

Renormalization Group Running of Newton's G : The Static Isotropic Case

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ABSTRACT

Corrections are computed to the classical static isotropic solution of general relativity, arising from non-perturbative quantum gravity effects. A slow rise of the effective gravitational coupling with distance is shown to involve a genuinely non-perturbative scale, closely connected with the gravitational vacuum condensate, and thereby, it is argued, related to the observed effective cosmological constant. Several analogies between the proposed vacuum condensate picture of quantum gravitation, and non-perturbative aspects of vacuum condensation in strongly coupled non-abelian gauge theories are developed. In contrast to phenomenological approaches, the underlying functional integral formulation of the theory severely constrains possible scenarios for the renormalization group evolution of couplings. The expected running of Newton's constant G is compared to known vacuum polarization induced effects in QED and QCD. The general analysis is then extended to a set of covariant non-local effective field equations, intended to incorporate the full scale dependence of G , and examined in the case of the static isotropic metric. The existence of vacuum solutions to the effective field equations in general severely restricts the possible values of the scaling exponent ν .

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1 Introduction

Over the last few years evidence has mounted to suggest that quantum gravitation, even though plagued by meaningless infinities in standard weak coupling perturbation theory, might actually make sense, and lead to a consistent theory at the non-perturbative level. As is often the case in physics, the best evidence does not come from often incomplete and partial results in a single model, but more appropriately from the level of consistency that various, often quite unrelated, field theoretic approaches provide. While it would certainly seem desirable to obtain a closed form analytical solution for the euclidean path integral of quantum gravity, experience with other field theories suggests that this goal might remain unrealistic in the foreseeable future, and that one might have to rely in the interim on partial results and reasoned analogies to obtain a partially consistent picture of what the true nature of the ground state of non-perturbative gravity might be.

One aspect of quantum gravitation that has stood out for some time is the rather strident contrast between the naive picture one gains from perturbation theory, namely the possibility of an infinite set of counterterms, uncontrollable divergences in the vacuum energy of just about any field including the graviton itself, and typical curvature scales comparable to the Planck mass [1-3], and, on the other hand, the new insights gained from non-perturbative approaches, which avoid reliance on an expansion in a small parameter (which does not exist in the case of gravity) and which would suggest instead a surprisingly rich phase structure, non-trivial ultraviolet fixed points [4-8] and genuinely non-perturbative effects such as the appearance of a gravitational condensate. The existence of non-perturbative vacuum condensates does not necessarily invalidate the wide range of semi-classical results [9-11] obtained in gravity so far, but re-interprets the gravitational background fields as suitable quantum averages, and further adds to the effective gravitational Lagrangian the effects of the (finite) scale dependence of the gravitational coupling, in a spirit similar to the Euler-Heisenberg corrections to electromagnetism.

Perhaps the goals that are sometimes set for quantum gravity and related extensions, that is, to explain and derive, from first principles, the values of Newton's constant and the cosmological constant, are placed unrealistically high. After all, in other well understood quantum field theories like QED and QCD the renormalized parameters (α , α_S , ...) are fixed by experiments, and no really compelling reason exists yet as to why they should take on the actual values observed in laboratory experiments. More specifically in the case of gravity, Feynman has given elaborate

arguments as to why quantities such as Newton's constant (and therefore the Planck length) might have cosmological origin, and therefore unrelated to any known particle physics phenomenon [1].

In this paper we will examine a number of issues connected with the renormalization group running of gravitational couplings. We will refrain from considering more general frameworks (higher derivative couplings, matter fields etc.), and will focus instead on basic aspects of the pure gravity theory by itself. Our presentation is heavily influenced by the numerical and analytical results from the lattice theory of quantum gravity (LQG), which have, in our opinion, helped elucidate numerous details of the non-perturbative phase structure of quantum gravity, and allowed a first determination of the scaling dimensions directly in $d = 4$. The lattice provides a well defined ultraviolet regulator, reduces the continuum functional integral to a finite set of convergent integrals, and allows statistical field theory methods, including numerical ones, to be used to explore the nature of ground state averages and correlations.

The scope of this paper is therefore to explore the overall consistency of the picture obtained from the lattice, by considering a number of core issues, one of which is the analogy with a much better understood class of theories, non-abelian gauge theories and QCD (Sec. 2). We will argue that, once one takes for granted a set of basic lattice results, it is possible to discuss a number of general features without having to explicitly resort to specific aspects of the lattice cutoff or the lattice action. For example, it is often sufficient to assume that a cutoff Λ is operative at very short distances, without having to involve in the discussion specific aspects of its implementation. In fact the use of continuum language, in spite of its occasional ambiguities when it comes to the proper, regulated definition of quantum entities, provides a more transparent language for presenting and discussing basic results.

The second aspect we wish to investigate in this paper is the nature of the rather specific predictions about the running of Newton's constant G . A natural starting point is the solution of the non-relativistic Poisson equation (Sec. 3), whose solutions for a point source can be investigated for various values of the exponent ν . We will then show that a scale dependence of G can be consistently embedded in a relativistic covariant framework, whose consequences can then be worked out in detail for specific choices of metrics (Sec. 4). For the static isotropic metric, we then derive the leading quantum correction and show that, unexpectedly, it seems to restrict the possible values for the exponent ν , in the sense that in some instances no consistent solution to the effective non-local field equations can be found unless ν^{-1} is an integer.

To check the overall consistency of the results, a slightly different approach to the solution of the static isotropic metric is discussed in Sec. 5, in terms of an effective vacuum density and pressure.

Again it appears that unless the exponent ν is close to $1/3$, a consistent solution cannot be obtained. At the end of the paper we add some general comments on two subjects we discussed previously. We first make the rather simple observation that a running of Newton's constant will slightly distort the gravitational wave spectrum at very long wavelengths (Sec. 6). We then return to the problem (Sec. 7) of finding solutions of the effective non-local field equations in a cosmological context [12], wherein quantum corrections to the Robertson-Walker metric and the basic Friedman equations are worked out, and discuss some of the simplest and more plausible scenarios for the growth (or lack thereof) of the coupling at very large distances, past the deSitter horizon. Sec. 8 contains our conclusions.

2 Vacuum Condensate Picture of Quantum Gravitation

The lattice theory of quantum gravity provides a well defined and regularized framework in which non-perturbative quantum aspects can be systematically investigated.

Let us recall here some of the main results of the lattice quantum gravity (LQG) approach, and their relationship to related approaches.

(i) The theory is formulated via a discretized Feynman functional integral [13-26]. Convergence of the euclidean lattice path integral *requires* in dimensions $d > 2$ a positive bare cosmological constant $\lambda_0 > 0$ [20]. The need for a bare cosmological constant is in line with renormalization group results in the continuum, which also imply that radiative corrections will inevitably generate a non-vanishing λ term.

(ii) The lattice theory in four dimensions is characterized by *two phases*, one of which appears for G less than some critical value G_c , and can be shown to be physically unacceptable as it describes a collapsed manifold with dimension $d \simeq 2$. The quantum gravity phase for which $G > G_c$ can be shown instead to describe smooth four-dimensional manifolds at large distances, and remains therefore physically viable. The continuum limit is taken in the standard way, by having the bare coupling G approach G_c . The two phase structure persists in three dimensions [24], and even at $d = \infty$ [15], whereas in two dimensions one has only one phase [23].

(iii) The presence of two phases in the lattice theory is consistent with the continuum $2 + \epsilon$ expansion result, which also predicts the existence of two phases above dimensions $d = 2$. The presence of a nontrivial ultraviolet fixed point in the continuum above $d = 2$, with nontrivial scaling dimensions, relates to the existence of a phase transition in the lattice theory. The lattice

results further suggest that the weakly coupled phase is in fact non-perturbatively *unstable*, with the manifold collapsing into a two-dimensional degenerate geometry. The latter phase, if it had existed, would have described gravitational screening.

(iv) One key quantity, the critical exponent ν , characterizing the non-analyticity in the vacuum condensates at G_c , is naturally related to the derivative of the beta function at G_c in the $2 + \epsilon$ expansion. The value $\nu \simeq 1/3$ in four dimensions, found by numerical evaluation of the lattice path integral, is close but somewhat smaller than the lowest order ϵ expansion result $\nu = 1/(d - 2)$. An analysis of the strongly coupled phase of the lattice theory further gives $\nu = 0$ at $d = \infty$ [15].

(v) The genuinely non-perturbative scale ξ , specific to the strongly coupled phase of gravity for which $G > G_c$, can be shown to be related to the vacuum expectation value of the curvature via $\langle \mathcal{R} \rangle \sim 1/\xi^2$, and is therefore presumably macroscopic [34]. It is naturally identified with the physical (scaled) cosmological constant λ ; ξ therefore appears to play a role analogous to the non-perturbative scaling violation parameter $\Lambda_{\overline{MS}}$ of QCD.

(vi) The existence of a non-trivial ultraviolet fixed point (a phase transition in statistical mechanics language) implies a scale dependence for Newton's constant in the physical, strongly coupled phase $G > G_c$. To leading order in the vicinity of the fixed point the scale dependence is determined by the exponent ν , and the overall size of the corrections is set by the condensate scale ξ . Thus in the strongly coupled phase, gravitational vacuum polarization effects should cause the physical Newton's constant to grow slowly with distances.

2.1 Non-Trivial Fixed Point and Scale Dependence of $G(\mu^2)$

This section will establish basic notation and provide some key results and formulas, some of which will be discussed further in the following sections. For more details the reader is referred to the more recent papers [13, 15, 12].

For the running gravitational coupling one has in the vicinity of the ultraviolet fixed point

$$G(k^2) = G_c \left[1 + a_0 \left(\frac{m^2}{k^2} \right)^{\frac{1}{2\nu}} + O((m^2/k^2)^{\frac{1}{\nu}}) \right] \quad (2.1)$$

with $m = 1/\xi$, $a_0 > 0$ and $\nu \simeq 1/3$ [13]. We have argued that the quantity G_c in the above expression should in fact be identified with the laboratory scale value, $\sqrt{G_c} \sim \sqrt{G_{phys}} \sim 1.6 \times 10^{-33} cm$, the reason being that the scale ξ can be very large. Indeed in [34, 14, 12] it was argued that ξ should be of the same order as the scaled cosmological constant λ . Quantum corrections on the r.h.s are therefore quite small as long as $k^2 \gg m^2$, which in real space corresponds to the "short distance" regime $r \ll \xi$.

The above expression diverges as $k^2 \rightarrow 0$, and the infrared divergence needs to be regulated. A natural infrared regulator exists in the form of $m = 1/\xi$, and therefore a properly infrared regulated version of the above expression is

$$G(k^2) \simeq G_c \left[1 + a_0 \left(\frac{m^2}{k^2 + m^2} \right)^{\frac{1}{2\nu}} + \dots \right] \quad (2.2)$$

with $m = 1/\xi$ the (tiny) infrared cutoff. Then in the limit of large k^2 (small distances) the correction to $G(k^2)$ reduces to the expression in Eq. (2.1), namely

$$G(k^2) \underset{k^2/m^2 \rightarrow \infty}{\sim} G_c \left[1 + a_0 \left(\frac{m^2}{k^2} \right)^{\frac{1}{2\nu}} \left(1 - \frac{1}{2\nu} \frac{m^2}{k^2} + \dots \right) + \dots \right] \quad (2.3)$$

whereas its limiting behavior for small k^2 (large distances) is now given by

$$G(k^2) \underset{k^2/m^2 \rightarrow 0}{\sim} G_\infty \left[1 - \left(\frac{a_0}{2\nu(1+a_0)} + \dots \right) \frac{k^2}{m^2} + O(k^4/m^4) \right] \quad (2.4)$$

implying that the gravitational coupling approaches the finite value $G_\infty = (1 + a_0 + \dots) G_c$, independent of $m = 1/\xi$, at very large distances $r \gg \xi$. At the other end, for large k^2 (small distances) one has, from either Eqs. (2.2) or (2.1),

$$G(k^2) \underset{k^2/m^2 \rightarrow \infty}{\sim} G_c \quad (2.5)$$

meaning that the gravitational coupling approaches the ultraviolet (UV) fixed point value G_c at “short distances” $r \ll \xi$. Since the theory is formulated with an explicit ultraviolet cutoff, the latter must appear somewhere, and indeed $G_c = \Lambda^{-2} \tilde{G}_c$, with the UV cutoff of the order of the Planck length $\Lambda^{-1} \sim 1.6 \times 10^{-33} \text{cm}$, and \tilde{G}_c a dimensionless number of order one. Note though that in Eqs. (2.1) or (2.2) the cutoff does not appear explicitly, it is “absorbed” into the definition of G_c .

The non-relativistic, static Newtonian potential is defined as

$$\phi(r) = (-M) \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} G(\mathbf{k}^2) \frac{4\pi}{\mathbf{k}^2} \quad (2.6)$$

and therefore proportional to the 3 – d Fourier transform of

$$\frac{4\pi}{\mathbf{k}^2} \rightarrow \frac{4\pi}{\mathbf{k}^2} \left[1 + a_0 \left(\frac{m^2}{\mathbf{k}^2} \right)^{\frac{1}{2\nu}} + \dots \right] \quad (2.7)$$

But, as we mentioned before, proper care has to be exercised in providing a properly infrared regulated version of the above expression, which, from Eq. (2.2), reads

$$\frac{4\pi}{(\mathbf{k}^2 + \mu^2)} \rightarrow \frac{4\pi}{(\mathbf{k}^2 + \mu^2)} \left[1 + a_0 \left(\frac{m^2}{\mathbf{k}^2 + m^2} \right)^{\frac{1}{2\nu}} + \dots \right] \quad (2.8)$$

where the limit $\mu \rightarrow 0$ should be taken at the end of the calculation. We wish to emphasize here that the regulators $\mu \rightarrow 0$ and m are quite distinct. The distinction originates in the condition that m arises due to strong infrared effects and renormalization group properties in the quantum regime, while μ has nothing to do with quantum effects: it is required to make the Fourier transform of the classical, Newtonian $4\pi/\mathbf{k}^2$ well defined. This is an important issue to keep in mind, and to which we will return later.

2.2 Renormalization group properties of $G(\mu^2)$

This section will discuss the relationship between the running of the coupling G , the renormalization group beta function $\beta(G)$, the lattice coupling $G(\Lambda)$ (the bare coupling, or equivalently, the running coupling at the scale of the cutoff Λ), and the parameter $m = 1/\xi$. Differentiation of Eq. (2.1) with respect to $k \rightarrow \Lambda$ gives

$$\Lambda \frac{\partial G(\Lambda)}{\partial \Lambda} \equiv \beta(G) = -\frac{1}{\nu}(G - G_c) + \dots \quad (2.9)$$

Here and in the following, unless stated otherwise, G will refer to the dimensionless gravitational coupling, i.e. $G_{phys} = \Lambda^{2-d} G(\Lambda)$. In four dimensions, on laboratory scales, $\sqrt{G_{phys}} \simeq 10^{-33} \text{cm}$. Therefore the exponent $\frac{1}{\nu} \equiv -\beta'(G_c)$ is related to the derivative of the beta function for G evaluated at the fixed point. Here the dots account for higher order terms not included in either Eq. (2.1) or Eq. (2.2).

The above scaling form for $G(k^2)$ and the non-perturbative exponent ν are determined as follows [34, 13]. Scaling around the fixed point originates in the divergence of the correlation length $\xi = 1/m$, related to the bare (lattice) couplings by

$$m \underset{G(\Lambda) \rightarrow G_c}{\sim} \Lambda \left[\frac{G(\Lambda) - G_c}{a_0 G_c} \right]^\nu \quad (2.10)$$

where Λ is the ultraviolet cutoff (the inverse lattice spacing). The continuum limit is approached in the standard way by having $G \rightarrow G_c$ and Λ very large, with m kept fixed. This last equation is recognized as being just Eq. (2.1) here with the scale $k^2 \rightarrow \Lambda^2$, and solved for the renormalization group invariant m (the inverse of the physical correlation length ξ) in terms of the bare (lattice) coupling $G(\Lambda)$, at the ultraviolet cutoff scale Λ . Scaling arguments in the vicinity of the non-trivial ultraviolet fixed point then allow one to determine the scaling behavior of correlation functions from the critical exponents characterizing the singular behavior of local averages. Since the physical quantity $m = 1/\xi$ is kept fixed and is not supposed to depend on the ultraviolet cutoff Λ , which is sent to infinity, one requires that $G(\Lambda)$ change in accordance to

$$\Lambda \frac{d}{d\Lambda} m(\Lambda, G(\Lambda)) = 0 \quad (2.11)$$

It is useful to introduce the dimensionless function $F(G)$ via

$$m \equiv \xi^{-1} = \Lambda F(G) \quad (2.12)$$

By differentiation of the renormalization-group invariant quantity m , one then obtains an expression for the Callan-Symanzik beta function $\beta(G)$ in terms of F . From its definition

$$\Lambda \frac{\partial}{\partial \Lambda} G(\Lambda) = \beta(G(\Lambda)) \quad , \quad (2.13)$$

one has

$$\beta(G) = - \frac{F(G)}{\partial F(G)/\partial G} \quad . \quad (2.14)$$

One concludes that the knowledge of the dependence of m on G , encoded in the function $F(G)$, implies a specific form for the β function. In terms of the function $\beta(G)$ the result of Eq. (2.10) is then equivalent to

$$\beta(G) \underset{G \rightarrow G_c}{\sim} -\frac{1}{\nu} (G - G_c) + O((G - G_c)^2) \quad (2.15)$$

with $\beta'(G_c) = -1/\nu$. In general one therefore expects the scaling behavior in the vicinity of the fixed point

$$m \sim \Lambda \exp\left(-\int^G \frac{dG'}{\beta(G')}\right) \underset{G \rightarrow G_c}{\sim} \Lambda |G - G_c|^{-1/\beta'(G_c)} \quad . \quad (2.16)$$

where here \sim indicates up to a constant of proportionality. The main conclusion is that the function $F(G)$ determines the beta function $\beta(G)$, which in turn determines the scale evolution of the coupling (obtained from Eq. (2.13), for any μ ,

$$\mu \frac{\partial}{\partial \mu} G(\mu) = \beta(G(\mu)) \quad , \quad (2.17)$$

The latter can in principle be integrated in the vicinity of the fixed point, and leads to a definite relationship between the relevant coupling G , the renormalization-group invariant (cutoff independent) quantity $m = 1/\xi$, and the arbitrary sliding scale $\mu^2 = k^2$, as outlined in the preceding section.

Let us add a comment on the so-called corrections to scaling in the vicinity of the fixed point at G_c . We note that whereas Eq. (2.2) follows from the result Eq. (2.10) and Eq. (2.9), the infrared regulated running coupling of G in Eq. (2.2) is equivalent to assuming the following correction to scaling to Eq. (2.10),

$$m \underset{G(\Lambda) \rightarrow G_c}{\sim} \frac{\Lambda \left(\frac{G(\Lambda) - G_c}{a_0 G_c}\right)^\nu}{\sqrt{1 - \left(\frac{G(\Lambda) - G_c}{a_0 G_c}\right)^{2\nu}}} \simeq \Lambda \left(\frac{G(\Lambda) - G_c}{a_0 G_c}\right)^\nu \left[1 + \frac{1}{2} \left(\frac{G(\Lambda) - G_c}{a_0 G_c}\right)^{2\nu} + \dots\right] \quad (2.18)$$

which gives by differentiation the previously quoted result, Eq. (2.1), plus a small corrections close to G_c

$$\beta(G) = -\frac{1}{\nu}(G - G_c) \frac{G}{G_c} \left[1 - \left(\frac{G - G_c}{a_0 G} \right)^{2\nu} \right] + \dots \quad (2.19)$$

where the dots account for higher order terms not included in Eq. (2.2), and, implicitly, in Eq. (2.1) as well.

2.3 Lattice gravity determination of the universal exponent ν

This section summarizes the connection between the lattice regularized quantum gravity path integral Z , the singular part of the corresponding free energy F (which determines the scaling behavior in the vicinity of the fixed point), and the universal critical exponent ν determining the scale dependence of the gravitational coupling in Eq. (2.2). For more details the reader is referred to [13], and references therein.

An important alternative to analytic analyses in the continuum is an attempt to solve quantum gravity directly via numerical simulations. The underlying idea is to perform the gravitational functional integral by discretizing the action on a space-time lattice, and then evaluate the partition function Z by summing over a suitable finite set of representative field configurations. In principle such a method, given enough configurations and a fine enough lattice, can provide an arbitrarily accurate solution to the original quantum gravity theory.

In practice there are several important factors to consider, which effectively limit the accuracy that can be achieved today in a practical calculation. Perhaps the most important one is the enormous amounts of computer time that such calculations can use up. This is particularly true when correlations of operators at fixed geodesic distance are evaluated. Another practical limitation is that one is mostly interested in the behavior of the theory in the vicinity of the critical point at G_c , where the correlation length ξ can be quite large and significant correlations develop both between different lattice regions, as well as among representative field configurations, an effect known as critical slowing down. Finally, there are processes which are not well suited to a lattice study, such as problems with several different length (or energy) scales. In spite of these limitations, the progress in lattice field theory has been phenomenal in the last few years, driven in part by enormous advances in computer technology, and in part by the development of new techniques relevant to the problems of lattice field theories.

In practice the exponent ν in Eqs. (2.10) or (2.18) (and therefore also in Eqs. (2.1) and (2.2), which follow from these) is determined from the singularities that arise in the free energy $F =$

$-\frac{1}{V} \ln Z$, with the euclidean path integral for pure quantum gravity Z defined as

$$Z = \int [d g_{\mu\nu}] e^{-\lambda_0 \int d^d x \sqrt{g} + \frac{1}{16\pi G} \int d^d x \sqrt{g} R} . \quad (2.20)$$

in the presence of a divergent correlation length $\xi \rightarrow \infty$ in the vicinity of the fixed point at G_c . This is the scaling hypothesis, the basis of many important results of statistical field theory [38].

On purely dimensional grounds, for the singular part of the free energy one has $F_{sing}(G) \sim \xi^{-d}$. Standard arguments then give, assuming a divergence $\xi \sim (G - G_c)^{-\nu}$ close to G_c , for the first derivative of F (here proportional to the average curvature $\mathcal{R}(G)$)

$$-\frac{1}{V} \frac{\partial}{\partial G} \ln Z \sim \frac{\langle \int dx \sqrt{g} R(x) \rangle}{\langle \int dx \sqrt{g} \rangle} \underset{G \rightarrow G_c}{\sim} A_{\mathcal{R}} (G - G_c)^{d\nu-1} . \quad (2.21)$$

An additive constant could appear as well, but the evidence up to now points to this constant being zero [13]). Similarly, for the second derivative of F , proportional to the fluctuation in the scalar curvature $\chi_{\mathcal{R}}(G)$, one has

$$-\frac{1}{V} \frac{\partial^2}{\partial G^2} \ln Z \sim \frac{\langle (\int dx \sqrt{g} R)^2 \rangle - \langle \int dx \sqrt{g} R \rangle^2}{\langle \int dx \sqrt{g} \rangle} \underset{G \rightarrow G_c}{\sim} A_{\chi_{\mathcal{R}}} (G - G_c)^{-(2-d\nu)} . \quad (2.22)$$

The above curvature fluctuation is related to the connected scalar curvature correlation at zero momentum,

$$\chi_{\mathcal{R}}(k) \sim \frac{\int dx \int dy \langle \sqrt{g} R(x) \sqrt{g} R(y) \rangle_c}{\langle \int dx \sqrt{g} \rangle} \quad (2.23)$$

and a divergence in the scalar curvature fluctuation is indicative of long range correlations, corresponding to scale invariance and the presence of massless modes.

From the computation of such averages one can determine by standard methods the numerical values for ν , G_c and a_0 to reasonably good accuracy [13]. It is often advantageous to express results in the cutoff (lattice) theory in terms of physical (i.e. cutoff independent) quantities. By the latter we mean quantities for which the cutoff dependence has been re-absorbed, or restored, in the relevant definition. Thus, for example, Eqs. (2.1) and (2.2) will include an overall factor of Λ^{-2} if they refer to the dimensionful, physical Newton's constant; the cutoff is still present, but is "hidden" in the definition of physical quantities, and cannot be set equal to infinity as the dimensionless fixed point value G_c is a finite number.

As an example, the result equivalent to Eq. (2.21), relating the vacuum expectation value of the local scalar curvature (computed therefore for infinitesimal loops) to the physical correlation length ξ , is

$$\frac{\langle \int dx \sqrt{g} R(x) \rangle}{\langle \int dx \sqrt{g} \rangle} \underset{G \rightarrow G_c}{\sim} \text{const.} \cdot (l_P^2)^{1-\frac{1}{2\nu}} \left(\frac{1}{\xi^2} \right)^{\frac{d}{2}-\frac{1}{2\nu}} , \quad (2.24)$$

and which is simply obtained from Eqs. (2.10) and (2.21). Matching of dimensionalities in this last equation has been restored by supplying appropriate powers of the Planck length $l_P = \sqrt{G_{phys}}$. For $\nu = 1/3$ the result of Eq. (2.24) becomes particularly simple [13, 14]

$$\frac{\langle \int dx \sqrt{g} R(x) \rangle}{\langle \int dx \sqrt{g} \rangle} \underset{G \rightarrow G_c}{\sim} \text{const.} \frac{1}{l_P \xi} \quad (2.25)$$

The naive estimate, based on a simple dimensional argument, would have suggested the (incorrect) result $\sim 1/l_P^2$. This shows that ν can also play the role of an anomalous dimension, giving the magnitude of the deviation from naive dimensional arguments. From the divergences of the free energy F one can determine the universal exponent ν appearing for example in Eqs. (2.1) and (2.2), but not the amplitude a_0 . The latter requires a direct determination of $m = 1/\xi$ in terms of bare lattice quantities (as in Eq. (2.10)), which we discuss next.

There are several correlation functions one can compute to extract ν and a_0 directly, either through the decay of euclidean invariant correlations at fixed geodesic distance [25], or, equivalently, from the correlations of Wilson lines associated with the propagation of heavy spinless particles [34]. In either case one expects the scaling result of Eq. (2.10) close to the fixed point, namely

$$m \simeq A_m \Lambda |k - k_c|^\nu, \quad \Lambda \rightarrow \infty, \quad k \rightarrow k_c, \quad m \text{ fixed}, \quad (2.26)$$

with the bare coupling $k(\Lambda) \equiv 1/(8\pi G(\Lambda))$, and A_m a calculable numerical constant. Detailed knowledge of $m(k)$ allows one to independently estimate the exponent ν , but the method is generally extremely time consuming (due to the appearance of geodesic distances in the correlation functions), and therefore so far not very accurate.

But, more importantly, from the knowledge of the dimensionless constant A_m one can estimate from first principles the value of a_0 in Eqs. (2.1) and (2.2). The first lattice results gave $A_m \simeq 0.56$ [25] and $A_m \simeq 0.87$ [34], with some significant uncertainty in both cases (perhaps by as much as an order of magnitude, due to the difficulties inherent in computing correlations at fixed geodesic distance), which then, combined with the more recent estimate $k_c \simeq 0.0636$ and $\nu \simeq 0.335$ in four dimensions [13], gives $a_0 = 1/(k_c A_m^{1/\nu}) \simeq 42$. The rather surprisingly large value for a_0 appears here perhaps as a consequence of the relatively small value of the lattice k_c in four dimensions. A new determination of a_0 with significantly reduced errors would clearly be desirable.

The direct numerical determinations of $k_c = 1/8\pi G_c$ in $d = 3$ and $d = 4$ space-time dimensions are in fact quite close to the analytical prediction of the lattice $1/d$ expansion [15],

$$k_c = \frac{\lambda_0^{\frac{d-2}{d}}}{d^3} \left[\frac{2}{d} \frac{d! 2^{d/2}}{\sqrt{d+1}} \right]^{2/d} \quad (2.27)$$

The latter gives for a bare $\lambda_0 = 1$ the estimate $k_c = \sqrt{3}/(16 \cdot 5^{1/4}) = 0.0724$ in $d = 4$, to be compared to the direct determination of $k_c = 0.0636(11)$ of [13], and $k_c = 2^{5/3}/27 = 0.118$ in $d = 3$, to be compared with the direct determination $k_c = 0.112(5)$ in [24] (these estimates will be compared later in Fig. 1).

2.4 How many independent bare couplings for pure gravity?

In this section it will be argued that the scaling behavior of pure quantum gravity is determined by *one* dimensionless combination of λ_0 and G only. We will then argue that the only sensible scenario, from a renormalization group point of view, is one in which the scaled cosmological constant λ is kept fixed, and only G is allowed to run, as in Eq. (2.2).

At first it might appear that in pure gravity one has two independent couplings (λ_0 and G), but in reality a simple scaling argument shows that there can only be one, which can be taken to be a suitable dimensionless ratio [20]. Consider the (euclidean) Einstein-Hilbert action with a cosmological term in d dimensions

$$I_E[g] = \lambda_0 \Lambda^d \int dx \sqrt{g} - \frac{1}{16\pi G_0} \Lambda^{d-2} \int dx \sqrt{g} R \quad (2.28)$$

Here λ_0 is the bare cosmological constant and G_0 the bare Newton's constant, both measured in units of the cutoff (we follow customary notation used in cutoff field theories, and denote by Λ the ultraviolet cutoff, not to be confused with the scaled cosmological constant)³. Convergence of the lattice regulated euclidean path integral requires $\lambda_0 > 0$ [20]. The natural expectation is for the bare microscopic, dimensionless couplings to have magnitudes of order one in units of the cutoff, $\lambda_0 \sim G_0 \sim O(1)$.

Next one rescales the metric so as to obtain a unit coefficient for the cosmological constant term,

$$g'_{\mu\nu} = \lambda_0^{2/d} g_{\mu\nu} \quad g'^{\mu\nu} = \lambda_0^{-2/d} g^{\mu\nu} \quad (2.29)$$

to obtain

$$I_E[g] = \Lambda^d \int dx \sqrt{g'} - \frac{1}{16\pi G_0 \lambda_0^{\frac{d-2}{d}}} \Lambda^{d-2} \int dx \sqrt{g'} R' \quad (2.30)$$

The (euclidean) Feynman path integral, defined as

$$Z = \int [dg_{\mu\nu}] e^{-I_E[g]} \quad (2.31)$$

³We also deviate in this paper from the convention used in our previous work. Due to ubiquitous ultraviolet cutoff Λ , we reserve here the symbol λ_0 for the cosmological constant, and λ for the *scaled* cosmological constant $\lambda \equiv 8\pi G \cdot \lambda_0$

includes as well a functional integration over all metrics, with functional measure given for example by [27, 28]

$$\int [d g_{\mu\nu}] = \int \prod_x [g(x)]^{\frac{(d-4)(d+1)}{8}} \prod_{\mu \geq \nu} d g_{\mu\nu}(x) \quad (2.32)$$

Therefore under a rescaling of the metric the functional measure only picks up a multiplicative constant. It does not drop out when computing vacuum expectation values, such as the one in Eq. (2.24), but cannot give rise to singularities at finite G such as the ones in Eqs. (2.21) and (2.22).

Equivalently, one can view a rescaling of the metric as simply a redefinition of the ultraviolet cutoff Λ , $\Lambda \rightarrow \lambda_0^{1/d} \Lambda$. As a consequence, the non-trivial part of the gravitational functional integral over metrics only depends on λ_0 and G_0 through the *dimensionless* combination [20]

$$\tilde{G} \equiv G_0 \lambda_0^{(d-2)/d} \quad (2.33)$$

The existence of an ultraviolet fixed point is then entirely controlled by this dimensionless parameter only, both on the lattice [20, 13, 14] and in the continuum [44]: the non-trivial part of the functional integral only depends on this specific combination. One has the Ward identity for the singular part of the generating function,

$$\left\{ G_0 \frac{\partial}{\partial G_0} - \frac{d}{d-2} \lambda_0 \frac{\partial}{\partial \lambda_0} \right\} F_{sing}(G, \lambda_0, \dots) = 0 \quad (2.34)$$

Thus the individual scaling dimensions of the cosmological constant and of the gravitational coupling constant do not have separate physical meaning; only the relative scaling dimension, as expressed through their dimensionless ratio, is physical.

Physically, the parameter λ_0 controls the overall scale of the problem (the volume of space-time), while the G_0 term provides the necessary derivative or coupling term. Since the total volume of space-time is normally not considered a physical observable, quantum averages are computed by dividing out by the total space-time volume. For example, for the quantum expectation value of the Ricci scalar one has the expression of Eq. (2.24).

Without any loss of generality one can therefore fix the overall scale in terms of the ultraviolet cutoff, and set the bare cosmological constant λ_0 equal to one in units of the ultraviolet cutoff. ⁴

The addition of matter field does not change the conclusions of the previous discussion, it is just that additional rescalings are needed. Thus for a scalar field with action

$$I_S[\phi] = \frac{1}{2} \int dx \sqrt{g} \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m_0^2 \phi^2 + R \phi^2 \right) \quad (2.35)$$

⁴These considerations are not dissimilar from the case of a self-interacting scalar field, where one might want to introduce three couplings for the kinetic term, the mass term and the quartic coupling term, respectively. A simple rescaling of the field would then reveal that only two coupling ratios are in fact physically relevant.

and functional measure (for a single field)

$$\int [d\phi] = \int \prod_x [g(x)]^{1/4} d\phi(x) \quad (2.36)$$

the metric rescaling is to be followed by a field rescaling

$$\phi'(x) = \phi(x) \lambda_0^{1/d-1/2} \quad (2.37)$$

with the only surviving change being a rescaling of the bare mass $m_0 \rightarrow m_0/\lambda_0^{1/d}$. Again the scalar functional measure acquires an irrelevant multiplicative factor which does not affect quantum averages.

The same results are obtained if one considers a lattice regularized version of the original (euclidean) path integral of Eq. (2.28), which reads [20]

$$Z_L = \int [dl^2] e^{-I_L[l^2]} \quad (2.38)$$

with lattice Regge action [16]

$$I_L = \lambda_0 \sum_h V_h(l^2) - 2\kappa_0 \sum_h \delta_h(l^2) A_h(l^2) \quad (2.39)$$

and regularized lattice functional measure [18, 21, 20]

$$\int [dl^2] \equiv \int_0^\infty \prod_{ij} dl_{ij}^2 \prod_s [V_d(s)]^\sigma \Theta(l_{ij}^2) \quad (2.40)$$

with $k_0 = 1/(16\pi G_0)$, and Θ a function incorporating the effects of the triangle inequalities. As is customary in lattice field theory, the lattice ultraviolet cutoff is set equal to one (i.e. all lengths and masses are measured in units of the cutoff). Convergence of the euclidean lattice functional integral requires a positive bare cosmological constant $\lambda_0 > 0$ [20, 22]. One can show again by a trivial rescaling of the edges that, as in the continuum, non-trivial part of the lattice regularized path integral only depends, in the absence of matter, on the single dimensionless parameter $\tilde{G} \equiv G_0 \lambda_0^{(d-2)/d}$. Without loss of generality therefore the bare coupling λ_0 can be set equal to one⁵.

The question that remains open is then the following: which coupling should be allowed to run within the renormalization group framework? Since the path integral in four dimensions only

⁵In lattice quantum gravity, the average edge length $l_0 = \langle l^2 \rangle^{1/2}$ is, for non-singular measures, largely a function of λ_0 . In the large d limit an explicit formula can be given for the measure of Eq. (2.40) with $\sigma = 0$,

$$l_0^2 = \frac{1}{\lambda_0^{2/d}} \left[\frac{2}{d} \frac{d! 2^{d/2}}{\sqrt{d+1}} \right]^{2/d} \quad (2.41)$$

which agrees well with numerical estimates for finite d [15].

depends on the dimensionless ratio $\tilde{G}^2 = G_0^2 \lambda_0$ (which is expected to be scale dependent), one has several choices; for example G runs and the cosmological constant λ_0 is fixed. Alternatively, G runs and the *scaled* cosmological constant $\lambda \equiv G\lambda_0$ is kept fixed; or G is fixed and λ runs etc.

At first thought, it would seem that the coupling λ_0 should not be allowed to run, as the overall space-time volume should perhaps be considered fixed, not to be rescaled under a renormalization group transformation. After all, in the spirit of Wilson [4], a renormalization group transformation provides a description of the original physical system in terms of a new coarse-grained Hamiltonian, whose new operators are interpreted as describing averages of the original system on the finer scale, but within the *same* physical volume. The new effective Hamiltonian is then supposed to still describe the original physical system, but more economically, in terms of a reduced set of degrees of freedom.

These considerations are to some extent implicit in the correct definition of gravitational averages, for example in Eq. (2.24). Physical, observable averages such as the one in Eq. (2.24) in general have some rather non-trivial dependence on the bare coupling G_0 , more so in the presence of an ultraviolet fixed point. Renormalization in the vicinity of the ultraviolet fixed point invariably leads to the introduction of a new dynamically generated, non-perturbative scale for $G > G_c$.

It appears though that the correct answer is that the combination $\lambda \equiv 8\pi G \cdot \lambda_0$, corresponding to the *scaled* cosmological constant, should be kept *fixed*, while Newton's constant is allowed to run in accordance to the scale dependence obtained from \tilde{G} . The reasons for this choice are three-fold. First, in the weak field expansion it is the combination $\lambda \equiv G\lambda_0$ that appears as a mass-like term (and not λ_0 or G separately). A similar conclusion is reached if one just compares the appearance of the field equations for gravity to say QED (massive via the Higgs mechanism), or a self-interacting scalar field,

$$\begin{aligned}
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} &= 8\pi G T_{\mu\nu} \\
\partial^\mu F_{\mu\nu} + \mu^2 A_\nu &= 4\pi e j_\nu \\
\partial^\mu \partial_\mu \phi + m^2 \phi &= \frac{g}{3!} \phi^3
\end{aligned} \tag{2.42}$$

Secondly, the scaled cosmological constant represents a measure of physical curvature, as should be clear from how the scaled cosmological constant relates for example to the expectation values of the scalar curvature at short distances (i.e. for infinitesimally small loops, whose size is comparable to the cutoff scale),

$$\frac{\langle \int dx \sqrt{g(x)} R(x) \rangle}{\langle \int dx \sqrt{g(x)} \rangle} \simeq R_{class} = 4\lambda \tag{2.43}$$

in the case of pure gravity. But perhaps the most convincing argument for the scaled cosmological constant $\lambda \equiv 8\pi G\lambda_0$ to be kept fixed is given in the following section.

2.5 The value of ξ dilemma - small or large ?

In this section we will argue that the scale ξ , which determines the running of G according to Eq. (2.2), should be identified with the observed scaled cosmological constant λ .

The lattice quantum gravity result of Eq. (2.24) (and Eq. (2.25) for $\nu = 1/3$) suggests a deep relationship between the correlation length $\xi = 1/m$ determining the size of scale dependent corrections of Eqs. (2.1) and the curvature. Small averaged curvatures correspond to very large length scales ξ . In gravity, curvature is detected by parallel transporting vectors around closed loops. This requires the calculation of a path dependent product of (Lorentz) rotations, \mathbf{R}^α_β , elements of $SO(4)$ in the euclidean case. On the lattice, the above rotation is directly related to the path-ordered (\mathcal{P}) exponential of the integral of the lattice affine connection $\Gamma^\lambda_{\mu\nu}$ via

$$\mathbf{R}^\alpha_\beta = \left[\mathcal{P} e^{\int_{\text{between simplices}}^{\text{path}} \Gamma^\lambda dx_\lambda} \right]^\alpha_\beta . \quad (2.44)$$

as discussed clearly for example in [30, 31], and more recently in [32]. Now, in the strongly coupled gravity regime ($G > G_c$) large fluctuations in the gravitational field at short distances will be reflected in large fluctuations of the \mathbf{R} matrices, which deep in the strong coupling regime should be reasonably well described by a uniform (Haar) measure [15, 29].

Borrowing still from the analogy with Yang-Mills theories, one might therefore worry that the effects of large strong coupling fluctuations in the \mathbf{R} matrices might lead to a phenomenon similar to confinement in non-Abelian lattice gauge theories [35]. That this is *not* the case can be seen from the fact that the gravitational analog of the Wilson loop $W(\Gamma)$, defined here as a path-ordered exponential of the affine connection $\Gamma^\lambda_{\mu\nu}$ around a closed planar loop,

$$W(\Gamma) \sim \text{Tr} \mathcal{P} \exp \left[\int_C \Gamma^\lambda \dots dx_\lambda \right] \quad (2.45)$$

does *not* give the static gravitational potential. The static gravitational potential is determined instead from the correlation of (exponentials of) geodesic line segments, as in

$$\exp \left[-M_0 \int d\tau \sqrt{g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \right] \quad (2.46)$$

where M_0 is the mass of the heavy source, as discussed already in some detail in [33, 34]. Indeed a direct lattice calculation of the potential between heavy sources via the correlation of geodesic lines showed no sign of confinement [34]. Borrowing from the well-established results in non-abelian

lattice gauge theories with compact groups (and to which no exceptions are known), it is easy to show that the expected decay of near-planar Wilson loops with area A is then given by

$$W(\Gamma) \sim \exp \left[\int_{S(C)} R_{\cdot\cdot\mu\nu} A_C^{\mu\nu} \right] \sim \exp(-A/\xi^2) \quad (2.47)$$

[36], where A is the minimal physical area spanned by the near-planar loop. The rapid decay of the Wilson loop as a function of the area is then seen simply as a general and direct consequence of the disorder in the fluctuations of the \mathbf{R} matrices at strong coupling. One concludes therefore that the Wilson loop in gravity provides a measure of the magnitude of the large-scale, averaged curvature, operationally determined by the process of parallel-transporting test vectors around very large loops, and which therefore, from the above expression, is computed to be of the order $R \sim 1/\xi^2$.

One important assumption in the above result is the identification of the correlation length ξ in Eq. (2.47) with the correlation length ξ in Eqs. (2.10) and (2.24). This is the scaling hypothesis, at the basis of most statistical field theory [37, 38, 39] : one assumes that *all* critical behavior in the vicinity of G_c is determined by one correlation length, which diverges (in lattice units) at the critical point at G_c in accordance with Eq. (2.10). As can be seen from Eq. (2.10), the scale ξ is genuinely non-perturbative, as in non-abelian gauge theories. To determine the actual physical value of ξ some physical input is needed, as the underlying theory *cannot* fix it : the ratio of the physical Newton's constant to ξ^2 can be as small as one desires, provided the bare coupling G is very close to its fixed point value G_c .

In conclusion, the above arguments and in particular the result of Eq. (2.47), suggest once more the identification of ξ with the large scale curvature, the most natural candidate being the (scaled) cosmological constant,

$$\lambda_{phys} \simeq \frac{1}{\xi^2} \quad \text{or} \quad \xi = \frac{1}{\sqrt{\lambda_{phys}}} \quad (2.48)$$

This relationship, taken at face value, implies a very large, cosmological value for $\xi \sim 10^{28} \text{cm}$, given the present bounds on λ_{phys} or H_0 . Other closely related possibilities may exist, such as an identification of ξ with the Hubble constant (as measured today) determining the macroscopic expansion rate of the universe via the correspondence

$$\xi \simeq 1/H_0 \quad , \quad (2.49)$$

Since this quantity is presumably time-dependent, a possible scenario would be one in which $\xi^{-1} = H_\infty = \lim_{t \rightarrow \infty} H(t) = \sqrt{\Omega_\lambda} H_0$ with $H_\infty^2 = \frac{8\pi G}{3} \lambda_0 = \frac{\lambda}{3}$, where λ_0 is the observed cosmological

constant, and for which the horizon radius is $R_\infty = H_\infty^{-1}$.

Should Newton's constant run with energy, and if so, according to what law? Newton's constant enters the field equations, after multiplication by G itself, as the coefficient of the $T_{\mu\nu}$ term

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (2.50)$$

In line with the previous discussion, the running of the (scaled) cosmological term is according to the rule $\lambda \rightarrow 1/\xi^2$, i.e. no scale dependence. Since, as emphasized in Sec. 2.4, the gravitational path integral only depends in a non-trivial way on the dimensionless combination $G\sqrt{\lambda_0}$ (see for example Eqs. (2.30) and (2.31) and related discussion), Newton's constant itself G can be decomposed uniquely into non-running and running parts, in the following way

$$G \equiv \frac{1}{G\lambda_0} \cdot (G\sqrt{\lambda_0})^2 \rightarrow \frac{1}{\xi^2} \cdot [(G\sqrt{\lambda_0})(\mu^2)]^2 \quad (2.51)$$

where the running of the second term can be directly deduced from either Eqs. (2.22) or (2.21) (both only depend on the combination $G\sqrt{\lambda_0}$), or Eq. (2.10), which is related to the previous two by the scaling assumption for the free energy F .

In conclusion, the modified Einstein equations, incorporating the proposed quantum running of G , read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G(\square) T_{\mu\nu} \quad (2.52)$$

with $\lambda \simeq \frac{1}{\xi^2}$, and only $G(\mu^2)$ on the r.h.s. scale-dependent. The precise meaning of $G(\square)$ is given in section 4. .

2.6 Non-perturbative gravitational vacuum condensate

In this section we will point out the deep relationship (well understood in strongly coupled non-abelian gauge theories) between the non-perturbative scale ξ appearing in Eqs. (2.1) and (2.2), and the non-perturbative vacuum condensate of Eqs. (2.24) and (2.48), which is a measure of curvature. The principal, and in our opinion inescapable, conclusion of the results of Eqs. (2.24) and (2.47) is that the scale ξ appearing in Eqs. (2.1) and (2.2) is related to curvature, and must be *macroscopic* for the theory to be consistent. How can quantum effects propagate to such large distances and give such drastic modifications to gravity? The answer to this paradoxical question presumably lies in the fact that gravitation is carried by a massless particle whose interactions cannot be screened, on any length scale.

It is worth pointing out here that the gravitational vacuum condensate, which only exists in the strong coupling phase $G > G_c$, and which is proportional to the curvature, is genuinely non-

perturbative. If one uses a shorthand notation \mathcal{R} for it, then one can summarize the result of Eq. (2.48) as

$$\mathcal{R} \simeq (10^{-30} eV)^2 \sim \xi^{-2} \quad (2.53)$$

where the condensate is, according to Eqs. (2.10) and (2.26) (relating ξ to $|G - G_c|$), and Eqs. (2.47) and (2.48) (relating \mathcal{R} to ξ), non-analytic at $G = G_c$,

$$\mathcal{R} \sim |G - G_c|^{2\nu} \quad (2.54)$$

The non-perturbative curvature scale ξ then corresponds to a non-vanishing graviton vacuum condensate of order $\xi^{-1} \sim 10^{-30} eV$, extraordinarily tiny compared to the QCD color condensate ($\Lambda_{\overline{MS}} \simeq 220 MeV$) and the electro-weak Higgs condensate ($v \simeq 250 GeV$). But as previously emphasized, the quantum gravity theory cannot provide, at least in its present framework, a value for the non-perturbative curvature scale ξ , which ultimately can only be fixed by phenomenological input, either by Eq. (2.1) or, equivalently, by Eq. (2.48). The main message here is that the scale in those two equations is one and the same.

Pursuing the analogy with strongly coupled Yang-Mills theories and QCD, we note that there the non-perturbative gluon vacuum condensate depends in a nontrivial way on the corresponding confinement scale parameter [40],

$$\alpha_S \langle F_{\mu\nu} \cdot F^{\mu\nu} \rangle \simeq (250 MeV)^4 \sim \xi^{-4} \quad (2.55)$$

with $\xi_{QCD}^{-1} \sim \Lambda_{\overline{MS}}$. The above condensate is not the only one that appears in QCD, another important non-vanishing vacuum condensate being the fermionic one [42]

$$(\alpha_S)^{4/\beta_0} \langle \bar{\psi} \psi \rangle \simeq -(230 MeV)^3 \sim \xi^{-3} \quad (2.56)$$

2.7 Quantum gravity near two dimensions

The result of Eqs. (2.1) and Eq. (2.9) are almost identical in overall structure to what one finds in $2 + \epsilon$ dimensions, if one allows for a different value of exponent ν as one transitions from two dimensions to the physical case of four dimensions. In this section we will explore their relationship, and the lessons one learns from similar field theory models, such as the non-linear sigma model above two dimensions. But the second major ingredient which non-perturbative lattice studies provide [20, 26], besides the existence of a phase transition between two geometrically rather distinct phases, is that the weakly coupled small G phase is pathological, in the sense that the theory becomes unstable, with the four-dimensional lattice collapsing into a tree-like two-dimensional structure for

$G < G_c$. Indeed the lattice theory close to the transition at G_c is “on the verge” of becoming two-dimensional [26], but only to the extent that the effects of higher derivative terms and conformal anomaly contributions can be ignored at short distances ⁶.

To one loop the $d = 2 + \epsilon$ result for the gravitational beta function was computed some time ago, and reads

$$\beta(G) = (d - 2)G - \beta_0 G^2 - \dots, \quad (2.58)$$

It exhibits the celebrated non-trivial ultraviolet fixed point of $2 + \epsilon$ quantum gravity at $G_c = (d - 2)/\beta_0$ (one finds $\beta_0 > 0$ for pure gravity). Furthermore, the physics of it bears some striking similarity to the non-linear sigma model, to be discussed below. The latter is also perturbatively non-renormalizable above two dimensions, but can be constructed by a suitable double expansion in the coupling g and $\epsilon = d - 2$ [48].

In gravity the corresponding running of G is obtained by integrating Eq. (2.58)

$$G(k^2) = \frac{G_c}{1 \pm (m^2/k^2)^{(d-2)/2}} \quad (2.59)$$

The choice of $+$ or $-$ sign is determined from whether one is to the left ($+$), or to right ($-$) of the fixed point, in which case the effective $G(k^2)$ decreases or, respectively, increases as one flows away from the ultraviolet fixed point at G_c . Physically they represent a screening and an anti-screening solution. It is noteworthy that the invariant mass scale m arises as an arbitrary integration constant of the renormalization group equations. While in the continuum both phases, and therefore both signs, seem acceptable (giving rise to both a “Coulomb” or “spin wave” phase, and a strong coupling phase), the euclidean lattice rules out the small $G < G_c$ phase as pathological, in the sense that the lattice collapses into a two-dimensional branched polymer [20, 13] ⁷.

Thus the smooth phase with $G > G_c$ emerges as the only physically acceptable phase in $d = 3$ and $d = 4$ [24, 13]. These arguments suggest therefore that in the renormalization group solution Eq. (2.59) only the $-$ sign is meaningful and physical, corresponding to an infrared growth of the

⁶One way of determining coarse aspects of the underlying geometry is to compute the effective dimension in the scaling regime, for example by considering how the number of points within a thin shell of geodesic distance between τ and $\tau + \Delta$ scales with the geodesic distance itself [26]. For distances a few multiples of the average lattice spacing one finds

$$N(\tau) \underset{\tau \rightarrow \infty}{\sim} \tau^{d_v}, \quad (2.57)$$

with $d_v = 3.1(1)$ for $G > G_c$ (the smooth phase) and $d_v \simeq 1.6(2)$ for $G < G_c$ (the rough phase). One concludes that in the rough phase the lattice tends to collapse into a degenerate tree-like configuration, whereas in the smooth phase the effective dimension of space-time is consistent with four. Higher derivative terms tend to affect these results at very short distances, where they tend to make the geometry smoother [20, 47].

⁷The collapse stops at $d = 2$ because the gravitational action becomes a topological invariant.

coupling for $G > G_c$, the gravitational anti-screening solution given by

$$G(k^2) \simeq G_c \left[1 + \left(\frac{m^2}{k^2} \right)^{(d-2)/2} + \dots \right] \quad (2.60)$$

The above expression has in fact exactly the same structure as the lattice result of Eq. (2.1). Even physically the result makes sense, as one would expect that gravity cannot be screened (as would happen for the “+” sign choice). Repeating some of the general arguments presented in the previous sections, and replacing $k^2 \rightarrow \Lambda^2$ in the above equation, where Λ is the ultraviolet cutoff, one can solve, for $G > G_c$, for the genuinely non-perturbative, dynamically generated mass scale m in terms of the coupling at the cutoff (lattice) scale, as in Eq. (2.16). It should be noted here that Eq. (2.16) is essentially the same as Eq. (2.10), and that Eq. (2.59) is essentially the same as Eq. (2.1).

The derivative of the beta function at the fixed point gives the exponent ν . From Eq. (2.58) one has therefore close to two dimensions [44]

$$\beta'(G_c) = -1/\nu = (2 - d) \quad (2.61)$$

Recently the above results have been extended to two loops, giving close to two dimensions ⁸

$$\beta(G) = (d - 2)G - \frac{2}{3}(25 - n_s)G^2 - \frac{20}{3}(25 - n_s)G^3 + \dots, \quad (2.62)$$

for n_s massless real scalar fields minimally coupled to gravity. After solving the equation $\beta(G_c) = 0$ to determine the location of the ultraviolet fixed point, one finds

$$\begin{aligned} G_c &= \frac{3}{2(25 - n_s)}(d - 2) - \frac{45}{2(25 - n_s)^2}(d - 2)^2 + \dots \\ \nu^{-1} &= -\beta'(G_c) = (d - 2) + \frac{15}{25 - n_s}(d - 2)^2 + \dots \end{aligned} \quad (2.63)$$

which gives, for pure gravity without matter ($n_s = 0$) in four dimensions, to lowest order $\nu^{-1} = 2$, and $\nu^{-1} \approx 4.4$ at the next order. Unfortunately in general the convergence properties of the $2 + \epsilon$ expansion for some better understood field theories, such as the non-linear sigma model, are not too encouraging, at least when compared for example to well-established results obtained by other means directly in $d = 3$ [51, 52]. This somewhat undesirable state of affairs is usually ascribed to

⁸For a while there was considerable uncertainty, due in part to the kinematic singularities which arise in gravity close to two dimensions, about the value of the graviton contribution to β_0 , which was quoted originally in [43] as $38/3$, in [44] as $2/3$, and more recently in [45, 46] as $50/3$. As discussed in [8], the original expectation was that the graviton contribution should be $d(d - 3)/2 = -1$ times the scalar contribution close to $d = 2$. Direct numerical estimates in $d = 3$ give $\nu^{-1} \simeq 1.67$ [24] and are therefore in much better agreement with the larger, more recent value for β_0 .

the suspected existence of renormalon-type singularities $\sim e^{-c/G}$ close to two dimensions, which could possibly arise in gravity as well. At the quantitative level, the results of the $2 + \epsilon$ expansion for gravity remain therefore somewhat limited, and obtaining the three- or four-loop corrections could represent a daunting task. But one should not overlook the fact that there are quantum field theory models which have in some qualitative respects astonishingly similar behavior in $2 + \epsilon$ dimensions, and are at the same time very well understood. Some analogies might therefore remain helpful.

The non-linear $O(N)$ sigma model is one example, studied extensively in the context of the $2 + \epsilon$ expansion, and solved exactly in the large N limit [38]. The model is not perturbatively renormalizable above two dimensions. Yet in both approaches it exhibits a non-trivial ultraviolet fixed point at g_c (a phase transition in statistical mechanics language), separating a weak coupling massless ordered phase from a massive, strong coupling phase. Therefore the correct continuum limit has to be taken in the vicinity of the non-trivial ultraviolet fixed point. Perhaps one of the most striking aspects of the non-linear sigma model above two dimensions is that all particles are massless in perturbation theory, yet they all become massive in the strong coupling phase $g > g_c$, with masses proportional to a non-perturbative scale m [49].

The second example of perturbatively non-renormalizable theory is the chirally-invariant self-coupled fermion above two dimensions [38]. It too exhibits a non-trivial ultraviolet fixed point above two dimensions, and can be studied perturbatively via a double $2 + \epsilon$ expansion. And it too can be solved exactly in the large N limit. With either method, one can show that the model is characterized by two phases, a weak coupling phase where the fermions stay massless and chiral symmetry is unbroken, and a strong coupling phase in which chiral symmetry is spontaneously broken, a fermion condensate arises, and a mass scale is generated non-perturbatively.

2.8 Other determinations of the exponent ν

Let us conclude this discussion by mentioning the remaining methods which have been used to estimate ν .

Recently some approximate renormalization group results have been obtained in the continuum based on an Einstein-Hilbert action truncation. In the limit of vanishing bare cosmological constant the result $\nu^{-1} = 2d(d-2)/(d+2) = 2.667$ was given in $d = 4$ [53]. In the cited work the sensitivity of the numerical answer for the exponent to the choice of gauge fixing term and to the specific shape of the momentum cutoff were systematically investigated as well. More details about the procedure can be found in the quoted references. The more recent results extend earlier calculations for the

exponent ν done by similar operator truncation methods, described in detail in [55] and references therein. A quantitative comparison of these continuum results with the lattice answer for ν^{-1} , and with the $2 + \epsilon$ estimates, can be found in Figs. 1 and 2. The overall the agreement seems reasonable, given the uncertainties inherent in the various estimates. There is also some connection between the just mentioned lattice and continuum renormalization group results and the ideas in [56], but in this last reference fractional exponents, such as the ones in Eq. (2.2) and (2.59), were not considered.

Both the lattice and the continuum renormalization group analysis can be extended to dimensions greater than four. In [14] it was suggested, based on a simple geometric argument, that $\nu = 1/(d - 1)$ for large d , and in the large- d limit one can prove that $\nu = 0$ [15]. Moreover, for the lattice theory in finite dimensions one finds no phase transition in $d = 2$ [23], $\nu \approx 0.60$ in $d = 3$ [24] and $\nu \approx 0.33$ in $d = 4$ [13, 14], which leads to the more or less constant sequence $(d - 2)\nu = 1, 0.60$ and 0.66 in the three cases respectively, which would suggest $\nu \sim 1/d$ for large d . On the other hand, in the same large d limit, in [53] the value $\nu \simeq 1/2d$ was obtained, again performing a renormalization group analysis truncated to the Einstein-Hilbert action with a cosmological term in the continuum. One cannot help noticing some reasonable agreement between the estimates for ν coming from the lattice, and the corresponding results for ν using the truncated renormalization group expansion in the continuum, as well as some degree of compatibility with the $2 + \epsilon$ expansion close to $d = 2$. For a more quantitative comparison, see again Figs. 1 and 2.

One possible advantage of the continuum renormalization group calculations is that, since they can be performed to a great extent analytically, they allow greater flexibility in exploring various scenarios, such as additional invariant operators in the action, measure contributions, or varying dimensions. At the same time the absence of a reliable estimate for the errors involved in the truncation leaves the method with uncertainties, which might be hard to quantify until an improved calculation is performed with an extended operator basis.

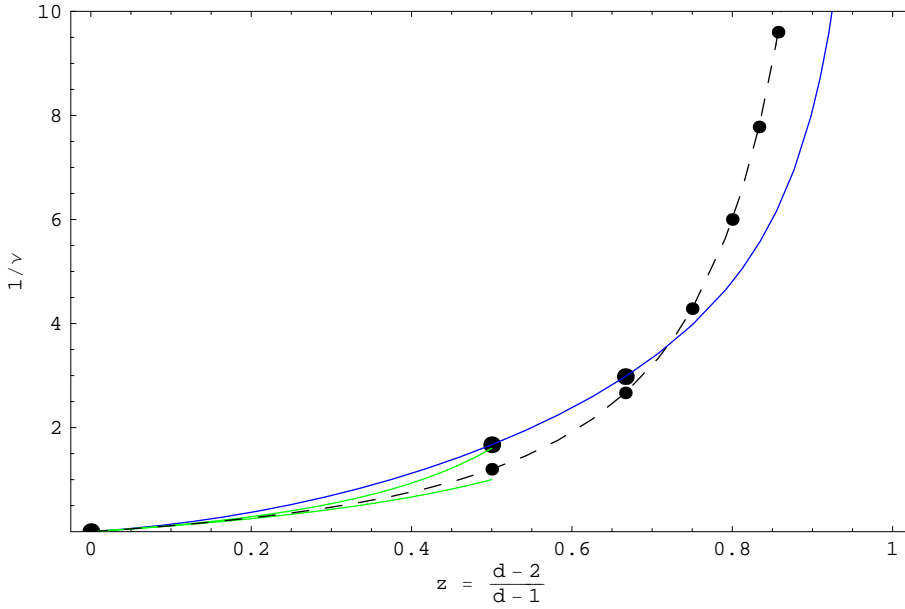


Fig. 1. Universal gravitational exponent $1/\nu$ of Eqs. (2.1) and (2.2) as a function of the dimension (the abscissa here is $z = (d - 2)/(d - 1)$, where d is the space-time dimension, which maps $d = 2$ to $z = 0$ and $d = \infty$ to $z = 1$). The larger circles at $d = 3$ and $d = 4$ are the lattice gravity results of [24, 13], interpolated (continuous curve) using the exact lattice results $1/\nu = 0$ in $d = 2$ [23], and $\nu = 0$ at $d = \infty$ [15]. The smaller black dots (connected by the dashed curve) are the recent results of [53, 54], obtained from a continuum renormalization group study around the non-trivial fixed point of the gravitational action in d dimensions in the limit $\lambda \rightarrow 0$. The two curves close to the origin are the $2 + \epsilon$ expansion for $1/\nu$ to one loop (lower curve) and two loops (upper curve) [46].

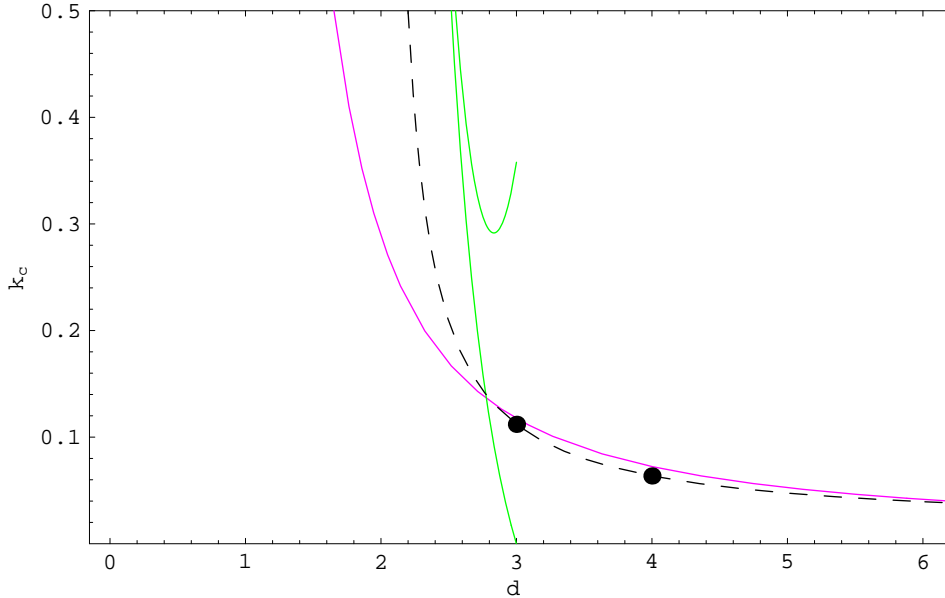


Fig. 2. Critical point $k_c = 1/(8\pi G_c)$ in units of the cutoff (as it appears for ex. in Eqs. (2.10) and (2.22)) as a function of dimension. The abscissa is the space-time dimension d . The large circles at $d = 3$ and $d = 4$ are the Regge lattice results of [24, 13], interpolated (dashed curve) using the additional lattice results $1/k_c = 0$ in $d = 2$ [23]. The lower continuous curve is the large- d lattice result of Eq. (2.27). The two upper continuous curves are the two-loop $2 + \epsilon$ expansion result in the continuum of Eq. (2.63) [46], whose prediction becomes uncertain as d approaches the value three.

2.9 The running of $\alpha(\mu^2)$ in QED and QCD

QED and QCD provide two invaluable illustrative cases where the running of the gauge coupling with energy is not only theoretically well understood, but also verified experimentally. This section is intended to provide analogies and distinctions between the two theories, and later gives justification, based on the structure of infrared divergences in QCD, for the transition between the running of the gravitational coupling as given in Eq. (2.1), and its infrared regulated version of Eq. (2.2). Most of the results found in this section are well known, but the purpose here is provide a contrast (and in some instances, a close relationship) with the gravitational case.

In QED the non-relativistic static Coulomb potential is affected by the vacuum polarization contribution due to electrons (and positrons) of mass m . To lowest order in the fine structure constant, the contribution is from a single Feynman diagram involving a fermion loop. One finds

for the vacuum polarization contribution $\omega_R(\mathbf{k}^2)$ at small \mathbf{k}^2 the well known result [57]

$$\frac{e^2}{\mathbf{k}^2} \rightarrow \frac{e^2}{\mathbf{k}^2[1 + \omega_R(\mathbf{k}^2)]} \sim \frac{e^2}{\mathbf{k}^2} \left[1 + \frac{\alpha}{15\pi} \frac{\mathbf{k}^2}{m^2} + O(\alpha^2) \right] \quad (2.64)$$

which, for a Coulomb potential with a charge centered at the origin of strength $-Ze$ leads to well-known Uehling [58, 59] δ -function correction

$$V(r) = \left(1 - \frac{\alpha}{15\pi} \frac{\Delta}{m^2} \right) \frac{-Ze^2}{4\pi r} = \frac{-Ze^2}{4\pi r} - \frac{\alpha}{15\pi} \frac{-Ze^2}{m^2} \delta^{(3)}(\mathbf{x}) \quad (2.65)$$

It is not necessary though to resort to the small- \mathbf{k}^2 approximation, and in general a static charge of strength e at the origin will give rise to a modified potential

$$\frac{e}{4\pi r} \rightarrow \frac{e}{4\pi r} Q(r) \quad (2.66)$$

with

$$Q(r) = 1 + \frac{\alpha}{3\pi} \ln \frac{1}{m^2 r^2} + \dots \quad m r \ll 1 \quad (2.67)$$

for small r , and

$$Q(r) = 1 + \frac{\alpha}{4\sqrt{\pi}} \frac{1}{(mr)^{3/2}} e^{-2mr} + \dots \quad m r \gg 1 \quad (2.68)$$

for large r . Here the normalization is such that the potential at infinity has $Q(\infty) = 1$ ⁹. The reason we have belabored on this example is to show that the screening vacuum polarization contribution would have dramatic effects in QED if for some reason the particle running through the fermion loop diagram had a much smaller (or even close to zero) mass.

There are two interesting aspects of the (one-loop) result of Eqs. (2.67) and (2.68). The first one is that the exponentially small size of the correction at large r is linked with the fact that the electron mass m_e is not too small: the range of the correction term is $\xi = 2\hbar/mc = 0.78 \times 10^{-10} \text{cm}$, but would have been much larger if the electron mass had been a lot smaller. In fact in the limit of zero electron mass the correction progressively extends out all the way to infinity. Ultimately the fact that the Uehling correction is not important in atomic physics is largely due to the fact that its value is already quite small on atomic scales, comparable to the Bohr radius $a_0 = \hbar/me^2 = 0.53 \times 10^{-8}$.

The second interesting aspect is that the correction is divergent at small r due to the log term, giving rise to a large unscreened charge close to the origin; and of course the smaller the electron mass m , the larger the correction. Also, it should be noted that the observed laboratory value for the effective electromagnetic charge is normalized so that $Q(r)$ is $Q(r = \infty) = 1$, whereas in gravity

⁹The running of the fine structure constant has recently been verified experimentally at LEP, the scale dependence of the vacuum polarization effects gives a fine structure constant changing from $\alpha(0) \sim 1/137.036$ at atomic distances to about $\alpha(m_{Z_0}) \sim 1/128.978$ at energies comparable to the Z^0 boson mass, in good agreement with the theoretical prediction [60].

the laboratory value corresponds to the “short distance” limit, $G(r = 0)$, due to the very large size of ξ in Eq. (2.2).

In QCD (and related Yang-Mills theories) radiative corrections are also known to alter significantly the behavior of the static potential at short distances. The changes in the potential are best expressed in terms of the running strong coupling constant $\alpha_S(\mu)$, whose scale dependence is determined by the celebrated beta function of $SU(3)$ QCD with n_f light fermion flavors [61]

$$\mu \frac{\partial \alpha_S}{\partial \mu} = 2\beta(\alpha_S) = -\frac{\beta_0}{2\pi} \alpha_S^2 - \frac{\beta_1}{4\pi^2} \alpha_S^3 - \frac{\beta_2}{64\pi^3} \alpha_S^4 - \dots \quad (2.69)$$

with $\beta_0 = 11 - \frac{2}{3}n_f$, $\beta_1 = 51 - \frac{19}{3}n_f$, and $\beta_2 = 2857 - \frac{5033}{9}n_f + \frac{325}{27}n_f^2$. The solution of the renormalization group equation Eq. (2.69) then gives for the running of $\alpha_S(\mu)$

$$\alpha_S(\mu) = \frac{4\pi}{\beta_0 \ln \mu^2 / \Lambda_{\overline{MS}}^2} \left[1 - \frac{2\beta_1}{\beta_0^2} \frac{\ln[\ln \mu^2 / \Lambda_{\overline{MS}}^2]}{\ln \mu^2 / \Lambda_{\overline{MS}}^2} + \dots \right] \quad (2.70)$$

The non-perturbative scale $\Lambda_{\overline{MS}}$ appears as an integration constant of the renormalization group equations, and is therefore - by construction - scale independent. The physical value of $\Lambda_{\overline{MS}}$ cannot be fixed from perturbation theory alone, and needs to be determined by experiment ¹⁰.

In QCD one determines experimentally from Eq. (2.70) (for example, from the size of scaling violations in deep inelastic scattering) $\Lambda_{\overline{MS}} \simeq 220 \text{ MeV}$, which not surprisingly is close to a typical hadronic scale. But from a purely theoretical standpoint one could envision other scenarios, where this scale would take on a completely different value. The point is that nothing in QCD itself determines the absolute magnitude of this scale: only its size in relation to other physical observables such as hadron masses is theoretically calculable within QCD.

In principle one can solve for $\Lambda_{\overline{MS}}$ in terms of the coupling at any scale, and in particular at the cutoff scale Λ , obtaining

$$\Lambda_{\overline{MS}} = \Lambda \exp\left(-\int^{\alpha_S(\Lambda)} \frac{d\alpha'_S}{2\beta(\alpha'_S)}\right) = \Lambda \left(\frac{\beta_0 \alpha_S(\Lambda)}{4\pi}\right)^{\beta_1/\beta_0^2} e^{-\frac{2\pi}{\beta_0 \alpha_S(\Lambda)}} [1 + O(\alpha_S(\Lambda))] \quad (2.71)$$

It should be clear by now that this expression is the analog of Eqs. (2.10) and (2.16) in the gravitational case. In lattice QCD this is usually taken as the definition of the running strong coupling constant $\alpha_S(\mu)$. It then leads to an effective potential between quarks and anti-quarks of the form

$$V(\mathbf{k}^2) = -\frac{4}{3} \frac{\alpha_S(\mathbf{k}^2)}{\mathbf{k}^2} \quad (2.72)$$

¹⁰Alternatively, the integration constant in Eq. (2.69) can be fixed by specifying the value of α_S at some specific energy scale, usually chosen to be the Z^0 mass.

and the leading logarithmic correction makes the potential appear softer close to the origin, $V(r) \sim 1/(r \ln r)$.

When the QCD result is contrasted with the QED answer of Eqs. (2.64) and (2.65) it appears that the infrared small \mathbf{k}^2 singularity in Eqs. (2.72) is quite serious. An analogous conclusion is reached when examining Eqs. (2.70) : the coupling strength $\alpha_S(\mathbf{k}^2)$ diverges in the infrared due to the singularity at $k^2 = 0$. In order to avoid a meaningless divergent answer, the uncontrolled growth in $\alpha_S(\mathbf{k}^2)$ needs therefore to be regulated (or self-regulated) by the dynamically generated QCD infrared cutoff $\Lambda_{\overline{MS}}$ [64] (as indeed happens in simpler theories, such as the non-linear sigma model, which can be solved exactly in the large N limit, where this mechanism can therefore be explicitly demonstrated [38]). To lowest order in the coupling this implies

$$V(\mathbf{k}^2) = -\frac{4}{3} \frac{4\pi}{\beta_0 \mathbf{k}^2 \ln\left(\frac{\mathbf{k}^2}{\Lambda_{\overline{MS}}^2}\right)} \rightarrow -\frac{4}{3} \frac{4\pi}{\beta_0 \mathbf{k}^2 \ln\left(\frac{\mathbf{k}^2 + \Lambda_{\overline{MS}}^2}{\Lambda_{\overline{MS}}^2}\right)} \quad (2.73)$$

The resulting potential can then be evaluated for small \mathbf{k}^2

$$V(\mathbf{k}^2) \underset{\mathbf{k}^2/\Lambda_{\overline{MS}}^2 \rightarrow 0}{\sim} -\frac{4}{3} \frac{4\pi}{\beta_0 \mathbf{k}^2 (\mathbf{k}^2/\Lambda_{\overline{MS}}^2)} \quad (2.74)$$

giving, after Fourier transforming back to real space, for large r ,

$$V(r) \underset{r \rightarrow \infty}{\sim} -\frac{4}{3} \frac{4\pi}{\beta_0} \Lambda_{\overline{MS}}^2 \cdot \left(-\frac{1}{8\pi} r\right) + O(r^0) \simeq +\sigma r \quad (2.75)$$

The desired linear potential, here with string tension σ , is then indeed recovered at large distances. In fact the interpolating potential of Eq. (2.73) is remarkably successful in describing non-relativistic QCD bound states, as discussed in [64, 65], incorporating correctly to some extent the leading short- and large-distance QCD corrections ¹¹.

What is rather remarkable in this context is that the removal of a serious infrared divergence in $\alpha_S(\mathbf{k}^2)$, by the replacement of Eq. (2.73), has in fact caused an even stronger infrared divergence in $V(\mathbf{k}^2)$ itself, which leads though, in this instance, to precisely the desired result of Eq. (2.75), namely a confining potential at large distances, as expected on the basis of non-perturbative arguments [35]. From $\alpha_S(\mathbf{k}^2)$ given in Eq. (2.73) one can calculate the corresponding beta function

$$\beta(\alpha_S) = -\frac{\beta_0}{4\pi} \alpha_S^2 \left(1 - e^{-\frac{4\pi}{\beta_0 \alpha_S}}\right) + O(\alpha_S^3) \quad (2.76)$$

which now exhibits a non-analytic (renormalon) contribution at $\alpha_S = 0$, not detectable to any order in perturbation theory [62, 63].

¹¹A similar infrared regularization was done in Eq. (2.2), $(m^2/\mathbf{k}^2)^{1/2\nu} = \exp\left[-\frac{1}{2\nu} \ln(\mathbf{k}^2/m^2)\right] \rightarrow \exp\left\{-\frac{1}{2\nu} \ln[(\mathbf{k}^2 + m^2)/m^2]\right\}$

The very fruitful analogy between strongly coupled non-abelian gauge theories and strongly coupled gravity can be pushed further [15], by developing other aspects of the correspondence which naturally lend themselves to such a comparison. In gravity the analog to the Wilson loop $W(\Gamma)$ of non-abelian gauge theories

$$W(\Gamma) \sim \text{Tr } \mathcal{P} \exp \left[\int_C A_\mu dx^\mu \right] \quad (2.77)$$

exists as well (see Eq. (2.45)), defined in the gravitational case as the path-ordered exponential of the affine connection $\Gamma_{\mu\nu}^\lambda$ around a closed planar loop.

Furthermore, it is known that in QCD, due to the non-trivial strong coupling dynamics, there arise several non-perturbative condensates. Thus the gluon condensate is related to the confinement scale via [40]

$$\alpha_S \langle F_{\mu\nu} \cdot F^{\mu\nu} \rangle \simeq (250 \text{MeV})^4 \sim \xi^{-4} \quad (2.78)$$

with $\xi_{QCD}^{-1} \sim \Lambda_{\overline{MS}}$. Another important non-vanishing non-perturbative condensate in QCD is the fermionic one [41, 42]

$$(\alpha_S)^{4/\beta_0} \langle \bar{\psi} \psi \rangle \simeq -(230 \text{MeV})^3 \sim \xi^{-3} \quad (2.79)$$

and represents a key quantity in discussing the size of chiral symmetry breaking, and the light quark masses.

3 Poisson's Equation with Running G

Given the running of G from either Eq. (2.2), or Eq. (2.1) in the large \mathbf{k} limit, the next step is naturally a solution of Poisson's equation with a point source at the origin, in order to determine the structure of the quantum corrections to the gravitational potential in real space. The more complex solution of the fully relativistic problem will then be addressed in the following sections. In the limit of weak fields the relativistic field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (3.1)$$

give for the ϕ field (with $g_{00}(x) \simeq -(1 + 2\phi(x))$)

$$(\Delta - \lambda) \phi(x) = 4\pi G \rho(x) - \lambda \quad (3.2)$$

which would suggest that the scaled cosmological constant λ acts like a mass term $m = \sqrt{\lambda}$. For a point source at the origin, the first term on the r.h.s is just $4\pi M G \delta^{(3)}(\mathbf{x})$. The solution for $\phi(r)$

can then be obtained simply by Fourier transforming back to real space Eq. (2.6), and, up to an additive constant, one has

$$\phi(r) = -M G \frac{e^{-m r}}{r} - \frac{a_0 m M G}{2^{\frac{1}{2}(\frac{1}{\nu}-1)} \Gamma(1 + \frac{1}{2\nu}) \sqrt{\pi}} (m r)^{\frac{1}{2}(\frac{1}{\nu}-1)} K_{\frac{1}{2}(\frac{1}{\nu}-1)}(m r) \quad (3.3)$$

where $K_n(x)$ is the modified Bessel function of the second type. The behavior of $\phi(r)$ would then be Yukawa-like $\phi(r) \sim \text{const. } e^{-m r}/r$ and thus rapidly decreasing for large r .

But the reason why both of the above results are in fact *incorrect* (assuming of course the validity of general coordinate invariance at very large distances $r \gg 1/\sqrt{\lambda}$) is that the exact solution to the field equations in the static isotropic case with a λ term gives

$$-g_{00} = B(r) = 1 - \frac{2 M G}{r} - \frac{\lambda}{3} r^2 \quad (3.4)$$

showing that the λ term definitely does *not* act like a mass term in this context.

Therefore the zeroth order contribution to the potential should be taken to be proportional to $4\pi/(\mathbf{k}^2 + \mu^2)$ with $\mu \rightarrow 0$, as already indicated in fact in Eq. (2.8). Also, proper care has to be exercised in providing an appropriate infrared regulated version of $G(\mathbf{k}^2)$, and therefore $V(\mathbf{k}^2)$, which from Eq. (2.8) reads

$$\frac{4\pi}{(\mathbf{k}^2 + \mu^2)} \left[1 + a_0 \left(\frac{m^2}{\mathbf{k}^2 + m^2} \right)^{\frac{1}{2\nu}} \right] \quad (3.5)$$

and where the limit $\mu \rightarrow 0$ is intended to be taken at the end of the calculation.

There are in principle two equivalent ways to compute the potential $\phi(r)$, either by inverse Fourier transform of the above expression, or by solving Poisson's equation $\Delta\phi = 4\pi\rho$ with $\rho(r)$ given by the inverse Fourier transform of the correction to $G(k^2)$, as given later in Eq. (3.17). Here we will first use the first, direct method.

3.1 Large r limit

The zeroth order term gives the standard Newtonian $-MG/r$ term, while the correction in general is given by a rather complicated hypergeometric function. But for the special case $\nu = 1/2$ one has for the Fourier transform of the correction to $\phi(r)$

$$a_0 m^{1/\nu} \frac{4\pi}{\mathbf{k}^2 + \mu^2} \frac{1}{(\mathbf{k}^2 + m^2)^{\frac{1}{2\nu}}} \rightarrow a_0 m^2 \frac{e^{-\mu r} - e^{-m r}}{r(m^2 - \mu^2)} \underset{\mu \rightarrow 0}{\sim} a_0 m^2 \frac{1 - e^{-m r}}{m^2 r} \quad (3.6)$$

giving for the complete quantum-corrected potential

$$\phi(r) = -\frac{M G}{r} [1 + a_0 (1 - e^{-m r})] \quad (3.7)$$

For this special case the running of $G(r)$ is particularly transparent,

$$G(r) = G_\infty \left(1 - \frac{a_0}{1 + a_0} e^{-mr} \right) \quad (3.8)$$

with $G_\infty \equiv (1 + a_0) G$ and $G \equiv G(0)$. G therefore increases slowly from its value G at small r to the larger value $(1 + a_0) G$ at infinity. Figure 3. provides a schematic illustration of the behavior of G as a function of r .

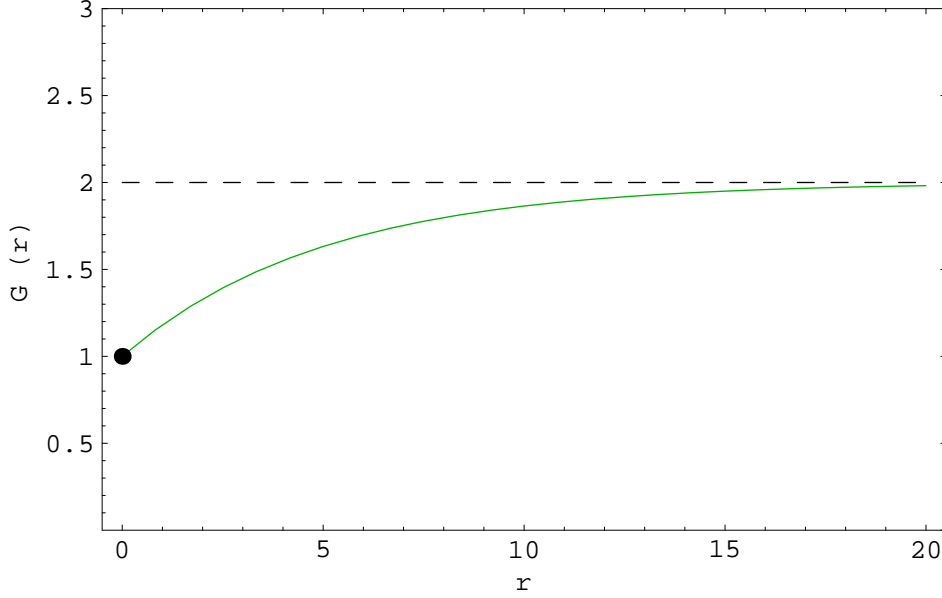


Fig. 3. Schematic scale dependence of the gravitational coupling $G(r)$ from Eq. (3.8), here for $\nu = 1/2$. The gravitational coupling rises initially like a power of r , and then approaches the asymptotic value $G_\infty = (1 + a_0)G$ for large r . The behavior for other values of $\nu > 1/3$ is similar.

Returning to the general ν case, one can expand for small \mathbf{k} to get the correct large r behavior,

$$\frac{1}{(\mathbf{k}^2 + \mu^2)(\mathbf{k}^2 + m^2)^{\frac{1}{2\nu}}} \simeq \frac{1}{m^{\frac{1}{\nu}}} \frac{1}{(\mathbf{k}^2 + \mu^2)} - \frac{1}{2\nu m^2} \frac{1}{(\mathbf{k}^2 + m^2)^{\frac{1}{2\nu}}} + \dots \quad (3.9)$$

After Fourier transform, one obtains the previous answer for $\nu = 1/2$, whereas for $\nu = 1/3$ one finds

$$-\frac{MG}{r} \left[1 + a_0 \left(1 - \frac{3mr}{\pi} K_0(mr) \right) \right] \quad (3.10)$$

and for general ν

$$-\frac{MG}{r} \left[1 + a_0 \left(1 - \frac{2^{\frac{1}{2}(3-\frac{1}{\nu})} mr}{2\nu \sqrt{\pi} \Gamma(\frac{1}{2\nu})} (mr)^{-\frac{1}{2}(3-\frac{1}{\nu})} K_{\frac{1}{2}(3-\frac{1}{\nu})}(mr) \right) \right] \quad (3.11)$$

Using the asymptotic expansion of the modified Bessel function $K_n(x)$ for large arguments, $K_n(z) \sim \sqrt{\pi/2} z^{-1/2} e^{-z} (1 + O(1/z))$, one finally obtains in the large r limit

$$\phi(r) \underset{r \rightarrow \infty}{\sim} -\frac{MG}{r} \left[1 + a_0 \left(1 - c_l (mr)^{\frac{1}{2\nu}-1} e^{-mr} \right) \right] \quad (3.12)$$

with $c_l = 1/(\nu 2^{\frac{1}{2\nu}} \Gamma(\frac{1}{2\nu}))$.

3.2 Small r limit

In the small r limit one finds instead, using again Fourier transforms, for the correction for $\nu = 1/3$

$$(-MG) a_0 m^2 \frac{r^2}{3\pi} \left[\ln\left(\frac{mr}{2}\right) + \gamma - \frac{5}{6} \right] + O(r^3) \quad (3.13)$$

In the general case the complete leading correction to the potential $\phi(r)$ for small r (and $\nu > 1/3$) has the structure $(-\text{const.})(-MG)a_0 m^{\frac{1}{\nu}} r^{\frac{1}{\nu}-1}$. Note that the quantum correction always vanishes at short distances $r \rightarrow 0$, as expected from the original result of Eqs. (2.1) or (2.2) for $k^2 \rightarrow \infty$ ¹².

The same result can be obtained via a different, but equivalent, procedure, in which one solves directly the radial Poisson equation for $\phi(r)$. First, for a point source at the origin, $4\pi MG \delta^{(3)}(\mathbf{x})$, with

$$\delta^{(3)}(\mathbf{x}) = \frac{1}{4\pi} 2 \frac{\delta(r)}{r^2} \quad (3.14)$$

one sets $\Delta\phi(r) \rightarrow r^{-1} d^2/dr^2 [r\phi(r)]$ in radial coordinates. In the $a_0 \neq 0$ case one then needs to solve $\Delta\phi = 4\pi\rho$, or in the radial coordinate for $r > 0$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = 4\pi G \rho_m(r) \quad (3.15)$$

with the source term ρ_m determined from the inverse Fourier transform of the correction term in Eq. (2.2), namely

$$a_0 M \left(\frac{m^2}{\mathbf{k}^2 + m^2} \right)^{\frac{1}{2\nu}} \quad (3.16)$$

One finds

$$\rho_m(r) = \frac{1}{8\pi} c_\nu a_0 M m^3 (mr)^{-\frac{1}{2}(3-\frac{1}{\nu})} K_{\frac{1}{2}(3-\frac{1}{\nu})}(mr) \quad (3.17)$$

with

$$c_\nu \equiv \frac{2^{\frac{1}{2}(5-\frac{1}{\nu})}}{\sqrt{\pi} \Gamma(\frac{1}{2\nu})} \quad (3.18)$$

The vacuum polarization density ρ_m has the property

$$4\pi \int_0^\infty r^2 dr \rho_m(r) = a_0 M \quad (3.19)$$

¹²At very short distances $r \sim l_P$ other quantum corrections come into play, which are not properly encoded in Eq. (2.1), which after all is supposed to describe the universal running in the *scaling region* $l_P \ll r \ll \xi$. Furthermore, higher derivative terms could also have important effects at very short distances.

where the standard integral $\int_0^\infty dx x^{2-n} K_n(x) = 2^{-n} \sqrt{\pi} \Gamma\left(\frac{3}{2} - n\right)$ has been used. Note that the vacuum polarization distribution is singular close to $r = 0$, just as in QED, Eq. (2.67).

The $r \rightarrow 0$ result for $\phi(r)$ (discussed in the following, as an example, for $\nu = 1/3$) can then be obtained by solving the radial equation for $\phi(r)$,

$$\frac{1}{r} \frac{d^2}{dr^2} [r \phi(r)] = \frac{2 a_0 M G m^3}{\pi} K_0(m r) \quad (3.20)$$

where the (modified) Bessel function is expanded out to lowest order in r , $K_0(m r) = -\gamma - \ln\left(\frac{m r}{2}\right) + O(m^2 r^2)$, giving

$$\phi(r) = -\frac{M G}{r} + a_0 M G m^3 \frac{r^2}{3 \pi} \left[-\ln\left(\frac{m r}{2}\right) - \gamma + \frac{5}{6} \right] + O(r^3) \quad (3.21)$$

where the two integration constants are matched to the large r solution of Eq. (3.11). Note again that the vacuum polarization density $\rho_m(r)$ has the expected normalization property

$$4 \pi \int_0^\infty r^2 dr \frac{a_0 M m^3}{2 \pi^2} K_0(m r) = \frac{2 a_0 M m^3}{\pi} \cdot \frac{\pi}{2 m^3} = a_0 M \quad (3.22)$$

so that the total enclosed additional “charge” is indeed just $a_0 M$, and $G_\infty = G_0(1 + a_0)$ (see for comparison also Eq. (3.11)). Using then the same method for general $\nu > \frac{1}{3}$, one finds for small r (using the expansion of the modified Bessel function $K_n(x)$ for small arguments as given later in Eq. (5.24))

$$\rho_m(r) \underset{r \rightarrow 0}{\sim} \frac{|\sec\left(\frac{\pi}{2\nu}\right)|}{4\pi \Gamma\left(\frac{1}{\nu} - 1\right)} a_0 M m^{\frac{1}{\nu}} r^{\frac{1}{\nu}-3} \equiv A_0 r^{\frac{1}{\nu}-3} \quad (3.23)$$

and from it the general result

$$\phi(r) \underset{r \rightarrow 0}{\sim} -\frac{M G}{r} + a_0 M G c_s m^{\frac{1}{\nu}} r^{\frac{1}{\nu}-1} + \dots \quad (3.24)$$

with $c_s = \nu |\sec\left(\frac{\pi}{2\nu}\right)| / \Gamma\left(\frac{1}{\nu}\right)$.

4 Relativistic Field Equations with Running G

Solutions to Poisson’s equation with a running G provide some insights into the structure of the quantum corrections, but a complete analysis requires the study of the full relativistic field equations, which will be discussed next in this section. A set of relativistic field equations incorporating the running of G is obtained by doing the replacement [12]

$$G(k^2) \rightarrow G(\square) \quad (4.1)$$

with the d'Alembertian \square intended to correctly represent invariant distances, and incorporating the running of G as expressed in either Eqs. (2.1) or (2.2),

$$G \rightarrow G(\square) = G \left[1 + a_0 \left(\frac{m^2}{-\square + m^2} \right)^{\frac{1}{2\nu}} + \dots \right] \quad (4.2)$$

For the use of \square to express the running of couplings in gauge theories the reader is referred to the references in [66]. Here the \square operator is defined through the appropriate combination of covariant derivatives

$$\square \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu \quad (4.3)$$

and whose explicit form depends on the specific tensor nature of the object it is acting on, as in the case of the energy-momentum tensor

$$\square T^{\alpha\beta\dots}_{\gamma\delta\dots} = g^{\mu\nu} \nabla_\mu \left(\nabla_\nu T^{\alpha\beta\dots}_{\gamma\delta\dots} \right) \quad (4.4)$$

Thus on scalar functions one obtains the fairly simple result

$$\square S(x) = \frac{1}{\sqrt{g}} \partial_\mu g^{\mu\nu} \sqrt{g} \partial_\nu S(x) \quad (4.5)$$

whereas on second rank tensors one has the significantly more complicated expression $\square T_{\alpha\beta} \equiv g^{\mu\nu} \nabla_\mu (\nabla_\nu T_{\alpha\beta})$. In general the invariant operator appearing in the above expression, namely

$$A(\square) = a_0 \left(\frac{m^2}{-\square} \right)^{1/2\nu} \quad (4.6)$$

or its infrared regulated version

$$A(\square) = a_0 \left(\frac{m^2}{-\square + m^2} \right)^{1/2\nu} \quad (4.7)$$

has to be suitably defined by analytic continuation from positive integer powers; the latter can be often be done by computing \square^n for positive integer n , and then analytically continuing to $n \rightarrow -1/2\nu$. In the following the above analytic continuation from positive integer n will always be understood. Usually it is easier to work with the expression in Eq. (4.6), and then later amend the final result to include the infrared regulator, if needed.

One is therefore lead to consider the effective field equations of Eq. (2.52), namely

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8 \pi G (1 + A(\square)) T_{\mu\nu} \quad (4.8)$$

with $A(\square)$ given by Eq. (4.7) and $\lambda \simeq 1/\xi^2$, as well as the trace equation

$$R - 4 \lambda = -8 \pi G (1 + A(\square)) T \quad (4.9)$$

Being manifestly covariant, these expressions at least satisfy some of the requirements for a set of consistent field equations incorporating the running of G , and can then be easily re-cast in a form similar to the classical field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G \tilde{T}_{\mu\nu} \quad (4.10)$$

with $\tilde{T}_{\mu\nu} = (1 + A(\square)) T_{\mu\nu}$ defined as an effective, gravitationally dressed, energy-momentum tensor. Just like the ordinary Einstein gravity case, in general $\tilde{T}_{\mu\nu}$ might not be covariantly conserved a priori, $\nabla^\mu \tilde{T}_{\mu\nu} \neq 0$, but ultimately the consistency of the effective field equations demands that it be exactly conserved in consideration of the Bianchi identity satisfied by the Einstein tensor [12]. The ensuing new covariant conservation law

$$\nabla^\mu \tilde{T}_{\mu\nu} \equiv \nabla^\mu [(1 + A(\square)) T_{\mu\nu}] = 0 \quad (4.11)$$

can be then be viewed as a constraint on $\tilde{T}_{\mu\nu}$ (or $T_{\mu\nu}$) which, for example, in the specific case of a perfect fluid, implies a definite relationship between the density $\rho(t)$, the pressure $p(t)$ and the metric components [12].

From now on, we will set the cosmological constant $\lambda = 0$, and its contribution can then be added at a later stage. As long as one is interested in static isotropic solutions, one takes for the metric the most general form

$$ds^2 = -B(r) dt^2 + A(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (4.12)$$

and for the energy momentum tensor the perfect fluid form

$$T_{\mu\nu} = \text{diag} [B(r) \rho(r), A(r) p(r), r^2 p(r), r^2 \sin^2 \theta p(r)] \quad (4.13)$$

with trace $T = 3p - \rho$. The trace equation then only involves the (simpler) scalar d'Alembertian, acting on the trace of the energy-momentum tensor

To the order one is working here, the above effective field equations should be equivalent to

$$\frac{1}{1 + A(\square)} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = (1 - A(\square) + \dots) \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 8\pi G T_{\mu\nu} \quad (4.14)$$

where the running of G has been moved over to the ‘‘gravitational’’ side, and later expanded out, assuming the correction to be small. For the vacuum solutions, the r.h.s. is zero for $r \neq 0$, and one can re-write the last equation simply as

$$\frac{1}{8\pi G (1 + A(\square))} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0 \quad (4.15)$$

4.1 Dirac delta function source

A mass point source is most suitably described by a Dirac delta function. The delta function at the origin can be represented for example as

$$\delta(r) = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\pi (r^2 + \epsilon^2)} \quad (4.16)$$

As an example, the derivative operator $(d/dr)^n$ acting on the delta function would give, for small ϵ ,

$$(-1)^n \Gamma(n+2) \pi^{-1} \epsilon r^n (r^2 + \epsilon^2)^{-n-1} \quad (4.17)$$

which can be formally analytically continued to a fractional value $n = -1/\nu$

$$(-1)^{-1/\nu} \Gamma\left(2 - \frac{1}{\nu}\right) \pi^{-1} \epsilon r^{-1/\nu} (r^2 + \epsilon^2)^{\frac{1}{\nu}-1} \quad (4.18)$$

and then re-expressed as an overall factor in front of the original δ function

$$(-1)^{-1/\nu} r^{-1/\nu} (r^2 + \epsilon^2)^{\frac{1}{\nu}} \Gamma\left(2 - \frac{1}{\nu}\right) \delta(r) \rightarrow (-1)^{-1/\nu} r^{1/\nu} \Gamma\left(2 - \frac{1}{\nu}\right) \delta(r) \quad (4.19)$$

The procedure achieves in this case the desired result, namely the multiplication of the original delta function by a power $r^{1/\nu}$. Note also that if one just takes the limit $\epsilon \rightarrow 0$ at fixed r one always gets zero for any $r > 0$, so close to the origin r has to be sent to zero as well to get a non-trivial result.

The above considerations suggests that one should be able to write for the point source at the origin

$$T_{\mu\nu}(r) = \text{diag}[B(r)\rho(r), 0, 0, 0] \quad (4.20)$$

with (in spherical coordinates)

$$\rho = M \delta^{(3)}(\mathbf{x}) \rightarrow M \frac{1}{4\pi} 2 \frac{\delta(r)}{r^2} \quad (4.21)$$

and the delta function defined as a suitable limit of a smooth function, and with vanishing pressure $p(r) = 0$.

Next we will consider as a warmup the trace equation

$$R = -8\pi G(\square)T \equiv -8\pi G(1 + A(\square))T = +8\pi G(1 + A(\square))\rho \quad (4.22)$$

where we have used the fact that the point source at the origin is described just by the density term. One then computes the repeated action of the invariant d'Alembertian on T ,

$$\square(-8\pi GT) = \square(+8\pi G\rho) = \frac{16G\pi\rho'}{rA} - \frac{4G\pi A'\rho'}{A^2} + \frac{4G\pi B'\rho'}{AB} + \frac{8G\pi\rho''}{A} \quad (4.23)$$

or, using the explicit form for $\rho(r)$,

$$\square(-8\pi GT) = -\frac{8GM\epsilon(-6AB - rA'B + rAB')}{\pi(r^2 + \epsilon^2)^3 A^2 B} + O(\epsilon^3) \quad (4.24)$$

In view of the rapidly escalating complexity of the problem it seems sensible to expand around the Schwarzschild solution, and set therefore

$$A(r)^{-1} = 1 - \frac{2MG}{r} + \frac{\sigma(r)}{r} \quad (4.25)$$

and

$$B(r) = 1 - \frac{2MG}{r} + \frac{\theta(r)}{r} \quad (4.26)$$

where the correction to the standard solution are parametrized here by the two functions $\sigma(r)$ and $\theta(r)$, both assumed to be “small”, i.e. proportional to a_0 as in Eqs. (4.6) or (4.7), with a_0 considered a small parameter. Then for the scalar curvature, to lowest order in σ and θ , one has

$$\frac{GM(\sigma - \theta) - (2GM - r)(GM\theta' + (3GM - 2r)\sigma' + (2GM - r)r\theta'')}{r^2(r - 2MG)^2} \quad (4.27)$$

To simplify the problem even further, we will assume that for $2MG \ll r \ll \xi$ (the “physical” regime) one can set

$$\sigma(r) = -a_0 MG c_\sigma r^\alpha \quad (4.28)$$

and

$$\theta(r) = -a_0 MG c_\theta r^\beta \quad (4.29)$$

This assumption is in part justified by the form of the non-relativistic correction of Eqs. (3.13). Then for $\alpha = \beta$ (the equations seem impossible to satisfy if α and β are different) one obtains for the scalar curvature

$$R = 0 + \alpha(2c_\sigma + (\alpha - 1)c_\theta) a_0 MG r^{\alpha-3} + O(a_0^2) \quad (4.30)$$

A first result can be obtained in the following way. Since in the ordinary Einstein case one has for a perfect fluid $R = -8\pi GT = +8\pi G(\rho - 3p)$, and since $\rho_m(r) \sim r^{\frac{1}{\nu}-3}$ from Eq. (3.17) in the same regime, one concludes that a solution is given by

$$\alpha = \frac{1}{\nu} \quad (4.31)$$

which also seems consistent with the Poisson equation result of Eq. (3.24).

Next one needs the action of \square^n on the point source (here hidden in T). To lowest order one has for the source term

$$8\pi G \rho = MG \frac{4}{\pi} \frac{\epsilon}{r^2(r^2 + \epsilon^2)} \quad (4.32)$$

The d'Alembertian then acts on the source term and gives

$$\square^n (8\pi G \rho) \rightarrow \frac{(2n+2)!}{2} M G \frac{4}{\pi} r^{-2n-3} \left(\frac{\epsilon}{r}\right) \quad (4.33)$$

which can then be analytically continued to $n = -\frac{1}{2\nu}$, resulting in

$$(\square)^{-\frac{1}{2\nu}} (8\pi G \rho) \rightarrow \frac{\Gamma(3 - \frac{1}{\nu})}{2} M G \frac{4}{\pi} r^{\frac{1}{\nu}-3} \left(\frac{\epsilon}{r}\right) \quad (4.34)$$

After multiplying the above expression by a_0 , consistency with l.h.s. of the trace equation, Eq. (4.30), is achieved to lowest order in a_0 provided again $\alpha = 1/\nu$. To zeroth order in a_0 , the correct solution is of course already built into the structure of Eqs. (4.25) and (4.26). Also note that setting $\epsilon = 0$ would give nonsensical results, and in particular the effective density would be zero for $r \neq 0$, in disagreement with the result of Eq. (3.17), $\rho_m(r) \sim r^{\frac{1}{\nu}-3}$ for small r .

The next step up would be the consideration of the action of \square on the point source, as it appears in the full effective field equations of Eq. (4.8), with again $T_{\mu\nu}$ described by Eq. (4.20). One perhaps surprising fact is the generation of an effective pressure term by the action of \square , suggesting that both terms should arise in the correct description of vacuum polarization effects,

$$\begin{aligned} (\square T_{\mu\nu})_{tt} &= -\frac{\rho B'^2}{2AB} + \frac{2B\rho'}{rA} - \frac{BA'\rho'}{2A^2} + \frac{B'\rho'}{2A} + \frac{B\rho''}{A} \\ (\square T_{\mu\nu})_{rr} &= -\frac{\rho B'^2}{2B^2} \end{aligned} \quad (4.35)$$

and $(\square T_{\mu\nu})_{\theta\theta} = (\square T_{\mu\nu})_{\varphi\varphi} = 0$. A similar effect, namely the generation of an effective vacuum pressure term in the field equations by the action of \square , is seen also in the Robertson-Walker metric case [12].

4.2 Effective trace equation

To check the overall consistency of the approach, consider the set of effective field equations that are obtained when the operator $(1 + A(\square))$ appearing in Eqs. (4.8) and (4.9) is moved over to the gravitational side, as in Eq. (4.15). Since the r.h.s of the field equations then vanishes for $r \neq 0$, one has apparently reduced the problem to one of finding vacuum solutions of a modified, non-local field equation.

Let us first look at the simpler trace equation, valid again for $r \neq 0$. If we denote by δR the lowest order variation (that is, of order a_0) in the scalar curvature over the ordinary vacuum solution $R = 0$, then one needs to find solutions to

$$\frac{1}{8\pi G A(\square)} \delta R = 0 \quad (4.36)$$

On a generic scalar function $F(r)$ one has the following action of the covariant d'Alembertian \square :

$$\square F(r) = -\frac{A'F'}{2A^2} + \frac{B'F'}{2AB} + \frac{2F'}{rA} + \frac{F''}{A} \quad (4.37)$$

The Ricci scalar is complicated enough, even in this simple case, and equal to

$$\frac{B'^2}{2AB^2} + \frac{A'B'}{2A^2B} - \frac{2B'}{rAB} + \frac{2A'}{rA^2} - \frac{B''}{AB} - \frac{2}{r^2A} + \frac{2}{r^2} \quad (4.38)$$

The action of the covariant d'Alembertian on it produces the rather formidable expression

$$\begin{aligned} & \frac{5B'^4}{2A^2B^4} + \frac{3A'B'^3}{A^3B^3} - \frac{5B'^3}{rA^2B^3} + \frac{3A'^2B'^2}{A^4B^2} - \frac{6A'B'^2}{rA^3B^2} - \frac{5A''B'^2}{4A^3B^2} - \frac{6B''B'^2}{A^2B^3} + \frac{B'^2}{r^2A^2B^2} \\ & + \frac{7A'^3B'}{2A^5B} - \frac{9A'^2B'}{rA^4B} - \frac{A'B'}{r^2A^3B} - \frac{13A'A''B'}{4A^4B} + \frac{4A''B'}{rA^3B} - \frac{23A'B''B'}{4A^3B^2} + \frac{9B''B'}{rA^2B^2} + \frac{A^{(3)}B'}{2A^3B} \\ & + \frac{5B^{(3)}B'}{2A^2B^2} - \frac{2B'}{r^3AB} + \frac{2B'}{r^3A^2B} + \frac{14A'^3}{rA^5} - \frac{4A'^2}{r^2A^4} + \frac{2B''^2}{A^2B^2} + \frac{2A'}{r^3A^2} - \frac{6A'}{r^3A^3} + \frac{2A''}{r^2A^3} \\ & - \frac{19A'^2B''}{4A^4B} + \frac{8A'B''}{rA^3B} + \frac{2A'B''}{A^3B} + \frac{2A^{(3)}}{rA^3} + \frac{3A'B^{(3)}}{A^3B} - \frac{4B^{(3)}}{rA^2B} - \frac{B^{(4)}}{A^2B} + \frac{4}{r^4A} - \frac{4}{r^4A^2} \end{aligned} \quad (4.39)$$

To lowest order in the functions σ and θ , from Eq. (4.27), the scalar curvature is given by

$$(GM(\sigma - \theta) - (2GM - r)(GM\theta' + (3GM - 2r)\sigma' + (2GM - r)r\theta'')) / (r^2(r - 2MG)^2) \quad (4.40)$$

After having the d'Alembertian \square act on this expression, one obtains the still formidable (here again to lowest order in σ and θ) result

$$\begin{aligned} & (\theta^{(4)}r^7 + 2\sigma^{(3)}r^6 - 8GM\theta^{(4)}r^6 - 4\sigma''r^5 + GM\theta^{(3)}r^5 - 15GM\sigma^{(3)}r^5 + 24G^2M^2\theta^{(4)}r^5 + 4\sigma'r^4 \\ & + 3GM\theta''r^4 + 31GM\sigma''r^4 - 6G^2M^2\theta^{(3)}r^4 + 42G^2M^2\sigma^{(3)}r^4 - 32G^3M^3\theta^{(4)}r^4 - 12GM\theta'r^3 \\ & - 28GM\sigma'r^3 - 14G^2M^2\theta''r^3 - 94G^2M^2\sigma''r^3 + 12G^3M^3\theta^{(3)}r^3 - 52G^3M^3\sigma^{(3)}r^3 + 16G^4M^4\theta^{(4)}r^3 \\ & + 12GM\theta r^2 - 12GM\sigma r^2 + 48G^2M^2\theta'r^2 + 96G^2M^2\sigma'r^2 + 20G^3M^3\theta''r^2 + 132G^3M^3\sigma''r^2 \\ & - 8G^4M^4\theta^{(3)}r^2 + 24G^4M^4\sigma^{(3)}r^2 - 24G^2M^2\theta r + 24G^2M^2\sigma r - 64G^3M^3\theta'r - 160G^3M^3\sigma'r \\ & - 8G^4M^4\theta''r - 72G^4M^4\sigma''r + 16G^3M^3\theta - 16G^3M^3\sigma + 32G^4M^4\theta' + 96G^4M^4\sigma') / \\ & \left((2GM - r)^3r^5 \right) \end{aligned} \quad (4.41)$$

Higher powers of the d'Alembertian \square then lead to even more complicated expressions, with increasingly higher derivatives of $\sigma(r)$ and $\theta(r)$. Assuming a power law correction, as in Eqs. (4.28) and (4.29), with $\alpha = \beta$, as in Eq. (4.30),

$$R = a_0 M G \alpha (2c_\sigma + c_\theta(\alpha - 1)) r^{\alpha-3} + O(a_0^2) \quad (4.42)$$

and then

$$\begin{aligned}\square R &\rightarrow a_0 M G r^{\alpha-5} (2c_\sigma + c_\theta (\alpha - 1)) \alpha (\alpha - 2) (\alpha - 3) \\ \square^2 R &\rightarrow a_0 M G r^{\alpha-7} (2c_\sigma + c_\theta (\alpha - 1)) \alpha (\alpha - 2) (\alpha - 3) (\alpha - 4) (\alpha - 5)\end{aligned}\quad (4.43)$$

and so on, and for general $n \rightarrow +\frac{1}{2\nu}$

$$\square^n R \rightarrow a_0 M G (2c_\sigma + c_\theta (\alpha - 1)) \frac{\alpha \Gamma(2 + \frac{1}{\nu} - \alpha)}{\Gamma(2 - \alpha)} r^{\alpha - \frac{1}{\nu} - 3} \quad (4.44)$$

Therefore the only possible power solution for $r \gg MG$ is $\alpha = 0, 2 \dots \frac{1}{\nu} + 1$, with c_σ and c_θ unconstrained to this order.

4.3 Full effective field equations

Next we examine the full effective field equations (as opposed to just their trace part) as in Eq. (4.15) with $\lambda = 0$,

$$\frac{1}{8\pi G (1 + A(\square))} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0 \quad (4.45)$$

valid for $r \neq 0$. If one denotes by $\delta G_{\mu\nu} \equiv \delta \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right)$ the lowest order variation (that is, of order a_0) in the Einstein tensor over the ordinary vacuum solution $G_{\mu\nu} = 0$, then one has

$$\frac{1}{8\pi G A(\square)} \delta \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0 \quad (4.46)$$

again for $r \neq 0$. Here the covariant d'Alembertian operator

$$\square = g^{\mu\nu} \nabla_\mu \nabla_\nu \quad (4.47)$$

acts on a second rank tensor,

$$\begin{aligned}\nabla_\nu T_{\alpha\beta} &= \partial_\nu T_{\alpha\beta} - \Gamma_{\alpha\nu}^\lambda T_{\lambda\beta} - \Gamma_{\beta\nu}^\lambda T_{\alpha\lambda} \equiv I_{\nu\alpha\beta} \\ \nabla_\mu (\nabla_\nu T_{\alpha\beta}) &= \partial_\mu I_{\nu\alpha\beta} - \Gamma_{\nu\mu}^\lambda I_{\lambda\alpha\beta} - \Gamma_{\alpha\mu}^\lambda I_{\nu\lambda\beta} - \Gamma_{\beta\mu}^\lambda I_{\nu\alpha\lambda}\end{aligned}\quad (4.48)$$

and would thus seem to require the calculation of as many as 1920 terms, of which many fortunately vanish by symmetry. In the static isotropic case the components of the Einstein tensor are given by

$$\begin{aligned}G_{tt} &= \frac{A'B}{rA^2} - \frac{B}{r^2A} + \frac{B}{r^2} \\ G_{rr} &= -\frac{A}{r^2} + \frac{B'}{rB} + \frac{1}{r^2} \\ G_{\theta\theta} &= -\frac{B'^2 r^2}{4AB^2} - \frac{A'B'r^2}{4A^2B} + \frac{B''r^2}{2AB} - \frac{A'r}{2A^2} + \frac{B'r}{2AB} \\ G_{\varphi\varphi} &= \sin^2 \theta G_{\theta\theta}\end{aligned}\quad (4.49)$$

Using again the expansion of Eqs. (4.25) and (4.26) one obtains for the tt , rr and $\theta\theta$ components of the Einstein tensor, to lowest order in σ and θ ,

$$\begin{aligned}
G_{tt} &\simeq \frac{(2GM - r) \sigma'}{r^3} \\
G_{rr} &\simeq \frac{-\theta + \sigma + (-2GM + r) \theta'}{r(-2GM + r)^2} \\
G_{\theta\theta} &\simeq \frac{(-GM + r) (\theta - \sigma + (2GM - r) (\theta' - \sigma')) + r(-2GM + r)^2 \theta''}{2(-2GM + r)^2} \\
G_{\varphi\varphi} &= \sin^2 \theta G_{\theta\theta}
\end{aligned} \tag{4.50}$$

After acting with \square on this expression one finds a rather complicated result. Here we will only list $(\square G)_{tt}$:

$$\begin{aligned}
&\frac{6BA^3}{rA^5} + \frac{2BA^2}{r^2A^4} - \frac{4B'A^2}{rA^4} - \frac{2BA'}{r^3A^3} - \frac{6BA''A'}{rA^4} + \frac{B''A'}{rA^3} + \frac{6B}{r^4A} \\
&- \frac{6B}{r^4A^2} - \frac{4B'}{r^3A} + \frac{4B'}{r^3A^2} - \frac{BA''}{r^2A^3} + \frac{2B'A''}{rA^3} + \frac{B''}{r^2A} - \frac{B''}{r^2A^2} + \frac{BA^{(3)}}{rA^3}
\end{aligned} \tag{4.51}$$

To lowest order in $\sigma(r)$ and $\theta(r)$ one finds the slightly simpler expressions

$$\begin{aligned}
(\square G)_{tt} = & -(-2G^2M^2\theta + 2G^2M^2\sigma - 4G^3M^3\theta' + 2G^2M^2r\theta' - 28G^3M^3\sigma' + 38G^2M^2r\sigma' \\
& -16GMr^2\sigma' + 2r^3\sigma' + 24G^3M^3r\sigma'' - 32G^2M^2r^2\sigma'' + 14GMr^3\sigma'' \\
& -2r^4\sigma'' - 8G^3M^3r^2\sigma^{(3)} + 12G^2M^2r^3\sigma^{(3)} - 6GMr^4\sigma^{(3)} + r^5\sigma^{(3)})/ \\
& (r^6(-2GM + r))
\end{aligned} \tag{4.52}$$

$$\begin{aligned}
(\square G)_{rr} = & (6G^2M^2\theta - 4GMr\theta - 6G^2M^2\sigma + 4GMr\sigma + 12G^3M^3\theta' - 14G^2M^2r\theta' \\
& +4GMr^2\theta' - 12G^3M^3\sigma' + 6G^2M^2r\sigma' + 4GMr^2\sigma' - 2r^3\sigma' + 8G^3M^3r\theta'' \\
& -12G^2M^2r^2\theta'' + 6GMr^3\theta'' - r^4\theta'' + 4G^2M^2r^2\sigma'' - 4GMr^3\sigma'' + r^4\sigma'' \\
& -8G^3M^3r^2\theta^{(3)} + 12G^2M^2r^3\theta^{(3)} - 6GMr^4\theta^{(3)} + r^5\theta^{(3)})/ \\
& (r^4(-2GM + r)^3)
\end{aligned} \tag{4.53}$$

$$\begin{aligned}
(\square G)_{\theta\theta} = & (24G^3M^3\theta - 36G^2M^2r\theta + 16GMr^2\theta - 24G^3M^3\sigma + 36G^2M^2r\sigma - 16GMr^2\sigma \\
& +48G^4M^4\theta' - 96G^3M^3r\theta' + 68G^2M^2r^2\theta' - 16GMr^3\theta' + 16G^4M^4\sigma' - 32G^3M^3r\sigma' \\
& +28G^2M^2r^2\sigma' - 16GMr^3\sigma' + 4r^4\sigma' + 8G^4M^4r\theta'' - 12G^3M^3r^2\theta'' + 10G^2M^2r^3\theta''
\end{aligned}$$

$$\begin{aligned}
& -5GM r^4 \theta'' + r^5 \theta'' - 24G^4 M^4 r \sigma'' + 52G^3 M^3 r^2 \sigma'' - 46G^2 M^2 r^3 \sigma'' + 19GM r^4 \sigma'' \\
& -3r^5 \sigma'' - 24G^4 M^4 r^2 \theta^{(3)} + 44G^3 M^3 r^3 \theta^{(3)} - 30G^2 M^2 r^4 \theta^{(3)} + 9GM r^5 \theta^{(3)} - r^6 \theta^{(3)} \\
& +8G^4 M^4 r^2 \sigma^{(3)} - 20G^3 M^3 r^3 \sigma^{(3)} + 18G^2 M^2 r^4 \sigma^{(3)} - 7GM r^5 \sigma^{(3)} + r^6 \sigma^{(3)} \\
& +16G^4 M^4 r^3 \theta^{(4)} - 32G^3 M^3 r^4 \theta^{(4)} + 24G^2 M^2 r^5 \theta^{(4)} - 8GM r^6 \theta^{(4)} + r^7 \theta^{(4)} / \\
& \left(2r^3 (-2GM + r)^3 \right) \tag{4.54}
\end{aligned}$$

with the $\varphi\varphi$ component proportional to the $\theta\theta$ component. If one again assumes that the corrections are given by a power, as in Eqs. (4.28) and (4.29), with $\alpha = \beta$, then one has to zeroth order

$$\begin{aligned}
G_{tt} &= a_0 M G c_\sigma \alpha r^{\alpha-3} \\
G_{rr} &= -a_0 M G (c_\sigma + c_\theta(\alpha - 1)) r^{\alpha-3} \\
G_{\theta\theta} &= -\frac{1}{2} a_0 M G (c_\sigma + c_\theta(\alpha - 1)) (\alpha - 1) r^{\alpha-1} \tag{4.55}
\end{aligned}$$

with the $\varphi\varphi$ component again proportional to the $\theta\theta$ component. Applying \square on the above Einstein tensor one then gets

$$\begin{aligned}
(\square G)_{tt} &= a_0 G M c_\sigma \alpha(\alpha - 2)(\alpha - 3) r^{\alpha-5} \\
(\square G)_{rr} &= -a_0 G M (c_\sigma + c_\theta(\alpha - 1)) \alpha(\alpha - 3) r^{\alpha-5} \\
(\square G)_{\theta\theta} &= -\frac{1}{2} a_0 G M (c_\sigma + c_\theta(\alpha - 1)) \alpha(\alpha - 3)^2 r^{\alpha-3} \tag{4.56}
\end{aligned}$$

again with the $\varphi\varphi$ component proportional to the $\theta\theta$ component. Applying \square again one obtains

$$\begin{aligned}
(\square^2 G)_{tt} &= a_0 G M c_\sigma \alpha(\alpha - 2)(\alpha - 3)(\alpha - 4)(\alpha - 5) r^{\alpha-7} \\
(\square^2 G)_{rr} &= -a_0 G M (c_\sigma + c_\theta(\alpha - 1)) \alpha(\alpha - 2)(\alpha - 3)(\alpha - 5) r^{\alpha-7} \\
(\square^2 G)_{\theta\theta} &= -\frac{1}{2} a_0 G M (c_\sigma + c_\theta(\alpha - 1)) \alpha(\alpha - 2)(\alpha - 3)(\alpha - 5)^2 r^{\alpha-5} \tag{4.57}
\end{aligned}$$

and so on, and for general $n \rightarrow +\frac{1}{2\nu}$

$$\begin{aligned}
(\square^n G)_{tt} &\rightarrow a_0 G M c_\sigma \frac{\Gamma(2 + \frac{1}{\nu} - \alpha)}{(\alpha - 1) \Gamma(-\alpha)} r^{\alpha-3-\frac{1}{\nu}} \\
(\square^n G)_{rr} &\rightarrow -a_0 G M (c_\sigma + c_\theta(\alpha - 1)) \frac{\Gamma(2 + \frac{1}{\nu} - \alpha)}{(\alpha - 1) \left(\alpha - \frac{1}{\nu}\right) \Gamma(-\alpha)} r^{\alpha-3-\frac{1}{\nu}} \\
(\square^n G)_{\theta\theta} &\rightarrow -\frac{1}{2} a_0 G M (c_\sigma + c_\theta(\alpha - 1)) \frac{\left(\alpha - 1 - \frac{1}{\nu}\right) \Gamma(2 + \frac{1}{\nu} - \alpha)}{(\alpha - 1) \left(\alpha - \frac{1}{\nu}\right) \Gamma(-\alpha)} r^{\alpha-1-\frac{1}{\nu}} \tag{4.58}
\end{aligned}$$

Inspection of the above results reveals a common factor $1/\Gamma(-\alpha)$, which would allow only integer powers $\alpha = 0, 1, 2, \dots$, but the additional factor of $1/(\alpha - 1)$ excludes $\alpha = 1$ from being a solution.

Even for α close to $1/\nu$ (as expected on the basis of the non-relativistic expression of Eq. (3.24), as well as from Eq. (4.31)) $\nu \sim 1/\alpha - \epsilon$ only integer values $\alpha = 2, 3, 4 \dots$ are allowed. For the covariant divergences $\nabla^\mu (\square^n G)_{\mu\nu}$ one has

$$\nabla^\mu (\delta G)_{\mu r} = 0 \quad (4.59)$$

and, at the next order,

$$\nabla^\mu (\square G)_{\mu r} = 2 a_0 G^2 M^2 \alpha (\alpha - 3) (c_\sigma + c_\theta(1 - \alpha)) r^{\alpha-7} \quad (4.60)$$

with the other components vanishing identically, and

$$\nabla^\mu (\square^2 G)_{\mu r} = 4 a_0 G^2 M^2 \alpha (\alpha - 3) (\alpha - 5)^2 (c_\sigma + c_\theta(1 - \alpha)) r^{\alpha-9} \quad (4.61)$$

again with the other components of the divergence vanishing identically.

In general the problem of finding a complete general solution to the effective field equations by this method lies in the difficulty of computing arbitrarily high powers of \square on general functions such as $\sigma(r)$ and $\theta(r)$, which eventually involve a large number of derivatives. Assuming for these functions a power law dependence on r simplifies the problem considerably, but also restricts the kind of solutions that one is likely to find. More specifically, if the solution involves (say for small r , but still with $r \gg 2MG$) a term of the type $r^\alpha \ln mr$, as in Eqs. (3.21), (5.35) and (5.38) for $\nu \rightarrow 1/3$, then this method will have to be dealt with very carefully. This is presumably the reason why in some of the Γ -function coefficients encountered here one finds a power solution (in fact $\alpha = 3$) for ν close to a third, but one gets indeterminate expression if one sets exactly $\alpha = 1/\nu = 3$.

5 The Quantum Vacuum as a Fluid

The discussion of Sec. 3 suggests that the quantum correction due to the running of G can be described, at least in the non-relativistic limit of Eq. (2.2) as applied to Poisson's equation, in terms of a vacuum energy density $\rho_m(r)$, distributed around the static source of strength M in accordance with the result of Eqs. (3.17) and Eq. (3.19). These expressions, in turn, can be obtained by Fourier transforming back to real space the original result for $G(k^2)$ of Eq. (2.2).

Furthermore, it was shown in Sec. 4 (and was discovered in [12] as well, see for example Eq. (7.8) later in this paper) that a manifestly covariant implementation of the running of G , via the $G(\square)$ given in Eqs. (4.2) and (4.7), will induce a non-vanishing effective pressure term. This result can be seen clearly, in the case of the static isotropic metric, for example from the result of Eq. (4.35).

We will therefore, in this section, consider a relativistic perfect fluid, with energy-momentum tensor

$$T_{\mu\nu} = [p + \rho] u_\mu u_\nu + g_{\mu\nu} p \quad (5.1)$$

which in the static isotropic case reduces to Eq. (4.13),

$$T_{\mu\nu} = \text{diag}[B(r)\rho(r), A(r)p(r), r^2 p(r), r^2 \sin^2 \theta p(r)] \quad (5.2)$$

and gives a trace $T = 3p - \rho$.

The tt , rr and $\theta\theta$ components of the field equations then read

$$-\lambda B(r) + \frac{A'(r)B(r)}{rA(r)^2} - \frac{B(r)}{r^2 A(r)} + \frac{B(r)}{r^2} = 8\pi G B(r)\rho(r) \quad (5.3)$$

$$\lambda A(r) - \frac{A(r)}{r^2} + \frac{B'(r)}{rB(r)} + \frac{1}{r^2} = 8\pi G A(r)p(r) \quad (5.4)$$

$$-\frac{B'(r)^2 r^2}{4A(r)B(r)^2} + \lambda r^2 - \frac{A'(r)B'(r)r^2}{4A(r)^2 B(r)} + \frac{B''(r)r^2}{2A(r)B(r)} - \frac{A'(r)r}{2A(r)^2} + \frac{B'(r)r}{2A(r)B(r)} = 8G\pi r^2 p(r) \quad (5.5)$$

with the $\varphi\varphi$ component equal to $\sin^2 \theta$ times the $\theta\theta$ component.

Energy conservation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$[p(r) + \rho(r)] \frac{B'(r)}{2B(r)} + p'(r) = 0 \quad (5.6)$$

and forces a definite relationship between $B(r)$, $\rho(r)$ and $p(r)$. The three field equations and the energy conservation equation are, as usual, not independent, because of the Bianchi identity.

It seems reasonable to attempt to solve the above equations (usually considered in the context of relativistic stellar structure [67]) with the density $\rho(r)$ given by the $\rho_m(r)$ of Eqs. (3.17), (3.18) and (3.19).

This of course raises the question of how the relativistic pressure $p(r)$ should be chosen, an issue that the non-relativistic calculation did not have to address. We will argue below that covariant energy conservation completely determines the pressure in the static case, leading to consistent equations and solutions (note that in particular it would not be consistent to take $p(r) = 0$).

Since the function $B(r)$ drops out of the tt field equation, the latter can be integrated immediately, giving

$$A(r)^{-1} = 1 + \frac{c_1}{r} - \frac{\lambda}{3} r^2 - \frac{8\pi G}{r} \int_0^r dx x^2 \rho(x) \quad (5.7)$$

which suggests the introduction of a function $m(r)$

$$m(r) \equiv 4\pi \int_0^r dx x^2 \rho(x) \quad (5.8)$$

It also seems natural in our case to identify $c_1 = -2MG$, which of course corresponds to the solution with $a_0 = 0$ ($p = \rho = 0$) (equivalently, the point source at the origin of strength M could be included as an additional δ -function contribution to $\rho(r)$).

Next, the rr field equation can be solved for $B(r)$,

$$B(r) = \exp \left\{ c_2 - \int_{r_0}^r dy \frac{1 + A(y) (\lambda y^2 - 8\pi G y^2 p(y) - 1)}{y} \right\} \quad (5.9)$$

with the constant c_2 again determined by the requirement that the above expression for $B(r)$ reduce to the standard Schwarzschild solution for $a_0 = 0$ ($p = \rho = 0$), giving $c_2 = \ln(1 - 2MG/r_0 - \lambda r_0^2/3)$. The last task left therefore is the determination of the pressure $p(r)$.

Using the rr field equation, $B'(r)/B(r)$ can be expressed in term of $A(r)$ in the energy conservation equation, which results in

$$2r p'(r) - \left[1 + A(r) (\lambda r^2 - 8\pi G r^2 p(r) - 1) \right] (p(r) + \rho(r)) = 0 \quad (5.10)$$

Inserting the explicit expression for $A(r)$, from Eq. (5.7), one obtains

$$p'(r) + \frac{\left(8\pi G r^3 p(r) + 2MG - \frac{2}{3}\lambda r^3 + 8\pi G \int_{r_0}^r dx x^2 \rho(x) \right) (p(r) + \rho(r))}{2r \left(r - 2MG - \frac{\lambda}{3} r^3 - 8\pi G \int_0^r dx x^2 \rho(x) \right)} = 0 \quad (5.11)$$

which is usually referred to as the equation of hydrostatic equilibrium. From now on we will focus only the case $\lambda = 0$. Then

$$p'(r) + \frac{(8\pi G r^3 p(r) + 2MG + 8\pi G \int_0^r dx x^2 \rho(x)) (p(r) + \rho(r))}{2r (r - 2MG - 8\pi G \int_0^r dx x^2 \rho(x))} = 0 \quad (5.12)$$

The last equation, a non-linear differential equation for $p(r)$, can be solved to give the desired solution $p(r)$, which then, by equation Eq. (5.9), determines the remaining function $B(r)$.

In our case though it will be sufficient to solve the above equation for small a_0 , where a_0 (see Eq. (2.2) and Eq. (3.17)) is the dimensionless parameter which, when set to zero, makes the solution revert back to the classical one.

It will also be convenient to pull out of $A(r)$ and $B(r)$ the Schwarzschild solution part, by introducing the small corrections $\sigma(r)$ and $\theta(r)$ (already defined before in Eqs. (4.25) and (4.26)), both of which are expected to be proportional to the parameter a_0 . One has

$$\sigma(r) = -8\pi G \int_0^r dx x^2 \rho(x) \equiv -2m(r)G \quad (5.13)$$

and

$$\theta(r) = \exp \left\{ c_2 + \int_{r_0}^r dy \frac{1 + 8\pi G y^2 p(y)}{y - 2MG - 8\pi G \int_0^y dx x^2 \rho(x)} \right\} + 2MG - r \quad (5.14)$$

Again, the integration constant c_2 needs to be chosen here so that the normal Schwarzschild solution is recovered for $p = \rho = 0$.

To order a_0 the resulting equation for $p(r)$, from Eq. (5.12), is

$$\frac{MG(p(r) + \rho(r))}{r(r - 2MG)} + p'(r) \simeq 0 \quad (5.15)$$

Note that in regions where $p(r)$ is slowly varying, $p'(r) \sim 0$, one has $p \simeq -\rho$, i.e. the fluid contribution is acting like a cosmological constant term with $\sigma(r) \sim \theta(r) \sim -(\rho/3)r^3$.

The last differential equation can then be solved for $p(r)$,

$$p_m(r) = \frac{1}{\sqrt{1 - \frac{2MG}{r}}} \left(c_3 - \int_{r_0}^r dz \frac{MG\rho(z)}{z^2 \sqrt{1 - \frac{2MG}{z}}} \right) \quad (5.16)$$

where the constant of integration has to be chosen so that when $\rho(r) = 0$ (no quantum correction) one has $p(r) = 0$ as well. Because of the singularity in the integrand at $r = 2MG$, we will take the lower limit in the integral to be $r_0 = 2MG + \epsilon$, with $\epsilon \rightarrow 0$.

To proceed further, one needs the explicit form for $\rho_m(r)$, which was given in Eqs. (3.17), (3.18) and (3.19). The required integrands involve for general ν the modified Bessel function $K_n(x)$, and can be therefore a bit complicated. But in some special cases the general form of the density ρ_m of Eq. (3.17)

$$\rho_m(r) = \frac{1}{8\pi} c_\nu a_0 M m^3 (mr)^{-\frac{1}{2}(3-\frac{1}{\nu})} K_{\frac{1}{2}(3-\frac{1}{\nu})}(mr) \quad (5.17)$$

reduces to a relatively simple expression, which we will list here. For $\nu = 1$ one has

$$\rho_m(r) = \frac{1}{2\pi^2} a_0 M m^3 \frac{1}{mr} K_1(mr) \quad (5.18)$$

whereas for $\nu = 1/2$ one has

$$\rho_m(r) = \frac{1}{4\pi} a_0 M m^3 \frac{1}{mr} e^{-mr} \quad (5.19)$$

and for $\nu = 1/3$

$$\rho_m(r) = \frac{1}{2\pi^2} a_0 M m^3 K_0(mr) \quad (5.20)$$

and finally for $\nu = 1/4$

$$\rho_m(r) = \frac{1}{8\pi} a_0 M m^3 e^{-mr} \quad (5.21)$$

Note that $\rho_m(r)$ diverges at small r for $\nu \geq 1/3$

Here we will limit our investigation to the small r ($mr \ll 1$) and large r ($mr \gg 1$) behavior. Since $m = 1/\xi$ is very small, the first limit appears to be of greater physical interest.

5.1 Small r limit

For small r the density $\rho_m(r)$ has the following behavior (see Eq. (3.17)),

$$\rho_m(r) \underset{r \rightarrow 0}{\sim} A_0 r^{\frac{1}{\nu}-3} \quad (5.22)$$

for $\nu > 1/3$, with

$$A_0 \equiv \frac{c_k c_\nu}{8\pi} a_0 M m^{\frac{1}{\nu}} = \frac{|\sec(\frac{\pi}{2\nu})|}{4\pi \Gamma(\frac{1}{\nu} - 1)} a_0 M m^{\frac{1}{\nu}} \quad (5.23)$$

where the dimensionless positive constant c_k is determined from the small x behavior of the modified Bessel function $K_n(x)$,

$$x^{\frac{1}{2}(\frac{1}{\nu}-3)} K_{\frac{1}{2}(3-\frac{1}{\nu})}(x) \underset{x \rightarrow 0}{\sim} -\frac{2^{\frac{1}{2}(1-\frac{1}{\nu})} \pi \sec(\frac{\pi}{2\nu})}{\Gamma(\frac{1}{2\nu} - \frac{1}{2})} x^{\frac{1}{\nu}-3} \equiv c_k x^{\frac{1}{\nu}-3} \quad (5.24)$$

valid for $\nu > 1/3$, and c_ν is given in Eq. (3.18). For $\nu < 1/3$ $\rho_m(r) \sim \text{const. } a_0 M m^3$, independent of r . For $\nu = 1/3$ the expression for $\rho_m(r)$ in Eq. (5.20) should be used instead.

Therefore in this limit, with $\frac{1}{3} < \nu < 1$, one has

$$m(r) \simeq 4\pi\nu A_0 r^{\frac{1}{\nu}} \quad (5.25)$$

and, from the definition of $\sigma(r)$,

$$\sigma(r) \simeq -2m(r)G = -8\pi\nu G A_0 r^{\frac{1}{\nu}} \quad (5.26)$$

and finally

$$A^{-1}(r) = 1 - \frac{2MG}{r} - 2a_0 M G c_s m^{\frac{1}{\nu}} r^{\frac{1}{\nu}-1} + \dots \quad (5.27)$$

with the constant $c_s = \nu |\sec(\frac{\pi}{2\nu})| / \Gamma(\frac{1}{\nu} - 1)$. For $\nu = 1/3$ the last contribution is indistinguishable from a cosmological constant term $-\frac{\lambda}{3}r^2$, except for the fact that the coefficient here is quite different, being proportional to $\sim a_0 M G m^3$.

To determine the pressure, we suppose that it as well has a power dependence on r in the regime under consideration, $p_m(r) = c_p A_0 r^\gamma$, where c_p is a numerical constant, and then substitute $p_m(r)$ into the pressure equation Eq. (5.15). This gives, past the horizon $r \gg 2MG$,

$$(2\gamma - 1) c_p M G r^{\gamma-1} - c_p \gamma r^\gamma - M G r^{1/\nu-4} \simeq 0 \quad (5.28)$$

giving the same power $\gamma = 1/\nu - 3$ as for $\rho(r)$, $c_p = -1$ and surprisingly also $\gamma = 0$, implying that in this regime only $\nu = 1/3$ gives a consistent solution. Again, the resulting correction is quite similar

to what one would expect from a cosmological term, with an effective $\lambda_m/3 \simeq 8\pi\nu a_0 M G m^{\frac{1}{\nu}}$. One then has for ν near $1/3$

$$p_m(r) = A_0 c_p r^{\frac{1}{\nu}-3} + \dots \quad (5.29)$$

and thus from Eq. (5.15)

$$B(r) = 1 - \frac{2MG}{r} - 2a_0 M G c_s m^{\frac{1}{\nu}} r^{\frac{1}{\nu}-1} + \dots \quad (5.30)$$

Both the result for $A(r)$ in Eq. (5.27), and the above result for $B(r)$ are, for $r \gg 2MG$, consistent with a gradual slow increase in G in accordance with the formula

$$G \rightarrow G(r) = G \left(1 + a_0 c_s m^{\frac{1}{\nu}} r^{\frac{1}{\nu}} + \dots\right) \quad (5.31)$$

We note here that both expressions for $A(r)$ and $B(r)$ have some similarities with the approximate non-relativistic (Poisson equation) result of Eq. (3.24), with the correction proportional to a_0 agreeing roughly in magnitude (but not in sign).

The case $\nu = 1/3$ requires a special treatment, since the coefficient c_k in Eq. (5.24) diverges as $\nu \rightarrow 1/3$. Starting from the expression for $\rho_m(r)$ for $\nu = 1/3$ in Eq. (5.20),

$$\rho_m(r) = \frac{1}{2\pi^2} a_0 M m^3 K_0(mr) \quad (5.32)$$

one has for small r

$$\rho_m(r) = -\frac{a_0}{2\pi^2} M m^3 \left(\ln \frac{mr}{2} + \gamma\right) + \dots \quad (5.33)$$

and therefore from Eq. (5.14),

$$\sigma(r) = \frac{4a_0 M G m^3}{3\pi} r^3 \ln(mr) + \dots \quad (5.34)$$

and consequently

$$A^{-1}(r) = 1 - \frac{2MG}{r} + \frac{4a_0 M G m^3}{3\pi} r^2 \ln(mr) + \dots \quad (5.35)$$

From Eq. (5.15) one can then obtain an expression for the pressure $p_m(r)$, and one finds

$$p_m(r) = \frac{a_0 M m^3 \log(mr)}{2\pi^2} - \frac{a_0 M m^3 \log\left(r + r\sqrt{1 - \frac{2MG}{r}} - MG\right)}{2\pi^2 \sqrt{1 - \frac{2MG}{r}}} + \frac{a_0 M m^3}{\pi^2} + \frac{a_0 M m^3 c_3}{2\pi^2 \sqrt{1 - \frac{2MG}{r}}} \quad (5.36)$$

where c_3 is again an integration constant. Here we will be content with the $r \gg 2MG$ limit of the above expression, which we shall write therefore as

$$p_m(r) = \frac{a_0}{2\pi^2} M m^3 \ln(mr) + \dots \quad (5.37)$$

After performing the required r integral in Eq. (5.14), and evaluating the resulting expression in the limit $r \gg 2MG$, one obtains an expression for $\theta(r)$, and consequently from it

$$B(r) = 1 - \frac{2MG}{r} + \frac{4a_0MGm^3}{3\pi} r^2 \ln(mr) + \dots \quad (5.38)$$

The expressions for $A(r)$ and $B(r)$ are, for $r \gg 2MG$, consistent with a gradual slow increase in G in accordance with the formula

$$G \rightarrow G(r) = G \left(1 + \frac{a_0}{3\pi} m^3 r^3 \ln \frac{1}{m^2 r^2} + \dots \right) \quad (5.39)$$

and therefore consistent as well with the original result of Eqs. (2.1) or (2.2), namely that the classical laboratory value of G is obtained for $r \ll \xi$. In fact it is reassuring that the renormalization properties of $G(r)$ as inferred from $A(r)$ are the same as what one finds from $B(r)$. Note that the correct relativistic small r correction of Eq. (5.39) agrees roughly in magnitude (but not in sign) with the approximate non-relativistic, Poisson equation result of Eq. (3.21).

One further notices some similarities, as well as some rather substantial differences, with the corresponding QED result of Eq. (2.67). In the gravity case, the correction vanishes as r goes to zero: in this limit one is probing the bare mass, unencumbered by its virtual graviton vacuum polarization cloud. On the other hand, in the QED case, as one approaches the source one is probing the bare charge, unscreened by the electron's vacuum polarization cloud, and whose magnitude diverges logarithmically for small r .

It should be recalled here that neither function $A(r)$ or $B(r)$ are directly related to the relativistic potential for particle orbits, which is given instead by the combination

$$V_{eff}(r) = \frac{1}{2A(r)} \left[\frac{l^2}{r^2} - \frac{1}{B(r)} + 1 \right] \quad (5.40)$$

where l is proportional to the orbital angular momentum of the test particle [68].

Furthermore, from the metric of Eqs. (5.27) and (5.30) one finds for $\nu \rightarrow 1/3$ the following results for the curvature invariants

$$\begin{aligned} R^2 &= 1024A_0^2G^2\pi^2 \\ R_{\mu\nu}R^{\mu\nu} &= 256A_0^2G^2\pi^2 \\ R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} &= 16G^2 \left(\frac{32\pi^2A_0^2}{3} + \frac{3M^2}{r^6} \right) \end{aligned} \quad (5.41)$$

which are non-singular at $r = 2MG$, and again consistent with an effective mass density around the source $m(r) \propto r^3$.

5.2 Large r limit

For large r one has instead, from Eq. (3.17) for $\rho_m(r)$,

$$\rho_m(r) \underset{r \rightarrow \infty}{\sim} A_0 r^{\frac{1}{2\nu}-2} e^{-mr} \quad (5.42)$$

with $A_0 = 1/\sqrt{128\pi} c_\nu a_0 M m^{1+\frac{1}{2\nu}}$. In the same limit, the integration constants is chosen so that the solution for $A(r)$ and $B(r)$ at large r corresponds to a mass $M' = (1+a_0)M$ (see the expression for the integrated density in Eq. (3.19)), or equivalently

$$\sigma(r) \sim \theta(r) \underset{r \rightarrow \infty}{\sim} -2 a_0 M G \quad (5.43)$$

On then recovers a result similar to the non-relativistic expression of Eqs. (3.7), (3.8) and (3.12), with $G(r)$ approaching the constant value $G_\infty = (1+a_0)G$, up to exponentially small corrections in mr at large r .

In conclusion, it appears that a solution to relativistic static isotropic problem of the running gravitational constant can be found, provided that the exponent ν in either Eq. (2.2) or Eq. (4.8) is close to one third. This last result seems to be linked with the fact that the running coupling term acts in some way like a local cosmological constant term, for which the r dependence of the vacuum solution for small r is fixed by the nature of the Schwarzschild solution with a cosmological constant term¹³.

6 Distortion of the Gravitational Wave Spectrum

A scale-dependent gravitational constant $G(k^2)$ will cause slight distortions in the spectrum of gravitational radiation at extremely low frequencies, to some extent irrespective of the nature of the perturbations that cause them. From the field equations with $\lambda = 0$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \quad (6.1)$$

one obtains in the weak field limit with harmonic gauge condition

$$\square h_{\mu\nu} = 8\pi G \bar{T}_{\mu\nu} \quad (6.2)$$

¹³In $d \geq 4$ dimensions the Schwarzschild solution to Einstein gravity with a cosmological term is [69] $A^{-1}(r) = B(r) = 1 - 2MGc_d r^{3-d} - \frac{2\lambda}{(d-2)(d-1)} r^2$, with $c_d = 4\pi\Gamma(\frac{d-1}{2})/(d-2)\pi^{\frac{d-1}{2}}$, which would suggest, in analogy with the results for $d = 4$ given in this section, that in $d \geq 4$ dimensions only $\nu = 1/(d-1)$ is possible. This last result would also be in agreement with the exact value $\nu = 0$ found at $d = \infty$ [15]

with as usual

$$\bar{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T_{\lambda}{}^{\lambda} \quad (6.3)$$

Density perturbations $\delta\rho(\mathbf{x}, t)$ will enter the r.h.s. of the field equations and give rise to gravitational waves with Fourier components

$$h_{\mu\nu}(k) = -8\pi G \frac{1}{k^2} \bar{T}_{\mu\nu}(p, \rho)(k) \quad (6.4)$$

giving for the power spectrum of transverse traceless (gravitational wave) modes

$$P_{TT}(k^2) \simeq k^3 |h_{TT}(k)|^2 = (8\pi)^2 G^2 \frac{1}{k} |\bar{T}(p, \rho)(k)|^2 \quad (6.5)$$

A scale dependent gravitational constant, with variation in accordance with Eq. (2.2),

$$G \rightarrow G(k^2) \quad (6.6)$$

would affect the spectrum of very long wavelength modes via

$$P_{TT}(k^2) \simeq k^3 |h_{\mu\nu}(k)|^2 = (8\pi)^2 G^2(k^2) \frac{1}{k} |\bar{T}_{\mu\nu}(p, \rho)|^2 \quad (6.7)$$

Specifically, according to the expression in Eq. (2.2) for the running of the gravitational constant,

$$\frac{G(k^2)}{G} \simeq 1 + a_0 \left(\frac{m^2}{k^2 + m^2} \right)^{\frac{1}{2\nu}} + \dots \quad (6.8)$$

one has for the tensor power spectrum

$$P_{TT}(k^2) \simeq k^3 |h_{\mu\nu}(k)|^2 = (8\pi)^2 G^2 \frac{1}{k} \left[1 + a_0 \left(\frac{m^2}{k^2 + m^2} \right)^{\frac{1}{2\nu}} \right]^2 |\bar{T}_{\mu\nu}(p, \rho)|^2 \quad (6.9)$$

with the expression in square brackets varying perhaps by as much as an order of magnitude from short wavelengths $k \gg 1/\xi$, to very long wavelengths $k \sim 1/\xi$.

7 Quantum Cosmology - An Addendum

In this section we will discuss briefly what modifications are expected when one uses Eq. (2.2) instead of Eq. (2.1) in the effective field equations. In [12] cosmological solutions within the Friedmann-Robertson-Walker (FRW) framework were discussed, starting from the quantum effective field equations of Eq. (2.52),

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G (1 + A(\square)) T_{\mu\nu} \quad (7.1)$$

with $A(\square)$ defined in either Eq. (4.6) or Eq. (4.7), and applied to the standard Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right\} \quad (7.2)$$

It should be noted that there are *two* quantum contributions to this set of equations. The first one arises because of the presence of a non-vanishing cosmological constant $\lambda \simeq 1/\xi^2$, as in Eq. (2.48), originating in the non-perturbative vacuum condensate of the curvature. As in the case of standard FRW cosmology, this is the dominant contributions at large times t , and gives an exponential expansion of the scale factor.

The second contribution arises because of the running of G for $t \ll \xi$ in the effective field equations,

$$G(\square) = G (1 + A(\square)) = G \left[1 + a_0 (\xi^2 \square)^{-\frac{1}{2\nu}} + \dots \right] \quad (7.3)$$

with $\nu \simeq 1/3$ and a_0 a calculable coefficient of order one (see Eqs. (2.1) and (2.2)).

In the simplest case, namely for a universe filled with non-relativistic matter ($p=0$), the effective Friedmann equations then have the following appearance [12]

$$\begin{aligned} \frac{k}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)} &= \frac{8\pi G(t)}{3} \rho(t) + \frac{1}{3\xi^2} \\ &= \frac{8\pi G}{3} \left[1 + c_\xi (t/\xi)^{1/\nu} + \dots \right] \rho(t) + \frac{1}{3\xi^2} \end{aligned} \quad (7.4)$$

for the tt field equation, and

$$\frac{k}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)} + \frac{2\ddot{a}(t)}{a(t)} = -\frac{8\pi G}{3} \left[c_\xi (t/\xi)^{1/\nu} + \dots \right] \rho(t) + \frac{1}{\xi^2} \quad (7.5)$$

for the rr field equation. The running of G appropriate for the RW metric, and appearing explicitly in the first equation, is described by

$$G(t) = G \left[1 + c_\xi \left(\frac{t}{\xi} \right)^{1/\nu} + \dots \right] \quad (7.6)$$

(with c_ξ of the same order as a_0 of Eq. (2.1) [12]). Note that the running of $G(t)$ induces as well an effective pressure term in the second (rr) equation.¹⁴

One can therefore talk about an effective density

$$\rho_{eff}(t) = \frac{G(t)}{G} \rho(t) \quad (7.7)$$

¹⁴We wish to emphasize that we are *not* talking here about models with a time-dependent value of G . Thus, for example, the value of $G \simeq G_c$ at laboratory scales should be taken to be constant throughout most of the evolution of the universe.

and an effective pressure

$$p_{eff}(t) = \frac{1}{3} \left(\frac{G(t)}{G} - 1 \right) \rho(t) \quad (7.8)$$

with $p_{eff}(t)/\rho_{eff}(t) = \frac{1}{3}(G(t)-G)/G(t)$ ¹⁵. Within the FRW framework, the gravitational vacuum polarization term behaves therefore in some ways (but not all) like a positive pressure term, with $p(t) = \omega \rho(t)$ and $\omega = 1/3$, which is therefore characteristic of radiation. One could therefore visualize the gravitational vacuum polarization contribution as behaving like ordinary radiation, in the form of a dilute virtual graviton gas: a radiative fluid with an equation of state $p = \frac{1}{3}\rho$. It should be emphasized though that the relationship between density $\rho(t)$ and scale factor $a(t)$ is very different from the classical case.

The running of $G(t)$ in the above equations follows directly from the basic result of Eq. (2.1) (with the dimensionless constant c_ξ proportional to a_0 , with a numerical coefficient of order one given in magnitude in [12]), but transcribed, by explicitly computing the action of the covariant d'Alembertian $\square \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ on $T_{\mu\nu}$, for the RW metric. In other words, following the more or less unambiguously defined sequence $G(k^2) \rightarrow G(\square) \rightarrow G(t)$. At the same time, the discussion of Sec. 1 underscores the fact that for large times $t \gg \xi$ the form of Eq. (2.1), and therefore Eq. (7.6), is no longer appropriate, due to the spurious infrared divergence of Eq. (2.1) at small k^2 . Indeed from Eq. (2.2), the infrared regulated version of the above expression should read instead

$$G(t) \simeq G \left[1 + c_\xi \left(\frac{t^2}{t^2 + \xi^2} \right)^{\frac{1}{2\nu}} + \dots \right] \quad (7.9)$$

with $\xi = m^{-1}$ the (tiny) infrared cutoff. Of course it reduces to the expression in Eq. (7.6) in the limit of small times t , but for very large times $t \gg \xi$ the gravitational coupling, instead of unphysically diverging, approaches a constant, finite value $G_\infty = (1 + a_0 + \dots) G_c$, independent of ξ . The modification of Eq. (7.9) should apply whenever one considers times for which $t \ll \xi$ is not valid. But since $\xi \sim 1\sqrt{\lambda}$ is of the order the size of the visible universe, the latter regime is largely of academic interest, and was therefore not discussed much in [12].

It should be noted that the effective Friedman equations of Eqs. (7.4) and (7.5) also bear a superficial degree of resemblance to what might be obtained in some scalar-tensor theories of gravity, where the gravitational Lagrangian is postulated to be some singular function of the scalar curvature [70, 71]. Indeed in the FRW case one has, for the scalar curvature in terms of the scale

¹⁵Strictly speaking, the above results can only be proven if one assumes that the pressure's time dependence is given by a power law, as discussed in detail in [12]. In the more general case, the solution of the above equations for various choices of ξ and a_0 has to be done numerically.

factor,

$$R = 6 \left(k + \dot{a}^2(t) + a(t) \ddot{a}(t) \right) / a^2(t) \quad (7.10)$$

and for $k = 0$ and $a(t) \sim t^\alpha$ one has

$$R = \frac{6 \alpha (2 \alpha - 1)}{t^2} \quad (7.11)$$

which suggests that the quantum correction in Eq. (7.4) is, at this level, nearly indistinguishable from an inverse curvature term of the type $(\xi^2 R)^{-1/2\nu}$, or $1/(1 + \xi^2 R)^{1/2\nu}$ if one uses the infrared regulated version. The former would then correspond to an effective gravitational action

$$I_{eff} \simeq \frac{1}{16\pi G} \int dx \sqrt{g} \left(R + \frac{f \xi^{-\frac{1}{\nu}}}{(R)^{\frac{1}{2\nu}-1}} - 2\lambda \right) \quad (7.12)$$

with f a numerical constant of order one, and $\lambda \simeq 1/\xi^2$. But this superficial resemblance is seen here more as an artifact, due to the particularly simple form of the RW metric, with the coincidence of several curvature invariants not expected to be true in general.

8 Conclusions

In this paper we have examined a number of basic issues connected with the renormalization group running of the gravitational coupling. The scope of this paper was to explore the overall consistency of the picture obtained from the lattice, by considering a number of basic issues, one of which is the analogy, or contrast, with a much better understood class of theories, namely QED on the one hand, and non-abelian gauge theories and QCD on the other.

The starting point for our discussion of the renormalization group running of G (Sec. 2) is Eq. (2.1) (valid at short distances $k \gg m$, or, equivalently $r \ll \xi$), and its improved infrared regulated version of Eq. (2.2). The scale dependence for G obtained from the lattice is remarkably similar to the result of the $2 + \epsilon$ expansion in the continuum, as in Eq. (2.59), with two important differences: only the strong coupling phase $G > G_c$ is physical, and for the exponent one has $\nu \simeq 1/3$ in four dimensions. The similarity between the two results in part also originates from the fact that in both cases the renormalization group properties of G are inferred (implicitly, in the $2 + \epsilon$ case) from the requirement that the non-perturbative scale of Eq. (2.10) be treated as an invariant.

Inspection of the quantum gravitational functional integral Z of Eq. (2.20) reveals that its singular part can only depend of the dimensionless combination $\lambda_0 G^2$, up to an overall factor

which cannot affect the non-trivial scaling behavior around the fixed point, since it is analytic in the couplings. This then leaves the question open of which coupling(s) run and which ones do not.

The answer in our opinion is possibly quite simple, and is perhaps best inferred from the nature of the Wilson loop of Eq. (2.47): the appropriate renormalization scheme for quantum gravity is one in which G runs with scale according to the prediction Eq. (2.2), and the scaled cosmological constant λ is kept fixed, as in Eqs. (2.48) and (2.51). Since the scale ξ is related to the observable curvature at large scales, it is an almost inescapable conclusion of these arguments that it must be macroscopic. Furthermore, it is genuinely non-perturbative and non-analytic in G , as seen for example from Eq. (2.54), and represents the effects of the gravitational vacuum condensate which makes its appearance in the strongly coupled phase $G > G_c$.

Another aspect we have investigated in this paper is the nature of the quantum corrections to the gravitational potential $\phi(r)$ in real space, arising from the scale dependence of Newton's constant G . The running is originally formulated in momentum space (see Eq. (2.2)), since it originates in the momentum dependence of G as it arises on the lattice, or in the equivalent renormalization group equations, Eqs. (2.9) or (2.17). The solution $\phi(r)$ to the non-relativistic Poisson equation for a point source is given in Eq. (3.21) of Sec. 3 for various values of the exponent ν . The solution is obtained by first computing the effective vacuum polarization density $\rho_m(r)$ of Eq. (3.17), and then using it as a source term in Poisson's equation. Already in the non-relativistic case, the value $\nu = 1/3$ appears to stand out, since it leads to logarithmic corrections at short distances $r \ll \xi$.

A relativistic generalization of the previous results was worked out in Secs. 4 and 5. First it was shown that the scale dependence of G can be consistently embedded in a relativistic covariant framework using the d'Alembertian \square operator, leading to a set of nonlocal effective field equations, Eq. (4.8). The consequences can then be worked out in some detail for the static isotropic metric (Sec. 4), at least in a regime where $2MG \ll r \ll \xi$, and under the assumption of a power law correction (otherwise the problem becomes close to intractable). One then finds that the structure of the leading quantum correction severely restricts the possible values for the exponent ν , in the sense that no consistent solution to the effective non-local field equations, incorporating the running of G , can be found unless ν^{-1} is an integer.

A somewhat different approach to the solution of the static isotropic metric was then discussed in Sec. 5, in terms of the effective vacuum density of Eq. (3.17), and a vacuum pressure chosen so as to satisfy a covariant energy conservation for the vacuum polarization contribution. The main result is the derivation from the relativistic field equations of an expression for the metric coefficients $A(r)$ and $B(r)$, given in Eqs. (5.35) and (5.38). For $\nu = 1/3$ it implies for the running

of G in the region $2MG \ll r \ll \xi$ the result of Eq. (5.39),

$$G(r) = G \left(1 + \frac{a_0}{3\pi} m^3 r^3 \ln \frac{1}{m^2 r^2} + \dots \right) \quad (8.1)$$

indicating therefore a gradual, very slow increase in G from the “laboratory” value $G \equiv G(r = 0)$. For the actual values of the parameters appearing in the above expression one expects that m is related to the curvature on the largest scales, $m^{-1} = \xi \sim 10^{28} \text{cm}$, and that $a_0 \sim O(10)$. From the nature of the solution for $A(r)$ and $B(r)$ one finds again that unless the exponent ν is close to $1/3$, a consistent solution of the field equations cannot be found. Note that for very large $r \gg \xi$ the growth in $G(r)$ saturates and the value $G_\infty = (1 + a_0)G$ is obtained, in accordance with the original formula of Eq. (2.2) for $k^2 \simeq 0$. A natural comparison is with the QED result of Eq. (2.67).

At the end of the paper we have added some remarks on the solution of the gravitational wave equation with a running G . We find that a running Newton’s constant will slightly distort the gravitational wave spectrum at very long wavelengths (Sec. 6), according to Eq. (6.9). Regarding the problem of finding solutions of the effective non-local field equations in a cosmological context [12], wherein quantum corrections to the Robertson-Walker metric and the basic Friedman equations (Eqs. (7.4) and (7.5)) are worked out, we have discussed some of the simplest and more plausible scenarios for the growth (or lack thereof) of the coupling at very large distances, past the de Sitter horizon.

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References

- [1] R. P. Feynman, ‘*Lectures on Gravitation*’, 1962-1963, edited by F. B. Morinigo and W. G. Wagner, California Institute of Technology (Pasadena, 1971).
- [2] G. ’t Hooft and M. Veltman, *Ann. Inst. Poincaré* **20** 69 (1974).
- [3] S. Deser and P. van Nieuwenhuizen, *Phys. Rev.* **D10** 401,410 (1974); S. Deser, H. S. Tsao and P. van Nieuwenhuizen, *Phys. Rev.* **D 10**, 3337 (1974).

- [4] K. G. Wilson, *Rev. Mod. Phys.* **47** 773 (1975); **55** 583 (1983); K. G. Wilson and M. Fisher, *Phys. Rev. Lett.* **28** 240 (1972).
- [5] K. G. Wilson, *Phys. Rev.* **D 7**, 2911 (1973).
- [6] G. Parisi, *Lett. Nuovo Cim.* **6** 450 (1973); *Nucl. Phys.* **B100** 368 (1975), *ibid.* **254** 58 (1985); ‘On Non-renormalizable Interactions’, in *New Development in Quantum Field Theory and Statistical Mechanics*, (Cargese 1976, M. Levy and P. Mitter eds. , Plenum Press, New York 1977).
- [7] K. Symanzik, *Comm. Math. Phys.* **45** 79-98 (1975).
- [8] S. Weinberg, in ‘*General Relativity - An Einstein Centenary Survey*’, edited by S. W. Hawking and W. Israel, (Cambridge University Press, 1979).
- [9] S. W. Hawking, in ‘*General Relativity - An Einstein Centenary Survey*’, edited by S. W. Hawking and W. Israel, (Cambridge University Press, 1979);
- [10] J. B. Hartle and S. W. Hawking, *Phys. Rev.* **D 28**, 2960 (1983).
- [11] J. B. Hartle, lectures delivered at the *Theoretical Advanced Study Institute in Elementary Particle Physics*, Yale University, 1985, vol. 2, p. 471-566.
- [12] H. W. Hamber and R. M. Williams, *Phys. Rev.* **D 72** 044026 (2005); *Mod. Phys. Lett. A* **21**, 735 (2006).
- [13] H. W. Hamber, *Phys. Rev.* **D45** 507 (1992); *Phys. Rev.* **D61** 124008 (2000).
- [14] H. W. Hamber and R. M. Williams, *Phys. Rev.* **D 70**, 124007 (2004).
- [15] H. W. Hamber and R. M. Williams, *Phys. Rev.* **D73**, 044031 (2006).
- [16] T. Regge, *Nuovo Cimento* **19** 558 (1961).
- [17] M. Roček and R. M. Williams, *Phys. Lett.* **104B** 31 (1981); *Z. Phys.* **C21** 371 (1984).
- [18] J. Cheeger, W. Müller and R. Schrader, in ‘*Unified Theories Of Elementary Particles*’, *Heisenberg Symposium*, München 1981, p. 176-188, (Springer, New York, 1982).
- [19] T. D. Lee, in ‘*Discrete Mechanics*’, 1983 Erice International School of Subnuclear Physics, vol. 21 (Plenum Press, New York 1985), and references therein.

- [20] H. W. Hamber and R. M. Williams, *Nucl. Phys.* **B248** 392 (1984); **B260** 747 (1985); *Phys. Lett.* **157B** 368 (1985); *Nucl. Phys.* **B269** 712 (1986).
- [21] J. B. Hartle, *J. Math. Phys.* **26** 804 (1985); **27** 287 (1986); **30** 452 (1989).
- [22] B. Berg, *Phys. Rev. Lett.* **55** 904 (1985); *Phys. Lett.* **B176** 39 (1986).
- [23] H. W. Hamber and R. M. Williams, *Nucl. Phys.* **B267** 482 (1986).
- [24] H. W. Hamber and R. M. Williams, *Phys. Rev.* **D47** 510 (1993).
- [25] H. W. Hamber, *Phys. Rev.* **D50** 3932 (1994).
- [26] H. W. Hamber, *Nucl. Phys. B* **400**, 347 (1993).
- [27] B. DeWitt, *Phys. Rev.* **160** 1113 (1967); in ‘General Relativity - An Einstein Centenary Survey’, edited by S. W. Hawking and W. Israel, Cambridge University Press (1979).
- [28] C. W. Misner, *Rev. Mod. Phys.* **29** 497 (1957); L. D. Faddeev and V. N. Popov, *Sov. Phys. Usp.* **16** 777-788 (1974); *Usp. Fiz. Nauk.* **111** 427-450 (1973).
- [29] V. N. Popov and L. D. Faddeev, Fermilab-pub-72-057-T, in “50 years of Yang-Mills theory”, edited by G. ’t Hooft, (World Scientific, Singapore, 2004) pp. 40-64.
- [30] M. Caselle, A. D’Adda and L. Magnea, *Phys. Lett.* **B 232**, 457 (1989).
- [31] J. Fröhlich, ‘Regge Calculus and Discretized Gravitational Functional Integrals’, I. H. E. S. preprint 1981 (unpublished); *Non-Perturbative Quantum Field Theory: Mathematical Aspects and Applications*, Selected Papers (World Scientific, Singapore, 1992), pp. 523-545.
- [32] H. W. Hamber and G. Kagal, *Class. and Quant. Grav.* **21**, 5915 (2004), and references therein.
- [33] G. Modanese, *Phys. Rev.* **D 47**, 502 (1993); *Phys. Rev.* **D 49**, 6534 (1994).
- [34] H. W. Hamber and R. M. Williams, *Nucl. Phys.* **B435** 361 (1995); *Phys. Rev.* **D59** 064014 (1999).
- [35] K. G. Wilson, *Phys. Rev.* **D 10**, 2445 (1974); ‘Quarks And Strings On A Lattice’, in *New Phenomena In Subnuclear Physics*, Erice 1975 (Plenum Press, New York, 1977).

- [36] See, for example, M. E. Peskin and D. V. Schroeder, ‘*An Introduction to Quantum Field Theory*’, sec. 22.1 (Addison Wesley, Reading, Massachusetts, 1995).
- [37] G. Parisi, ‘*Statistical Field Theory*’, Addison Wesley, New York (1988).
- [38] See, for example, J. Zinn-Justin, ‘*Quantum Field Theory and Critical Phenomena*’, (Oxford University Press, 3rd edition, 1996).
- [39] E. Brezin, J. C. Le Guillou and J. Zinn-Justin, ‘*Field Theoretical Approach To Critical Phenomena*’, in “Phase Transitions and Critical Phenomena”, edited by C. Domb and M. Green (Academic Press, London 1976), Vol.6, pp. 125-247.
- [40] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. B147 (1979), p. 385; p. 448; p. 519; V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov preprint ITEP-42 Moscow (1981).
- [41] A. Armoni, M. Shifman and G. Veneziano, Phys. Lett. B **579**, 384 (2004); A. Armoni, G. Shore and G. Veneziano, Nucl. Phys. B **740**, 23 (2006).
- [42] H. W. Hamber and G. Parisi, Phys. Rev. D **27**, 208 (1983); Phys. Rev. Lett. **47**, 1792 (1981).
- [43] H. S. Tsao, Phys. Lett. B **68**, 79 (1977).
- [44] R. Gastmans, R. Kallosh and C. Truffin, Nucl. Phys. **B133** 417 (1978); S. M. Christensen and M. J. Duff, Phys. Lett. **B79** 213 (1978).
- [45] H. Kawai and M. Ninomiya, Nucl. Phys. **B336** 115 (1990); H. Kawai, Y. Kitazawa and M. Ninomiya, Nucl. Phys. **B393** 280 (1993) and **B404** 684 (1993); Y. Kitazawa and M. Ninomiya, Phys. Rev. **D55** 2076 (1997).
- [46] T. Aida and Y. Kitazawa, Nucl. Phys. **B491** 427 (1997).
- [47] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **104B** 377 (1981) and **106B** 63 (1981); Nucl. Phys. **B201** 469 (1982); I. G. Avramidy and A. O. Barvinsky, Phys. Lett. **159B** 269 (1985).
- [48] E. Brezin and J. Zinn-Justin, Phys. Rev. Lett. **36** 691 (1976); Phys. Rev. **D14** 2615 (1976); Phys. Rev. **B14**, 3110 (1976); E. Brezin and S. Hikami, LPTENS-96-64 (Dec 1996) [cond-mat/9612016].
- [49] W. A. Bardeen, B. W. Lee and R. E. Shrock, Phys. Rev. D **14**, 985 (1976).

- [50] F. David, Commun. Math. Phys. **81**, 149 (1981).
- [51] S. Hikami and E. Brezin, *J. Phys.* **A 11**, 1141 (1978).
- [52] H. Kleinert, *Phys. Lett.* **A 264**, 357 (2000).
- [53] D. Litim, CERN-Th-2003-299, *Phys. Rev. Lett.* **92** 201301 (2004); CERN-Th-2005-256 (hep-th/0606044) (2006).
- [54] P. Fischer and D. F. Litim, CERN-Th-2005-257 (hep-th/0606135) (Feb. 2006); CERN-Th-2006-066 (hep-th/0602203) (June 2006) .
- [55] M. Reuter and F. Saueressig, *Phys. Rev.* **D65** 065016 (2002); O. Lauscher and M. Reuter, *Class. Quant. Grav.* **19** 483 (2002).
- [56] A. M. Polyakov, in the proceedings of the Les Houches Summer School on *Gravitation and Quantizations*, Session 57, Aug. 1992, J. Zinn-Justin and B. Julia eds. (North-Holland, 1995) (hep-th/9304146).
- [57] See, for example, C. Itzykson and J. B. Zuber, ‘*Quantum Field Theory*’, (McGraw-Hill, New York, 1980).
- [58] E. A. Uehling, *Phys. Rev.* **48**, 55-63 (1935).
- [59] E. Wichmann and N. Kroll, *Phys. Rev.* **96**, 232-234 (1954).
- [60] A. D. Martin, J. Outhwaite and M. G. Ryskin, *Phys. Lett. B* **492**, 69 (2000).
- [61] We follow the conventions of the Particle Data Group, pdg.lbl.gov, S. Eidelman et al., *Phys. Lett.* **B 592**, 1 (2004).
- [62] Y. L. Dokshitzer, G. Marchesini and B. R. Webber, *Nucl. Phys. B* **469**, 93 (1996); A. V. Manohar and M. B. Wise, *Phys. Lett. B* **344**, 407 (1995); R. Akhoury and V. I. Zakharov, *Phys. Lett. B* **357**, 646 (1995); B. R. Webber, *Nucl. Phys. Proc. Suppl.* **71**, 66 (1999).
- [63] M. Beneke, *Phys. Rept.* **317**, 1 (1999); M. Beneke and V. M. Braun, *Nucl. Phys. B* **454**, 253 (1995).
- [64] J. L. Richardson, *Phys. Lett. B* **82**, 272 (1979).

- [65] E. Eichten and F. Feinberg, *Phys. Rev. D* **23**, 2724 (1981); W. Buchmuller and S. H. H. Tye, *Phys. Rev. D* **24**, 132 (1981); U. Ellwanger, M. Hirsch and A. Weber, *Eur. Phys. J. C* **1**, 563 (1998).
- [66] G. A. Vilkovisky, *Nucl. Phys.* **B 234** (1984) 125; A. O. Barvinsky and G. A. Vilkovisky, *Nucl. Phys.* **B 282**, 163 (1987); **B 333**, 471 (1990); **B 333**, 512 (1990); A. O. Barvinsky, Y. V. Gusev, V. V. Zhytnikov and G. A. Vilkovisky, Print-93-0274 (Manitoba); A. O. Barvinsky and G. A. Vilkovisky, *Phys. Rept.* **119**, 1 (1985).
- [67] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, (W. H. Freeman and Co. Publishers, New York, 1973).
- [68] J. B. Hartle, “Gravity: an Introduction to Einstein’s General Relativity”, (Addison-Wesley, New York, 2002).
- [69] R. C. Myers and M. J. Perry, *Annals Phys.* **172**, 304 (1986); D. Y. Xu, *Class. Quant. Grav.* **5** (1988) 871.
- [70] S. Capozziello, S. Carloni and A. Troisi, in “*Recent Research Developments in Astronomy and Astrophysics*” -RSP/AA/21-2003, [astro-ph/0303041]; S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, *Phys. Rev. D* **70**, 043528 (2004); E. E. Flanagan, *Phys. Rev. Lett.* **92**, 071101 (2004).
- [71] J. W. Moffat, *JCAP* **0603**, 004 (2006); *JCAP* **0505**, 003 (2005).