where one considers an infinite number of triangles, $N_\Delta \to \infty$, which corresponds to $n \to \infty$ in Eq. (153). In Figs. 4 and 5, we display the behavior of the wave function in Eq. (153), both with and without the curvature contribution in the Wheeler-DeWitt equation. One notices that when the curvature term is included ($\beta \neq 0$), the peak in the wave function shifts away from the origin. This is largely expected, based on the contribution from the repulsive Coulomb term in the wave equation of Eq. (134).

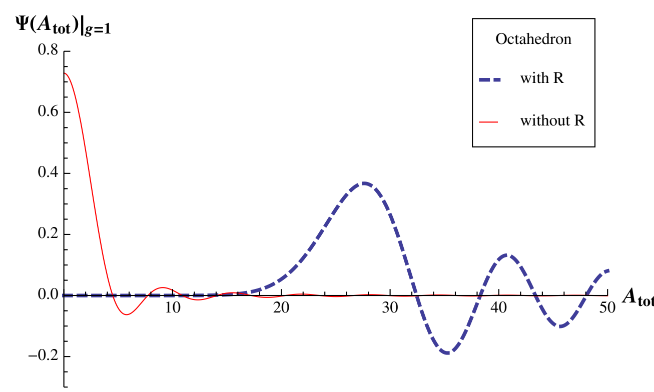

FIG. 4 (color online). Wave Function Ψ vs total area for the octahedron lattice, with and without curvature contribution. The wave function is shown here for $g = \sqrt{G} = 1$, a value chosen here for illustration purposes. The relevant expression for the wave function is given in Eq. (153). We refer to the text for further details on how the wave function was obtained and what its domain of validity is. The wave functions shown here have been properly normalized. Note that with a nonzero curvature term, the peak in the wave function moves away from the origin.

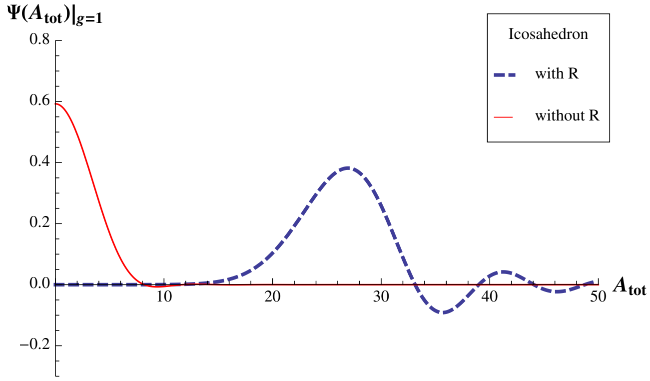

FIG. 5 (color online). Same wave function Ψ as in Fig. 4, but now for the icosahedron lattice.

The average total area can then be computed from the above wave function, as the ground state expectation value

$$\langle A\rangle = \frac{\langle\Psi|A|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{\int d\mu[g]A(g)|\Psi(g)|^2}{\int d\mu[g]|\Psi(g)|^2}, \tag{154}$$

where g here, is the three-metric, and $d\mu[g]$ denotes a functional integration over all three-metrics. In our case we use the measure

$$\int d\mu[g] \rightarrow \int_0^\infty dA_{\text{tot}}, \tag{155}$$

which then gives, in terms of the scaled area variable x,

$$\langle A_{\text{tot}}\rangle = g\frac{\int_0^\infty dxx\cdot|\Psi(x)|^2}{\int_0^\infty dx|\Psi(x)|^2}. \tag{156}$$

In the absence of a curvature term in the Wheeler-DeWitt equation ($\beta = 0$), the average area can easily be computed analytically in terms of Bessel function integrals, and the result is

$$\langle A_{\text{tot}}\rangle = g\cdot\frac{\pi(4n-1)\Gamma(4n-2)}{2^{8n-5}\Gamma(n)^4}. \tag{157}$$

Note that the average area diverges as $n \rightarrow \frac{1}{2}$, which corresponds to the tetrahedron; this entirely spurious divergence prevents us from using the tetrahedron lattice in plotting and numerically extrapolating the remaining two lattices (octahedron and icosahedron) to the infinite lattice limit. For the octahedron, one finds $\langle A_{\text{tot}}\rangle = 15g/\pi$, for the icosahedron $\langle A_{\text{tot}}\rangle = 21879g/3920\pi$, and in the large n limit $\langle A_{\text{tot}}\rangle = \sqrt{2n/\pi}g + \mathcal{O}(1/\sqrt{n})$.

One finds that in the presence of a curvature term ($\beta \neq 0$), the resulting integrals are significantly more complicated. We have, therefore, resorted to a number of tools, which include an analytic expansion in β, the use of known asymptotic expansions for the wave function at large arguments, and an exact numerical integration of the resulting integrals. Let us first discuss here the expansion in β. It is known that the Coulomb wave functions can be expanded in terms of spherical Bessel functions (Neumann expansion) [27–29], so that one has

$$F_l(\eta,\rho) = \frac{2^{l+1}}{\sqrt{\pi}}\Gamma\left(l+\frac{3}{2}\right)C_l(\eta)\rho\sqrt{\frac{\pi}{2\rho}}\cdot\left\{\sum_{k=l}^{\infty} b_k(\eta)J_{k+\frac{1}{2}}(\rho)\right\} \tag{158}$$

with coefficients $b_k(\eta)$ given by a simple recursion relation. When written out explicitly, the expression in curly brackets involves

$$J_{l+\frac{1}{2}}(x) + \frac{2l+3}{l+1}\eta\cdot J_{l+\frac{3}{2}}(x) + \frac{2l+5}{l+1}\eta^2\cdot J_{l+\frac{5}{2}}(x) + \cdots, \tag{159}$$

with additional terms linear in η reappearing at higher orders. That the above expansion is a bit problematic is not entirely surprising, given the modified asymptotic behavior of the Coulomb wave functions for $\eta \neq 0$. In the following, in order to provide initially some insight into the effects of the η (or β) term on the wave function Ψ, we will include the first correction as a perturbation, and drop the rest. Later on, higher order corrections can be included as additional contributions. With this truncation, the Coulomb wave function in Eq. (144) becomes

$$F_l(\eta,\rho) = \frac{2^{l+1}}{\sqrt{\pi}}\Gamma\left(l+\frac{3}{2}\right)C_l(\eta)\rho\sqrt{\frac{\pi}{2\rho}} \times\left[J_{l+\frac{1}{2}}(\rho) + \eta\frac{2l+3}{l+1}J_{l+\frac{3}{2}}(\rho) + \cdots\right] \tag{160}$$

with the last term treated as a perturbation, giving for the wave function itself [see Eq. (135)]

$$\Psi(x) \simeq e^{-ix}{}_1F_1\left(n+\frac{1}{2}-i\beta, 2n+1, 2ix\right) = \frac{1}{x^n}\left[J_n(x) + \beta\frac{2n+2}{n+\frac{1}{2}}J_{n+1}(x) + \cdots\right], \tag{161}$$

again up to an overall wave function normalization constant $\tilde{\mathcal{N}}$. Note that if m Bessel function terms are kept in Eq. (161), beyond the zeroth order, strong coupling, term involving $J_n(x)$, then the resulting expansion in β contains terms up to β^m. One finds to lowest order ($m = 1$)

$$\frac{1}{\tilde{\mathcal{N}}^2} = \frac{\Gamma(n)}{2\Gamma(n+\frac{1}{2})\Gamma(2n+\frac{1}{2})} + \frac{4^{1-n}(n+1)\beta}{(2n+1)\Gamma(n+1)^2} + \cdots. \tag{162}$$

From the above expressions, the average area can then be computed as some still rather complicated function,

$$\langle A_{\text{tot}}\rangle = g\left\{\frac{\pi(4n-1)\Gamma(4n-2)}{2^{8n-5}\Gamma(n)^4} + \frac{4(n+1)\beta}{2n+1}\left[1 - \frac{4^{1-2n}\Gamma(n-\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(2n+\frac{1}{2})^2}{n^2\Gamma(n)^6}\right] + \cdots\right\}. \tag{163}$$

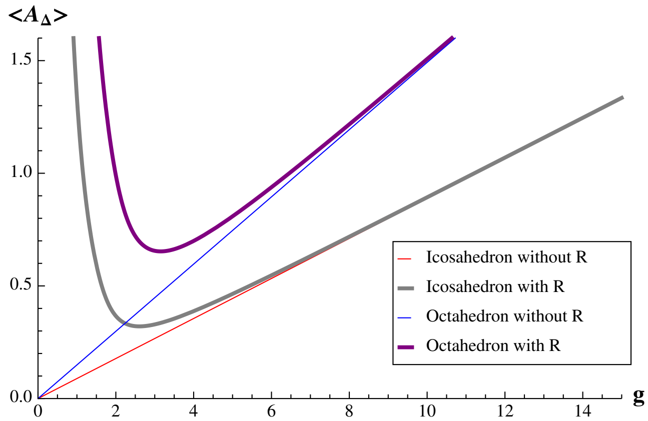

FIG. 6 (color online). Average area of a single triangle vs $g = \sqrt{G}$ for the octahedron and the icosahedron configurations. The average area was calculated using the expression in Eq. (156). Note the qualitative change when one includes the curvature term, with a minimum appearing at $g \sim \mathcal{O}(1)$.

Additional terms can later be included in the Bessel function expansion of Eq. (158), so as to obtain more accurate values for the averages; this will be done later.

Figure 6 shows the exact value of the average area for a single triangle $\langle A_\Delta \rangle = \langle A_{\rm tot} \rangle / N_\Delta$ as a function of the coupling g, obtained by doing the integral in Eq. (156) numerically, with the wave function given in Eq. (153). One noteworthy aspect is that a qualitative change seems to occur when one includes the curvature term: a well defined minimum occurs at $g \sim 1$, which would suggest the appearance of some sort of phase transition. Doing the integrals numerically, one finds a minimum in the average area of a triangle at $g_c \approx 3.1$ for the octahedron and at $g_c \approx 2.6$ for the icosahedron. On the other hand, using the lowest order Bessel function expansion of Eq. (161) for the octahedron ($n = 3/2$), one finds a minimum at $g_c = 2.683$ and for the icosahedron ($n = 9/2$) at $g_c = 2.271$. Adding one more Bessel function correction term then gives $g_c = 3.135$ and $g_c = 2.637$ for the two cases, respectively, which suggests that the expansion is converging.

The limit of a large number of triangles $N_\Delta \to \infty$ corresponds to taking the parameter n in Eq. (153) to infinity, since $n \equiv \frac{1}{4}(N_\Delta - 2)$. From the lowest order Bessel function expansion, one obtains the following analytic expression for the average total area

$$\langle A_{\rm tot} \rangle = g \cdot \sqrt{\frac{2n}{\pi}} \left[1 + \frac{3}{16n} + \mathcal{O}\left(\frac{1}{n^2}\right) \right] + \frac{2(\pi - 2)g}{\pi} \beta + \cdots, \tag{164}$$

with $\beta \equiv 4\pi/g^3$ [see Eq. (50)]. In this limit the resulting function of g has, again, a well-defined minimum at

$$g_c^3 = \frac{8(\pi - 2)\sqrt{2\pi}}{\sqrt{n}} \tag{165}$$

or $g_c \simeq 2.839/n^{1/6}$ for large n with one Bessel function correction term. With two Bessel function correction terms in Eq. (161), one finds $g_c \simeq 3.276/n^{1/6}$, which again suggests that the expansion is slowly converging. Using the exact wave function to do the integrals numerically, one finds for the minimum $g_c \simeq 3.309/n^{1/6}$, which is close to the above answer. Interestingly enough, the above result would suggest that in the limit of infinitely many lattice points, the critical point g_c actually moves to the origin, indicating a phase transition located at exactly $g = 0$ ($G = 0$) in the infinite volume ($n \to \infty$) limit (see further discussion later). We note here that the average area for a single triangle is obtained by simply dividing the average total area by the total number of triangles $N_\Delta = 4n + 2$, which then gives in the same limit of large n and strong coupling

$$\langle A_\Delta \rangle = \frac{g}{2\sqrt{2\pi n}} + \mathcal{O}\left(\frac{1}{n}\right). \tag{166}$$

Quite generally, the average of the area per site in the lattice theory (the spatial volume per site) appears to be well defined mainly due to our wave function normalization choices and, consequently, can be explicitly calculated without any leftover ambiguity.

As will be discussed further below in more detail, the estimate for the critical point given in Eq. (165) is also in good agreement with a previous variational estimate. In Ref. [1] the quantum-mechanical variational (Rayleigh-Ritz) method was used to find an approximation for the ground state wave function, using as variational wave function a correlated (Jastrow-Slater) product of single-triangle wave functions. There it was found, from the roots of the equation $\langle \Psi | H | \Psi \rangle = 0$, that the variational parameters are almost purely imaginary for strong coupling (large $G > G_c$), whereas for weak enough coupling (small $G < G_c$), they become real. This abrupt change in behavior of the wave function at G_c then suggested the presence of a phase transition. With the notation used in this paper, the result of Ref. [1] reads $g_c^3 \sim 1/N_\Delta$, in qualitative agreement with the result of Eq. (165), in the sense that both calculations point to a critical point $G_c = 0$ in the infinite volume limit.

Let us now make some additional comments which should help clarify the interpretation of the previous results. It is well known that if there is some sort of continuous phase transition in the lattice theory, the latter is generally associated with a divergent correlation length in the vicinity of the critical point. In our case it is clear that at strong coupling (large g), the correlation length is small (of order one) in units of lattice spacing. This can be seen from the fact that (a) the coupling term in the Wheeler-DeWitt equation is due mainly to the curvature term, which is small for large g and (b) that the ground state wave function is of the form of a correlated product in the same limit [see Eq. (152)]. Then, as the effects of the curvature term are included, the correlation length starts to grow due to the additional coupling between edge variables. The previous calculation would then suggest that the point of divergence is located at $g = 0$. It is, of course, essential that one looks at the limit of infinitely many triangles, $N_\Delta \to \infty$, since no continuous phase transition can occur in a system with a finite number of degrees of freedom.

It is also of interest here to discuss how the above (Lorentzian) results relate to what is known about the corresponding Euclidean lattice theory in three dimensions, which was studied in some detail in Ref. [24]. There, a phase transition was found between two phases, with the weak coupling phase $G < G_c$ exhibiting a sort of pathological behavior, whereby the lattice collapses into what geometrically could be described as a branched polymer. This is clearly a nonperturbative phenomenon that cannot be seen from perturbation theory in G. In the Euclidean formulation, average volumes are obtained as suitable derivatives of $\log Z_{\rm latt}$ with respect to the bare cosmological constant λ_0, where $Z_{\rm latt}$ is the lattice path integral

$$Z_{\rm latt} = \int [dl^2] e^{-I_{\rm latt}(l^2)} \tag{167}$$

with, in four dimensions, the action given by

$$I_{\rm latt} = \lambda_0 \sum_h V_h(l^2) - k \sum_h \delta_h(l^2) A_h(l^2) \tag{168}$$

and h denoting a hinge [more details can be found in Ref. [24]]. Similarly, the average curvature can also be obtained as a derivative of $\log Z$ with respect to $k \equiv 1/(8\pi G)$. More importantly, a nonanalyticity in Z, as induced by a phase transition, is expected to show up in local averages as well. From the above expression for $Z_{\rm latt}$ exact sum rules can be derived relating various averages [30]. In the case of the three-dimensional Euclidean theory, the sum rule reads

$$2\lambda_0 \left\langle \sum_T V_T \right\rangle - k \left\langle \sum_h \delta_h l_h \right\rangle - C_0 = 0, \tag{169}$$

where the first term contains a sum over all lattice tetrahedra, and the second term involves a sum over all lattice hinges (just edges in this case). The quantity C_0, here, is a constant that solely depends on how the lattice is put together (i.e., on the local coordination number, or incidence matrix).

In Ref. [24] it was found that the average curvature goes to zero at some g_c with a characteristic universal exponent δ,

$$\left\langle \sum_h \delta_h l_h \right\rangle = -R_0 |g - g_c|^\delta \tag{170}$$

and that the curvature fluctuation diverges in the same limit. From the sum rule in Eq. (169), one then deduces that the average volume in the Euclidean theory has a singularity of the type

$$\left\langle \sum_T V_T \right\rangle = V_0 - V_1 |g - g_c|^\delta \tag{171}$$

with the same exponent $\delta \simeq 0.77$. The latter is related by standard universality and scaling arguments [31–33] (see Ref. [19] for details specific to the gravity case) to the correlation length exponent ν by $\nu = (1 + \delta)/d$ in d dimensions. To compare to the Lorentzian theory discussed in this paper, one notes that the three-dimensional Euclidean theory corresponds to the $(2 + 1)$-dimensional Wheeler-DeWitt theory, so that the average volume in the above discussion should be taken to correspond to an average area in our case.[3] To conclude, the results for the average area suggest the existence of a phase transition in the Lorentzian theory located at $g = 0$. In the next sections we will present a further test of this hypothesis, based on physical observables that can establish directly and unambiguously the location of the phase transition point.

VIII. AREA FLUCTUATION, FIXED POINT AND CRITICAL EXPONENT

Another quantity that can be obtained readily from the wave function Ψ is the fluctuation in the total area

$$\chi_A = \frac{1}{N_\Delta} \{ \langle (A_{\rm tot})^2 \rangle - \langle A_{\rm tot} \rangle^2 \}. \tag{172}$$

The latter is related to the fluctuations in the individual triangles by

$$\chi_A = N_\Delta \{ \langle A_\Delta^2 \rangle - \langle A_\Delta \rangle^2 \} \tag{173}$$

with the usual definition of averages, such as the one given in Eq. (154).

Generally, for a field $\phi(x)$ with renormalized mass m and correlation length $\xi = m^{-1}$, wave function renormalization constant Z, and (Euclidean) propagator

$$\langle \phi(x) \phi(0) \rangle = \int \frac{d^d p}{(2\pi)^d} e^{-ip\cdot x} \frac{Z}{p^2 + m^2}, \tag{174}$$

one has for $\Phi \equiv \int_x \phi(x)$

[3]It should be noted that in the case of the lattice Wheeler-DeWitt equation of Eqs. (20) and (21) and, generally, in any lattice Hamiltonian continuous-time formulation, the lattice continuum limit along the time direction has already been taken. This is due to the fact that one can view the resulting $2 + 1$ theory as originating from one where there exist initially two lattice spacings, a_t and a. The first one is relevant for the time direction and the second one for the spatial directions. In the present lattice formulation, the limit $a_t \to 0$ has already been taken; the only limit left is $a \to 0$, which requires the existence of an ultraviolet fixed point of the renormalization group.

$$\langle\Phi^2\rangle = \int_{x,y}\langle\phi(x)\phi(y)\rangle = V\int_x\langle\phi(x)\phi(0)\rangle = V\frac{Z}{m^2} = VZ\xi^2. \tag{175}$$

Thus, the field fluctuation probes the propagator at zero momentum, which in turn is directly related to the renormalized mass (and thus ξ) for the field in question. If the field Φ acquires a nonzero expectation value, the above result is modified to

$$\frac{1}{V}\{\langle\Phi^2\rangle - \langle\Phi\rangle^2\} = \frac{Z}{m^2} = Z\xi^2, \tag{176}$$

involving instead the connected propagator. In the gravity case, the quantity $A_{\rm tot}$ plays the role of Φ; if the fluctuation diverges ($\xi \longrightarrow \infty$), then one has a phase transition or an ultraviolet fixed point in quantum field theory language [17,30].

Without the curvature term in the Wheeler-DeWitt equation [$\beta = 0$ for the wave function Ψ in Eq. (161)], the area fluctuation does not diverge, even when n is large and is simply proportional to g^2. In this case one finds

$$\begin{aligned}\chi_A(\beta = 0) &= \frac{4n-1}{16}\bigg[\frac{2n-1}{2n^2-n-1} \\ &\quad - \frac{\pi^2(4n-1)\Gamma(4n-2)^2}{2^{16n-13}(2n+1)\Gamma(n)^8}\bigg]g^2 \\ &\sim \frac{\pi-2}{4\pi}g^2 + \mathcal{O}\Big(\frac{1}{n}\Big).\end{aligned} \tag{177}$$

Note the spurious singularity for the special case of the tetrahedron, $n = 1/2$. When the curvature term is taken into account, one finds, from the full wave function Ψ in Eq. (161) and in the limit of large n,

$$\chi_A = \Big(1 - \frac{2}{\pi}\Big)\frac{g^2}{4} + 2(4-\pi)\sqrt{\frac{2}{n\pi}}\frac{1}{g} + \cdots. \tag{178}$$

Note that the fluctuations now appear to diverge as $g \longrightarrow 0$ (see also Fig. 7). Furthermore, χ_A is nonanalytic in the original Newton's coupling $G = g^2$, which suggests that perturbation theory in G is useless. A divergence of the fluctuations as $g \longrightarrow 0$ implies that in this limit, the correlation length diverges in lattice units, signaling the emergence of a massless excitation.

Just as for the case of the average curvature [Eq. (169)], an exact sum rule can be derived in the (Euclidean) lattice path integral formulation, relating the local volume fluctuations to the local curvature fluctuations. In the three-dimensional Euclidean path integral theory, the following exact identity holds for the fluctuations [30]

$$\begin{aligned}4\lambda_0^2\bigg[\Big\langle\Big(\sum_h V_h\Big)^2\Big\rangle - \Big\langle\sum_h V_h\Big\rangle^2\bigg] - k^2\bigg[\Big\langle\Big(\sum_h \delta_h l_h\Big)^2\Big\rangle \\ - \Big\langle\sum_h \delta_h l_h\Big\rangle^2\bigg] - 2N_1 = 0,\end{aligned} \tag{179}$$

where N_1 is the number of edges on the lattice (further exact sum rules can be derived by considering even higher

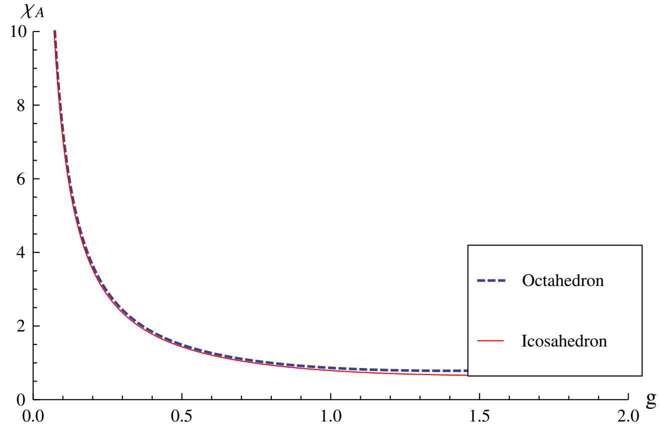

FIG. 7 (color online). Area fluctuation χ_A vs $g = \sqrt{G}$ for the octahedron and icosahedron, computed from Eq. (172). Note the divergence for small g.

derivatives of the free energy $\ln Z_L$ with respect to the parameters λ_0 and k). Since the last equation relates the fluctuation in the curvature to fluctuations in the volumes, it also implies a relationship between their singular (divergent) parts.[4]

According to the sum rule of Eq. (179), a divergence in the curvature fluctuation

$$\chi_R \sim \Big\langle\Big(\sum_h \delta_h l_h\Big)^2\Big\rangle - \Big\langle\sum_h \delta_h l_h\Big\rangle^2 \tag{180}$$

for the three-dimensional (Euclidean) theory generally implies a corresponding divergence in the volume fluctuation

$$\chi_V \sim \Big\langle\Big(\sum_h V_h\Big)^2\Big\rangle - \Big\langle\sum_h V_h\Big\rangle^2 \tag{181}$$

for the same theory. In our case a divergence is expected in 2 + 1 dimensions of the form

$$\chi_A \sim_{g\to g_c} |g - g_c|^{-\alpha} \tag{182}$$

with exponent $\alpha \equiv 1 - \delta = 2 - 3\nu$, where δ is the universal curvature exponent defined previously in Eq. (170) and ν the correlation length exponent. The latter is defined in the usual way [31,32] through

$$\xi \sim_{g\to g_c} |g - g_c|^{-\nu}, \tag{183}$$

where ξ is the invariant gravitational correlation length. The scaling relations among various exponents (ν, δ, α) are

[4]We noted previously that in our Hamiltonian formulation, the lattice continuum limit along the time direction has already been taken. This results in two lattice spacings, one for the time and one for the space directions, denoted here respectively by a_t and a, with the first lattice spacing already sent to zero. It is then relatively straightforward to relate volumes between the two formulations, such as $V \simeq a_t A$. Relating curvatures (for example, $2R$ in the 2 + 1 theory vs the Ricci scalar R in the original three-dimensional theory) in the two formulations is obviously less easy, due to the presence of derivatives along the time direction.

rather immediate consequences of the scaling assumption for the singular part of the free energy, $F_{\rm sing} \sim \xi^{-d}$ in the vicinity of a critical point (for more detailed discussion see, for example, Refs. [19,31,32]). The preceding argument then implies, via scaling, that a determination of α provides a direct estimate for the correlation length exponent ν defined in Eq. (183). Note that based on the results so far, one would be inclined to conclude that for 2 + 1 gravity the critical point $g_c \to 0$ as $n \to \infty$. Equation (182) can then be rewritten either as

$$\chi_A \sim_{g \to g_c} \xi^{\alpha/\nu} \tag{184}$$

or, in a finite volume with linear lattice dimensions $L \sim N_0^{1/d} \sim \sqrt{N_\Delta} \sim \sqrt{n}$ (since $N_\Delta = 4n + 2$), as

$$\chi_A \sim_{g \to g_c} L^{\alpha/\nu} \sim n^{1/\nu - 3/2}, \tag{185}$$

since, for a very large box and g very close to the critical point g_c, the correlation length saturates to its maximum value $\xi \sim L$. Hence, the volume- or n-dependence of χ provides a clear and direct way to estimate the critical correlation length exponent ν defined in Eq. (183).

IX. RESULTS FOR ARBITRARY EULER CHARACTERISTIC χ

The results of the previous sections refer to regular triangulations of the sphere ($\chi = 2$) and the torus ($\chi = 0$) in 2 + 1 dimensions. It would seem that one has enough information at this point to reconstruct the same type of answers for arbitrary χ. In particular, one has for the parameter β [see Eqs. (48) and (51)]

$$\beta = \frac{2\pi\chi}{g^3}, \tag{186}$$

relevant for the wave functions in Eqs. (135) or (147). For the average total area, one then finds, using the wave function expansion in Eq. (161),

$$\langle A_{\rm tot}\rangle = g\left\{\frac{2^{1-2n}\Gamma(n-\frac{1}{2})\Gamma(2n+\frac{1}{2})}{\Gamma(n)^3} + \frac{8(n+1)\pi\chi[1 - \frac{4^{1-2n}\Gamma(n-\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(2n+\frac{1}{2})^2}{n^2\Gamma(n)^6}]}{g^3(2n+1)} + \cdots\right\}. \tag{187}$$

In the large n limit, one obtains for the average area of a single triangle

$$\langle A_\Delta\rangle = \frac{g}{2\sqrt{2\pi n}}\left[1 - \frac{5}{16n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right] + \frac{(\pi-2)\chi}{g^2 n}\left[1 + \frac{1}{4n(\pi-2)} + \cdots\right] \tag{188}$$

and for the average total area

$$\langle A_{\rm tot}\rangle \sim \sqrt{\frac{2n}{\pi}}g + \frac{4(\pi-2)\chi}{g^2} + \cdots. \tag{189}$$

For the area fluctuation defined in Eq. (173), one finds in the same large n limit

$$\chi_A = \left(1 - \frac{2}{\pi}\right)\frac{g^2}{4} + \mathcal{O}\left(\frac{1}{n}\right) + (4-\pi)\sqrt{\frac{2}{n\pi}}\frac{\chi}{g} + \cdots. \tag{190}$$

Again, note that the fluctuation appears to diverge as $g \to 0$, which implies that this is the more interesting limit, so from now on we will focus specifically on this limit. It is clear from the analytic expression for $\langle A_{\rm tot}\rangle$ in Eqs. (187) or (188) that as $n \to \infty$, the gravitational coupling $g(n)$, to this order in the Bessel expansion, has to scale like

$$g(n) \sim \frac{1}{\sqrt{n}}, \tag{191}$$

so that the expression for $\langle A_{\rm tot}\rangle$ scales like n or N_Δ, with the expression for $\langle A_\Delta\rangle$ staying finite.

The result of Eq. (190) for χ_A then implies

$$\chi_A \sim \frac{1}{g\sqrt{n}} \sim n^0 \tag{192}$$

in the same limit $n \to \infty$. In view of Eqs. (187) and (185) with $n \sim N_\Delta \sim L^2$, this would imply $2/\nu - 3 = 0$, and thus for the universal critical exponent ν, itself, $\nu = \frac{2}{3} = 0.666$ to first order ($m = 1$) in the Bessel function expansion of Eq. (161) and $\nu = \frac{17}{10} = 0.588$ to the next order ($m = 2$) in the same expansion.

With some additional work one can, in fact, completely determine the asymptotic behavior of various averages for large β (small g) and large n. First, one notes that when m Bessel functions are included in the expansion for the wave function given in Eq. (161), beyond the leading order one at strong coupling, one obtains a wave function which contains powers of β up to β^m. For a given fixed m, one then finds for the average area per triangle the following asymptotic result

$$\langle A_\Delta\rangle \sim \frac{1}{g^{3m-1}n^{\frac{m+1}{2}}}, \tag{193}$$

up to terms which contain higher powers of $1/n$ (making these less relevant in the limit $n \to \infty$) and also up to terms which are less singular in g for small g. The requirement that the average area per triangle be finite as $n \to \infty$, then requires that the coupling g, itself, should scale with n according to

$$g(n) \sim \frac{1}{n^{\frac{m+1}{2(3m-1)}}}. \tag{194}$$

For the area fluctuation, itself, one then computes in the same limit

$$\chi_A \sim \frac{1}{g^{3m-2} n^{\frac{m}{2}}}, \tag{195}$$

again to leading order in $1/n$ and $1/g$. The requirement that $g(n)$ scale according to Eq. (194), then implies from Eq. (195) that the area fluctuation diverges in the limit $n \to \infty$ as

$$\chi_A(n) \sim n^{\frac{m-1}{3m-1}}. \tag{196}$$

By comparing with Eqs. (184) and (185), one obtains immediately for the exponent

$$\frac{\alpha}{\nu} = \frac{2m-2}{3m-1}, \tag{197}$$

and, therefore, from the scaling relation $\alpha = 2 - 3\nu$, finally,

$$\nu = \frac{6m-2}{11m-5}. \tag{198}$$

One can now take the limit $m \to \infty$ [infinite number of Bessel functions retained in the expansion of Eq. (161)], which leads to the exact result for the correlation length exponent of 2 + 1 dimensional quantum gravity

$$\nu = \frac{6}{11} = 0.5454\ldots. \tag{199}$$

The derivation shows that the exponent ν does not seem to depend on the Euler characteristic χ or, therefore, on the boundary conditions.[5] Furthermore, one can compare the above value for ν with the (numerically exact) Euclidean three-dimensional quantum gravity result obtained over twenty years ago in Ref. [24], namely $\nu \simeq 0.59(2)$. It would, of course, be of great interest to repeat the above Euclidean lattice calculation in order to refine the estimate and improve on the statistical and systematic uncertainty. The exponent ν is expected to represent a universal quantity, independent of short-distance regularization details and, therefore, characteristic of gravity's universal scaling properties on distances much larger than the lattice cutoff. As such, it should apply equally to both the Lorentzian and the Euclidean formulation, and our results are consistent with this conclusion. Moreover, in 3 + 1 dimensions the exponent ν is a key physical quantity as it determines the power for the running of the gravitational constant G [34] and, for the Euclidean theory, it is known [30] that the universal scaling exponent is consistent with $\nu = 1/3$.

It is perhaps worthwhile at this point to compare with other attempts at determining the critical exponent ν in three-dimensional gravity. The latest and best results for quantum gravity in the perturbative diagrammatic $2 + \epsilon$ continuum expansion using the background field method [35,36] give in $d = 3$ ($\epsilon = 1$ and central charge $c = 1$)

$$\nu^{-1} = 1 + \frac{3}{5} + \ldots \tag{200}$$

to two-loop order and, therefore, $\nu \approx 0.625$, with a substantial uncertainty of about fifty percent (which can be estimated for example by comparing the one- and two-loop results). On the other hand, truncated renormalization group calculations for gravity directly in three dimensions [37,38] give to lowest order in the truncation (i.e., with the inclusion of the cosmological and Einstein-Hilbert terms only), the estimate

$$\nu^{-1} = \frac{2d(d-2)}{d+2} \tag{201}$$

and, therefore, in $d = 3$ the value $\nu \approx 0.833$. This last result is also affected by a rather substantial uncertainty (again as much as fifty percent), which can be estimated, for example, by including curvature-squared terms in the truncated expansion. Nevertheless, and in light of the uncertainties associated with the various methods, it is very encouraging to note that widely different calculations (on the lattice and in the continuum) give values for the universal scaling exponent ν that are roughly in the same ballpark.

From Eq. (199) one obtains the fractal dimension for a gravitational path in 2 + 1 dimensions

$$\nu^{-1} = d_F = \frac{11}{6} = 1.8333\ldots. \tag{202}$$

This is slightly smaller than the value for a free scalar field $d_F = 2$, corresponding to the Brownian motion (or Wiener path) value. It is closer to the value expected for a dilute branched polymer in the same dimension [39,40], and the best match at this point seems to be the $O(n)$ vector model for $n = -1$. The exact value $\nu = 6/11$ for 2 + 1 gravity would then suggest a connection between the ground state properties of quantum gravity and the geometry of dilute branched polymers in the same dimension.

In light of the results obtained so far, it is possible to make a number of additional observations. First, note from Eq. (194) that as $n \to \infty$, the critical point (or renormalization group ultraviolet fixed point) moves to $g = 0$

$$g(n) \sim_{m\to\infty} \frac{1}{n^{1/6}}. \tag{203}$$

For comparison, a variational calculation based on correlated product (Slater-Jastrow) wave functions [1] in 2 + 1 dimensions gave

$$g_c^3 = \frac{4\pi\chi}{N_\Delta \sqrt{\sigma_0(\sigma_0 - 2)}}, \tag{204}$$

where $\sigma_0 > 2$ was a parameter associated there with the choice of functional measure over edges. The variational result of Eq. (204) can be compared directly with the result of Eqs. (165) and (203), for $\chi = 2$ and $N_\Delta = 2n + 2$. Thus, in both treatments the limiting value for the critical

[5] One might wonder if the value for ν is affected by the choice of normalization in Eqs. (56) and (155). It is easy to check that at least the inclusion of a weight factor A^m, with m integer, does not change the result given in Eq. (199).

point for g in $2+1$ dimensions is zero, $g_c \to 0$ as the number of triangles $N_\Delta \to \infty$.

Physically, this last result implies that there is no weak coupling phase ($g < g_c$, or in terms of Newton's constant $G < G_c$): the only surviving phase for gravity in three dimensions is the strongly coupled one ($g > g_c$ or $G > G_c$). Furthermore, the correlation length ξ of Eq. (183) is finite for $g > 0$ and diverges at $g = 0$. In particular, the weak field expansion, which assumes g small, is expected to have zero radius of convergence.[6] In a sense this is a welcome result, as in the Euclidean theory the weak coupling phase was found to be pathological and thus physically unacceptable in both three [24] and four dimensions [17,30]. It would seem, therefore, that the Euclidean and Lorentzian lattice results are ultimately completely consistent: quantum gravity in $2+1$ dimensions always resides in the strong coupling, gravitational antiscreening phase; the weak coupling, gravitational screening phase is physically excluded. In addition, the exact value for ν determines, through standard renormalization group arguments, the scale dependence of the gravitational coupling in the vicinity of the ultraviolet fixed point [34].[7]

X. SUMMARY AND CONCLUSIONS

In this paper we have discussed the form of the gravitational wave function that arises as a solution of the lattice Wheeler-DeWitt equation [Eqs. (20), (21), and (34)] for finite lattices. The main result was the wave function Ψ given in Eqs. (135), (147), and (153) with strong coupling limit (curvature term absent) corresponding to the choice of parameter $\beta = 0$.

To summarize, and for the purpose of the following discussion, the wave function Ψ given in Eq. (153) can be written in the most general form as

$$\Psi \sim e^{-ix} {}_1F_1(a, b, 2ix) \tag{205}$$

up to an overall normalization constant $\tilde{N}$, and with parameters related to various geometric invariants

$$a \equiv \frac{1}{4} N_\Delta - \frac{\sqrt{2}\pi i}{\sqrt{\lambda} G} \chi = \frac{1}{4} N_\Delta - \frac{i}{2\sqrt{2\lambda} G} \int d^2 y \sqrt{g} R$$

$$b \equiv \frac{1}{2} N_\Delta \qquad x \equiv \frac{\sqrt{2\lambda}}{G} A_{\mathrm{tot}} = \frac{\sqrt{2\lambda}}{G} \int d^2 y \sqrt{g}. \tag{206}$$

In the above definitions one can trade, if one so desires, the total number of triangles N_Δ for the total area

$$N_\Delta = \frac{1}{\langle A_\Delta \rangle} A_{\mathrm{tot}} = \frac{1}{\langle A_\Delta \rangle} \int d^2 y \sqrt{g}. \tag{207}$$

Use has been made of the relationship between various coupling constants (g, G, β, $\tilde{\lambda}$, λ) to reexpress the wave function ψ in slightly more general terms, as a function of the original couplings λ and G appearing in the original form of the Wheeler-DeWitt equation [see for example Eqs. (42) and (44)]. We did show that an equivalent form for the wave function Ψ can be given in terms of Coulomb wave functions [see Eq. (147)], with argument

$$\beta \equiv \frac{\sqrt{2}\pi \chi}{\sqrt{\lambda} G} = \frac{1}{2\sqrt{2\lambda} G} \int d^2 x \sqrt{g} R \tag{208}$$

and x defined as in Eq. (206).

The above wave function is exact in the limit of large areas and completely independent of the weak field expansion. Nevertheless, it is only correct to some low order in the same expansion in the limit of small areas. This situation was interpreted as follows. For large areas one has a very large number of triangles, and the short distance details of the lattice setup play a vanishingly small role in this limit. One recognizes this limit as being relevant for universal scaling properties, including critical exponents. For small areas, on the other hand, a certain sensitivity to the short distance properties of the lattice regularization persists, and thus a universal behavior is, not unexpectedly, hard to achieve. In any case this last limit, in the absence of a truly fundamental and explicit microscopic theory, is always expected to be affected by short distance details of the regularization, no matter what its ultimate nature might be (a lattice of some sort, dimensional regularization, or an invariant continuum momentum cutoff, etc.)

In principle, any well-defined diffeomorphism-invariant average can be computed using the above wave functions. This will involve, at some point, the evaluation of a vacuum expectation value of some operator $\tilde{O}(g)$

$$\langle \Psi | \tilde{O}(g) | \Psi \rangle = \frac{\int d\mu[g] \tilde{O}(g_{ij}) |\Psi[g_{ij}]|^2}{\int d\mu[g] |\Psi[g_{ij}]|^2}, \tag{209}$$

where $d\mu[g]$ is the appropriate functional measure over three-metrics g_{ij}. Evaluating such an average is, in general, nontrivial, as it requires the computation of a (Euclidean) lattice path integral in one dimension less

$$\langle \Psi | \tilde{O}(g) | \Psi \rangle = \mathcal{N} \int d\mu[g] \tilde{O}(g_{ij}) \exp\{-S_{\mathrm{eff}}[g]\} \tag{210}$$

[6]These circumstances are perhaps unfamiliar in the gravity context but are nevertheless rather similar to what happens in gauge theories, including compact quantum electrodynamics in $2+1$ dimensions [41]. There, the theory always resides in the strong coupling or disordered phase, with a finite correlation length which eventually diverges at zero charge.

[7]Specifically, the universal exponent ν is related to the behavior of the Callan-Symanzik beta function for Newton's constant G in the vicinity of the ultraviolet fixed point by $\beta'(G)|_{G=G_c} = -1/\nu$. Integration of the renormalization group equations for G then determines the scale dependence of $G(\mu)$ or $G(\Box)$ in the vicinity of the ultraviolet fixed point. Concretely, ν determines the exponent in the running of G. One finds $G(\Box) \sim (\xi^2 \Box)^{-1/2\nu}$, with $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ the covariant d'Alembertian and ξ the renormalization group invariant correlation length. A broader discussion of renormalization group methods as they apply to quantum gravity can be found, for example, in Ref. [19].

with $S_{\text{eff}}[g] \equiv -\ln|\Psi[g_{ij}]|^2$ and $\mathcal{N}$ a normalization constant. The operator $\tilde{\mathcal{O}}(g)$, itself, can be local, or nonlocal as in the case of the gravitational Wilson loop discussed in Ref. [42]. Note that the statistical weights have many zeros corresponding to the nodes of the wave function Ψ, and that S_{eff} is infinite there.

In the previous sections we have shown that the wave function allows one to calculate a number of useful and physically relevant averages and fluctuations, which were later extrapolated to the infinite volume limit of infinitely many triangles. It was found that these diffeomorphism-invariant observables point in $2+1$ dimensions to the existence of a fixed point (a phase transition in statistical field theory language) at the origin, $G_c = 0$. One concludes, therefore, that the weak coupling (or gravitational screening) phase has completely disappeared in the lattice nonperturbative formulation and that the theory resides in the strong coupling phase only. By contrast, in the Euclidean theory it was found in Ref. [24] that the weak coupling or gravitational screening phase exists but is pathological, corresponding to a degenerate branched polymer. A similar set of results is found in the four-dimensional Euclidean theory, where the weak coupling, gravitational screening phase also describes a branched polymer.[8]

The calculations presented in this paper and in Ref. [1] can be regarded, therefore, as consistent with the conclusions reached earlier from the Euclidean framework, and no new surprises arise when considering the Lorentzian $2+1$ theory. Furthermore, we have emphasized before that the results obtained point at a nonanalyticity in the coupling at $G = 0$, signaling a strong vacuum instability of quantum gravitation in this dimension. In view of these results, it is, therefore, not surprising that calculations that rely on the weak field, semiclassical or small G expansion run into serious trouble and uncontrollable divergences very early on. Such an expansion does not seem to exist if the nonperturbative lattice results presented here are taken seriously. The correct physical vacuum apparently cannot in any way be obtained as a small perturbation of flat or near-flat spacetime.

Let us add here a few final comments aimed at placing the present work in the context of previous calculations for the same theory. A number of attempts have been made over the years to obtain an estimate for the gravitational wave functional $\Psi[g]$ in the absence of sources. These generally have relied on the weak field expansion in the continuum, as originally done in Refs. [8,9]. Thus, for example, one finds in $3+1$ dimensions

$$\Psi[h^{TT}] = \mathcal{N} \exp\left\{-\frac{1}{4}\int d^3\mathbf{k}\, k h_{ik}^{TT}(\mathbf{k}) h_{ik}^{TT*}(\mathbf{k})\right\}, \quad (211)$$

[8]The nature of solutions to the lattice Wheeler-DeWitt equation in $3+1$ dimensions will be discussed in a separate publication [43].

where $h_{ik}^{TT}(\mathbf{k})$ is the Fourier amplitude of transverse-traceless modes for the linearized gravitational field in four dimensions. The above wave functional describes a collection of harmonic oscillator wave functions, one for each of the infinitely many physical modes of the linearized gravitational field. As in the case of the electromagnetic field, the ground state wave functional can be expressed equivalently in terms of first derivatives of the field potentials (the corresponding $\mathbf{B}$ field for gravity), without having to mention Fourier amplitudes, as

$$\Psi[h^{TT}] = \mathcal{N} \exp\left\{-\frac{1}{8\pi^2}\int d^3\mathbf{x}\int d^3\mathbf{y}\, \frac{h_{ik,l}^{TT}(\mathbf{x}) h_{ik,l}^{TT*}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\right\}. \quad (212)$$

Clearly both of the above expressions represent only the leading term in an expansion involving infinitely many terms in the metric fluctuation h_{ij} (and since they apply to an expansion about flat space, the cosmological constant term does not appear either). Now, in $2+1$ dimensions, the above expressions become meaningless, since there cannot be any transverse-traceless modes. The only expectation that remains true is that the wave functional should still depend on physical degrees of freedom only: it should be a function of the intrinsic geometry of 3-space and should not change under a general coordinate change.

If one restricts oneself to local terms, a number of invariants are possible in $2+1$ dimensions. In principle, the wave function could depend on, besides the total area

$$A_{\text{tot}} = \int d^2x\sqrt{g} \quad (213)$$

and curvature

$$4\pi\chi = \int d^2x\sqrt{g}R, \quad (214)$$

other invariants such as

$$r_n = \int d^2x\sqrt{g}R^n \quad (215)$$

with n an integer. The latter result follows from the fact that in $d = 2$, both the Riemann and Ricci tensors only have one component, related to the scalar curvature,

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}R(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}), \qquad R_{\mu\nu} = \frac{1}{2}Rg_{\mu\nu}. \quad (216)$$

Nonlocal terms are possible as well, involving inverse powers of the covariant d'Alembertian $\Box$, but these do not seem to play a significant role in the lattice theory.

Now, the relevant Euclidean theory for the present work is, of course, gravity in three $(2+1)$ dimensions. But in three dimensions the Riemann and Ricci tensor have the same number of algebraically independent components (6) and are related to each other by

$$R^{\mu\nu}{}_{\lambda\sigma} = \epsilon^{\mu\nu\kappa}\epsilon_{\lambda\sigma\rho}\left(R^{\rho}{}_{\kappa} - \frac{1}{2}\delta^{\rho}{}_{\kappa}\right). \tag{217}$$

The field equations then imply, using Eq. (217), that the Riemann tensor is completely determined by the matter distribution implicit in $T_{\mu\nu}$,

$$R_{\mu\nu\rho\sigma} = 8\pi G[g_{\mu\rho}T_{\nu\sigma} + g_{\nu\sigma}T_{\mu\rho} + g_{\mu\sigma}T_{\nu\rho} - g_{\nu\rho}T_{\mu\sigma} + T(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma})]. \tag{218}$$

In empty space $T_{\mu\nu} = 0$, which then implies for zero cosmological constant the vanishing of Riemann there,

$$R_{\mu\nu\rho\sigma} = 0. \tag{219}$$

As a result in three dimensions, classical spacetime is locally flat everywhere outside a source, gravitational fields do not propagate outside matter, and two bodies cannot experience any gravitational force: they move uniformly on straight lines. There cannot be any gravitational waves either: the Weyl tensor, which carries information about gravitational fields not determined locally by matter, vanishes identically in three dimensions.

What seems rather puzzling at first is that the Newtonian theory seems to make perfect sense in $d = 3$. This can only mean that the Newtonian theory is *not* recovered in the weak field limit of the relativistic theory. To see this explicitly, it is sufficient to consider the trace-reversed form of the field equations,

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{d-2}g_{\mu\nu}T\right) \tag{220}$$

with $T = T^{\lambda}{}_{\lambda}$, in the weak field limit. In the linearized theory, with $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, and in the gauge $\nabla_{\lambda}h^{\lambda}{}_{\mu} - \frac{1}{2}\nabla_{\mu}h^{\lambda}{}_{\lambda} = 0$, one obtains the wave equation

$$\Box h_{\mu\nu} = -16\pi G\left(\tau_{\mu\nu} - \frac{1}{d-2}\eta_{\mu\nu}\tau\right) \tag{221}$$

with $\tau_{\mu\nu}$ the linearized stress tensor. After neglecting the spatial components of $\tau_{\mu\nu}$ in comparison to the mass density τ_{00}, and assuming that the fields are quasistatic, one obtains a Poisson equation for h_{00},

$$\nabla^2 h_{00} = -16\pi G\frac{d-3}{d-2}\tau_{00}. \tag{222}$$

In four dimensions this is equivalent to Poisson's equation for the Newtonian theory when one identifies the metric with the Newtonian field ϕ in the usual way via $h_{00} = -2\phi$. But in three dimensions such a correspondence is obstructed by the fact that, from Eq. (222), the nonrelativistic Newtonian coupling appearing in Poisson's equation is given by

$$G_{\text{Newton}} = \frac{2(d-3)}{(d-2)}G \tag{223}$$

and the mass density τ_{00} completely decouples from the gravitational field h_{00}. As a result, the linearized theory in three dimensions fails to reproduce the Newtonian theory.

In a complementary way one can show that gravitational waves are not possible either in the linearized theory in three dimensions. Indeed, the counting of physical degrees of freedom for the d-dimensional theory goes as follows. The metric $g_{\mu\nu}$ has $\frac{1}{2}d(d+1)$ independent components; the Bianchi identity and the coordinate conditions reduce this number to $\frac{1}{2}d(d+1) - d - d = \frac{1}{2}d(d-3)$, which gives indeed the correct number of physical degrees of freedom (two) corresponding to a massless spin two particle in $d = 4$, and no physical degrees of freedom in $d = 3$ (and minus one degree of freedom in $d = 2$, which is in fact incorrect). Nevertheless, investigations of quantum two-dimensional gravity have uncovered the fact that there can be surviving degrees of freedom in the quantum theory, at least in two dimensions. The usual treatment of two-dimensional gravity [44] starts from the metric in the conformal gauge $g_{\mu\nu}(x) = e^{\phi}(x)\tilde{g}_{\mu\nu}$, where $\tilde{g}_{\mu\nu}$ is a reference metric, usually taken to be the flat one. The conformal gauge-fixing then implies a nontrivial Faddeev-Popov determinant, which, when exponentiated, results in an effective Liouville action, with a potential term coming from the cosmological constant contribution. One would, therefore, conclude from this example that gravity in the functional integral representation needs a careful treatment of the conformal degree of freedom, since, in general, its dynamics cannot be assumed to be trivial. The calculations presented in this paper show that this is, indeed, the case in 2 + 1 dimensions as well.

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