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Quantum Field Theory Methods in the Study of Gravitation

DISSERTATION

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for the degree of

DOCTOR OF PHILOSOPHY

in Physics

by

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DEDICATION

to my parents

and dedicated to Nature

TABLE OF CONTENTS

LIST OF FIGURES

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Wheeler DeWitt Equation in $3+1$ *Dimensions*
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Scale Dependent Newton's Constant G in the Conformal Newtonian Gauge
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ABSTRACT OF THE DISSERTATION

Quantum Field Theory Methods in the Study of Gravitation

By

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Doctor of Philosophy in Physics

University of California, Irvine, 2013

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Some further results that support the phase structure of quantum gravitation with a nontrivial ultraviolet fixed point are presented. The approaches explored here are based on quantum field theories and lattice discretization. Finally consequences of this phase structure in quantum gravitation are studied in the context of cosmology.

There are a number of problems in science which have, as a common characteristic, that complex microscopic behavior underlies macroscopic effects. In simple cases the microscopic fluctuation average out when larger scales are considered, and the averaged quantities satisfy classical continuum equations. Hydrodynamics is a standard example of this, where atomic flucturaions average out and the classical hydrodynamic equations emerge. Unfortunately, there is a much more difficult class of problems where flucturations persist out to macroscopic wavelengths, and fluctuations on all intermediate length scales are important too. In this last category are the problems of fully developed turbulent fluid flow, critical phenomena, and elementary-particle physics. In fully developed turbulence in the atmosphere, global air circulation becomes unstable, leading to eddies on a scale of thousands of miles. These eddies break down into smaller eddies, which in turn break down, until chaotic motions on all length scales down to millimeters have been excited.

K. Wilson

"The renormalization group and critical phenomena,"

Rev. Mod. Phys. **55**, 583-600 (1983),

Nobel Lecture 1982

Chapter 1

Introduction

In 4 spacetime dimensions (more precisely above 2 spacetime dimensions), since the gravitational theory is not perturbatively renormalizable in a conventional and naive sense, quantizing gravity using quantum field theory has been a difficult task and requires understanding of subtle issues.

For example, when compared to more naive expectations, if a theory admits a nontrivial ultraviolet fixed point of the renormalization group, it can be shown to radically alter the short and long distance behavior of the theory. The renormalization group origin of such fixed points was discussed in detail by Wilson for scalar and self coupled fermion theories [1, 2, 3, 4, 5]. The general field theoretic methods were later extended to gravity, where they are referred to as the nontrivial fixed point scenario or asymptotic safety [6]. Encouragingly, such nontrivial fixed points are well studied and well understood in statistical field theory, where they generally depict phase transitions between ordered and disordered ground states, or between weakly coupled and condensed states.

The key example in this context is nonlinear sigma model. I will review the relevant aspects of nonlinear sigma model in Section 1.2. Later, we discuss how the nonlinear sigma model gives a relevant idea to gravity. We also discuss some enlightening similarities with non Abelian gauge theories including QCD (Section 1.1), in which we find dynamically generated mass like scale, which appear to be related with color (quark in QCD) vacuum condensate. We then consider explicitly quantum gravity in $2+\epsilon$ dimensions using quantum field theory in Section 1.3, which connects us to higher dimensions *i.e.*, 4 dimensions in Section 1.5. After analyzing nonlinear sigma model and QCD in analogies with quantum gravity, one notices an important role that the cosmological constant plays in the theory of gravitation. Some crucial aspects are reviewed in Section 1.4.

We emphasize here that in nature, one observes that phase transitions may be the most generic feature of the physical systems. Especially if a physical system is interacting, and if lasted over a vast scale, it is not likely to avoid phase transitions. The gravitational theory which appears to span from the order of cosmological constant to the order of Planck scale, one is hard to imagine that gravity will not experience any phase transitions. Furthermore many systems also exhibit condensate formations, such as quantum chromodynamics and theories of spontaneously broken symmetry as in Higgs mechanisms. In fact quantum field theories which let us study the system with infinite degrees of freedom, are suitable for studying these phase structures of a theory. Its core concept, the Wilson's renormalization group studies, give explicitly the dependence of coupling constant on the scale, *i.e.,* β function. With the β function, one is ready to analyze what types of phase transition a theory exhibits.

It is important to note that up to today, quantum field theory together with Wilsonian renormalization group yields us the most sophisticated way to analyze the systems with many degrees of freedom. Quantum field theory successfully described qualitatively and quantitatively quantum electrodynamics [1], quantum chromodynamics which make up the Standard Model as well as condensed matter systems that belong to the universal class of nonlinear sigma model and Ising model.

[1] *e.g.,* in QED, $\alpha = \frac{e^2}{2\pi}$ is measured to be $1/137.035999084(51)$ at the energy scale of mass of electron and $1/127.916(15)$ at the mass of Z boson in agreement with the prediction from the qunatum field theory [7].

Therefore in this thesis, we employ approaches quantum field theories in conjunction with statistical field theory, lattice field theories to further explore the phase structures of quantum gravity. Furthermore, we take on the lattice discretization to have a better control on nonperturbative effects.

1.1 QCD

The renormalization group β function allows us to directly see how the coupling flows with scale, therefore lets us explore interesting physics how the the different scales talk to each other in a given theory. The β function of Yang-Mills theory is a significant one, for example giving a surprising result, when first computed by Gross, Politzer, and Wilczek, of an asymptotically free β function [8, 9, 10, 11]. If the origin (of the $\beta(g)$ function, *i.e.,* at $\beta = 0$ and $g = 0$) is an ultraviolet fixed point, we say that the theory is asymptotically free. For Yang-Mills theory with sufficiently small number of fermions [2] , one obtains β function that is negative.

$$\beta(\alpha_s \equiv \frac{g^2}{4\pi}) \equiv \frac{\partial \alpha_s}{\partial \ln \mu} = -a\ \alpha^2 + \mathcal{O}(\alpha^3)\,, \qquad \text{where } \ a > 0\,, \tag{1.1}$$

which gave us for Yang-Mills theory $a = \frac{11 - \frac{2}{3}\,n_f}{2\,\pi}$ where n_f is a number of fermions in the theory.

One can integrate the both sides of the above equation to obtain the expression for scale

[2]For a non Abelian gauge theory with fermion content, one writes its Lagrangian as $\mathcal{L} = \bar{\psi}_i(i\not{\partial} - m)\psi_i = gA^a_\mu \bar{\psi}_i \gamma^\mu t^a_{ij} \psi_j - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$, where the field strength $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$, where f^{abc} is the structure constant which satisfies $\left[t^a, t^b\right] = i f^{abc} t^c$ with t_a's being the generators of the symmetry group and A^a_μ is the gauge field of the theory.

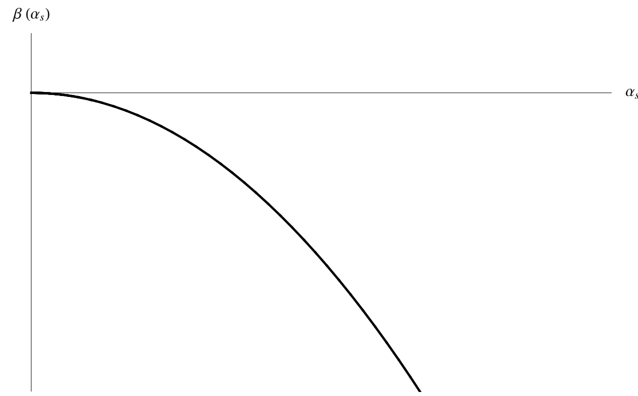

Figure 1.1: Qualitative picture of $\beta(\alpha_s)$ as given by Eq. (1.1). The asymptotic free β function exhibit UV fixed point, and is always negative and corresponds to the flow of α_s such that α_s increases (resp. decreases) toward IR (resp. UV), and to antiscreening.

dependent g,

$$\begin{aligned}
\int -\frac{1}{a}\frac{1}{\alpha_s(\mu')^2}\partial\alpha_s &= \int \partial \ln \mu' \\
\frac{1}{a}\;\frac{1}{\alpha_s(\mu')}\Bigg|_{\alpha_s(m_0)}^{\alpha_s(\mu)} &= \ln \mu'|_{m_0}^{\mu} \\
\frac{1}{a}\;\frac{1}{\alpha_s(\mu)} &= \ln\frac{\mu}{m_0} + \frac{1}{a}\;\frac{1}{\alpha_s(m_0)} \\
\frac{1}{a}\;\frac{1}{\alpha_s(\mu)} &= \ln\frac{\mu}{m_0} + \ln\frac{m_0}{m} \\
\alpha_s(\mu) &= \frac{1}{a\ln\frac{\mu}{m}}
\end{aligned} \tag{1.2}$$

up to the leading order in μ, where μ is the arbitrary sliding momentum scale and m is the infrared mass scale (as it should be $m < \mu$ by construction) and is basically coming from the integration constant as one can see in the intermediate steps. This function has a pole which is apparent from its form which includes ln, located at $\mu = m$. As plotted in Fig. 1.2, g increases as the momentum becomes small for $m < \mu$; therefore the perturbation theory around $g = 0$ becomes unreliable at some scale around m. This nonperturbative scale is by construction renormalization group invariant and does not run with scale:

$$\mu\frac{d}{d\mu}m \equiv 0\,, \tag{1.3}$$

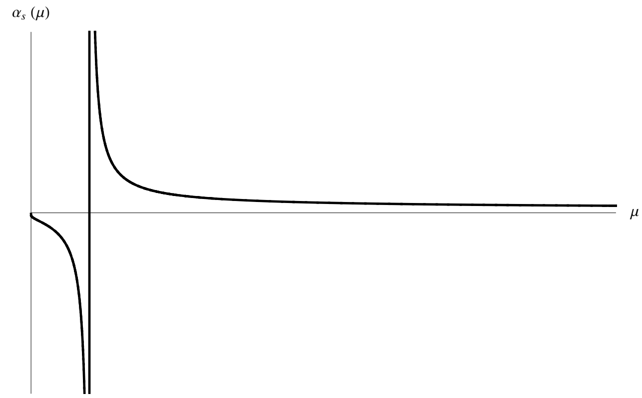

Figure 1.2: Qualitative but literal picture of $\alpha_s(\mu)$ vs. μ as given by Eq. (1.2). Notice that it clearly shows an existence of a pole at $\mu \sim m$ in Eq. (1.2), and the curve should be only valid up to around that scale of pole from the above. We note that $\Lambda_{QCD} \sim m$. Since of course we know that $\alpha_s \geq 0$, only the smooth curve that is above 0 of α_s is the physical one.

and because of this nonperturbative nature, it cannot be determined with renormalization group, but needs to be determined by experiment. This dynamically generated scale corresponds to Λ_{QCD} for quantum chromodynamics (and in this case, the positive constant a above is $a = \frac{11-\frac{2}{3}\,n_f}{2\,\pi}$ with $n_f = 6$ for QCD, which is a $SU(3)$ non Abelian gauge theory), is experimentally determined to be $\Lambda_{QCD} \sim 220MeV$, which corresponds to the scale of gluon condensate or glueball. Therefore, in gauge theories, if the negative β function possesses UV fixed point, then it is natural to obtain the dynamically generated nonperturbative scale which corresponds to the presence of a gauge field condensate.

1.2 Nonlinear Sigma Model

I will briefly review nonlinear sigma model. It is an interesting and very rich theory which also describes spontaneous symmetry breaking and where one can explore in different ways including double expansion in g and ϵ and large N expansion. Here we focus on the double expansion in g and ϵ approach. We have a field theory that has $O(N)$ symmetry.

$$S(\phi) = \frac{1}{2g}\int d^d x \partial_\mu \phi^a(x) \partial^\mu \phi^a(x) \,. \tag{1.4}$$

Note that the coupling g in this theory has $2-d$ mass dimensions, and above 2 dimensions, therefore, this theory is naïvely nonrenormalizable: in 2 dimensions, one obtains the coupling constant to be dimensionless, and the theory becomes trivially renormalizable. Nonlinear sigma model is a field theory with N component field ϕ^a with a constraint such that $\phi^2(x) = 1$, therefore one obtains its partition function to be,

$$Z = \int \mathcal{D}\phi \prod_x \delta\left(\phi^a(x)\phi^a(x) - 1\right)\, e^{-S(\phi)} \,, \tag{1.5}$$

where we are using Euclidean signature.

One obtains in $2+\epsilon$ expansion, the β function of the form,

$$\beta(g) = \epsilon\, g + \beta^{(2)}(g) \,, \tag{1.6}$$

where $\beta^{(2)}(g) = -\frac{N-2}{2\pi} g^2 + \mathcal{O}\left(g^3\right)$ is the β function obtained for 2 dimensions. The β function is shifted by the contribution of the mass dimension of g, *i.e.*, $\epsilon = d-2$. One can find the standard procedure of finding the β function of g for example using Callan Symanzik equations and calculating 2 - point function for the π field (where $\phi^a = (\pi^1, \cdots, \pi^{N-1}, \sigma)$ in [see for example [12]].

Now for its appearance to be more general, we define a and b to be both positive and write

$$\beta(g) = \widetilde{a}\, g - \widetilde{b}\, g^2 \,, \tag{1.7}$$

and its shape is given in Fig. 1.3. Then one notices that the critical point occurs at

$$g_c = \frac{\widetilde{a}}{\widetilde{b}} \,. \tag{1.8}$$

Therfore, for the nonlinear sigma model in $2+\epsilon$ expansion as given above in Eq. 1.6, we identify

$$\widetilde{a} = d - 2 \equiv \epsilon \,, \qquad \widetilde{b} = \frac{N-2}{2\pi} \,. \tag{1.9}$$

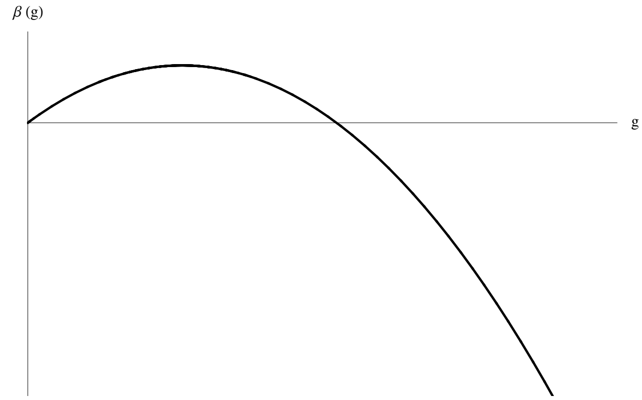

Figure 1.3: Qualitative picture of $\beta(g)$ as given by Eq. (1.7). A nontrivial ultraviolet fixed point g_c is located where the $\beta(g) = 0$. The right side of g_c where $\beta < 0$ corresponds to the flow of g such that g increases (resp. decreases) toward IR (resp. UV), and to disordered phase. The left side of g_c where $\beta > 0$ corresponds to the flow of g such that g decreases (resp. increases) toward IR (resp. UV), and to ordered phase. When one studies in large N limit by decomposing N field into 1 σ field, and $N-1$ π fields and can perform a saddle point approximation, one notices easily that the ordered phase has massless Goldstone bosons (π's).

[3] By integrating both sides, one can obtain an explicit expression for the coupling g which depends on the momentum scale μ;

$$
\begin{aligned}
\int \partial g \left(\frac{1/\widetilde{a}}{g} + \frac{\widetilde{b}/\widetilde{a}}{\widetilde{a} - \widetilde{b}\, g} \right) &= \int \partial \ln \mu' \\
\left[\frac{1}{\widetilde{a}} \ln |g(\mu')| - \frac{1}{\widetilde{a}} \ln \left| \frac{\widetilde{a}}{\widetilde{b}} - g(\mu') \right| \right] \Bigg|_m^{\mu} &= \ln \left| \frac{\mu}{m} \right| \\
\frac{g(\mu)}{\frac{\widetilde{a}}{\widetilde{b}} - g(\mu)} &= \pm b^{-1} \left(\frac{\mu}{m} \right)^{\widetilde{a}} \\
g(\mu) &= \frac{g_c}{1 \pm b \left(\frac{m^2}{\mu^2} \right)^{\widetilde{a}/2}} \,,
\end{aligned}
\tag{1.10}
$$

where I absorbed the integration constant (which involve m, $\widetilde{a}$, and $\widetilde{b}$) to b, and identified $g_c = \frac{\widetilde{a}}{\widetilde{b}}$ and where $+$ is for $g < g_c$, $-$ is for $g > g_c$.

[3] For $N = 2$, in $d = 2$ dimensions, β function vanishes. This is not a trivial check; one can have two fields that are represented as $\cos\theta$ and $\sin\theta$, which gives a Lagrangian of a free field theory.

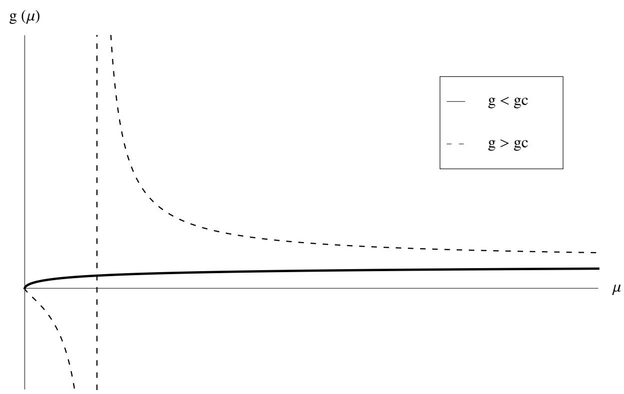

Figure 1.4: Qualitative but literal picture of $g(\mu)$ vs. μ as given by Eq. (1.10). We plot for strong coupling regime ($g > g_c$, $\beta < 0$) and weak coupling regime ($g < g_c$, $\beta > 0$). Notice that for $g > g_c$, the smooth curve with $g > 0$ should be valid. Both $g > g_c$ and $g < g_c$ curves approach a critical value g_c as $\mu \to \infty$.

We can define a universal critical exponent ν as

$$\nu \equiv -\left.\frac{\partial\beta(g)}{\partial g}^{-1}\right|_{g=g_c} = -\frac{1}{\beta'(g_c)}\,. \tag{1.11}$$

Note that one can write in the vicinity of the critical point g_c,

$$\beta(g) \sim \beta'(g_c)\,|g-g_c| + \cdots \tag{1.12}$$

Of course from the construction of invariant mass, one can reexpress the renormalization group invariance of mass as

$$\begin{aligned} m &\sim \Lambda\, e^{-\ln\Lambda} \\ &= \Lambda\, e^{-\int_g d\tilde{g}\,\beta(\tilde{g})^{-1}} \\ &= \Lambda\, e^{-\int_g d\tilde{g}\,(\beta'(g_c)\,|\tilde{g}-g_c|)^{-1}} \\ &= \Lambda\, e^{\,\nu\ln|g-g_c|} \\ &= \Lambda\,|g-g_c|^{\nu}\,. \end{aligned} \tag{1.13}$$

The continuum eld theory predictions as in nonlinear sigma model doubly expanded in ϵ and g, generally agree, for distances much larger than the cuto scale, with lattice results, and, perhaps more importantly, with high precision experiments on systems belonging to the same universality class of the $O(N)$ model [13]. In other words, what is significant in nonlinear sigma model in $2 + \epsilon$ dimensions study is that it gives a qualitatively correct picture for the physical systems in 3 dimensions. For example, $O(2)$ nonlinear sigma model describes the phase transition of (share the same universality class with) superfluid Helium, where the critical exponent of specific heat $\alpha = 2 - 3\nu$ was measured to be $-0.0127(3)$ to a very good accuracy [14]. Theoretical predictions for the $N = 2$ model include the four-loop $4 - \epsilon$ continuum result $\alpha = 0.01126(10)$ (Kleinert, 2000 [15]), a lattice Monte Carlo estimate $\alpha = 0.0146(8)$ (Campostrini *et al*, 2001 [16]), the lattice variational renormalization group prediction $\alpha = 0.0125(39)$(Hamber, 1981 [17]), and $\alpha = -0.011(4)$ obtained from Borel summation of the ϵ expansion [18, 19]).

1.3 Gravitation in $2 + \epsilon$ Expansion Approach

We emphasize here that quantum field theory studies in $2 + \epsilon$ dimensions can be robust enough to describe qualitatively the same physical theory in higher dimensions, as we saw an example, nonlinear sigma model.

We outline here the main steps and results to obtain the renormalization group β function of the gravity in $2 + \epsilon$ dimensions, using the double expansion in ϵ and around the flat metric. Since the mass dimension of the Newton's gravitational coupling G is $2 - d$, where d is the dimension of the spacetime, in 2 spacetime dimensions, the theory appears perturbative renormalizable. We note here that the nonlinear sigma models also has its coupling g to have $2 - d$ mass dimensions. Despite the fact that gravitational action reduces to a topological invariant in 2 dimensions $\int d^2x \sqrt{g}\, R$ corresponding to just Euler characteristic, we will find its use to study this theory perturbatively as a double serious in $\epsilon = d - 2$ and in G. The original works for scalar field theories were studied long time ago in [20, 21, 22] and later the

$2+\epsilon$ expansion for pure gravity was studied in [23], which we outline here the key steps to illuminate the relevant discussions.

In the presence of an explicit renormalization scale parameter μ, the Callan Symanzik β function for pure gravity is obtained by requiring the independence of the effective coupling G from the original renormalization scale μ. One obtains to one loop order

$$\beta(G) \equiv \mu \frac{\partial}{\partial \mu} G(\mu) = \epsilon G - \widetilde{c} G^2 + \mathcal{O}(G^3, \epsilon G^2)\,, \tag{1.14}$$

where $\epsilon = d - 2$. $\widetilde{c} = \frac{2}{3} \cdot 19$ in absence of matter was calculated for the 2 - dimensional case considering one-loop divergences associated with $\sqrt{g}R$ terms which one will see it appearing in Sec. 1.4.2 and in [21, 24]. Depending on whether one is on the right ($G > G_c$) or on the left ($G < G_c$) of the nontrivial ultraviolet fixed point at

$$G_c = \frac{\epsilon}{\widetilde{c}} + \mathcal{O}(\epsilon^2) \tag{1.15}$$

the coupling will either flow to increasingly larger values of G, or flow toward the Gaussian fixed point at $G = 0$ respectively. In the following we refer to the two phases as the *strong coupling* and *weak coupling* phase, respectively. Notice that already observing the form of β function, which exhibits UV fixed point and where $\beta < 0$ is possible. Recalling that gravity being gauge theory, it suggests the presence of a gravitational condensate in $\beta < 0$ phase, as discussed in QCD (Sec. 1.1).

The running of G as a function of a sliding momentum scale μ in pure gravity is obtained from integrating Eq. (1.14), which is exactly the same form as in nonlinear sigma model Eq. (1.7) with $\widetilde{a} = \epsilon$, $\widetilde{b} = \widetilde{c}$ and $b = c_0$, therefore obtaining the same picture as Fig. 1.3. It then gives just as in Eq. (1.10),

$$G(\mu) = \frac{G_c}{1 \pm c_0 \left(\frac{m^2}{\mu^2}\right)^{(d-2)/2}}\,, \tag{1.16}$$

where c_0 is a positive constant which absorbed a part of integration constant including $\widetilde{c}$, and $m \equiv \xi^{-1}$ is a mass scale that arises as an integration constant of the renormalization group equations, and therefore its qualitative picture is given in Fig. 1.4. The μ^2 dependent contribution on the *right hand side* in Eq. (1.16) is the quantum correction, which we assume it to be small in this perturbative framework. The choice of $+$ or $-$ sign is determined from whether one is smaller ($+$) or bigger ($-$) than G_c, in which case the effective $G(\mu)$ decreases or increases as one flows away from the ultraviolet fixed point toward lower momenta (larger distances). Physically the two solutions represent screening ($G < G_c$) and antiscreening ($G > G_c$). While in the above continuum perturbative calculation both phases (and therefore both signs) are acceptable, the Euclidean and Lorentzian (which we will cover in this thesis in Chapter 2, Section 2.6.11) lattice results rule out the weak coupling phase as pathological, in the sense that there the lattice collapses into a 2 dimensional degenerate object [25, 26, 27, 28]. Notice that the μ^2 dependent quantum correction in Eq. (1.16) involves a new physical, renormalization group invariant scale $\xi = m^{-1}$ which cannot be perturbatively fixed, and whose size determines the distance scale relevant for quantum effects. In terms of the bare coupling which is at the cutoff scale, *i.e.,* $G(\Lambda)$, it is given by

$$\xi^{-1}(G) \equiv m = \Lambda \cdot A_m e^{-\int^{G(\Lambda)} \frac{dG'}{\beta(G')}} \overset{G \to G_c}{\sim} A_m \Lambda |G(\Lambda) - G_c|^{1/\epsilon} \tag{1.17}$$

One can proceed further with Eq. 1.16 and Eq. 1.17. Manipulating Eq. 1.16, one gets for bare coupling $g \equiv g(\Lambda)$,

$$\frac{G_c}{G(\Lambda)} = 1 - c_0 \left(\frac{m^2}{\Lambda^2} \right)^{\epsilon/2} , \tag{1.18}$$

which can be manipulated further to obtain either

$$m = \Lambda \left(\frac{G_c}{c_0} \right)^{1/\epsilon} \left| \frac{1}{G_c} - \frac{1}{G} \right|^{1/\epsilon} \tag{1.19}$$

or utilizing the fact that bare $\sim$ critical, and $\left|\frac{1}{G} - \frac{1}{G_c}\right| = \left|\frac{G_c - G}{G G_c}\right| \sim \left|\frac{G_c - G}{G_c^2}\right|$,

$$m = \Lambda \left(\frac{1}{c_0 G_c} \right)^{1/\epsilon} |G - G_c|^{1/\epsilon} . \tag{1.20}$$

Therefore, one notices that the calculable constant relating m to G in Eq. (1.20) uniquely determines the coefficient c_0 in Eq. (1.16). Additionally, we note

$$A_m = \frac{1}{c_0 G_c}^{1/\epsilon} . \tag{1.21}$$

This argument is a general one for a theory with β function in the form Eq. (1.14) and therefore with a running coupling in the form Eq. (1.16) . Thus, one realizes here that this holds for nonlinear sigma model in $2 + \epsilon$ dimensions as well as quantum gravity in $2 + \epsilon$ dimensions as discussed here.

1.4 The Role of Cosmological Constant in the Renormalization group

Here, I summarize the key aspects that support the idea that the renormalization group invariant scale ξ is a gravitational condensate which therefore should be identified with the observed cosmological condensate [29].

1.4.1 Gravitational Functional Integral

We recall simple properties of the functional integral for gravity as already studied long time ago as in [30, 31, 32, 33, 34, 35, 36, 25]. This is to let us recall basic facts and to lead us to see that the bare cosmological constant can be scaled out of the problem as it reflects the rescaling of the field which is unphysical (and which is a standard procedure also in quantum field theory and renormalization procedure).

The Euclidean Feynman path integral for pure Einstein gravity with a cosmological constant term can be written as

$$Z = \int \mathcal{D}g_{\mu\nu} \, e^{-\int d^dx \mathcal{L}} \,, \tag{1.22}$$

where

$$\mathcal{L} = -\frac{1}{16\pi G}\sqrt{g}R + \lambda_0\sqrt{g} \,, \tag{1.23}$$

where λ_0 is the unscaled cosmological constant as compared to the scaled cosmological constant λ; $\mathcal{L} = -\frac{1}{16\pi G}\sqrt{g}R + \lambda_0\sqrt{g} = -\frac{1}{16\pi G}\sqrt{g}(R - 2\lambda)$. We note here that the mass dimension of the G and λ_0 are $2-d$ and d respectively. The state sum in the partition function involves a functional integration over all metrics with suitably regularized measure,

$$\int \mathcal{D}g_{\mu\nu} \equiv \int \prod_x g(x)^{\sigma/2} \prod_{\mu\geq\nu} dg_{\mu\nu}(x)\,, \tag{1.24}$$

where σ is some real number. [4] Here, we focus on the action given above for Einstein gravity with a cosmological constant, *i.e.,* we will not consider geometries with boundaries as in such cases, the action will have additional terms. Under the rescaling of the metric (therefore gravitational field in quantum field theory language)

$$g_{\mu\nu} = \omega g'_{\mu\nu}\,, \tag{1.25}$$

one obtains the Lagragian of Eq. (1.23) to now be

$$\mathcal{L} = -\frac{1}{16\pi G}\omega^{d/2-1}\sqrt{g'}R' + \lambda_0\omega^{d/2}\sqrt{g'}\,, \tag{1.26}$$

with which then one identifies the rescaling of 2 bare couplings,

$$G \to G\omega^{-d/2+1} \qquad \lambda_0 \to \lambda_0\omega^{d/2}\,, \tag{1.27}$$

which means that the dimensionless combination $G^d\lambda_0^{d-2}$ is left unchanged under the rescaling of the metric. Therefore one recognizes that only the dimensionless combination has the physical meaning in pure gravity; it seems meaningless to discuss the renormalizations of G and λ_0 separately. Furthermore, one sees more clearly the analogous discussion with rescaling

[4] In function space, volume element $\sqrt{G}$ can be defined after one specifies a form of metric, and the functional measure takes a form $\int \mathcal{D}g_{\mu\nu} = \int \prod_x \sqrt{G\,[g(x)]}\prod_{\mu\geq\nu} dg_{\mu\nu}(x)$. DeWitt [37, 38] defined a ultralocal supermetric $G^{\mu\nu,\rho\sigma}\,(g(x)) = \frac{1}{2}\sqrt{g(x)}\,(g^{\mu\rho}(x)g^{\nu\sigma}(x) + g^{\mu\sigma}(x)g^{\nu\rho}(x) + \alpha g^{\mu\nu}(x)g^{\rho\sigma}(x))$ with $\alpha \neq -2/d$, such that it gives an invariant form for metric deformations; $||\delta g||^2 = \int d^dx\delta g_{\mu\nu}(x)G^{\mu\nu,\rho\sigma}\,(g(x))\,\delta g_{\rho\sigma}(x)$. This DeWitt supermetric gives the functional measure to be $\int \mathcal{D}g_{\mu\nu} = \int \prod_x\,[g(x)]^{(d-4)(d+1)/8}\prod_{\mu\geq\nu} dg_{\mu\nu}(x) \xrightarrow{d\to 4} \int \prod_x \prod_{\mu\geq\nu} dg_{\mu\nu}(x)$. The DeWitt construction is not the unique one to define a functional measure. MIsner [39] started without $\sqrt{g(x)}$, *i.e.,* $G^{\mu\nu,\rho\sigma}\,(g(x)) = \frac{1}{2}\,(g^{\mu\rho}(x)g^{\nu\sigma}(x) + g^{\mu\sigma}(x)g^{\nu\rho}(x) + \alpha g^{\mu\nu}(x)g^{\rho\sigma}(x))$ with $\alpha \neq -2/d$, which gives $\int \mathcal{D}g_{\mu\nu} = \int \prod_x\,[g(x)]^{-(d+1)/2}\prod_{\mu\geq\nu} dg_{\mu\nu}(x) \xrightarrow{d\to 4} \int \prod_x\,[g(x)]^{-5/2}\prod_{\mu\geq\nu} dg_{\mu\nu}(x)$. Indeed the Dewitt and Misner measure can be regarded as two special cases of a slightly more general supermetric G with prefactor $\sqrt{g}^{1-\omega}$, with $\omega = 0$ and $\omega = 1$ respectively, which leads to more general form of the functional measure given in Eq. 1.24 [25].

of the field of a scalar field theory, [5] if one chooses the scale (which one is entitled to do without any loss of generality) $\omega = \lambda_0^{-2/d}$, such that the volume term should have a unit coefficient;

$$\mathcal{L} = -\frac{1}{16\pi G \lambda_0^{1-2/d}} \sqrt{g'} R' + \sqrt{g'}\,. \tag{1.28}$$

This is to verify that the value of λ_0 is arbitrary as it is nothing more than the volume term and directly relating to the rescaling of metric, therefore to negate the arbitrariness of the rescaling which confuses the renormalization issues more, it is clever to fix the volume term such that it has a unit coefficient as above. In fact from above, one can conclude that the only coupling that matters for pure gravity in 4 spacetime dimensions is to the combination $G\sqrt{\lambda_0}$ and without the loss of generality, one can set $\lambda_0 = 1$ in units of some UV cutoff (*i.e.,* to write everything in units of λ_0). We explicitly take this unit system in the studies of Wheeler DeWitt equations on lattice later on in Chapter 2.

[5] For example, let's consider ϕ^4 theory, which has the Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4$, where ϕ_0, m_0 and λ_0 are bare parameters of the theory and note also this λ_0 is nothing to do with the notation that we are using for the cosmological constant. Then we can eliminate the shift in field strength which appear for example in 2 - point function in a form of a residue, $\int d^4x < \Omega | T\phi(x)\phi(0)|\Omega >= \frac{iZ}{p^2-m^2} + \cdots$ where $\cdots$ are terms we do not care as they are regular at $p^2 = m^2$ and m is the physical mass of the scalar field. The rescaling of the field is $\phi \to \phi_r = Z^{-1/2}\phi$. With this, there is no more confusing factors of multiples of Z floating around in a scattering amplitude (which is necessarily to compute for radiative corrections and therefore renormalization group calculations). This way, one is able to calculate scattering amplitudes now simply as the sum of connected diagrams according to the Feynman rules. In fact, the rescaling of the field is redundant in a physical sense and can be absorbed into the renormalizations of physical parameters such as the coupling constants and masses, as it is obvious in the following. After rescaling of the field, the Lagrangian in terms of the renormalized parameters and with the counter terms which absorb the unobservable infinities of the theory becomes $\mathcal{L} = \frac{1}{2}\partial_\mu\phi_r\partial^\mu\phi_r - \frac{1}{2}m^2\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 + \frac{1}{2}\delta_z\partial_\mu\phi_r\partial^\mu\phi_r - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4$, where we absorbed Z such that $Z = 1 + \delta_z$, $m_0^2 Z = m^2 + \delta_m$, and $\lambda_0 Z^2 = \lambda + \delta_\lambda$ from $\mathcal{L} = \frac{1}{2}Z\partial_\mu\phi_r\partial^\mu\phi_r - \frac{1}{2}m_0^2 Z\phi_r^2 - \frac{\lambda_0}{4!}Z^2\phi_r^4$. This procedure is standard and one can find for example in [12]. It is exactly the similar argument that we discussed that cosmological constant term can be scaled out of the problem and irrelevant for discussion of the renormalization of physical parameters, as it is precisely casted as the rescaling of the metric (the field in quantum field theoretic treatment.)

1.4.2 Gauge Dependence in the Renormalization of the Cosmological Constant

Perturbation theory lets us systematically track the gauge dependence of renormalization effects. This is a very useful tool to keep track of what is physical and what is an entirely spurious gauge artifact. We review in this section the work done in [40] to point out the relation between the gauge invariance and the two coupling constants G and λ_0 in the bare Lagrangian.

Unfortunately Einstein gravity is not perturbatively renormalizable around Minkowski metric in 4 spacetime dimensions, however, one can go down in dimensions, then one is able to perform perturbative calculations and therefore to address some of the key issues; at least one can see how to resolve the problem to recover the gauge invariance even in lower dimensions which should not be restricted to the lower dimensions.

Gravitational part of the Lagrangian is

$$\mathcal{L} = -\frac{\mu^{\epsilon}}{16\pi G}\sqrt{g}R\,, \tag{1.29}$$

where we took G to be the mass dimensionless coupling and take out the part that depends on the momentum, *i.e.,* μ^{ϵ}, where μ is an arbitrary momentum scale. We perturb the metric around the classical flat background metric linearly,

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}\,, \tag{1.30}$$

therefore where $\delta_{\mu\nu}$ is the classical flat background field and $h_{\mu\nu}$ is the small quantum fluctuation.

We shall add a gauge fixing term, for example by taking the form of a background harmonic gauge condition.

$$\mathcal{L}_{gf} = \frac{1}{2}\alpha\sqrt{g}\delta_{\nu\rho}(\nabla_{\mu}h^{\mu\nu} - \frac{1}{2}\beta\delta^{\mu\nu}\nabla_{\mu}h)(\nabla_{\lambda}h^{\lambda\rho} - \frac{1}{2}\beta\delta^{\lambda\rho}\nabla_{\lambda}h) \tag{1.31}$$

where contraction is done with the background metrics, *e.g.*, $h^{\mu\nu} = \delta^{\mu\alpha}\delta^{\nu\beta}h_{\alpha\beta}$, $h = \delta^{\mu\nu}h_{\mu\nu}$ and ∇_μ is the covariant derivative with respect to the background metric $\delta_{\mu\nu}$. To accommodate the gauge fixing term, one needs to introduce Faddeev Popov ghost, Ψ_μ, then the Lagrangian becomes $\mathcal{L} + \mathcal{L}_{gf} + \mathcal{L}_{ghost}$, where $\mathcal{L}$ includes the gravitational Lagrangian in Eq. (1.29), and also cosmological constant term *i.e.*, $\mathcal{L}_{cc} = \lambda_0\sqrt{g}$. One obtains an expression for the graviton propagator,

$$\begin{aligned} < h_{\mu\nu}(k)h_{\alpha\beta}(-k) > &= 1/k^2(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}) - \frac{2}{d-2}\frac{1}{k^2}\delta_{\mu\nu}\delta_{\alpha\beta} \\ &- \left(1 - \frac{1}{\alpha}\right)\frac{1}{k^4}(\delta_{\mu\alpha}k_\nu k_\beta + \delta_{\nu\alpha}k_\nu k_\beta + \delta_{\mu\beta}k_\nu k_\beta + \delta_{\nu\beta}k_\mu k_\alpha) \\ &+ \frac{1}{d-2}\frac{4(\beta-1)}{\beta-2}\frac{1}{k^4}(\delta_{\mu\nu}k_\alpha k_\beta + \delta_{\alpha\beta}k_\mu k_\nu) \\ &+ \frac{4(1-\beta)}{(\beta-2)^2}\left(2 - \frac{3-\beta}{\alpha} - \frac{2(1-\beta)}{d-2}\right)\frac{1}{k^6}k_\mu k_\nu k_\alpha k_\beta\,, \end{aligned} \tag{1.32}$$

where α and β are gauge dependent parameters as appeared in Eq. (1.31). We keep them as free parameters to keep track of gauge dependence/independence. Particularly, one will notice that the gauge parameter β is associated with the gauge freedom related to a rescaling of the metric $g_{\mu\nu}$. For the one loop divergences, one obtains,

$$\lambda_0 \to \lambda_0\left[1 - \frac{G}{\epsilon}\left(-\frac{8}{\alpha} + 8\frac{(\beta-1)^2}{(\beta-2)^2} + 4\frac{(\beta-1)(\beta-3)}{\alpha(\beta-2)^2}\right) + \frac{G}{\epsilon^2}\left(8\frac{(\beta-1)^2}{(\beta-2)^2}\right)\right], \tag{1.33}$$

and

$$\frac{\mu^\epsilon}{16\pi G} \to \frac{\mu^\epsilon}{16\pi G}\left[1 - \frac{G}{\epsilon}\left(\frac{2}{3}\cdot 19 + \frac{4(\beta-1)^2}{(\beta-2)^2}\right)\right]. \tag{1.34}$$

To put them together, we obtain the total Lagrangian to be

$$\begin{aligned} \mathcal{L} \to &-\frac{\mu^\epsilon}{16\pi G}\left[1 - \frac{G}{\epsilon}\left(\frac{2}{3}\cdot 19 + \frac{4(\beta-1)^2}{(\beta-2)^2}\right)\right]\sqrt{g}R \\ &+\lambda_0\left[1 - \frac{G}{\epsilon}\left(-\frac{8}{\alpha} + 8\frac{(\beta-1)^2}{(\beta-2)^2} + 4\frac{(\beta-1)(\beta-3)}{\alpha(\beta-2)^2}\right) + \frac{G}{\epsilon^2}\left(8\frac{(\beta-1)^2}{(\beta-2)^2}\right)\right]\sqrt{g}\,. \end{aligned} \tag{1.35}$$

However, one notices that the above expression for the total Lagrangian including the one loop radiative corrections are gauge dependent. By rescaling the metric,

$$\left[1 - \frac{G}{\epsilon}\left(-\frac{8}{\alpha} + 8\frac{(\beta-1)^2}{(\beta-2)^2} + 4\frac{(\beta-1)(\beta-3)}{\alpha(\beta-2)^2}\right) + \frac{G}{\epsilon^2}\left(8\frac{(\beta-1)^2}{(\beta-2)^2}\right)\right]\sqrt{g} = \sqrt{g'}\,, \tag{1.36}$$

which is equivalent to having

$$g_{\mu\nu} = \left[1 - \frac{G}{\epsilon}\left(-\frac{8}{\alpha} + 8\frac{(\beta-1)^2}{(\beta-2)^2} + 4\frac{(\beta-1)(\beta-3)}{\alpha(\beta-2)^2}\right) + \frac{G}{\epsilon^2}\left(8\frac{(\beta-1)^2}{(\beta-2)^2}\right)\right]^{-2/d} g'_{\mu\nu}\,, \quad (1.37)$$

we gain

$$\mathcal{L} \to -\frac{\mu^\epsilon}{16\pi G}\left[1 - \frac{2}{3}\cdot 19\,\frac{G}{\epsilon}\right]\sqrt{g'}R' + \lambda_0\sqrt{g'}\,, \quad (1.38)$$

instead of Eq. (1.35) and where we regained the cosmological term back into its standard from $\lambda_0\sqrt{g'}$. Eq. (1.38) now does not depend on the gauge, and one can finally read off the renormalization of Newton's constant

$$\frac{1}{G} \to \frac{1}{G}\left[1 - \frac{2}{3}\cdot 19\,\frac{G}{\epsilon}\right]. \quad (1.39)$$

The main point here is that only the renormalization of G can be gauge independent, therefore has physical meaning.

1.4.3 Dynamically Generated Emergent Infrared Mass Scale

We saw in the above two Sections 1.4.2 and 1.4.1, that the cosmological constant that appears in the gravitational action can be completely scaled out of the problem from the redefinition of the field. We also saw that in fact by separating G and λ_0 upon studying the renormalization group will only result in spurious gauge dependent effects. Therefore only G can be physically relevant to be studied to run with scale in the renormalization group computations. But it is important that we realized that there emerges dynamically generated emergent infrared mass-like scale in the problem m as in Eq. (1.16) and (1.17) . This scale is precisely similar to the nonperturbative infrared mass-like scale, Λ_{QCD}, in QCD where β function is negative, which we also observed appearing in the renormalization group equations Eq. (1.2) and (1.3). We claim that the scale similar to Λ_{QCD} arises in quantum gravity above 2 dimensions for the phase with $\beta < 0$ just like in QCD, and corresponds to the gauge field condensate, *i.e.*, gravitational field condensate, which can be identified with the physical *observed* cosmological constant value (in 4 dimensions).

Furthermore, there have been studies conducted to calculate gravitational Wilson loop on lattice based on Regge calculus [41, 42, 43]. Wilson loop is an inherently interesting quantity as it is a gauge invariant quantity of the theory, yet it is nonlocal as it is a quantity defined in a region of space enclosed by a loop. In fact all gauge invariant functions of A_μ can be thought of as combinations of Wilson loops for different choices of path. We first take a look at the Wilson loop for non Abelian gauge theories $SU(N)$ which gave us a confining potential for a static source in the fundamental representation at strong coupling. The Wilson loop [44] is defined for a large closed planar loop C,

$$W(C) = \left\langle Tr\ \mathcal{P} e^{ig \oint_C A_\mu(x) dx^\mu} \right\rangle , \tag{1.40}$$

where $A_\mu \equiv t_a A^a_\mu$ and t_a's are the generators of $SU(N)$ in the fundamental representation. Constructing a Wilson loop involves finding the expectation value of a product of rotation matrices around a loop; the natural procedure is to treat these rotation matrices as variables and to integrate over their product, weighted by the exponential of the action. To present what I mean here a bit more explicitly, we take a finite group transformation (rotation matrix) $U(x_2, x_1) = \mathcal{P} e^{ig \int_{x_1}^{x_2} A_\mu dx^\mu}$ or on lattice $U(v_2, v_1) = \mathcal{P} e^{ig \int A_\mu dx^\mu}$ which is the line element from one vertex v_1 to v_2, and integrate over U for each link of the lattice. By taking a path ordered product of the U matrices around a closed path $\langle \prod_P U \rangle$, one constructs a gauge invariant observable, the Wilson loop. In the pure gauge theory at strong coupling the leading contribution to the Wilson loop can be shown to follow an area law for sufficiently large loops

$$W(C) \overset{A \to \infty}{\sim} e^{-\frac{A(C)}{\xi^2}} , \tag{1.41}$$

where $A(C)$ is the minimal area spanned by the planar loop C [45]. The quantity ξ is the gauge field correlation length and in non Abelian gauge theories, whose inverse is understood to correspond to the lowest mass excitation, the scalar glueball. One can further examine what the above area law implies physically first by imagining that a pair color sources that are a distance R apart for a Euclidean time T, therefore represented by a large rectangular loop of width R and length T yielding the area $A = RT$. Recalling that in fact the *Action* $\sim$

$Energy \cdot Time$, the expression of Eq. 1.41 can be further deduced as

$$\left\langle e^{-ET} \right\rangle \sim \left\langle e^{-\frac{RT}{\xi^2}} \right\rangle . \tag{1.42}$$

Then we conclude that the static sources of gauge charge in strong coupling, are attracted to one another by a potential energy

$$V(R) \sim \frac{R}{\xi^2} \tag{1.43}$$

for a sufficiently large area, therefore R. This tells us that it requires more energy (E increases) to let the quarks apart further (as R increases), leading to the phenomenon, quark confinement [see for example, [12]].

Now we turn into the gravitational case. The rotation matrix appearing in the gravitational Wilson loop can be related classically to a well defined physical process which is that a vector is parallel transported around a large loop and the orientation of the starting and ending points are compared. Therefore Wilson loop provides information about the parallel transport of vectors, and therefore on the effective curvature, around large, near planar loops. The vector's rotation can then be directly related to an average curvature enclosed by the loop. The total rotation matrix $\mathbf{R}(C)$is given by a path ordered exponential of the integral of the affine connection $\Gamma^{\lambda}_{\mu\nu}$ via

$$R^{\alpha}_{\beta}(C) = \left[\mathcal{P} e^{\oint_C \Gamma^{\cdot}_{\lambda \cdot} dx^{\lambda}} \right]^{\alpha}_{\beta} \tag{1.44}$$

In such a description of the parallel transport process of a vector around a *very large loop*, one reexpresses the connection in terms of a suitable coarse grained, or semi classical Riemann tensor, by using Stokes' theorem [6]

$$R^{\alpha}_{\beta}(C) \sim \left[e^{\frac{1}{2} \int_{\Sigma(C)} R^{\cdot}_{\cdot \mu\nu} A^{\mu\nu}_{C}} \right]^{\alpha}_{\beta} \tag{1.45}$$

[6] It is important that we are taking a large loop and the Riemann tensor is that of the suitably coarse grained one, here for Stokes' theorem to be valid, once one understands the underlying principle of the Stokes' theorem. The idea is depicted in the diagram Fig. 1.5, which illustrates in an oriented tiling of a manifold, the interior paths are traversed in opposite directions; therefore their contribution to the path integral cancel each other pairwise. Thus, only the contribution from the boundary survives. This argument holds to a good approximation only if the tilings are sufficiently fine, or in another perspective, the area of the loop is large.

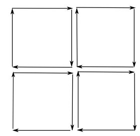

Figure 1.5: Stokes' Theorem.

with an area bivector associated with the loop C,

$$A_C^{\mu\nu} = \frac{1}{2} \oint dx^\mu x^\nu \,. \tag{1.46}$$

Since Wilson loop is a scalar expression, one needs to have the matrix expression Eq. **??** properly turn into a scalar expression; we shall take the trace after contracting the rotation matrix with properly chosen some bivector $\widetilde{U}_C$ that encode the *average* direction for the loop which is assumed to be planar after all [25], therefore, giving

$$W(C) \sim Tr \left(\widetilde{U}_C \, e^{\frac{1}{2} \int_{\Sigma(C)} R_{\cdot\cdot\mu\nu}^{\cdot\cdot} A_C^{\mu\nu}} \right) \,. \tag{1.47}$$

Then by expressing the properly coarse grained (semi classical) Riemann tensor,

$$R^\alpha_{\beta\mu\nu} = \bar{R} U^\alpha_\beta U_{\mu\nu} \,, \tag{1.48}$$

where one defines $\bar{R}$ to be some average curvature over the loop, and U's are area bivectors perpendicular to the loop. Then we obtain in the expression in Eq. 1.47,

$$Tr(\bar{R} U_C^2 A_C) = -2\bar{R} A_C \,, \tag{1.49}$$

letting us identify that the gravitational Wilson loop exhibits area law and furthermore, identify via Eq. 1.41,

$$\bar{R} \sim \frac{1}{\xi^2} \,, \tag{1.50}$$

at least in strong coupling limit. This result can be further interpreted: $\frac{1}{\xi^2}$ should be identified, up to a constant, with the scaled [7] cosmological constant λ, with the latter being

[7] The scaled cosmological constant is the one that has the same mass dimension as R. This is coming from the fact that one can write the Lagrangian as in $\mathcal{L} = -\frac{1}{16\pi G}\sqrt{g}(R - 2\lambda)$ (which gives our familiar Einstein field equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$), or $\mathcal{L} = -\frac{1}{16\pi G}\sqrt{g} R + \lambda_0 \sqrt{g}$, where the unscaled one appears and $\lambda = \frac{\lambda_0}{8\pi G}$.

regarded as a measure of the intrinsic curvature of the vacuum. [8] With this argument, one identifies the dynamically generated infrared mass like scale ξ that appears in the renormalization group equations should be identified with the observed cosmological constant λ.

1.5 Gravitation in 4 Spacetime Dimensions

Together with what we have explored in lower dimensions in Sections 1.3 and 1.4 and the extensive lattice calculations performed earlier in [25, 26, 28, 46, 47], we summarize here the qualitative (quantum field theory and renormalization group studies in lower dimensions as discussed in this Chapter mainly in Section 1.3) and quantitative (deduced from lattice calculations cited above).

The renormalization group studies and therefore renormalized G can be computed on the lattice directly in 4 dimensions [25, 26, 28, 46, 47], or in the continuum within the framework of the background field expansion applied to $2+\epsilon$ spacetime dimensions as we discussed earlier in Sections 1.3 and [6, 23, 40, 48, 49] and later using truncation methods (functional renormalization group equation method) applied in 4 dimensions [50].

The situation regarding the running of G is possibly most easily illustrated close and above 2 dimensions, where the gravitational coupling becomes mass dimensionless, $G \sim \Lambda^{2-d}$ with Λ the ultraviolet cutoff required to regularize the theory (a similar and completely parallel line of arguments and results can in fact be presented for the 4 dimensional lattice theory as well, but a discussion of renormalization on the lattice ends up being inevitably quite a bit less transparent [46, 47, 25]). We find the detailed discussion in Sec. 1.3 for gravity in above 2 (*i.e.*, $2+\epsilon$) dimensions. There the theory appears perturbatively renormalizable, so that the full machinery of covariant renormalization and of the renormalization group can

[8] Taking the trace of the Einstein field equation for vacuum, *i.e.*, $T_{\mu\nu} = 0$, it is automatic to note that the curvature corresponds to the scaled cosmological constant, obtaining $R = 4\lambda$.

in principle be applied, following Wilson's dimensional expansion method, now formulated as a double expansion in G and $\epsilon = d - 2$ [6, 23, 40, 48, 49]. Both here and on the lattice a renormalization of the bare cosmological constant, besides being gauge dependent, is also physically meaningless, as it can be reabsorbed by a trivial rescaling of the metric; the latter is needed in order to recover the proper normalization of the volume term in the path integral, thus avoiding spurious renormalization effects, as discussed in Section 1.4.1 and 1.4.2 and [25, 46, 47, 23, 40, 48, 49]

In momentum space we take the result corresponding to Eq. (1.16) with $\epsilon \to 1/\nu$ as $\nu^{-1} \equiv -\frac{\partial \beta}{\partial g}|_{g=g_c} \sim \epsilon + \mathcal{O}(\epsilon^2)$, to consider more general exponent ν that characterizes the 4 dimensions, and expanding it in Taylor series,

$$G(\mu^2) \;\simeq\; G_c \left[1 \pm c_0 \left(\frac{1}{\xi^2 \, \mu^2} \right)^{1/2\nu} + \, \dots \right] , \tag{1.51}$$

where G_c the value of G at the critical point (nontrivial UV fixed point) therefore given by:

$$\beta(G_c) = 0 \, , \tag{1.52}$$

ν is the universal critical exponent:

$$\nu^{-1} = - \left. \frac{\partial \beta(G)}{\partial G} \right|_{G=G_c} , \tag{1.53}$$

$\xi = m^{-1}$ the new, genuinely nonperturbative, gravity scale:

$$m = \xi^{-1} = A_m \; \mu e^{-\int^{G(\mu)} \frac{dG'}{\beta(G')}} \overset{G \to G_c}{\sim} A_m \; \Lambda |G(\Lambda) - G_c|^{\nu} \, , \tag{1.54}$$

and c_0 a positive (nonperturbative) constant, which can be determined once one knows ν, G_c and the relative constant A_m between integration constant of the renormalization group equation and the nonperturbative infrared scale m, *i.e.*,

$$c_0 = \frac{1}{A_m^{1/\nu} G_c} \, . \tag{1.55}$$

which was discussed earlier in Eq. (1.21). [9] Consequently the above expression for $G(\mu^2)$

[9] A properly infrared regulated version of the above expression, here with the choice of + sign, would read

$$G(\mu^2) \;\simeq\; G_0 \left[1 \, + \, c_0 \left(\frac{\xi^{-2}}{\mu^2 \, + \, \xi^{-2}} \right)^{1/2\nu} + \, \dots \right] . \tag{1.56}$$

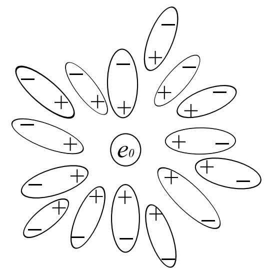

Figure 1.6: Vacuum polarization which occurs in QED. Virtual electron positron pairs surround the bare change and make the vacuum dielectric. This in turn, makes the apparent charge less than the true charge. We therefore have screening.

can be used whenever the full generality of the manifestly covariant expression in Eq. (1.51) is not really needed, for example when dealing with the Newtonian (nonrelativistic) limit.

The choice of + or − sign in Eq. (1.51) is ultimately determined from whether one is initially to the left (−), or to right (+) of the fixed point G_0, in which case the effective $G(\mu^2)$ decreases or, respectively, increases as one flows away from the ultraviolet fixed point towards lower momenta, or larger distances. Physically the two solutions represent of course gravitational screening ($G < G_0$) or antiscreening ($G > G_0$). One may argue which phase we possibly reside; there is an intuitive picture that we can provide. One notices that in quantum electrodynamics (QED), where one has opposite charges, the renormalization effects can be thought of as screening effect due to vacuum polarization. As depicted in Fig. 1.6, virtual e^+e^- pairs would turn the vacuum into a dielectric medium by lining up such that the apparent (effective, renormalized) charge is less than the true (bare) charge [see for example [12]]. The effect of the vacuum polarization is bigger at larger distances; as one proceeds in smaller distances, one starts to see the bare charge by shearing through the screening cloud of virtual electron positron pairs. One can clearly see the effect spelled out above in a form

Then for large distances $r \gg \xi$ the gravitational coupling no longer exhibits the spurious infrared divergence, but instead approaches the finite value $G_\infty \simeq (1 + c_0 + \dots)\, G_0$.

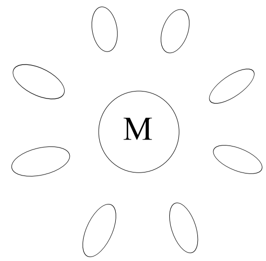

Figure 1.7: Vacuum polarization which occurs in quantum gravity. One expects gravitational coupling becomes bigger as you include more and more mass (energy). Therefore resulting in antiscreening.

of vacuum polarization in the expression for effective coupling constant α,

$$\alpha_{eff}(\mu^2) = \frac{\alpha}{1 + \frac{\alpha}{3\pi} \log \left| \frac{\mu^2}{A\, m^2} \right|}, \tag{1.57}$$

where $\alpha = \frac{e^2}{4\pi}$ and A is some irrelevant constant due to specific regularization scheme. The effective electric charge becomes larger at small distances (large momenta), *i.e.,* as we pass the polarization cloud.

For the gravitational case, as given in Fig. 1.7, the first thing one notices is that the charge here is mass or energy which are positive definite, and thus we do not have opposite charges. One then pictures that as you go further in distances, you include more mass, therefore, the effective mass should be bigger than the bare mass, *i.e.,* the coupling will increase as you go larger distances: this is opposite of what was happening in QED, and the effect is that of antiscreening. In fact screening phase corresponds to when β function is positive, whereas antiscreening phase to negative β function.

There had been further studies, on which phase the physical one corresponds to strong coupling phase. [51] We outline the result from [47, 46, 52] We note that already studies on small lattices form a rich structure for the ground state of quantum gravity to be discussed [53, 26, 27, 28, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 46, 64, 65]. It is found that simplicial quantum gravity in 4 dimensions exhibits a phase transition between *two phases*: a strong coupling phase, where the geometrical manifold is smooth at large scales and quantum fluctuations in the gravitational field are bounded and average out, and a weak coupling phase, where the geometrical manifodl is degenerate and spacetime collapses into a branched polymer, *i.e.,* a lower dimensional manifold. Only the smooth phase appears to be physically acceptable. Phrased in different terms, the two phases of quantized gravity found in [46, 64, 65] can loosely be described as having, in one phase (with $G < G_C$, the rough branched polymer-like phase),

$$< g_{\mu\nu} >= 0\,, \tag{1.58}$$

while, in the other (with $G > G_c$, the smooth phase),

$$< g_{\mu\nu} >\sim c\,\eta_{\mu\nu}\,, \tag{1.59}$$

with a vanishingly small negative average curvature in the vicinity of the critical point at G_c. The existence of a phase transition at a finite coupling G, which is associated in quantum field theory with the appearance of an ultraviolet fixed point of the renormalization group, implies in principle nontrivial, calculable nonperturbative scaling properties for correlations and effective coupling constants and, in particular, in the case at hand for Newton's gravitational constant.

One then conclude that it is the antiscreening from these lattice studies, *i.e.,* negative β function regime, which gives rise to strong coupling regime ($G > G_c$), that the gravitation that describes out universe should reside. This then gives (+) sign for Eq. (1.51) and we notice that with this, G increases with large distance scale or infrared (IR) energy momentum scale. One then notices that the quantum effects which one may naïvely think only to be able to see by probing small length scale or large momenta, should be probed instead at large distance scale. This is in fact the heart of what Wilson's renormalization group tells us: that it allows us to explore the relation between microscale and macroscale of the theory. The effect on the cosmology therefore due to the bahavior of this quantum gravity are explored in Chapter 3. Furthermore, possible evidences of physical phase are explored in Chapter 2 using lattice discretized Wheeler DeWitt equations.

It is crucial that the quantum correction involves a new physical, renormalization group invariant, scale ξ, whose value cannot be fixed by a perturbative calculation, and whose absolute size determines the comparison scale for the new nonlocal quantum effects. It should therefore be rightfully considered as the gravity analog of the celebrated gauge theory scaling violation parameter $\Lambda_{\overline{MS}}$. An explicit expression of this nonperturbative scale is given in Eq. (1.54) and it is then more or less a direct consequence of the renormalization group that the value of the constant A_m determines the coefficient c_0 in Eq. (1.51), as in Eq. (1.55). The nonperturbative lattice formulation of quantum gravity then allows an explicit and direct computation of A_m, and therefore of the coefficient c_0 in $G(\mu^2)$ [46, 47, 25].

Physically it would seem at first, based on renormalization group considerations alone, that the nonperturbative (renormalization group integration constant) scale ξ could in principle take any value, including a very small one - based on the naive estimate $\xi \sim L_P$ - which would then of course preclude any observable quantum effects in the foreseeable future. But recent results for the gravitational Wilson loop on the Euclidean lattice at strong coupling which was reviewed in Section 1.4.3, giving an area law, and their subsequent interpretation in light of the observed large scale semiclassical curvature [41, 42, 43], would suggest otherwise, namely that the nonperturbative scale ξ appears in fact to be related to *macroscopic curvature*. From astrophysical observation the average curvature on very large scales, or, stated in somewhat better terms, the measured physical cosmological constant λ, is very small. This in return means that the new scale ξ can be very large, even cosmological,

$$\frac{1}{\xi^2} \simeq \frac{\lambda}{3} \tag{1.60}$$

which would then give a more concrete quantitative estimate for the scale in the $G(\mu^2)$ of Eq. (1.51), namely $\xi \sim 1/\sqrt{\lambda/3} \sim 1.51 \times 10^{28}$cm. Indeed for quantum gravity no other suitable infrared cutoff presents itself, so that λ can almost be considered as the only natural candidate to take on the role of a (generally covariant) infrared regulator or graviton mass like parameter.

Finally let us mention here briefly for completeness that for a limited number of metrics it has been possible, after some considerable work, to find exact solutions, in some regime, to the above effective nonlocal field equations. One such case is the static isotropic metric, where in the limit $r \gg 2MG$ one can obtain an explicit solution for the metric coefficients $A(r) = 1/B(r)$, leading eventually to the rather simple result [66, 67]

$$G \;\to\; G(r) \;=\; G_0 \left(1 \,+\, \frac{c_0}{3\,\pi} \, m^3 \, r^3 \, \ln \, \frac{1}{m^2 \, r^2} \,+\, \dots \right) \tag{1.61}$$

with $m \equiv \xi^{-1}$, consistent with a gradual slow increase of $G(r)$ with distance. [10] One amusing aspect of the exact solution in the static isotropic case is that no consistent solution can be found unless $\nu = 1/3$ exactly in 4 dimensions, and similarly $\nu = 1/(d-1)$ in dimensions $d \geq 4$ [66, 67], lending further support, and independently of the lattice theory results, to this particular value for ν in 4 dimensions.

In my works I have therefore explored the properties associated with the quantum gravity which appear to exhibit a nontrivial UV fixed point, therefore exhibit 2 phases which contain weak coupling phase and strong coupling phase. In the studies of Wheeler De Witt equations on lattice, we find evidences of this existence of a nontrivial UV fixed point and furthermore to support the idea that the strong coupling phase is the physical phase (Chapter 2). Later in Chapter 3, we explore the evidence from cosmology to see if the gravitational coupling G is running as the renormalization group predicts and if the strong coupling phase is the physical phase which coincides with our universe that we observe.

[10] We have pointed out before that the result for $G(r)$ is in a number of ways reminiscent of the analogous QED result (known as the Uehling correction to the Coulomb potential in atoms)

$$Q \;\to\; Q(r) \;=\; Q \left(1 \,+\, \frac{\alpha}{3\,\pi} \, \ln \frac{1}{m^2 \, r^2} \,+\, \dots \right) \,. \tag{1.62}$$

In the gravity case the correction is not a log but a power, which is what one would naïvely expect from a perturbatively non renormalizable theory. In gravity, the infrared cutoff due in QED to the finite physical electron mass is naturally replaced by the physical cosmological constant; the magnitude of neither one of these two quantities can be predicted by the fundamental theory.

Chapter 2

Lattice Approach to Quantum Gravity

Discretization derives an intuitive method to quantization. Indeed it is a natural occurring and essential idea in the path integral formulation, as typical paths of quantum mechanical particles are highly irregular on a fine scale, and nowhere differentiable. In fact, one can think of lattice formulation to be a regularization method as it naturally introduces a UV cutoff, *i.e.,* inverse of which is a lattice spacing. The lattice formulation is, not surprisingly, an important and essential step towards a quantitative and controlled investigation of the physical content of the theory, especially for a theory like gravity where the interaction is highly nonlinear.

In lattice gravity, we expect spacetime to be discretized, as gravity is a manifestation of the structure of space and time, where, spacetime is built up from flat simplices, with curvature residing in the codimension 2 subspace. For example in 4 dimensions, the hinges, where the curvature lies, are triangles.

In this Chapter, we examine lattice approaches to quantum gravitation. In the first following Section 2.1, I will review the lattice formulation, focusing on the similarity to the path integral formalism. The further detail on lattice discretized description of gravity can be found for example in [25, 68]. One can develop the lattice formulation within a framework of covariant approach to quantum gravity, or alternatively one can take the canonical Hamiltonian approach. The rest of the Chapter is focused on the work in the Hamiltonian approach on lattice. Hamiltonian approach may be less attractive as compared to the covariant approach especially in the context of gravity, as one inherently already gives up the manifest general covariance by treating the time special. Nonetheless, quantum mechanics (which is based on the Hamiltonian formulation) is successful in its own account and gave us a enlightening ideas, therefore, it is not completely useless to consider canonical formulation and listen to what its results tell us.

2.1 Lattice Formulation

Approaches based on linearized perturbation methods have had moderate success so far, as the underlying theory is known not to be perturbatively renormalizable [20, 6]. In addition gravitational fields are themselves the source for gravitation already at the classical level, which leads to the problem of an intrinsically nonlinear theory where perturbative results are possibly of doubtful validity for sufficiently strong effective couplings. In general nonperturbative effects can give rise to a novel behavior in a quantum field theory and particularly, to the emergence of nontrivial fixed points of the renormalization group (a phase transition). It has been realized for some time that in general the universal low and high energy behavior of field theories is almost completely determined by the fixed point structure of the renormalization group flows [69, 70, 71, 72]. The situation bears some resemblance to the theory of strong interactions, QCD. There too, nonlinear effects play an important role, and end up restricting the validity of perturbative calculations to the high energy, short distance regime, where the effective gauge coupling are actually weak due to asymptotic freedom [9, 8, 11, 10]. For low energy properties, Wilson's discrete lattice formulation, combined with the renormalization group and computer simulations, has provided so far the only convincing evidence for quark confinement and chiral symmetry breaking, two phenomena which are invisible to any order in the weak coupling, perturbative expansion. A discrete lattice formulation can be applied to the problem of quantizing gravitation. Instead of continuum metric fields, one deals with gravitational degrees of freedom which live only on discrete spacetime points and interact locally with each other. In Regge's simplicial formulation of gravity [73, 74], one approximates the functional integration over continuous metrics by a discretized sum over piecewise linear simplicail geometries [26, 27, 75, 76, 28, 27, 56, 55]. In such a model, the role of the continuum metric is played by the edge lengths of the simplices, while curvature is described by a set of deficit angles, which can be computed in terms of given edge lengths. The simplicial lattice formulation of gravity is locally gauge invariant [77] and can be shown to contain perturbative gravitions in the lattice weak field expansion [26, 27], making it an attractive and faithful lattce regularization of the continuum theory. The discretized theory is restricted to a finite set of dynamical variables, once a set of suitable boundary conditions are imposed such as periodic or with some assigned boundary manifold. In the end the original continuum theory of gravity is recovered as the spacetime volume is sent large and the fundamental lattice spacing of the discrete theory is sent to zero.

Quantum fluctuations in the underlying geometry are represented in the discrete theory by fluctuations in the edge lengths, which can be modeled by a well defined, and numerically exact, stochastic process. In analogy with other field theory models studied by computer, calculations are usually performed in the Euclidean imaginary time framework, which is the only formulation amenable to a controlled numerical study for the time being. By a careful and systematic analysis of the lattice results, the critical exponents can be extracted and the scaling properties of invariant correlation functions determined from first principles.

We remark here that quantum field theories are quite outstanding in that a wide variety of physical properties can be determined from a relatively small set of universal quantities [78]: namely, the universal critical exponents in the vicinity of some fixed point (or line) of the renormalization group equations. In the lattice theory, the presence of a fixed point of phase transition is often inferred from the appearance of nonanalytic terms in invariant local averages. We will see an example of this in detail later in the Section 2.5.10.

We will describe in the following the lattice discretized description of gravity originally formulated by Regge. There, we express the Einstein theory in a simplicial decomposition of spacetime manifolds. The approach was readily available by use of the developed techniques in lattice gauge theories. It allows us to use powerful nonperturbative analytical techniques of statistical mechanics as well as numerical methods. It is also of its advantage to be able to formulate it in any dimensions including the physically relevant 4 dimensions.

We have infinite number of degrees of freedom in the continuum, whereas in the lattice it is restricted to finite number of variables which are the geodesic distances between neighboring points describing the Riemannian manifolds. We now have a manifold that is almost flat everywhere (except at the hinges where the curvature resides), called piecewise linear. The elementary building blocks for d - dimensional spacetime are d - dimensional simplexes. A 0-simplex is a point, 1-simplex an edge, 2-simplex a triangle, and 3-simplex a tetrahedron, *etc.* A d simplex is a d - dimensional object with $d+1$ vertices and $d(d+1)/2$ edges connecting them. The primary dynamical variables are edge lengths, and their values specify the shape and uniquely determine the relative angles. A simplicial complex are therefore viewed as a set of simplices glued together sharing hinges which are codimension 2 object. The relative position of lattice points is completely specified by the edge lengths and an incidence matrix which tells us which point is next to which point, giving us the metric structure.

The geometry of the interior of a d - simplex is flat, and therefore completely determined by the lengths of its $d(d+1)/2$ edges. Let $x^{\mu}(i)$ to be the μ^{th} coordinate of the i^{th} site. For each pairs of neighboring sites, i and j, the link length^2 is given by the expression

$$l_{ij}^2 = \eta_{\mu\nu} \left[x(i) - x(j)\right]^{\mu} \left[x(i) - x(j)\right]^{\nu} \ , \tag{2.1}$$

where $\eta_{\mu\nu}$ is the flat metric. Given a n - simplex, we denote the vertices by $0, 1, 2, 3, \ldots, n$ and edge length[2] by $l_{01}^2 = l_{10}^2, \ldots l_{0n}^2$. The simplex is spanned by the set of n vectors $e_1, \ldots e_n$ connecting the vertex 0 to other vertices. We assign the rest of the edges within the simplex with vectors $e_{ij} = e_i - e_j$ where $1 \leq i < j \leq n$. One notices that we have n independent vectors, and $n(n+1)/2$ invariants given by all the edge lengths[2] within a given simplex σ. We now erect a orthonormal (Lorentz) frame to the interior od n - simplex. The expansion coefficients relating this orthonormal frame to the one specified by the n directed edges of the simplex associated with the vectors e_i are the lattice analogue of the n - bein (or tetrad e_μ^α. Then for each n - simplex σ, one can define a metric

$$g_{ij}(\sigma) = e_i \cdot e_j \,, \tag{2.2}$$

and it is positive definite in the Euclidean space. (In components, one can write $g_{ij} = \eta_{ab} e_i^a e_j^b$.) One can express the metric in the edge lengths $l_{ij} = |e_i - e_j|$

$$g_{ij}(\sigma) = \frac{1}{2}\left(l_{0i}^2 + l_{0j}^2 - l_{ij}^2\right) \,. \tag{2.3}$$

Comparison with the standard expression for the invariant interval $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ confirms that the coordinates have been chosen along the n e_i directions. One can readily write the volume of the n - simplex (σ),

$$V_n(\sigma) = \frac{1}{n!}\sqrt{\det g_{ij}(\sigma)} \tag{2.4}$$

as well as the deficit angle at a given hinge h,

$$\delta(h) = 2\pi - \sum_{s \supset h} \theta(\sigma, h) \,, \tag{2.5}$$

where the sum extends over *all* simplexes meeting on h and where θ is dihedral angles associated with the faces of simplexes meeting at a given hinge. [1] It follows that the deficit angle δ is a measure of the curvature at h.

[1] With each *face* f (a codimension 1 object, therefore a tetrahedron in 4 dimensional manifolds, a triangle in 3 dimensions, an edge in 2 dimensions, a vertex in 1 dimension) of a n - simplex, one can associate a vector perpendicular to the face $\omega(f)_\alpha = \epsilon_{\alpha\beta_1\ldots\beta_{n-1}} e_{(1)}^{\beta_1} \ldots e_{(n-1)}^{\beta_{n-1}}$, where $e_{(1)}^{\beta_1} \ldots e_{(n-1)}^{\beta_{n-1}}$ are a set of oriented edges belonging to the face f, and $\epsilon_{\alpha_1\ldots\alpha_n}$ is the sign given by the permutation $(\alpha_1 \ldots \alpha_n)$. Then the volume

One then identifies that the Regge action for pure Einstein gravity with a cosmological constant and Regge scalar curvature term

$$I_{latt}(l^2) = \lambda_0 \sum_{\text{simplexes } \sigma} V_\sigma^{(d)} - \frac{1}{8\pi G} \sum_{\text{hinges h}} \delta_h V_h^{(d-2)} \,, \tag{2.6}$$

where $V_h^{(d-2)}$ is the volume of the hinge which is a codimension 2 object [see footnote 1]. and the lattice regularized Euclidean version of

$$Z_{latt}(\lambda_0, G) = \int \mathcal{D}[l^2] e^{-I_{latt}(l^2)} \,, \tag{2.7}$$

which corresponds to the continuum Euclidean Feynman path integral for gravity

$$Z_{cont} = \int \mathcal{D}[g_{\mu\nu}] e^{-\lambda_0 \int d^dx \sqrt{g} + \frac{1}{16\pi G} \int d^dx \sqrt{g} R} \tag{2.8}$$

We remark that looking at the lattice action, edges only couple with other edges which belong either to the same simplex or to a set of neighboring simplexes. Therefore the lattice action is still considered to be local just like the continuum action. $\mathcal{D}(l^2)$ is an appropriate functional integration measure over the edge lengths. One notes that looking at Eq. (2.3) and further identifying that the variation of the metric is given by $\delta g_{ij}(l^2) = \frac{1}{2}(\delta l_{0i}^2 + \delta l_{0j}^2 - \delta l_{ij}^2)$, for *one d - dimensional simplex* labeled by σ, integration over the metric is equivalent to an integration over the edge lengths. We have an identity

$$\left(\frac{1}{d!} \sqrt{\det g_{ij}(\sigma)} \right)^\zeta \prod_{i \geq j} dg_{ij}(\sigma) = \left(-\frac{1}{2} \right)^{\frac{d(d-1)}{2}} \left[V^{(d)}(l^2) \right]^\zeta \prod_{k=1}^{\frac{d(d+1)}{2}} dl_k^2 \,. \tag{2.9}$$

In fact it is apparent from the above expression that the fundamental degrees of freedom in continuum (metric) and discrete (edge lengths) match as there are $d(d+1)/2$ edges for each simplex, just as there are $d(d+1)/2$ independent components for the metric tensor in d

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of the face is given by $V^{(n-1)}(f) = \left(\sum_{\alpha=1}^n \omega_\alpha^2(f)\right)^{1/2}$. Similarly, one can define for a hinge (a codimension 2 object, therefore a triangle in 4 dimensions, an edge in 2 dimensions, a vertex in 1 dimensions), a hinge bivector $\omega(h)_{\alpha\beta} = \epsilon_{\alpha\beta\gamma_1\ldots\gamma_{n-2}} e_{(1)}^{\gamma_1} \ldots e_{(n-2)}^{\gamma_{n-2}}$ which in turn allows one to write the volume of the hinge as $V^{(n-2)}(h) = \frac{1}{(n-2)!} \left(\sum_{\alpha<\beta} \omega_{\alpha\beta}^2(h) \right)^{1/2}$. With this, one can express the dihedral angles as $\cos\theta(f,f') = \frac{\omega^{(n-1)}(f)\cdot\omega^{(n-1)}(f')}{V^{(n-1)}(f)\cdot V^{(n-1)}(f')} = \frac{\frac{1}{((n-1)!)^2}\det(e_i\cdot e_j')}{V^{(n-1)}(f)\cdot V^{(n-1)}(f')}$ where $e_i \cdot e_j' = g_{ij}$ and Eq. (2.3) can be used to compute it. For a triangle, the above expression for dihedral angles reduces to the familiar one: $\cos\theta_{12} = \frac{l_1^2+l_2^2-l_3^2}{2l_1l_2}$.

dimensions [79]. Here we are ignoring temporarily the triangle inequality constraint, which will further require all subdeterminants of g_{ij} to be positive, including the obvious restriction $l_k^2 > 0$.

The extension of the measure to *many* simplexes glues together at their common faces is then immediate. Firstly, we identify the relations between edges $l_k(\sigma)$ and $l'_k(\sigma')$ which are shared between simplexes σ and σ',

$$\int_0^\infty dl_k^2(\sigma) \int_0^\infty dl_{k'}^2(\sigma')\, \delta[l_k^2(\sigma) - l_{k'}^2(\sigma')] = \int_0^\infty dl_k^2(\sigma) \tag{2.10}$$

After summing over all simplexes in a problem, one derives the unique functional measure for simplicial geometries

$$\int \mathcal{D}[l^2] = \int_\epsilon^\infty \prod_\sigma [V^{(d)}(\sigma)]^\zeta \prod_{ij} dl_{ij}^2 \Theta[l_{ij}^2]\,. \tag{2.11}$$

(We omitted an irrelevant numerical constant.) $\Theta[l_{ij}^2]$ is a step function of the edge lengths, $= 1$ for the triangle inequalities and their higher dimensional analogues are satisfied and $= 0$ otherwise. ϵ is introduced as a ultraviolet cutoff of the theory for small edge lengths, and can be sent to zero if the measure is nonsingular for small edges. In 4 dimensions, for $\zeta =$ in the above measure corresponds to lattice analog of DeWitt measure, taking a simple form

$$\int \mathcal{D}[l^2] = \int_0^\infty \prod_{ij} dl_{ij}^2\ \Theta[l_{ij}^2]\,. \tag{2.12}$$

In numerical simulations of simplicial quantum gravity, lattice measures over the space of edge lengths[2] have been used extensively foe example in [26, 53, 55, 56]. One could take different measures for example by absorbing the cosmological constant λ_0 term in the measure, or working out from defining a discrete analog of the supermetric to yield a form analogous to the general form that encompass several construction of the continuum measure, $\int \mathcal{D}g_{\mu\nu} = \prod_x [g(x)]^{\widetilde{\zeta}/2} \prod_{\mu\geq\nu} dg_{\mu\nu}(x)$ where $\widetilde{\zeta}$ depends on the construction. The latter for the lattice version however gives nonlocal property, in spite of the fact that the original continuum measure is completely local. Further one can show that it disagrees with the continuum one already to the lowest order in the weak field expansion and would not qualify for an acceptable lattice measure [80].

2.2 Introduction to the Work on Lattice Hamiltonian for Quantum Gravity

For the rest of the Chapter, we present the work on the lattice Hamiltonian for quantum gravity. In this work we will focus on the Hamiltonian approach to gravity, which assumes from the beginning a metric with Lorentzian signature. In order to obtain useful insights regarding the nonperturbative ground state, a Hamiltonian lattice formulation was introduced based on the Wheeler DeWitt equation, where the quantum gravity Hamiltonian is written down in the position space representation. In [81] a general discrete Wheeler DeWitt equation was given for pure gravity, based on the simplicial lattice formulation originally developed by Regge and Wheeler. On the lattice the infinite - dimensional manifold of continuum geometries is replaced by a finite manifold of piecewise linear spaces, with solutions to the lattice equations then providing a suitable approximation to the continuum gravitational wave functional. The lattice equations were found to be explicit enough to allow the development of potentially useful practical solutions. As a result, a number of sample quantum gravity calculations were carried out in $2 + 1$ and $3 + 1$ dimensions. These were based mainly on the strong coupling expansion and on the Rayleigh Ritz variational method, the latter implemented using a set of correlated product (Slater Jastrow) wave functions.

We will explore the infrared structure of quantum gravity, by solving a lattice version of the Wheeler DeWitt equations. The nature of the wavefunction solutions is such that a finite correlation length emerges and cuts off naturally any infrared divergences. Properties of the lattice vacuum are consistent with the existence of an ultraviolet fixed point in G located at the origin, thus precluding the existence of a weak coupling perturbative phase. The correlation length exponent is determined exactly and found to be $\nu = 6/11$. It is possible that the well known ultraviolet divergences affecting the perturbative treatment of quantum gravity in 4 dimensions point to a fundamental vacuum instability of the full theory. If this is the case, then the correct identification of the true ground state for gravitation necessarily requires the introduction of a consistent nonperturbative cutoff. To this day the only known way to do this reliably in quantum field theory is via the lattice formulation.

Using the insights that we obtained from the studies in $2+1$ dimensions, we further increase the dimensions to 3 space and 1 time to study our real physical world using the discrete quantum Hamiltonian approach in a position representation, Wheeler DeWitt equation.

Physical properties of the vacuum state of quantum gravity are explored by solving a lattice version of the Wheeler DeWitt equation. We will see that the constraint of diffeomorphism invariance is strong enough to uniquely determine the structure of the vacuum wave functional in the limit of infinitely fine triangulations of the 3 - sphere. In the large fluctuation regime the nature of the wave function solution is such that a physically acceptable ground state emerges, with a finite nonperturbative correlation length naturally cutting off any infrared divergences. The location of the critical point in Newton's constant G_c, separating the weak from the strong coupling phase, is obtained, and it is inferred from the structure of the wave functional that fluctuations in the curvatures become unbounded at this point. Furthermore, investigations of the vacuum wave functional further indicate that for weak enough coupling, $G < G_c$, a pathological ground state with no continuum limit appears, where configurations with small curvature have vanishingly small probability. One is then lead to the conclusion that the weak coupling, perturbative ground state of quantum gravity is nonperturbatively unstable, and that gravitational screening cannot be physically realized in the lattice theory. These results we find extend to further arguments that they agree with the Euclidean lattice gravity results, and lend further support to the claim that the Lorentzian and Euclidean lattice formulations for gravity describe the same underlying nonperturbative physics.

As noted before, to this day the only known reliable nonperturbative method in quantum field theory is via the lattice formulation. Previous works on lattice quantum gravity have dealt almost exclusively with the Euclidean formulation in d dimensions, treated via the manifestly covariant Feynman path integral method. Indeed the latter is very well suited for numerical integration, and many analytical and numerical results have been obtained over the years within this framework. One notes here that the issue of their relationship to the Lorentzian theory has remained largely open, at least from the point of view of a rigorous treatment. The main supporting arguments for the Euclidean approach come from the fact that the above equivalence holds true for other field theories (no exceptions are known), and from the fact that in gravity itself it is rigorously true to all orders in the weak field expansion.

2.3 Continuum Wheeler DeWitt Equation

In my work, we took the canonical quantization of gravity, therefore we begin here with a simple summary of the classical canonical formalism [82] as derived by Arnowitt, Deser and Misner [83], to recall the main results and formulae and provide suitable references for expressions used later in the calculations.

The first step in developing a canonical formulation for gravity is to introduce a time-slicing of spacetime, by introducing a sequence of spacelike hypersurfaces labeled by a *continuous* time coordinate t. The invariant distance is then written as

$$ds^2 \equiv -d\tau^2 = g_{\mu\nu}\, dx^\mu dx^\nu = g_{ij}\, dx^i\, dx^j + 2g_{ij}\, N^i dx^j dt - (N^2 - g_{ij}\, N^i N^j) dt^2 \,, \quad (2.13)$$

where x^i $(i = 1, 2, 3)$ are coordinates on a 3 dimensional manifold and τ is the proper time, in units of $c = 1$.

Indices are raised and lowered with $g_{ij}(\mathbf{x})$ $(i, j = 1, 2, 3)$, which denotes the 3-metric on the given spacelike hypersurface, and $N(\mathbf{x})$ and $N^i(\mathbf{x})$ are the lapse and shift functions, respectively. We mark here 4 dimensional quantities by the prefix ^{4}, so that all un-marked quantities refer to 3 dimensions (and may occasionally be marked explicitly by a ^{3}). In terms of the original 4dimensional metric $^4g_{\mu\nu}$ one has

$$\begin{pmatrix} ^4g_{00} & ^4g_{0j} \\ ^4g_{i0} & ^4g_{ij} \end{pmatrix} = \begin{pmatrix} N_k N^k - N^2 & N_j \\ N_i & g_{ij} \end{pmatrix} , \tag{2.14}$$

which then gives for the spatial metric and the lapse and shift functions

$$g_{ij} = {}^4g_{ij} , \qquad N = \left(-{}^4g^{00}\right)^{-1/2} , \qquad N_i = {}^4g_{0i} . \tag{2.15}$$

For the volume element one has

$$\sqrt{-{}^4g} = N \sqrt{g} , \tag{2.16}$$

where the latter involves the determinant of the 3-metric, $g \equiv \det g_{ij}$. As usual g^{ij} denotes the inverse of the matrix g_{ij}.

A transition from the classical to the quantum description of gravity is obtained by promoting the metric g_{ij}, the conjugate momenta π^{ij}, the Hamiltonian density H and the momentum density H_i to quantum operators, with $\hat{g}_{ij}$ and $\hat{\pi}^{ij}$ satisfying canonical commutation relations. In particular the classical constraints now select a physical vacuum state $|\Psi\rangle$, such that in the source free case

$$\hat{H} \, |\Psi\rangle = 0 , \qquad \hat{H}_i \, |\Psi\rangle = 0 , \tag{2.17}$$

and in the presence of sources more generally

$$\hat{T} \, |\Psi\rangle = 0 , \qquad \hat{T}_i \, |\Psi\rangle = 0 , \tag{2.18}$$

where $\hat{T}$ and $\hat{T}_i$ now include matter contributions that should be added to $\hat{H}$ and $\hat{H}_i$. The first is the energy constraint, and the second is the momentum constraint. The momentum constraint involving $\hat{H}_i$ or, more generally $\hat{T}_i$, ensures that the state functional does not change under a transformation of coordinates x^i, so that Ψ depends only on the intrinsic geometry of the 3 space. We will see the explicit form that arise in continuum quantum

Hamiltonian (continuum Wheeler DeWitt equation) later. The Hamiltonian constraint is then the only remaining condition that the state functional must satisfy.

As in ordinary nonrelativistic quantum mechanics, one can choose different representations for the canonically conjugate operators $\hat{g}_{ij}$ and $\hat{\pi}^{ij}$. In the functional *position representation* one sets

$$\hat{g}_{ij}(\mathbf{x}) \;\to\; g_{ij}(\mathbf{x}) , \qquad \hat{\pi}^{ij}(\mathbf{x}) \;\to\; -i\hbar \cdot 16\pi G \cdot \frac{\delta}{\delta g_{ij}(\mathbf{x})} . \tag{2.19}$$

In this picture quantum states become wave functionals of the 3-metric $g_{ij}(\mathbf{x})$,

$$|\Psi\rangle \;\to\; \Psi\left[g_{ij}(\mathbf{x})\right] . \tag{2.20}$$

The 2 quantum constraint equations in Eq. (2.18) then become the Wheeler DeWitt equation [37, 31, 32, 33, 84]

$$\left\{ - 16\pi G \cdot G_{ij,kl} \frac{\delta^2}{\delta g_{ij}\, \delta g_{kl}} \;-\; \frac{1}{16\pi G} \sqrt{g} \left({}^3R \,-\, 2\lambda \right) \,+\, \hat{H}^{\phi} \right\} \Psi\left[g_{ij}(\mathbf{x})\right] \;=\; 0 , \tag{2.21}$$

and the momentum constraint listed below. Here $G_{ij,kl}$ is the inverse of the DeWitt supermetric, given by

$$G_{ij,kl} \;=\; \frac{1}{2} g^{-1/2} \left(g_{ik}g_{jl} + g_{il}g_{jk} + \alpha\, g_{ij}g_{kl} \right) , \tag{2.22}$$

with parameter α depending on the dimension of the space, and for 3 space dimensions, $\alpha = -1$. The 3 dimensional version of the DeWitt supermetric itself, $G^{ij,kl}(x)$ is given by

$$G^{ij,kl} \;=\; \frac{1}{2} \sqrt{g} \, \left(g^{ik}g^{jl} + g^{il}g^{jk} + \bar{\alpha}\, g^{ij}g^{kl} \right) , \tag{2.23}$$

with parameter α in Eq. (2.22) related to $\bar{\alpha}$ in Eq. (2.23) by $\bar{\alpha} = -2\alpha/(2+3\alpha)$, so that $\alpha = -1$ gives $\bar{\alpha} = -2$ (note that these parameters α and $\bar{\alpha}$ are dimension-dependent). In the position representation the diffeomorphism (or momentum) constraint reads

$$\left\{ 2\, i\, g_{ij}\, \nabla_k \frac{\delta}{\delta g_{jk}} \,+\, \hat{H}^{\phi}_i \right\} \Psi\left[g_{ij}(\mathbf{x})\right] \;=\; 0 , \tag{2.24}$$

Let us focus on the pure gravity case in the above momentum constraint, *i.e.,* only the first term. Notice that ∇_k is a covariant derivative. One can then see clearly that this constraint corresponds to saying that the gradient of Ψ on the superspace of $3-$ metric is zero along the direction of the gauge transformations *i.e.,* diffeomorphisms on the 3-space dimensional manifold. Note here that even if one considers only the Hamiltonian constraint, since the Hamiltonian and the momentum constraints are related by commutation relations, one necessarily obtains the solutions that are consistent with the momentum constraints, in other words, the form of the solutions one obtains from the Hamiltonian constraint should retain the spatial part of the diffeomorphism invariance. This can be used as a consistency check. In fact in our studies, we show that our solutions retain the spatial form of diffeomorphism invariance, *i.e.*, one sees that the solutions only depend on the geometric quantities that depend only on the whole geometry like total volume (collectively on edge lengths) and not on the edge lengths individually. $\hat{H}^\phi$ and $\hat{H}_i^\phi$ that appear in Eq. (2.21) and Eq. (2.24) are possible matter contributions. In the following, we shall set both of these to zero, as we will focus here almost exclusively on the pure gravitational case.

A number of basic issues need to be addressed before one can gain a full and consistent understanding of the dynamical content of the theory (see for example [85, 86, 87, 88, 89] as a small set of representative references). These include possible problems of operator ordering, and the specification of a suitable Hilbert space, which entails at some point a choice for the inner product of wave functionals, for example in the Schrödinger form

$$\langle \Psi | \Phi \rangle \ = \ \int \mathcal{D}\mu[g] \ \ \Psi^* \left[g_{ij} \right] \ \Phi \left[g_{ij} \right] \tag{2.25}$$

where $\mathcal{D}\mu[g]$ is some appropriate measure over the 3-metric g. Note also that the continuum Wheeler DeWitt equation contains, in the kinetic term, products of functional differential operators which are evaluated at the same spatial point $\mathbf{x}$. One would expect that such terms could produce $\delta^{(3)}(0)$ -type singularities when acting on the wave functional, which would then have to be regularized in some way. The lattice cutoff discussed below is one way to provide such an explicit ultraviolet regularization.

A strange property of the Wheeler DeWitt equation, which distinguishes it from the usual

Schrödinger equation $H\Psi = i\hbar\partial_t\Psi$, is the absence of an explicit time coordinate. As a result, the *right hand side* term of the Schrödinger equation is entirely absent. It is reflected by the diffeomorphism invariance of the underlying theory, which expresses now the fundamental quantum equations in terms of fields g_{ij}, and not coordinates just as in the notion of superspace as Wheeler originally proposed it [90].

2.4 Regge Discretized Wheeler DeWitt Equations

Here, since we are talking about Hamiltonian formulation, the spacetime manifold is $d+1$ dimensions, where spatial part is d dimensions. In constructing a discrete Hamiltonian for gravity one has to decide first what degrees of freedom one should retain on the lattice. One possibility, which we choose to pursue, is to use the Regge-Wheeler lattice discretization for gravity [91, 90], with edge lengths suitably defined on a random lattice as the primary dynamical variables. In fact the degrees of freedom for the edges and the independent component of the metric tensor are $D(D+1)/2$ and they match. This can be thought of as an advantage of the Regge calculus discretization method as the notion of continuum is already present to start with as compared to the method such as causal dynamical triangulations, where its method does not carry a notion of continuum spacetime to start with however expected to recover emergently. Even within Regge theory, several avenues for discretization are possible. One can discretize the theory from the very beginning, while it is still formulated in terms of an action, and introduce for it a lapse and a shift function, extrinsic and intrinsic discrete curvatures *etc*. Alternatively, one can directly discretize the continuum Wheeler DeWitt equation, a procedure that makes sense in the lattice formulation, as these equations are still given in terms of geometric objects, for which the Regge theory is very well suited. It is the latter approach which we will proceed to outline here.

The starting point for the following discussion will be the Wheeler DeWitt equation for pure gravity in the absence of matter, Eq. (2.21),

$$\left\{ -\,(16\pi G)^2\, G_{ij,kl}(\mathbf{x})\, \frac{\delta^2}{\delta g_{ij}(\mathbf{x})\, \delta g_{kl}(\mathbf{x})} \;-\; \sqrt{g(\mathbf{x})}\, \left({}^3R(\mathbf{x}) \,-\, 2\lambda \right) \right\} \Psi[g_{ij}(\mathbf{x})] \;=\; 0\,, \quad (2.26)$$

and the diffeomorphism constraint of Eq. (2.24),

$$\left\{ 2\, i\, g_{ij}(\mathbf{x})\, \nabla_k(\mathbf{x})\, \frac{\delta}{\delta g_{jk}(\mathbf{x})} \right\} \Psi[g_{ij}(\mathbf{x})] \;=\; 0\,. \quad (2.27)$$

Both of these equations express a constraint on the state $|\Psi\rangle$ at *every* $\mathbf{x}$, each of the form $\hat{H}(\mathbf{x})\,|\Psi\rangle = 0$ and $\hat{H}_i\,(\mathbf{x})|\Psi\rangle = 0$. It is then natural to view Eq. (2.26) as made up of 3 terms, the first one identified as a kinetic term for the metric degrees of freedom, the second one involving $-\sqrt{g}\,{}^3R$ and thus seen as a potential energy contribution (of either sign, due to the nature of the 3 - curvature ${}^3R$), and finally the cosmological constant term proportional to $+\,\lambda\,\sqrt{g}$ acting as a mass-like term. The kinetic term can be regarded as containing a Laplace-Beltrami-type operator acting on the 6 dimensional Riemannian manifold of positive definite metrics g_{ij}, with $G_{ij,kl}$ acting as its contravariant metric [31, 32, 33]. It was further shown in the quoted reference that the manifold in question has hyperbolic signature $-+++++$, with pure dilations of g_{ij} corresponding to timelike displacements within this manifold of metrics.

As discussed before, this momentum constraint corresponds to saying that the gradient of Ψ on the superspace of 3 - metric is zero along the direction of the gauge transformations *i.e.,* diffeomorphisms on the 3 - space dimensional manifold, as ∇_k is a covariant derivative. One can contemplate that the momentum constraint is encoded in the Hamiltonian constraint, since the Hamiltonian and the momentum constraints are related by commutation relations. Therefore, even if one considers only the Hamiltonian constraint, one necessarily obtains the solutions that are consistent with the momentum constraints, in other words, the form of the solutions one obtains from the Hamiltonian constraint should retain the spatial part of the diffeomorphism invariance. This can be used as a consistency check. In fact in our studies, we show that our solutions retain the spatial form of diffeomorphism invariance, *i.e.*, one sees that the solutions only depend on the geometric quantities that depend only on the whole geometry like total volume (collectively on edge lengths) and not on the edge lengths individually.

Next we turn to the lattice theory. Here we will follow the procedure outlined in [81] and discretize the continuum Wheeler DeWitt equation directly, a procedure that makes sense in the lattice formulation, as these equations are still formulated in terms of geometric objects, for which the Regge theory is very well suited. On a simplicial lattice [73, 74, 79, 92, 93, 26, 28, 53, 75] (see for example [25], and references therein, for a more complete discussion of the Regge Wheeler lattice formulation for gravity) one knows that deformations of the squared edge lengths are linearly related to deformations of the induced metric. In a given simplex σ, take coordinates based at a vertex 0, with axes along the edges from 0. The other vertices are each at unit coordinate distance from 0 (see Figure 2.1 as an example for this labeling, for a triangle and Figure 2.2 for a tetrahedron for later reference for the studies in $3 + 1$ dimensions.). In terms of these coordinates, the metric within the simplex is given by

$$g_{ij}(\sigma) \; = \; \frac{1}{2} \left(l_{0i}^2 + l_{0j}^2 - l_{ij}^2 \right) . \tag{2.28}$$

Note that in the following discussion only edges and volumes along the spatial direction are involved. It follows that one can introduce in a natural way a lattice analog of the DeWitt supermetric of Eq. (2.23), by adhering to the following procedure [94, 95]. First one writes for the supermetric in edge length space

$$\| \, \delta l^2 \, \|^2 \; = \; \sum_{ij} \, G^{ij}(l^2) \, \delta l_i^2 \, \delta l_j^2 \, , \tag{2.29}$$

with the quantity $G^{ij}(l^2)$ suitably defined on the space of squared edge lengths. By a straightforward exercise of varying the squared volume of a given simplex σ in d dimensions

$$V^2(\sigma) \; = \; \left(\tfrac{1}{d!} \right)^2 \det g_{ij}(l^2(\sigma)) \tag{2.30}$$

to quadratic order in the metric (on the *right hand side*), or in the squared edge lengths belonging to that simplex (on the *left hand side*), one is led to the identification

$$G^{ij}(l^2) \; = \; - \, d! \, \sum_{\sigma \supset i,j} \, \frac{1}{V(\sigma)} \, \frac{\partial^2 \, V^2(\sigma)}{\partial l_i^2 \, \partial l_j^2} \, , \tag{2.31}$$

which was given by Regge and Lund. It should be noted that in spite of the appearance of a sum over simplices σ, $G^{ij}(l^2)$ is local, since the sum over σ only extends over those simplices which contain either the i or the j edge. Note that this is so because this term only depends on the edges that are labeled by i and j, which belong to the simplex of your consideration, therefore in the end, in spite of the appearance, this term really only involves the edges that belong to the simplex that one is considering.

Now the curvature term is written as

$$\sqrt{g}\ {}^{(d)}R = \frac{2}{q} \sum_{h \supset \sigma} \delta_h\ {}^{(d-2)}V_h \tag{2.32}$$

where q is the coordination number (*i.e.,* how many simplices are meeting at a hinge), V_h is the volume of the hinge (co dimension 2 object). The curvature is summed over for all the hinges that belong to one simplex. For example, if one is approximating a 2 dimensional surface, then the building blocks are triangles, and there are 3 hinges (vertices in this case) for a given triangle (simplex in this dimension). One then notices that this curvature term now involves the edges that belong to the neighboring simplices, and therefore this is the term that contain nonlocal information, carrying the interaction with neighboring simplices.

Finally one is ready to write a lattice analog of the Wheeler DeWitt equation for pure gravity, which reads

$$\left\{ -\,(16\pi G)^2\, G_{ij}(l^2)\, \frac{\partial^2}{\partial l_i^2\, \partial l_j^2} \;-\; \sqrt{g(l^2)}\ \left[\, {}^{d}R(l^2) \;-\; 2\lambda \,\right] \right\} \Psi[\, l^2\,] \;=\; 0\,, \tag{2.33}$$

with $G_{ij}(l^2)$ the inverse of the matrix $G^{ij}(l^2)$ given above and where ${}^{d}R(l^2)$ is the spatial part of the curvature. The range of the summation over i and j and the appropriate expression for the scalar curvature, in this equation, are discussed below and made explicit in Eq. (2.34). Again, to summarize the key points discussed earlier, the kinetic term (the first term) only involves the edges that belong to the simplex that you are considering, therefore it is completely local (local in lattice sense). On the other hand, the curvature term (the second term) involved the edges that belong to the neighboring simplexes of the simplex that you are considering, therefore it is nonlocal as it starts involve the information of the neighboring points (in lattice). One can see this argument clearly in a concrete example which will be given later.

Eqs. (2.21) or (2.33) express a constraint equation at each *point* in space. We elaborate here more on this point. On the lattice, points in space are replaced by a set of edge labels i, with a few edges clustered around each vertex, in a way that depends on the dimensionality and the local lattice coordination number (which is the number of simplices meeting at a hinge (codimension 2 object)), therefore on how the lattice is put together. To be more specific, the first term in Eq. (2.33) contains derivatives with respect to edges i and j connected by a matrix element G_{ij} which is nonzero only if i and j are close to each other, essentially nearest neighbor. One would therefore expect that the first term could be represented by just a sum of edge contributions, all from within one (d)-simplex σ (a triangle in 2 dimensions, a tetrahedron in 3 dimensions). The second term containing ${}^dR(l^2)$ in Eq. (2.33) is also local in the edge lengths: it only involves a handful of edge lengths which enter into the definition of areas, volumes and angles around the point $\mathbf{x}$, and follows from the fact that the local curvature at the original point $\mathbf{x}$ is completely determined by the values of the edge lengths clustered around i and j. Apart from some geometric factors, it describes, through a deficit angle δ_h, the parallel transport of a vector around an elementary dual lattice loop. It should therefore be adequate to represent this second term by a sum over contributions over all $(d-2)$-dimensional hinges (vertices in $2+1$ dimensions, edges in $3+1$ dimensions) h attached to the simplex σ, giving therefore in 3 - space dimensions

$$\left\{ -(16\pi G)^2 \sum_{i,j \subset \sigma} G_{ij}\,(\sigma)\, \frac{\partial^2}{\partial l_i^2\, \partial l_j^2} \;-\; 2\, n_{\sigma h} \sum_{h \subset \sigma} l_h\, \delta_h \;+\; 2\lambda\, V_\sigma \right\} \Psi[\, l^2\,] \;=\; 0\,. \tag{2.34}$$

Here δ_h is the deficit angle at the hinge h, l_h the corresponding edge length, $V_\sigma = \sqrt{g(\sigma)}$ the volume of the simplex (tetrahedron in 3 spatial dimensions) labeled by σ. $G_{ij}\,(\sigma)$ is obtained either from Eq. (2.31), or from the lattice transcription of Eq. (2.22)

$$G_{ij,kl}(\sigma) \;=\; \frac{1}{2}\, g^{-1/2}(\sigma)\, \left[g_{ik}(\sigma) g_{jl}(\sigma) + g_{il}(\sigma) g_{jk}(\sigma) - \, g_{ij}(\sigma) g_{kl}(\sigma) \right]\,, \tag{2.35}$$

with the induced metric $g_{ij}(\sigma)$ within a simplex σ given in Eq. (2.28). The combinatorial factor $n_{\sigma h}$ ensures the correct normalization for the curvature term, since the latter has to give the lattice version of $\int \sqrt{g}\, {}^3R = 2 \sum_h \delta_h l_h$ (in 3 spatial dimensions) when summed over all simplices σ. The inverse of $n_{\sigma h}$ counts therefore the number of times the same hinge appears in various neighboring simplices, and consequently depends on the specific choice of underlying lattice structure; for a flat lattice of equilateral triangles in 2 dimensions $n_{\sigma h} = 1/6$. [2]

2.4.1 Choice of Coupling Constants

As in the Euclidean lattice theory of gravity, we will find it convenient here to factor out an overall irrelevant length scale from the problem, and set the (unscaled) cosmological constant equal to one as was done in [26, 28]. Indeed recall that the Euclidean path integral weight always contains a factor $P(V) \propto \exp(-\lambda_0 V)$ where $V = \int \sqrt{g}$ is the total volume on the lattice, and λ_0 is the unscaled cosmological constant. The choice $\lambda_0 = 1$ then fixes this overall scale once and for all, as the volume is set to be the same so that when we compare different theories, the quantities we are comparing are scaled properly. This procedure is similar to what we do when we scale the kinetic term in the field theory with redefinition of fields. Here, since $\lambda_0 = 2\lambda/16\pi G$ one then has $\lambda = 8\pi G$ in this system of units. Note here that both λ_0 and λ in this context are *bare* parameters as they describe the microphysics as appearing in Wheeler DeWitt equation which is a quantum mechanical equation. One can now rewrite the Wheeler DeWitt equation so that the kinetic term (the term involving the Laplacian) has unit coefficient, and write Eq. (2.26) as in units of λ_0. Two further notational simplifications will be done in the following. The first one is introduced in order to avoid lots of factors of 16π in many of the subsequent formulas. Consequently from now on we shall write $\boldsymbol{G}$ as a

[2] Instead of the combinatorial factor $n_{\sigma h}$ one could insert a ratio of volumes $V_{\sigma h}/V_h$ (where V_h is the volume per hinge [26, 28] and $V_{\sigma h}$ is the amount of that volume in the simplex σ), but the above form is simpler.

shorthand for $16\pi\, G$,

$$16\pi\, G \longrightarrow \boldsymbol{G}\,. \tag{2.36}$$

In this notation one then has $\lambda = \boldsymbol{G}/2$. The above notational choices then lead to a much more streamlined representation of the Wheeler DeWitt equation,

$$\left\{ - \boldsymbol{\Delta} + \frac{1}{\boldsymbol{G}} \sqrt{g} - \frac{1}{\boldsymbol{G}^2} \sqrt{g}\, R \right\} \Psi = 0\,. \tag{2.37}$$

Note that in the extreme strong coupling limit ($G \to \infty$) the kinetic term is the dominant one, followed by the volume (cosmological constant) term, and finally by the curvature term. Consequently, at least in the first approximation, the curvature R term can be neglected compared to the other 2 terms in this limit. A second notational choice will be dictated later on by the structure of the wave function solutions, which will commonly involve factors of $\sqrt{G}$. For this reason we will now define the new coupling g as

$$\boldsymbol{g} \;\equiv\; \sqrt{\boldsymbol{G}}\,, \tag{2.38}$$

so that $\lambda = \boldsymbol{g}^2/2$ (the latter $\boldsymbol{g}$ should not be confused with the square root of the determinant of the metric). We will use this notation heavily when we start calculating averages using the obtained wave function solutions.

Later on it will turn out convenient to define a parameter β for the triangulations of the sphere, defined as

$$\beta|_{S^2} \;\equiv\; \frac{2\sqrt{2}\,\pi}{\sqrt{\lambda}\,\boldsymbol{G}}\,. \tag{2.39}$$

Factors of 2π arise here because we are looking at various triangulations of the 2 - sphere. More generally, for a 2 dimensional closed manifold with arbitrary topology one has by the Gauss-Bonnet theorem

$$\int d^2x\, \sqrt{g}\, R \;=\; 4\pi\, \chi \tag{2.40}$$

with χ is the Euler characteristic of the manifold. The latter is related to the genus g (the number of handles) via $\chi = 2 - 2\,\mathrm{g}$ (note that for a discrete manifold in 2 dimensions, the Euler characteristic is a well defined quantity and one has the equivalent form due to Euler

$\chi = N_0 - N_1 + N_2$, where N_i denotes the number of simplices of dimension i). Thus it is clear that one can relate β and χ and for a general 2 dimensional manifold we will define

$$\beta \;=\; \frac{\sqrt{2}\,\pi\,\chi}{\sqrt{\lambda}\,\boldsymbol{G}}\,. \tag{2.41}$$

Equivalently, using in units of λ_0,

$$\sqrt{\lambda}\,\boldsymbol{G} \;=\; \frac{\sqrt{\boldsymbol{G}}}{\sqrt{2}}\cdot\boldsymbol{G}^2 \;=\; \frac{1}{\sqrt{2}}\,\boldsymbol{G}^{3/2} \;=\; \frac{1}{\sqrt{2}}\,\boldsymbol{g}^3\,, \tag{2.42}$$

one has simply

$$\beta|_{S^2} \;=\; \frac{4\pi}{\boldsymbol{g}^3} \tag{2.43}$$

for the sphere, and for the general topological manifold,

$$\beta \;=\; \frac{2\pi\,\chi}{\boldsymbol{g}^3}\,. \tag{2.44}$$

This generalization is an attempt to generalize our result to any topological manifold not limited to the spherical manifold. Later, we will see that our result (*e.g.,* critical exponent ν) does not depend on this parameter β therefore nor on χ, implying that the result is universal for any topological manifold.

2.5 Discretized Wheeler DeWitt Equation in $2+1$ dimensions

Here we extend the work initiated in [81], and show how exact solutions to the lattice Wheeler DeWitt equations can be obtained in $2+1$ dimensions, for arbitrary values of Newton's constant G. The procedure we follow is to solve the lattice equations exactly for several finite regular triangulations of the sphere, and then extend the result to an arbitrarily large number of triangles. One finds that for large enough areas the exact lattice wave functional depends on geometric quantities only, such as the total area and the total integrated curvature (which in $2+1$ dimensions is just proportional to the Euler characteristic). The regularity condition on the solutions of the wave equation at small areas is shown to play an essential role in constraining the form of the wave functional, which we eventually find to be expressible in closed form as a confluent hypergeometric function of the first kind. Later it will be shown that the resulting wave function allows an exact evaluation of a number of useful (and manifestly diffeomorphism invariant) averages, such as the average area of the manifold and its fluctuation.

From these results a number of suggestive physical results can be obtained, the first one of which is that the correlation length in units of the lattice spacing is found to be finite for all $G > 0$, and diverges at $G = 0$. Such a result can be viewed as consistent with the existence of an ultraviolet fixed point (or a phase transition in statistical field theory language) in G located at the origin, thus entirely precluding the existence of a weak coupling phase for gravity in $2+1$ dimensions. Simple renormalization group arguments would then suggest that gravitational screening is not physically possible in $2+1$ dimensions, and that gravitational antiscreening is the only physically realized option in this model. A second result that follows from our analysis is an exact determination of the critical correlation length exponent for gravity in $2+1$ dimensions, which is found to be $\nu = 6/11$. It is known that the latter determines, through standard renormalization group arguments, the scale-dependence of the gravitational coupling in the vicinity of the ultraviolet fixed point.

A short outline of this Section is as follows. In the previous Section 2.4, we introduced the lattice Wheeler DeWitt equation derived in [81], and Section 2.5.1 makes more explicit various quantities appearing in it. This section also discusses briefly the role of continuous lattice diffeomorphism invariance in the Regge framework as it applies to the present case of $2+1$ - dimensional gravity. Section 2.5.2 gives a detailed outline of the general method of solution for the lattice equations, and then gives the explicit solution for a number of regular triangulations of the sphere in Section 2.5.3, 2.5.4, 2.5.5, 2.5.6, and in torus in

Section 2.5.7. Later in Section 2.5.8, a general form of the wave function is given which covers all the previous discrete cases, and allows a subsequent study of the infinite volume limit. Section 2.5.9 focuses on one of the simplest diffeomorphism invariant averages that can be computed from the wave function, namely the average total area. A brief discussion follows on how the latter quantity relates to the corresponding averages computed in the Euclidean theory. Section 2.5.10 extends the calculation to the area fluctuation, and shows how the critical exponents (anomalous dimensions) of the 2+1 gravity theory can be obtained from the exact wave function solution, using some rather straightforward scaling arguments. Section 2.5.11discusses some simple physical implications that can be inferred from the values of the exact exponents, and the fact that quantum gravity in $2+1$ dimensions does not seemingly possess, in either the Euclidean or Lorentzian formulation, a weak coupling phase. Section 2.5.12 contains a summary of the results obtained so far.

2.5.1 Explicit Setup for the Lattice Wheeler DeWitt Equation in $2+1$ dimensions

We will now focus almost exclusively on the case of 2 space + 1 time dimensions. We first derive the relevant terms in the discrete Wheeler DeWitt equation for a simplex for a clear and concrete illustration. *The building blocks are triangles*, with vertices and squared edge lengths labelled as in Figure 2.1. We set $l_{01}^2 = a$, $l_{12}^2 = b$, $l_{02}^2 = c$. The components of the metric for coordinates based at vertex 0, with axes along the 01 and 02 edges, are

$$g_{11} \;=\; a\,, \qquad g_{12} \;=\; \frac{1}{2}(a+c-b)\,, \qquad g_{22} \;=\; c\,. \tag{2.45}$$

The area A of the triangle is given by

$$A^2 \;=\; \frac{1}{16}\left[\, 2(a\,b+b\,c+c\,a)-a^2-b^2-c^2\,\right]\,, \tag{2.46}$$

so the supermetric G^{ij}, according to Eq. (2.31), is

$$G^{ij} \;=\; \frac{1}{4\,A} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \tag{2.47}$$

Thus for the triangle we have

$$G_{ij} \, \frac{\partial^2}{\partial s_i \, \partial s_j} \;=\; -4A \left(\frac{\partial^2}{\partial a \, \partial b} \;+\; \frac{\partial^2}{\partial b \, \partial c} \;+\; \frac{\partial^2}{\partial c \, \partial a} \right), \tag{2.48}$$

and the Wheeler DeWitt equation is

$$\left\{ (16\pi G)^2 \, 4A \, \left(\frac{\partial^2}{\partial a \, \partial b} \;+\; \frac{\partial^2}{\partial b \, \partial c} \;+\; \frac{\partial^2}{\partial c \, \partial a} \right) \;-\; \frac{2}{q} \sum_{\sigma \supset h} \delta_h \;+\; 2\,\lambda\,A \right\} \Psi[\,s\,] \;=\; 0, \tag{2.49}$$

where the sum is over the 3 vertices h of the triangle.

The above form for the kinetic term holds true for all the lattice configurations in $2+1$ dimensions, however, to give explicit forms of the curvature term, one needs to consider a specific lattice configuration. Here, just for the purpose to illustrate, I will consider 4 simplices (triangles in $2+1$ dimensions) put together to approximate a 2 sphere. The 6 edges are labeled as a, b, c, d, e, and f, and the vertices 0 ($a \cap b \cap c$), 1 ($b \cap c \cap e$), 2 ($a \cap d \cap e$), and 3 ($d \cap e \cap f$), and we will consider the curvature term (and hence Wheeler DeWitt equation) for the simplex (or the triangle in this case) abc. Then the curvature term in $2+1$ dimensions becomes

$$\sum_{h \supset \sigma} \delta_h \;\; {}^{(d-2)}V_h \;\; \xrightarrow{d=2} \;\; \sum_{h \supset \triangle} \delta_h \;\; \xrightarrow{\triangle_{abc}} \;\; \delta_0 + \delta_1 + \delta_2 \,, \tag{2.50}$$

where

$$\begin{aligned}
\delta_0 &= 2\pi - \cos^{-1}\left(\frac{a+c-b}{2\,\sqrt{a}\sqrt{c}} \right) - \cos^{-1}\left(\frac{a+d-e}{2\,\sqrt{a}\sqrt{d}} \right) - \cos^{-1}\left(\frac{c+d-f}{2\,\sqrt{c}\sqrt{d}} \right) \\
\delta_1 &= 2\pi - \cos^{-1}\left(\frac{a+b-c}{2\,\sqrt{a}\sqrt{b}} \right) - \cos^{-1}\left(\frac{a+e-d}{2\,\sqrt{a}\sqrt{e}} \right) - \cos^{-1}\left(\frac{b+e-f}{2\,\sqrt{b}\sqrt{e}} \right) \\
\delta_2 &= 2\pi - \cos^{-1}\left(\frac{b+c-a}{2\,\sqrt{b}\sqrt{c}} \right) - \cos^{-1}\left(\frac{c+f-d}{2\,\sqrt{c}\sqrt{f}} \right) - \cos^{-1}\left(\frac{b+f-e}{2\,\sqrt{b}\sqrt{f}} \right) .
\end{aligned} \tag{2.51}$$

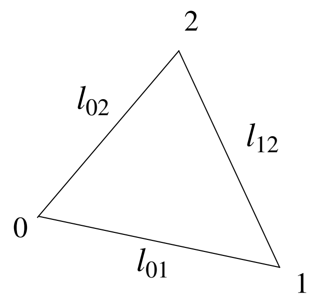

Figure 2.1: A triangle with labels.

Notice now that the curvature term involves the edges that do not belong to the triangle abc, meaning the curvature is the term that involves the information for interaction between the neighboring simplexes.

In principle, any solution of the Wheeler DeWitt equation should correspond to a possible quantum state of the universe. However, if one looks at the ordinary quantum mechanics like Schrodinger equation, one needs to employ boundary conditions to extract physically acceptable solutions. In the universe case, the situation is far less clear on what type of boundary conditions one needs to consider. Therefore, we decided to consider the simplest topology as a start, a sphere, or in other words, spherical boundary conditions, *i.e.,* triangulations of the 2 - sphere. We further take simpler consideration to only look at regular triangulations of the 2-sphere, *i.e.,* the tetrahedron (4 triangles), the octahedron (8 triangles) and the icosahedron (20 triangles). Note that these 3 are the only choices we have for the regular triangulation of the 2 - sphere. We later deduce the solutions in general form with parameters N_2, *i.e.,* number of triangles in a configuration, and further send N_2 to infinity for the infinite

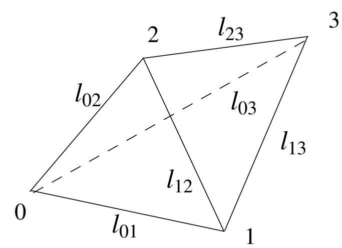

Figure 2.2: A tetrahedron with labels.

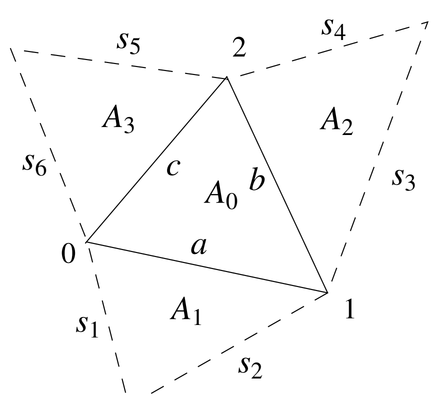

Figure 2.3: Neighbors of a given triangle. The picture illustrates the fact that the Laplacian $\mathbf{\Delta}(l^2)$ appearing in the kinetic term of the lattice Wheeler DeWitt equation (here in $2+1$ dimensions) contains edges a, b, c that belong both to the triangle in question, as well as to several neighboring triangles (here 3 of them) with squared edges denoted sequentially by $s_1 = l_1^2 \dots s_6 = l_6^2$.

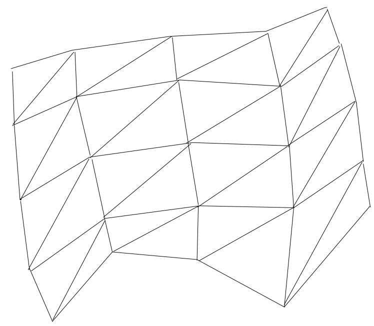

Figure 2.4: A small section of a suitable dynamical spatial lattice for quantum gravity in $2+1$ dimensions.

many triangulation of the 2 - sphere. However, of course, the question remains as to what type of the universe we are considering here, as well as the validity of only taking the limit of the solutions instead of the generalized Wheeler DeWitt equation. Nevertheless, as we will see, the results are concrete and at least complete in this particular triangulations. In particular, another point one may wonder is how much weight would this particular topology, the sphere, has as compared to other possible topologies of the manifold, for example torus. The solution we deduced later to be a more general case also explicitly included the topologically dependent parameter, Euler characteristics. The universal quantity that we obtained, the critical exponent in the end did not depend on this Euler characteristics, therefore the results seems to be independent of the topology of the manifold.

Discrete Wheeler DeWitt equations using the regular triangulation of the 2 -sphere were already studied in some detail in [96, 54]. A key aspect of the Regge theory is the presence of a continuous, local lattice diffeomorphism invariance, whose main aspects in regard to their relevance for the $3+1$ formulation of gravity were already addressed in some detail in [81] in the context of the lattice weak field expansion. Here we will add some remarks about how this local invariance manifests itself in the $2+1$ formulation, and in particular for the discrete triangulations of the sphere studied later on in this Chapter. Of some relevance is the presence of exact zero modes of the gravitational lattice action, reflecting a local lattice diffeomorphism invariance, present already on a finite lattice. Since the Einstein action is a topological invariant in 2 dimensions, the relevant action in this case has to be a curvature squared action supplemented by a cosmological constant term. Specifically, part of the results in [51, 96] can be summarized as follows. For a given lattice one finds for the counting of zero modes

$$\begin{aligned} \text{Tetrahedron } (N_0 = 4): &\qquad \text{2 zero modes} \\ \text{Octahedron}(N_0 = 6): &\qquad \text{6 zero modes} \\ \text{Icosahedron}(N_0 = 12): &\qquad \text{18 zero modes}\,. \end{aligned} \tag{2.52}$$

Thus if the number of zero modes for each regular triangulation of the sphere is denoted by $N_{z.m.}$, then the results can be re-expressed as

$$N_{z.m.} \;=\; 2\,N_0 \;-\; 6\,. \tag{2.53}$$

which agrees with the expectation that, *in the continuum limit*, $N_0 \to \infty$, $N_{z.m.}/N_0$ should approach the constant value D in D spacetime dimensions, the expected number of local parameters for a diffeomorphism. Similar estimates were obtained when looking at deformations of a flat lattice in various dimensions [96]. The case of near-flat space is obviously the simplest: by moving the location of the vertices around in flat space, one can find a different assignment of edge lengths which represents the same flat geometry. This then leads to the $D \cdot N_0$-parameter family of transformations for the edge lengths in flat space.

In general, lattice diffeomorphisms correspond to local deformations of the edge lengths about a vertex, which leave the local geometry physically unchanged, the latter being described by the values of local lattice operators corresponding to local volumes, curvatures etc. The lesson is that *the correct count of continuum zero modes will in general only be recovered asymptotically for large triangulations*, where N_0 is significantly larger than the number of neighbors to a point in D dimensions. I stress here that we are specifically talking here about *lattice diffeomorphism* which can be fully recovered to be exact diffeomorphism at the continuum limit, *i.e.*, at asymptotic limit for large number of triangulations. With these observations in mind, we can now turn to a discussion of the solution method for the lattice Wheeler DeWitt equation in $2+1$ dimensions.

One item that needs to be discussed at this point is the proper normalization of various terms (kinetic, cosmological and curvature) appearing in the lattice equation of Eq. (2.34). For the lattice gravity action in d dimensions one has generally the following correspondence

$$\int d^d x \sqrt{g} \quad \longleftrightarrow \quad \sum_\sigma V_\sigma \tag{2.54}$$

where V_σ is the volume of a simplex; in 2 dimensions it is simply the area of a triangle. The curvature term involves deficit angles in the discrete case,

$$\frac{1}{2} \int d^d x \sqrt{g}\, R \quad \longleftrightarrow \quad \sum_h V_h\, \delta_h \tag{2.55}$$

where δ_h is the deficit angle at the hinge h, and V_h the associated "volume of the hinge" [91]. In 4 dimensions the latter is the area of a triangle (usually denoted by A_h), whereas in 3 dimensions it is simply given by the length l_h of the edge labelled by h. In 2 dimensions $V_h = 1$. In this work we will focus almost exclusively on the case of $2+1$ dimensions; consequently the relevant formulas will be Eqs. (2.236) and (2.237) for dimension $d = 2$.

The continuum Wheeler DeWitt equation is local, as can be seen from Eq. (2.26). One can integrate the Wheeler DeWitt operator over all space and obtain

$$\left\{ -\, (16\pi\, G)^2 \int d^2 x\, \boldsymbol{\Delta}(g) + 2\, \lambda \int d^2 x \sqrt{g} - \int d^2 x \sqrt{g}\, R \right\} \Psi = 0 \tag{2.56}$$

with the super Laplacian on metrics defined as

$$\boldsymbol{\Delta}(g) \;\equiv\; G_{ij,kl}(\mathbf{x})\, \frac{\delta^2}{\delta g_{ij}(\mathbf{x})\, \delta g_{kl}(\mathbf{x})}\,. \tag{2.57}$$

In the discrete case one has one local Wheeler DeWitt equation for *each* triangle [see Eqs. (2.33) and (2.34)], which therefore takes the form

$$\left\{ -(16\pi\, G)^2\, \mathbf{\Delta}(l^2) - \frac{2}{q} \sum_{i \subset \Delta} \delta_i + 2\,\lambda\, A_\Delta \right\} \Psi = 0\,, \tag{2.58}$$

where $\mathbf{\Delta}(l^2)$ is the lattice version of the super Laplacian In the above expression $\mathbf{\Delta}(l^2)$ is a discretized form of the covariant (only in space) super Laplacian, acting locally on the space of l^2 variables. From Eqs. (2.48) and (2.71) one has explicitly

$$\mathbf{\Delta}(l^2) \;=\; -\,4\, A_\Delta \left(\frac{\partial^2}{\partial a\, \partial b} \;+\; \frac{\partial^2}{\partial b\, \partial c} \;+\; \frac{\partial^2}{\partial c\, \partial a} \right)\,. \tag{2.59}$$

Note that the curvature term involves 3 deficit anglesδ_i, associated with the 3 vertices of a triangle. Now, Eq. (2.71) applies to a single given triangle, with one equation to be satisfied at each triangle on the lattice. One can also construct the total Hamiltonian by simply summing over all triangles, which leads to

$$\left\{ -(16\pi\, G)^2 \sum_\Delta\, \mathbf{\Delta}(l^2) \;+\; 2\,\lambda \sum_\Delta A_\Delta \;-\; \kappa \sum_\Delta \sum_{i \subset \Delta} \delta_i \right\} \Psi = 0\,. \tag{2.60}$$

Summing over all triangles (Δ) is different from summing over all lattice sites (i), and the above equation is equivalent to

$$\left\{ -(16\pi\, G)^2 \sum_\Delta \mathbf{\Delta}(l^2) \;+\; 2\,\lambda \sum_\Delta A_\Delta \;-\; \kappa\, q\, \sum_i \delta_i \right\} \Psi = 0\,, \tag{2.61}$$

where q is the lattice coordination number, and is determined by how the lattice is put together (which vertices are neighbors to each other, or equivalently by the so called incidence matrix). Here q is the number of neighboring simplexes that share a given edge. For a flat triangular lattice $q = 6$, whereas for a tetrahedron, octahedron, and icosahedron one has $q = 3,\ 4,\ 5$ respectively. For proper normalization in Eq. (2.242) one requires

$$\int d^2x\, \sqrt{g} \quad \longleftrightarrow \quad \sum_\Delta A_\Delta \tag{2.62}$$

as well as

$$\frac{1}{2} \int d^2x\, \sqrt{g}\; R \longleftrightarrow \sum_i \delta_i\,. \tag{2.63}$$

This last correspondence allows one to fix the overall normalization of the curvature term

$$\kappa = \frac{2}{q} , \tag{2.64}$$

which then determines the relative weight of the local volume and curvature terms.

2.5.2 Outline of the General Method of Solution in 2+1 dimensions

It should be clear from the previous discussion that in the strong coupling limit (large G) one can, at least at first, neglect the curvature term, which can then be included at a later stage. This simplifies the problem quite a bit, as it is the curvature term that introduces complicated interactions between neighboring simplices (this is evident from the lattice Wheeler DeWitt equation of Eq. (2.34), where the deficit angles enter the curvature term only).

The general procedure for finding a solution will be as follows. First a solution will be found for *equilateral* squared edge lengths s. *Later this solution will be extended to determine whether it is consistent to higher order in the weak field expansion.* Consequently we shall write for the squared edge lengths

$$l_{ij}^2 = s\,(1 + \epsilon\, h_{ij}) , \tag{2.65}$$

with ϵ a small expansion parameter, and where $\mathcal{O}\,(h^2)$ is omitted. Therefore, for example, in Eq. (2.59) one has $a = s(1 + \epsilon\ h_a)$, $b = s(1 + \epsilon\ h_b)$ and $c = s(1 + \epsilon\ h_c)$. The resulting solution for the wave function will then be given by a suitable power series in the h variables. Nevertheless, in some rare cases (such as the single triangle case described below, or the single tetrahedron in $3+1$ dimensions [81]) one is lucky enough to find immediately an exact solution, without having to rely in any way on the weak field expansion.

To lowest order in h, a solution to the Wheeler DeWitt equation is readily found using the standard power series (Frobenius) method, appropriate for the study of quantum mechanical wave equations. In this method one first obtains the correct asymptotic behavior of the solution for small and large arguments, and later constructs a full solution by writing the remainder as a power series or polynomial in the relevant variable. Of some importance in the following is the correct determination of the wave functional Ψ for small and large areas, and to what extent the resulting wave function can be expressed in terms of invariants such as areas and curvatures, or powers thereof.

In the following we will see that the natural variable for displaying results is the scaled total area x, defined as

$$x \equiv \frac{\sqrt{2}\,\sqrt{\lambda}}{\boldsymbol{G}} A_{tot} = \frac{\sqrt{2}\,\sqrt{\lambda}}{\boldsymbol{G}} \sum_{\Delta} A_{\Delta} \, . \tag{2.66}$$

In the equilateral case the above x reduces to

$$x|_{equilateral} = \sqrt{2} \;\; N_{\Delta} \, \frac{\sqrt{\lambda}}{\boldsymbol{G}} \, A_{\Delta} \, . \tag{2.67}$$

We will look at a variety of 2 dimensional lattices, including the regular triangulations of the 2 - sphere given by the tetrahedron, octahedron and icosahedron, as well as the case of a triangulated torus with coordination number 6. *These are boundary conditions that we imposed on the spatial manifold.* Later on we will be interested in taking the thermodynamic limit (the infinite volume limit), defined in the usual way as

$$\begin{aligned} N_{\Delta} &\rightarrow \infty \, , \\ A_{tot} &\rightarrow \infty \, , \\ \frac{A_{tot}}{N_{\Delta}} &\rightarrow \text{const.} \, . \end{aligned} \tag{2.68}$$

It follows that this last ratio can be used to define a fundamental lattice spacing l_0, for example via $A_{tot}/N_{\Delta} = A_{\Delta} = \sqrt{3}\, l_0^2/4$.

The full solution of the quantum mechanical problem will in general require that the wave functions be properly normalized, as in Eq. (2.25). This will introduce at some stage wave function normalization factors $\mathcal{N}$ and $\tilde{\mathcal{N}}$, which will be fixed by the standard rules of quantum mechanics. If the wave function depends on the total area only, then the relevant requirement becomes

$$\int_0^\infty dA_{tot} \mid \Psi(A_{tot}) \mid^2 \; \equiv \; \frac{\boldsymbol{G}}{\sqrt{2}\,\sqrt{\lambda}} \int_0^\infty dx \mid \Psi(x) \mid^2 \; = \; 1\,. \tag{2.69}$$

As in nonrelativistic quantum mechanics, 2 solutions are expected, as we are dealing with second order differential equations; only one of which will be regular at the origin and thus satisfy the wave function normalizability requirement.

At this point it will be necessary to discuss each lattice separately in some detail. For each lattice geometry, we will break down the presentation into 4 separate items:

(a) Equilateral Case in the Strong Coupling Limit ($\epsilon = 0$). This section will find a solution in the extreme strong coupling limit (large G), without curvature term in the Wheeler DeWitt equation. The solution will not rely on the weak field expansion, and the results will be exact to zeroth order in the weak field expansion of Eq. (2.245). In this case the simplices are all taken to be equilateral, and the lattice edge lengths fluctuate together.

(b) Equilateral Case with Curvature Term ($\epsilon = 0$). Next the curvature term is included. The solution again will not rely on the weak field expansion, and all the triangles will be taken to be equilateral. The resulting solution will therefore be valid again (and exact) to zeroth order in the ϵ expansion parameter of Eq. (2.245).

(c) Large Area in the Strong Coupling Limit ($\epsilon \neq 0$). In this case we will look at nonzero local fluctuations in Eq. (2.245). The method of solution will now rely on the weak field expansion for large areas (large s), but nevertheless it will turn out that an exact solution can be found in this case. The resulting answer will be found to be correct to arbitrarily large order $\mathcal{O}(\epsilon^n)$, with n a positive integer.

(d) Small Area in the Strong Coupling Limit ($\epsilon \neq 0$). Finally we will look at the case of nonzero fluctuations [$\epsilon \neq 0$ in Eq. (2.245)] in the limit of small areas (small s). In this limit we will find that in general the solution can be written entirely in terms of invariants involving total areas and curvatures only up to order $\mathcal{O}(\epsilon)$ or $\mathcal{O}(\epsilon^2)$, depending on whether a further symmetrization of the problem is performed or not.

In the end, we will present the summary in Sec. (2.5.8).

2.5.3 Single Triangle Configuration

Here we will consider a lattice composed of only one simplex, *i.e.,* a triangle in $2+1$ dimensions. This is the simplest case that one can consider, however, the physics of it is not trivial. Obviously there will be no curvature term present in this lattice configuration. Without the curvature term, as we saw in the Eq. (2.37), this limit corresponds to the strong coupling limit ($G \to \infty$), and the edges start to fluctuate independently. Therefore, looking at this simplest configuration, one can probe the behaviors of the solutions deep in the strong coupling limit; in fact in this configuration, one can solve the equation *exactly.* From Eq. (2.71) the Wheeler DeWitt equation for a single triangle reads

$$\left\{ (16\pi G)^2 \, 4A_\Delta \left(\frac{\partial^2}{\partial a \, \partial b} + \frac{\partial^2}{\partial b \, \partial c} + \frac{\partial^2}{\partial c \, \partial a} \right) + 2 \, \lambda \, A_\Delta \right\} \Psi(\, a, b, c \,) \; = \; 0, \qquad (2.70)$$

where a, b, c are the 3 squared edge lengths for the given triangle, and A_Δ is the area of the same triangle. Note that for a single triangle there can be no curvature term. Equivalently one needs to solve

$$\left\{ \frac{\partial^2}{\partial a \, \partial b} + \frac{\partial^2}{\partial b \, \partial c} + \frac{\partial^2}{\partial c \, \partial a} + \frac{\lambda}{2 \, \boldsymbol{G}} \right\} \Psi(\, a, b, c \,) \; = \; 0 \, . \qquad (2.71)$$

For the single triangle configuration, the total area equals the area of the single triangle, $A_{tot} = A_\Delta$. Here we will define

$$x \; \equiv \; 2 \, \sqrt{2} \, \frac{\sqrt{\lambda}}{\boldsymbol{G}} \, A_\Delta \, , \quad \text{for single triangle only} \qquad (2.72)$$

so that the solution will be a function of this variable only. Note that in this case, and in this case only, we will deviate from the general definition of the variable x given in Eq. (2.246). One then finds the solution to Eq. (2.71) in the form

$$\Psi(a, b, c) \; = \; \Psi(x) \; = \; \mathcal{N} \, \frac{J_n(x)}{x^n} \qquad (2.73)$$

with

$$n \; = \; \frac{1}{2} \tag{2.74}$$

so that

$$\Psi(x) \; = \; \mathcal{N} \, \frac{J_{1/2} \left(2 \, \sqrt{2} \, \frac{\sqrt{\lambda}}{\boldsymbol{G}} \; A_{tot} \right)}{\left(2 \, \sqrt{2} \, \frac{\sqrt{\lambda}}{\boldsymbol{G}} \; A_{tot} \right)^{1/2}} \, . \tag{2.75}$$

The wave function normalization constant is given here by

$$\mathcal{N} \; = \; 2^{3/4} \, \frac{\lambda^{1/4}}{\sqrt{\boldsymbol{G}}} \, . \tag{2.76}$$

Note that the above solution is exact, and did not require in any way the weak field expansion. Two alternate forms of the wave function are

$$\begin{aligned} \Psi(A_{tot}) \; &= \; \tilde{\mathcal{N}} \, \frac{\sin \left(2 \, \sqrt{2} \, \frac{\sqrt{\lambda}}{\boldsymbol{G}} \, A_{tot} \right)}{2 \, \sqrt{2} \, \frac{\sqrt{\lambda}}{\boldsymbol{G}} \, A_{tot}} \\ &= \; \tilde{\mathcal{N}} \, \exp \left(- \, 2 \, \sqrt{2} \, i \, \frac{\sqrt{\lambda}}{\boldsymbol{G}} \, A_{tot} \right) \, {}_1F_1 \left(1, \; 2, \; 4 \, \sqrt{2} \, i \, \frac{\sqrt{\lambda}}{\boldsymbol{G}} \, A_{tot} \right) \, . \end{aligned} \tag{2.77}$$

Here ${}_1F_1(a, b, z)$ is the confluent hypergeometric functions of the first kind. The usefulness of the latter representation will become clearer later, when other lattices are considered and the curvature term is included. Expanding the solution for small area one obtains

$$\Psi(x) \; = \; \tilde{\mathcal{N}} \; \left[1 - \frac{x^2}{6} + \frac{x^4}{120} + \mathcal{O} \left(x^6 \right) \right] \tag{2.78}$$

which shows that it is indeed nonsingular, and thus normalizable and where $\tilde{\mathcal{N}} = \sqrt{\frac{2}{\pi}} \, \mathcal{N}$.

Note that what is significant here as one can see in Eq. (2.73), Eq. (2.75), and Eq. (2.77) is that the solutions were naïvely expected to depend individually on edge lengths, however, in the end it depended collectively on the edge lengths in a form of the area of the triangle, *i.e.*,

$$\Psi \left(\, a, b, c \, \right) \; = \; \Phi \left(\, A_{\Delta} \, \right) \, . \tag{2.79}$$

In fact one finds the following equivalent differential equation

$$A_{\Delta} \, \frac{d^2 \Phi}{d A_{\Delta}^2} \; + \; 2 \, \frac{d \Phi}{d A_{\Delta}} \; + \; 8 \, \frac{\lambda}{\boldsymbol{G}} \, A_{\Delta} \, \Phi \; = \; 0 \, . \tag{2.80}$$

This means that the solution retains the spatial diffeomorphism invariance which one sees in the momentum constraint. We will see later, the solutions with lattice configurations with more than one simplices (therefore with the curvature term present) reduces to this form of the solutions as in Eq. (2.73) (but of course with different values of n) in the strong coupling limit, suggesting that the solutions that we are obtaining are consistent.

In the limit of large areas a solution to the original differential equation is given either by the asymptotic behavior of the above Bessel (here just reduces to sine) function (J), or by the same limiting behavior for the corresponding Bessel function Y, or by the 2 corresponding Hankel functions (H).

$$\Psi \underset{x \to \infty}{\sim} \frac{1}{x} \, \exp\left(\pm\, i\, x\right) \sim \frac{1}{A_{tot}} \, \exp\left(\pm\, 2\sqrt{2}\, i\, \frac{\sqrt{\lambda}}{\boldsymbol{G}}\, A_{tot}\right) , \tag{2.81}$$

where the $\sim$ signs are used to indicate only that there are proportionality constants floating around to use equal signs. However, among those 4 solutions, only one is regular and therefore physically acceptable.

We will now summarize the key aspect of the results from the single simplex lattice configuration in $2+1$ dimensions. The calculation for a single triangle can be regarded as a useful starting point, and a basic stepping stone, for the strong coupling expansion in $1/G$. It shows the physical characteristics of the wave function solution deep in the strong coupling regime: for $G \to \infty$ the coupling term between different simplices, which is caused mainly by the curvature term, disappears and *one ends up with a completely decoupled problem, where the edge lengths in non-adjacent simplices fluctuate independently.*

2.5.4 Tetrahedron Configuration

In the case of the tetrahedron one has 4 triangles, 6 edges, and 4 vertices, and 3 neighboring triangles for each vertex. Let us discuss again here the various cases individually. See Fig. 2.5.

(a) Equilateral Case in the Strong Coupling Limit ($\epsilon = 0$)

Following Eq. (2.246), the scaled area variable cane be defined

$$x \;=\; \sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot} \;\left(=\; 4 \;\times\; \sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{\Delta}\right) \tag{2.82}$$

and the solution will be found later to be a function of this variable only. For equilateral triangles the wave function Ψ needs to satisfy

$$\Psi'' + \frac{2}{x}\,\Psi' + \Psi \;=\; 0\,. \tag{2.83}$$

The correct solution can be written in the form

$$\Psi(x) \;=\; \mathcal{N}\,\frac{J_n(x)}{x^n} \tag{2.84}$$

with

$$n \;=\; \frac{1}{2} \tag{2.85}$$

so that

$$\Psi(x) \;=\; \mathcal{N}\,\frac{J_{1/2}\left(\sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)}{\left(\sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)^{1/2}}\,. \tag{2.86}$$

The wave function normalization constant is given by

$$\mathcal{N} \;=\; 2^{1/4}\,\frac{\lambda^{\frac{1}{4}}}{\sqrt{\boldsymbol{G}}}\,. \tag{2.87}$$

Below are 2 equivalent forms of the same wave function

$$\begin{aligned}\Psi(A_{tot}) \;&=\; \tilde{\mathcal{N}}\,\frac{\sin\left(\sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)}{\sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}} \\ &=\; \tilde{\mathcal{N}}\,\exp\left(-\sqrt{2}\,i\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)\,{}_1F_1\left(1,\;2,\;2\sqrt{2}\,i\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)\end{aligned} \tag{2.88}$$

for the equilateral case. We first look at the case when $\epsilon = 0$ in Eq. (2.245), deep in the strong coupling region and without the curvature term. In the limit of small area one obtains

$$\Psi \;=\; \tilde{\mathcal{N}}\;\left[1 - \frac{x^2}{6} + \frac{x^4}{120} + \mathcal{O}\left(x^6\right)\right] \tag{2.89}$$

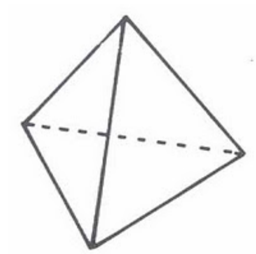

Figure 2.5: Tetrahedron configuration. The building blocks are triangles.

which again confirms that the wave function is regular at the origin and where $\tilde{\mathcal{N}} = \sqrt{\frac{2}{\pi}}\,\mathcal{N}$. Since one is solving a second order linear differential equation one expects 2 solutions; here one is singular and the other one is not, as is often the case in quantum mechanics. For the geometry of the tetrahedron, one solution can be written in terms of Bessel functions of the first kind (J)

$$\frac{J_{1/2}(x)}{\sqrt{x}} \;=\; \sqrt{\frac{2}{\pi}}\,\frac{\sin x}{x}\,. \tag{2.90}$$

The Bessel function of the second kind (Y) also satisfies the same differential equation, but since

$$\frac{Y_{1/2}(x)}{\sqrt{x}} \;=\; -\sqrt{\frac{2}{\pi}}\,\frac{\cos x}{x} \tag{2.91}$$

this second solution is not normalizable, and is therefore discarded on physical grounds. We shall see below that the same behavior at small x holds also for nonzero curvature term. Note that both of the above solutions are real. [3]

(b) Equilateral Case with Curvature Term ($\epsilon = 0$)

Next we include the effects of the curvature term. To zeroth order in weak field expansions, when all edges fluctuate in unison, one now needs to solve the ordinary differential equation

$$\Psi'' + \frac{2}{x}\,\Psi' - \frac{2\beta}{x}\,\Psi + \Psi \;=\; 0\,, \tag{2.92}$$

[3] There are also linear combinations of Bessel functions which give complex Hankel (H) functions. These satisfy the Wheeler DeWitt equation as well; however they are not physically acceptable since both are singular at the origin.

with $\beta = 2\sqrt{2}\,\pi / \left(\sqrt{\lambda}\,\boldsymbol{G}\right)$ as in Eq. (2.39). Since the deficit angle $\delta = \pi$ at each vertex, the curvature contribution for each triangle is $\kappa \cdot \pi \cdot 3$. In this case one has therefore

$$\kappa_{tetra} = 2 \cdot \frac{1}{3} \tag{2.93}$$

and therefore the solution is given by

$$\begin{aligned} \Psi &= \exp\left(-\sqrt{2}\,i\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)\,{}_1F_1\left(1 - i\,\frac{3\sqrt{2}\,\pi\,\kappa_{tetra}}{\boldsymbol{G}\sqrt{\lambda}},\ 2,\ 2\sqrt{2}\,i\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right) \\ &= \exp\left(-\sqrt{2}\,i\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)\,{}_1F_1\left(1 - i\,\frac{2\sqrt{2}\,\pi}{\boldsymbol{G}\sqrt{\lambda}},\ 2,\ 2\sqrt{2}\,i\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right) \end{aligned} \tag{2.94}$$

in the equilateral case, up to an overall normalization factor.

(c) Large Area in the Strong Coupling Limit ($\epsilon \neq 0$)

Next we look at the case $\epsilon \neq 0$ in Eq. (2.245). In the limit of large areas one finds that the 2 independent solutions reduce to

$$\Psi \underset{x \to \infty}{\sim} \exp\left(\pm\, i\, x\right) \ \sim\ \exp\left(\pm\sqrt{2}\,i\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right) \tag{2.95}$$

to all orders in ϵ. To show this, one sets $\Psi = e^{\alpha\,A_{tot}}$, where A_{tot} is a sum of the 4 triangle areas that make up the tetrahedron, and then expands the edge lengths in the usual way according to Eq. (2.245), by setting $a = s(1 + \epsilon\,h_a)$ etc. Here we are interested specifically in the limit when s is large and ϵ is small. One then finds that the *right hand side* of the lattice Wheeler DeWitt equation is given to $\mathcal{O}(\epsilon^n)$ by

$$\frac{e^{\,\alpha\sqrt{3}\,s}}{4}\,\frac{1}{2^n\,\sqrt{3}^{\,n}\,n!}\,\alpha^n\left(\alpha^2 + 2\sqrt{2}\,\frac{\lambda}{\boldsymbol{G}}\right)\epsilon^n\,s^n\left(\sum h\right)^n + \cdots \ . \tag{2.96}$$

One concludes that in this limit it is sufficient to have

$$\alpha^2 + 2\,\frac{\lambda}{\boldsymbol{G}^2} \ =\ 0, \tag{2.97}$$

or $\alpha = \pm\sqrt{2}\,i\sqrt{\lambda}/\boldsymbol{G}$, to obtain an exact solution in the limit $n \to \infty$. Note that in the strong coupling limit the 2 independent wave function solutions in Eq. (2.95) completely factorize as a product of single triangle contributions.

(d) Small Area in the Strong Coupling Limit $(\epsilon \neq 0)$

In the limit of small area, we have shown before that the solution reduces to a constant in the equilateral case $[\mathcal{O}(\epsilon^0)]$ for small x or small areas. Beyond the equilateral case one can write a general ansatz for the wave function in terms of geometric invariants

$$\Psi = \left(\prod_{\Delta} A_{\Delta} \right)^{\gamma_0} \left[1 + \gamma_2 \left(\sum_{\Delta} A_{\Delta} \right)^2 + \gamma_4 \left(\sum_{\Delta} A_{\Delta} \right)^4 + \cdots \right] , \tag{2.98}$$

and then expand the solution in ϵ for small s. To zeroth order in ϵ we had the solution $\Psi \sim J_n(x)/x^n$ with $x = \sqrt{2}\sqrt{\lambda}\, A_{tot} \big/ \boldsymbol{G}$ and $n = 1/2$. This gives in Eq. (2.319) $\gamma_0 = 0$, $\gamma_2 = -\frac{1}{3}\, \lambda \big/ \boldsymbol{G}^2$ and $\gamma_4 = \frac{1}{30}\, \left(\lambda \big/ \boldsymbol{G}^2 \right)^2$. To linear order $[\mathcal{O}(\epsilon)]$ one finds though that terms appear which cannot be expressed in the form of Eq. (2.319). But one also finds that, while these terms are nonzero if one uses the Hamiltonian density (the Hamiltonian contribution from just a single triangle), if one uses the sum of such triangle Hamiltonians then the resulting solution is symmetrized, and the corrections to Eq. (2.319) are found to be of order $\mathcal{O}(\epsilon^2)$. Then the wave function for small area is of the form

$$\Psi \sim 1 - \frac{1}{3}\, \frac{\lambda}{\boldsymbol{G}^2}\, A_{tot}^2 + \frac{1}{30} \left(\frac{\lambda}{\boldsymbol{G}^2} \right)^2 A_{tot}^4 + \ldots \tag{2.99}$$

up to terms $\mathcal{O}(\epsilon^2)$.

2.5.5 Octahedron Configuration

The discussion of the octahedron configuration proceeds in a way that is similar to what was done before for the tetrahedron. In the case of the octahedron one has 8 triangles, 12 edges and 6 vertices, with 4 neighboring triangles per vertex. See Fig. 2.6.

Again we will now discuss the various cases individually.

(a) Equilateral Case in the Strong Coupling Limit $(\epsilon = 0)$

Again we look first at the case $\epsilon = 0$ in Eq. (2.245), deep in the strong coupling region and without the curvature term. Following Eq. (2.246) we define the scaled area variable as

$$x \;=\; \sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\left(\;=\;8\times\sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{\Delta}\right) \tag{2.100}$$

and it is found that the solution is a function of this variable only. For equilateral triangles the wave function Ψ needs to satisfy

$$\Psi'' \,+\, \frac{4}{x}\,\Psi' \,+\, \Psi \;=\; 0\,. \tag{2.101}$$

The correct solution can be written in the form

$$\Psi(x) \;=\; \mathcal{N}\,\frac{J_n(x)}{x^n} \tag{2.102}$$

with

$$n \;=\; \frac{3}{2} \tag{2.103}$$

so that

$$\Psi(x) \;=\; \mathcal{N}\,\frac{J_{3/2}\left(\sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)}{\left(\sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)^{3/2}}\,. \tag{2.104}$$

The wave function normalization factor is given by

$$\mathcal{N} \;=\; 2\,\sqrt{30}\,\frac{\lambda^{1/4}}{\sqrt{\boldsymbol{G}}}\,. \tag{2.105}$$

Equivalent forms of the above wave function are

$$\begin{aligned}\Psi(A_{tot}) \;&=\; \mathcal{N}\,\frac{1}{2^{3/2}\,\Gamma\left(\frac{5}{2}\right)}\,\exp\left(-\sqrt{2}\,i\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)\,{}_1F_1\left(2,\;4,\;2\sqrt{2}\,i\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)\\ &=\; \mathcal{N}\,\sqrt{\frac{\pi}{2}}\left[-\frac{\cos\left(\sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)}{\left(\sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)^2}+\frac{\sin\left(\sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)}{\left(\sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)^3}\right]\,.\end{aligned} \tag{2.106}$$

These can be expanded for small A_{tot} or small x to give

$$\Psi \;=\; \mathcal{N}\,\frac{\sqrt{2}}{3\,\sqrt{\pi}}\,\left[1-\frac{x^2}{10}+\frac{x^4}{280}+\mathcal{O}(x^6)\right]\,. \tag{2.107}$$

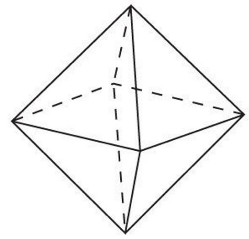

Figure 2.6: Octahedron configuration. The building blocks are triangles.

We note here again that both Bessel functions of the first (J) and second (Y) kind in principle give solutions for this case, as well as the two corresponding Hankel (H) functions. However, only the solution associated with the Bessel J function is regular near the origin.

(b) Equilateral Case with Curvature Term ($\epsilon = 0$)

Next we include the effects of the curvature term. Since here the deficit angle $\delta = 2\pi/3$ at each vertex, the curvature contribution for each equilateral triangle is $\kappa \cdot \frac{2\pi}{3} \cdot 3 = 2\pi\,\kappa$. For the octahedron one has in Eq. (2.244)

$$\kappa_{octa} \;=\; 2 \cdot \frac{1}{4} \,. \tag{2.108}$$

With the curvature term one finds

$$\begin{aligned}
\Psi(A_{tot}) \;&=\; \exp\left(-\sqrt{2}\, i\, \frac{\sqrt{\lambda}}{\boldsymbol{G}}\, A_{tot} \right)\; {}_1F_1\left(2 - i\, \frac{4\sqrt{2}\,\pi\,\kappa_{octa}}{\sqrt{\lambda}\,\boldsymbol{G}},\; 4,\; 2\sqrt{2}\, i\, \frac{\sqrt{\lambda}}{\boldsymbol{G}}\, A_{tot} \right) \\
&=\; \exp\left(-\sqrt{2}\, i\, \frac{\sqrt{\lambda}}{\boldsymbol{G}}\, A_{tot} \right)\; {}_1F_1\left(2 - i\, \frac{2\sqrt{2}\,\pi}{\sqrt{\lambda}\,\boldsymbol{G}},\; 4,\; 2\sqrt{2}\, i\, \frac{\sqrt{\lambda}}{\boldsymbol{G}}\, A_{tot} \right) \,,
\end{aligned} \tag{2.109}$$

up to a normalization constant.

(c) Large Area in the Strong Coupling Limit ($\epsilon \neq 0$)

In the limit of large areas the two independent solutions reduce to

$$\Psi \underset{x \,\to\, \infty}{\sim} \exp\left(\pm\, i\, x\right) \;\sim\; \exp\left(\pm\, \sqrt{2}\, i\, \frac{\sqrt{\lambda}}{\boldsymbol{G}}\, A_{tot} \right) \tag{2.110}$$

to all orders in ϵ. In other words, to $\mathcal{O}(\epsilon^n)$ with $n \to \infty$, as for the tetrahedron configuration. Note also that in the strong coupling limit the two independent wave function solutions again completely factorize as a product of single triangle contributions.

(d) Small Area in the Strong Coupling Limit $(\epsilon \neq 0)$

In the limit of small area the solution approaches a constant in the equilateral case. To go beyond the equilateral case, one can write again a general ansatz for the wave function, written in terms of geometric invariants as in Eq. (2.319). Then the solution can be expanded in ϵ for small s. To zeroth order in ϵ the solution is $\Psi \sim J_n(x)/x^n$ with $n = 3/2$. This gives in Eq. (2.319) $\gamma_0 = 0$, $\gamma_2 = -\frac{1}{5}\lambda/\boldsymbol{G}^2$ and $\gamma_4 = \frac{1}{70}\left(\lambda/\boldsymbol{G}^2\right)^2$. However, to linear order $\mathcal{O}(\epsilon)$ one finds again that linear terms in h appear which cannot be expressed in the form of Eq. (2.319). But one also finds that, while these terms are nonzero if one uses the Hamiltonian density (the Hamiltonian contribution from just a single triangle), if one uses the sum of such triangle Hamiltonians then the resulting solution is symmetrized, and the corrections to Eq. (2.319) are found to be of order $\mathcal{O}(\epsilon^2)$. Then the wave function for small area is of the form

$$\Psi \sim 1 - \frac{1}{5}\frac{\lambda}{\boldsymbol{G}^2} A_{tot}^2 + \frac{1}{70}\left(\frac{\lambda}{\boldsymbol{G}^2}\right)^2 A_{tot}^4 + \dots \tag{2.111}$$

up to terms of $\mathcal{O}(\epsilon)$.

2.5.6 Icosahedron Configuration

The discussion of the icosahedron proceeds in a way that is similar to what was done before for the other regular triangulations. Here one has 20 triangles, 30 edges and 12 vertices, with 5 neighboring triangles per vertex. Let us again discuss the various cases individually. See Fig. 2.7.

(a) Equilateral Case in the Strong Coupling Limit $(\epsilon = 0)$

Again we look first at the case $\epsilon = 0$ in Eq. (2.245), deep in the strong coupling region and

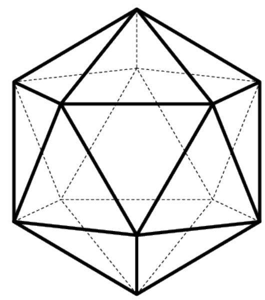

Figure 2.7: Icosahedron configuration. The building blocks are triangles.

without curvature term. Following Eq. (2.246) we define the scaled area variable as

$$x \;=\; \sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\;\left(\;=\;20\times\sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{\Delta}\right) \tag{2.112}$$

and a solution is found which is a function of this variable only. For equilateral triangles the wave function Ψ needs to satisfy

$$\Psi'' + \frac{10}{x}\,\Psi' + \Psi \;=\; 0\,. \tag{2.113}$$

A solution can then be found of the form

$$\Psi(x) \;=\; \mathcal{N}\,\frac{J_n(x)}{x^n} \tag{2.114}$$

with

$$n \;=\; \frac{9}{2} \tag{2.115}$$

so that

$$\Psi(x) \;=\; \mathcal{N}\,\frac{J_{9/2}\left(\sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)}{\left(\sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)^{9/2}}\,. \tag{2.116}$$

The wave function normalization factor is given by

$$\mathcal{N} \;=\; \frac{9\,\sqrt{12155}}{2^{1/4}}\,\frac{\lambda^{1/4}}{\sqrt{\boldsymbol{G}}}\,. \tag{2.117}$$

Below is an equivalent form of the same solution

$$\Psi(A_{tot}) \;=\; \mathcal{N}\,\frac{1}{2^{9/2}\,\Gamma\left(\frac{11}{2}\right)}\;\exp\left(-\sqrt{2}\,i\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)\;{}_1F_1\left(5,\;10,\;2\sqrt{2}\,i\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)\,. \tag{2.118}$$

For small area A_{tot} or small x one obtains

$$\Psi \;=\; \mathcal{N}\,\frac{1}{2^{9/2}\,\Gamma\left(\frac{11}{2}\right)}\,\left[1-\frac{x^2}{22}+\frac{x^4}{1144}+\mathcal{O}\left(x^6\right)\right] \tag{2.119}$$

which shows that the above solution is regular at the origin, and normalizable.

(b) Equilateral Case with Curvature Term ($\epsilon = 0$)

Next we include again the effects of the curvature term. Since now the deficit angle $\delta = \pi/3$ at each vertex, the curvature contribution for each triangle is $\kappa \cdot \frac{\pi}{3} \cdot 3 = \pi\,\kappa$. For the icosahedron one has in Eq. (2.244)

$$\kappa_{icosa} \;=\; 2 \cdot \frac{1}{5}\,. \tag{2.120}$$

Then with the curvature term included for equilateral triangles one obtains for equilateral triangles $[\mathcal{O}(\epsilon^0)]$

$$\begin{aligned}\Psi(A_{tot}) \;&\simeq\; \exp\left(-\sqrt{2}\,i\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)\;{}_1F_1\left(5-i\,\frac{5\sqrt{2}\,\pi\,\kappa_{icosa}}{\sqrt{\lambda}\,\boldsymbol{G}},\;10,\;2\sqrt{2}\,i\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)\\ &=\; \exp\left(-\sqrt{2}\,i\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)\;{}_1F_1\left(5-i\,\frac{2\sqrt{2}\,\pi}{\sqrt{\lambda}\,\boldsymbol{G}},\;10,\;2\sqrt{2}\,i\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)\,,\end{aligned} \tag{2.121}$$

up to an overall wave function normalization constant. Note that in this case one had to include a factor $A_{tot}/4A_{\Delta}$, which in the dodecahedron case equals five.

(c) Large Area in the Strong Coupling Limit ($\epsilon \neq 0$)

In the limit of large areas the two independent solutions reduce to

$$\Psi \underset{x\,\to\,\infty}{\sim} \exp\left(\pm\,i\,x\right) \;\sim\; \exp\left(\pm\,\sqrt{2}\,i\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right) \tag{2.122}$$

to all orders in the weak field expansion parameter ϵ, as for the tetrahedron and octahedron case. Note also that in the strong coupling limit the two independent wave function solutions again completely factorize as a product of single triangle contributions.

(d) Small Area in the Strong Coupling Limit ($\epsilon \neq 0$)

In the limit of small area the solution approaches a constant in the equilateral case. To go beyond the equilateral case, one can write again a general ansatz for the wave function, written in terms of geometric invariants as in Eq. (2.319). Then the solution in ϵ for small s. To zeroth order in ϵ the solution is $\Psi \sim J_n(x)/x^n$ with $n = 9/2$. This gives in Eq. (2.319) $\gamma_0 = 0$, $\gamma_2 = -\frac{1}{11}\lambda\big/\boldsymbol{G}^2$ and $\gamma_4 = \frac{1}{286}\left(\lambda\big/\boldsymbol{G}^2\right)^2$. But to linear order $\mathcal{O}\left(\epsilon\right)$ one finds again that linear terms in h appear which cannot be expressed in the form of Eq. (2.319). But one also finds that, while these terms are nonzero if one uses the Hamiltonian density (the Hamiltonian contribution from just a single triangle), if one uses the sum of such triangle Hamiltonians then the resulting solution is symmetrized, and the corrections to Eq. (2.319) are found to be of order $\mathcal{O}\left(\epsilon^2\right)$. Then the wave function for small area is of the form

$$\Psi \simeq 1 - \frac{1}{11}\,\frac{\lambda}{\boldsymbol{G}^2}\,A_{tot}^2 + \frac{1}{286}\left(\frac{\lambda}{\boldsymbol{G}^2}\right)^2 A_{tot}^4 + \dots , \tag{2.123}$$

up to terms of $\mathcal{O}(\epsilon)$.

2.5.7 Torus

Finally we will consider a regularly triangulated torus, which will consist here of an infinite lattice built out of triangles, with each triangle having twelve neighboring triangles. The torus topology is equivalent to requiring periodic boundary conditions in the 2 spatial directions. One could consider the same type of lattice but with some other sort of boundary condition, but we shall not pursue that aspect here.

Due to the local structure of the lattice Wheeler DeWitt equation in Eq. (2.71), it will not be necessary to include in the wave function triangles that are arbitrarily far apart. Instead it will be sufficient, in order to determine the overall structure of the solution, to include only those triangles that are affected in a nontrivial way by the interaction terms in the Wheeler DeWitt equation. In the present case this requires the consideration of one given triangle plus its twelve neighbors, giving a total of 13 triangles. Here we will also set as before $x \equiv \sqrt{2}\,\sqrt{\lambda}\,A_{tot}\big/\boldsymbol{G}^2$.

(a) Equilateral Case in the Strong Coupling Limit $(\epsilon = 0)$

For this case the relevant equation and its solution is largely in line with what was obtained for the previous cases. For equilateral triangles the wave function Ψ has to satisfy

$$\Psi'' + \frac{13}{2\,x}\,\Psi' + \Psi \;=\; 0\,. \tag{2.124}$$

The wave function can now be written as

$$\Psi(x) \;=\; \mathcal{N}\,\frac{J_n(x)}{x^n} \tag{2.125}$$

with here (due to our specific choice of sub-lattice)

$$n \;=\; \frac{11}{4} \tag{2.126}$$

so that

$$\Psi(x) \;=\; \mathcal{N}\,\frac{J_{11/4}\left(\sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)}{\left(\sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)^{11/4}}\,. \tag{2.127}$$

The wave function normalization constant is given in this case by

$$\mathcal{N} \;=\; 4\sqrt{\frac{30\,\Gamma\left(\frac{13}{4}\right)}{\Gamma\left(\frac{11}{4}\right)}}\,\left(\frac{\lambda}{2\,\boldsymbol{G}^2}\right)^{1/4}\,. \tag{2.128}$$

For the above wave function an equivalent form is

$$\Psi(A_{tot}) \;=\; \mathcal{N}\,\frac{1}{2^{11/4}\,\Gamma\left(\frac{15}{4}\right)}\,\exp\left(-\sqrt{2}\,i\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)\,{}_1F_1\left(\frac{13}{4},\,\frac{13}{2},\,2\sqrt{2}\,i\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot}\right)\,. \tag{2.129}$$

Expanding the above solution for small area one obtains

$$\Psi \;=\; \mathcal{N}\,\frac{1}{2^{11/4}\,\Gamma\left(\frac{15}{4}\right)}\,\left[1 - \frac{x^2}{15} + \frac{x^4}{570} + \mathcal{O}(x^6)\right]\,, \tag{2.130}$$

which shows the above solution is indeed regular at the origin.

(b) Equilateral Case with Curvature Term $(\epsilon = 0)$

In the case of the torus the curvature term is gives zero ($\chi = 0$), so there are no changes to the preceding discussion.

(c) Large Area in the Strong Coupling Limit ($\epsilon \neq 0$)

In the limit of large areas the two independent solutions reduce to

$$\Psi \underset{x \to \infty}{\sim} \exp\left(\pm i\, x\right) \;\sim\; \exp\left(\pm i \sqrt{2}\, \frac{\sqrt{\lambda}}{\boldsymbol{G}}\, A_{tot}\right) . \tag{2.131}$$

to all orders in ϵ. This is similar to what was found earlier for the other lattices. In particular, the two independent solutions again completely factorize as a product of single triangle contributions.

(d) Small Area in the Strong Coupling Limit ($\epsilon \neq 0$)

In the limit of small area, the regular solution approaches a constant and the discussion, and solution, is rather similar to the previous cases. Here one finds

$$\Psi \simeq 1 - \frac{2}{15} \frac{\lambda}{\boldsymbol{G}^2} A_{tot}^2 + \frac{2}{285} \left(\frac{\lambda}{\boldsymbol{G}^2}\right)^2 A_{tot}^4 + \ldots , \tag{2.132}$$

up to terms of $\mathcal{O}(\epsilon^2)$.

2.5.8 Asymptotic Solutions to Any Configuration

In this section we will summarize the results obtained so far for the various finite lattices considered (tetrahedron, octahedron, icosahedron, and regularly triangulated torus).

(a) Equilateral Case in the Strong Coupling Limit ($\epsilon = 0$)

It is rather remarkable that all of the previous cases (except the trivial case of a single triangle, which has no curvature) can be described by one single set of interpolating wave functions, where the interpolating variable is simply related to the overall lattice size (specifically, the number of triangles).

Indeed for equilateral triangles and in the absence of curvature, the wave function $\Psi(x)$ for all previous cases is a solution to the following equation

$$\Psi'' + \frac{2n+1}{x}\Psi' + \Psi = 0\,, \tag{2.133}$$

with parameter n given by

$$n = \frac{1}{4}\left(N_\Delta - 2\right) \tag{2.134}$$

where $N_\Delta \equiv N_2$ is the total number of triangles on the lattice. Thus

$$N_\Delta = 4\left(n + \frac{1}{2}\right) \tag{2.135}$$

and consequently

$$\begin{aligned}
n_{tetrahedron} &= \frac{1}{4}\,(4-2) = \frac{1}{2}\,, \\
n_{octahedron} &= \frac{1}{4}\,(8-2) = \frac{3}{2}\,, \\
n_{icosahedron} &= \frac{1}{4}\,(20-2) = \frac{9}{2}\,, \\
n_{torus} &= \frac{1}{4}\,(13-2) = \frac{11}{4}\,.
\end{aligned} \tag{2.136}$$

Note that for a single triangle one has $n = \frac{1}{2}$ as well, but the definition of the scaled area is different in that case.

Furthermore the differential equation in Eq. (2.133) describes, in spherical coordinates and with suitable choice of constants, the radial wave function for a free quantum particle in $D = 2n+2$ dimensions. Indeed recall that in D dimensions the Laplace operator in spherical coordinates has the form

$$\Delta\Psi = \frac{\partial^2\Psi}{\partial r^2} + \frac{D-1}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\,\Delta_{S^{D-1}}\,\Psi \tag{2.137}$$

where $\Delta_{S^{D-1}}$ is the Laplace-Beltrami operator on the $(D-1)$-sphere. In our case the wave function does not, to this order, depend on angles and therefore the last (angular variable) term does not contribute. The role of the angles is played in our case by the h variables, which to this order do not fluctuate.

A non singular, normalizable solution to Eq. (2.133) is then given by

$$\Psi(x) \;=\; \mathcal{N}\, \frac{J_n(x)}{x^n} \;=\; \tilde{\mathcal{N}}\, e^{-i\,x}\; {}_1F_1\left(n+\frac{1}{2},\; 2\,n+1,\; 2\,i\,x\right) \tag{2.138}$$

where $\mathcal{N}$ is the wave function normalization constant

$$\mathcal{N} \;\equiv\; 2\, \left[\frac{\Gamma(n+\frac{1}{2})\,\Gamma(2\,n+\frac{1}{2})}{\Gamma(n)}\right]^{1/2} \left(\frac{\lambda}{2\; \boldsymbol{G}^2}\right)^{1/4} , \tag{2.139}$$

and

$$\tilde{\mathcal{N}} \;\equiv\; \frac{1}{2^n\,\Gamma(n+1)}\, \mathcal{N}\,. \tag{2.140}$$

Here and in Eq. (2.138) ${}_1F_1(a,b;z)$ denotes the confluent hypergeometric function of the first kind, sometimes denoted also by $M(a,b,;z)$. In either form, the above wave function is real, in spite of appearances. The general asymptotic behavior of the solution $\Psi(x)$ is found from Eq. (2.133). For small x one has

$$\Psi(x) \;\sim\; x^{\alpha} \tag{2.141}$$

with index $\alpha = 0,\; -2\,n$. The latter solution is singular, and will be discarded. For large x one finds immediately

$$\Psi(x) \;\sim\; \frac{1}{x^{n+\frac{1}{2}}}\; \exp\left(\pm\; i\; x\right), \tag{2.142}$$

which is of course consistent with all the previous results. Indeed the other possible independent solution of Eq. (2.133) would be

$$\Psi(x) \;\simeq\; \frac{Y_n(x)}{x^n}\,, \tag{2.143}$$

where $Y_n(x)$ is a Bessel function of the second kind (or Neumann function). However, the latter leads to a wave function Ψ which is singular as $x \to 0$,

$$\Psi(x) \;\sim\; -\frac{1}{\pi}\,\Gamma(n)\,2^n\,x^{-2n} \tag{2.144}$$

and gives therefore a solution which is not normalizable. For completeness we record here the small x (small area) behavior of the normalized wave function in Eq. (2.138)

$$\Psi(x) \;\sim\; \mathcal{N}\, \frac{1}{2^n\,\Gamma(n+1)}, \tag{2.145}$$

and the corresponding large x (large area) behavior

$$\Psi(x) \; \sim \; \mathcal{N} \sqrt{\frac{2}{\pi}} \, \frac{1}{x^{n+\frac{1}{2}}} \, \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \,, \tag{2.146}$$

both of which reflect well known properties of the Bessel functions $J_n(x)$.

(b) Equilateral Case with Curvature Term ($\epsilon = 0$)

When the curvature term is included in the Wheeler DeWitt equation, and still in the limit of equilateral triangles, one obtains the following interpolating differential equation

$$\Psi'' + \frac{2n+1}{x} \Psi' - \frac{2\beta}{x} \Psi + \Psi = 0 \,, \tag{2.147}$$

which now describes the radial wave function for a quantum particle in $D = 2\,n{+}2$ dimensions, with a repulsive Coulomb potential proportional to 2β. The non-singular, normalizable solution is now given by

$$\Psi(x) \; = \; e^{-i\,x} \, {}_1F_1\left(n + \frac{1}{2} - i\,\beta, \, 2\,n + 1, \, 2\,i\,x\right) \,, \tag{2.148}$$

up to an overall wave function normalization constant $\tilde{\mathcal{N}}(n, \beta)$. The normalization constant can be evaluated analytically, but has a rather unwieldy form, and will not be recorded here. Note that the imaginary part (β) of the first argument in the confluent hypergeometric function of Eq. (2.148) depends on the topology (Euler characteristics, χ, here), but does not depend on the number of triangles (n). In view of the previous discussion the parameter n increases gradually as more triangles are included in the simplicial geometry. *For the regular triangulations of the sphere the total deficit angle (the sum of the deficit angles in a given simplicial geometry) is always 4π, so even if one writes for the wave functional $\Psi[A_{tot}, \delta_{tot}]$, the curvature contribution $\sum_h \delta_h$ is a constant and does not contribute in any significant way.* Note also that, in spite of appearances, the above wave function is still real for nonzero β. That $\Psi(x)$ in Eq. (2.148) is a real function can be seen, for example, from its definition via the power series expansion

$$\Psi(x) \; \simeq \; 1 + \frac{2\beta}{2n+1} \, x - \frac{1 + 2n - 4\beta^2}{4 + 12n + 8n^2} \, x^2 - \frac{\beta\,(5 + 6n - 4\beta^2)}{6\,(3 + 11n + 12n^2 + 4n^3)} \, x^3 + \mathcal{O}(x^4) \,, \tag{2.149}$$

and again up to an overall normalization factor $\mathcal{N}(n, \beta)$.

The general asymptotic behavior of the solution $\Psi(x)$ is again easily determined from Eq. (2.147). For small x one has

$$\Psi(x) \ \sim \ x^{\alpha} \tag{2.150}$$

with again $\alpha = 0, -2n$, and therefore independent of the curvature contribution involving β. The second solution is singular and will be discarded as before. For large x one finds immediately

$$\Psi(x) \ \sim \ \frac{1}{x^{n+\frac{1}{2}}} \ \exp\{ \pm i\,(x - \beta \ln x) \} \,, \tag{2.151}$$

which is of course consistent with all previous results. It also shows that the convergence properties of the wave function at large x are not affected by the β term. A second independent solution to Eq. (2.147) is given by

$$\Psi(x) \ \simeq \ e^{-i\,x}\, U \left(n + \frac{1}{2} - i\,\beta,\ 2\,n + 1,\ 2\,i\,x \right) \,, \tag{2.152}$$

where $U(a, b, ; z)$ is the confluent hypergeometric function of the second kind (sometimes referred to as Tricomi's function). This second solution is singular at the origin, leading to a wave function that is not normalizable, and will not be considered further here.

The asymptotic behavior of the regular solution for large argument z (discussed in standard quantum mechanics textbooks such as [97, 98], and whose notation we will follow here) can be obtained from the asymptotic form of the confluent hypergeometric function ${}_1F_1$, defined originally, for small z, by the series

$${}_1F_1(a, b, z) \ = \ 1 + \frac{az}{b\,1!} + \frac{a(a+1)z^2}{b(b+1)\,2!} + \cdots \,. \tag{2.153}$$

It is common procedure to then write ${}_1F_1(a, b, z) = W_1(a, b, z) + W_2(a, b, z)$, where W_1 and W_2 are separately solutions of the confluent hypergeometric equation

$$z \frac{d^2 F}{dz^2} + (b - z) \frac{dF}{dz} - a\,F \ = \ 0 \,. \tag{2.154}$$

Then an asymptotic expansion for ${}_1F_1$ (or M) is obtained from the following relations:

$$
\begin{aligned}
W_1(a,b,z) &= \frac{\Gamma(b)}{\Gamma(b-a)}\,(-z)^{-a}\,w(a,a-b+1,-z) \\
W_2(a,b,z) &= \frac{\Gamma(b)}{\Gamma(a)}\,e^{z}\,z^{a-b}\,w(1-a,b-a,z)
\end{aligned}
\tag{2.155}
$$

where

$$
w(\alpha,\beta,z) \underset{z\to\infty}{\sim} 1+\frac{\alpha\beta}{z\,1!}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{z^2\,2!}+\cdots\,, \tag{2.156}
$$

with the irregular (at the origin) solution given instead by the combination $G(a,b,z) = i\,W_1(a,b,z) - i\,W_2(a,b,z)$. One immediate and useful consequence of the above result is that, as anticipated before, the behavior of the regular solution close to the origin is not affected by the presence of the β (curvature) term. In other words, the wave function solution $\Psi(x)$ in Eq. (2.148) is always well behaved for small areas, and therefore leads to a perfectly acceptable, normalizable solution.

Furthermore, the combination and properties of arguments in the confluent hypergeometric function in Eq. (2.148), allows one to write it equivalently as a Coulomb wave function with (Sommerfeld) parameter η

$$
C_l(\eta)\,\rho^{l+1}\cdot e^{-i\rho}\,{}_1F_1\,(l+1-\,i\,\eta,\,2\,l+2,\,2\,i\,\rho) \;=\; F_l(\eta,\rho)\,, \tag{2.157}
$$

where $F_l(\eta,\rho)$ denotes the *regular* Coulomb wave function that arises in the solution of the quantum mechanical 3 dimensional Coulomb problem in spherical coordinates [97, 98]. The latter is a solution of the radial differential equation

$$
\frac{d^2 F_l}{d\,\rho^2} + \left[1-\frac{2\,\eta}{\rho}-\frac{l(l+1)}{\rho^2}\right] F_l \;=\; 0\,, \tag{2.158}
$$

with the actual radial wave function then given by $R_l(r) = F_l(kr)/r$. After comparing the above equation with Eq. (2.148) one then identifies $\rho = x$, $l = n - \frac{1}{2}$ and $\eta = \beta$. Thus $l = N_\Delta/4 - 1$ where N_Δ is the number of triangles on the lattice. The proportionality constant C_l in Eq. (2.157) is given by the (Gamow) parameter

$$
C_l(\eta) \;\equiv\; \frac{2^l\,e^{-\frac{\pi\,\eta}{2}}\,|\Gamma(l+1+\,i\,\eta)|}{\Gamma(2l+2)}\,. \tag{2.159}
$$

One then has immediately, from Eq. (2.148), an equivalent representation for the regular wave function as

$$\Psi(x) \ \simeq \ \left[C_{n-\frac{1}{2}}(\beta) \right]^{-1} \frac{1}{x^{n+\frac{1}{2}}} \, F_l(\beta, x) \,, \tag{2.160}$$

again up to an overall wave function normalization constant $\tilde{\mathcal{N}}(n, \beta)$. Again we note here that, on the other hand, the *irregular* Coulomb wave function [usually denoted by $G_l(\eta, \rho)$] is singular for small r, and will therefore not be considered here. Further relevant properties of the Coulomb wave function can be found in [97, 98, 99, 100, 101].

The known asymptotics of Coulomb wave function [99, 100, 101] allows one to derive the following result for the wave function Ψ for large x

$$\Psi(x) \ = \ \tilde{\mathcal{N}} \, \frac{1}{C_{n-\frac{1}{2}}(\beta) \cdot x^{n+\frac{1}{2}}} \ \sin \left[x - \beta \ \ln{(2x)} - \frac{(2\,n-1)\pi}{4} + \sigma_n \right] \tag{2.161}$$

with (Coulomb) phase shift

$$\sigma_n(\beta) \ = \ \arg \Gamma(n + \frac{1}{2} + i\beta) \,. \tag{2.162}$$

Also, from Eq. (2.159),

$$C_{n-\frac{1}{2}}(\beta) \ \equiv \ \frac{2^{\,n-\frac{1}{2}} \ e^{-\frac{\pi\beta}{2}} \ \left| \Gamma(n + \frac{1}{2} + i\,\beta) \right|}{\Gamma(2\,n+1)} \,. \tag{2.163}$$

It is easy to check that the above result correctly reduces to the asymptotic expression given earlier for Ψ in Eq. (2.146) in the limit $\beta = 0$. The structure of the wave function in Eq. (2.161) implies that the norm is still finite for $\beta \neq 0$, since the convergence properties of the wave function are not affected by the curvature term.

(c) Large Area in the Strong Coupling Limit ($\epsilon \neq 0$)

In the limit of large areas the two independent solutions reduce to

$$\Psi \ \underset{x \to \infty}{\sim} \ \exp{(\pm\, i\, x)} \tag{2.164}$$

where $x \ \propto \ A_{tot}$. This is true without assuming the weak field expansion, as was already the case before (see in particular Section 2.5.4 discussing the tetrahedron case).

Consequently in the strong coupling limit the two wave function solutions in Eq. (2.164) completely factorize as a product of single triangle contributions,

$$\Psi \;\simeq\; \prod_{\Delta} \exp\left(\pm\, 2\; i\; \frac{\sqrt{\lambda}}{\boldsymbol{G}}\; A_{\Delta} \right) \;, \tag{2.165}$$

again up to an overall normalization constant. The above result, anticipated in [81], was the basis for the variational treatment using correlated product wave functions. Note also, in view of the result of Eq. (2.146), that the correct solution, satisfying the required regularity condition for small areas, is actually a linear combination of the above factorized solutions.

(d) Small Area in the Strong Coupling Limit ($\epsilon \neq 0$)

In the limit of small area, we have shown before in all cases that the solution reduces to a constant in the equilateral case $\mathcal{O}\left(\epsilon^{0}\right)$ for small x or small areas. To linear order $\mathcal{O}\left(\epsilon\right)$ the general result is still that linear terms in h appear which cannot be expressed in the form of Eq. (2.319). But one also finds that, while these terms are nonzero if one uses the Hamiltonian density (the Hamiltonian contribution from just a single triangle), if one uses the sum of such triangle Hamiltonians then the resulting solution is symmetrized, and the corrections to Eq. (2.319) are found to be of order $\mathcal{O}\left(\epsilon^{2}\right)$. In other words, it seems that some residual lattice artifacts that survive at very short distances can be partially removed by a suitable coarse-graining procedure on the Hamiltonian density.

One might wonder what lattices correspond to values of n greater than $9/2$, which is the highest value attained for a regular triangulation of the sphere, corresponding to the icosahedron. For each of the 3 regular triangulations with N_0 sites one has for the number of edges $N_1 = \frac{q}{2}\, N_0$ and for the number of triangles $N_2 = (\frac{q}{2} - 1)N_0 + 2$, where q is the number of edges meeting at a vertex (the local coordination number). In the 3 cases examined before q was between 3 and 5, with 6 corresponding to the regularly triangulated torus. Note that for a sphere $N_0 - N_1 + N_2 = 2$ always. The interpretation of other, even non-integer, values of q is then clear. Additional triangulations of the sphere can be constructed by considering irregular triangulations, where now the parameter q is interpreted as an *average* coordination number. Of course the simplest example is a semi-regular lattice with N_a vertices with coordination number q_a and N_b vertices with coordination number q_b, such that $N_a + N_b = N_0$. Various irregular and random lattice were considered in detail some time ago in [93], and we refer the reader to this work for a clear exposition of the properties of these lattices.

We conclude this section by briefly summarizing the key properties of the gravitational wave function given in Eqs. (2.148) and (2.160), which from now on will be used as the basis for additional calculations. First we note that the above wave function is a function of the total area and total curvature only, and as such is manifestly diffeomorphism invariant and in accord with the spatial diffeomorphism constraint. While it was derived by looking at the discrete triangulations of the sphere, it contains a parameter n, related to the total number of triangles on the lattice by Eq. (2.134), that will allow us to go beyond the case of a finite lattice and investigate the physically meaningful, and presumably universal, infinite volume limit $n \to \infty$ [see Eq. (2.247)]. We have also shown that the above wave function is, in all cases, an exact solution of the full lattice Wheeler DeWitt equation of Eq. (2.34) in the limit of large areas, and to all orders in the weak field expansion. Again, this last case is most relevant for taking the infinite volume limit, defined previously in Eq. (2.247). Furthermore, the small area behavior of the wave function plays a crucial role in uniquely constraining, through the regularity condition, the correct choice of solution. In this last limit one also finds that the various individual lattice solutions agree with the universal form of Eqs. (2.148) and (2.160) only to a low order in the weak field expansion, *which is expected given the different short distance lattice artifacts of the regular triangulation solutions.* Nonetheless, knowledge of their behavior is completely adequate for extracting the most important physically relevant piece of information, namely the constraint on the wave function based on the stated regularity condition at small areas, which comes down to a simple integrability or power counting argument.

We would briefly like to provide a more physical interpretation of our boundary conditions that we employed for the spherical topology manifold, *i.e.,* the tetrahedron, octahedron, and icosahedron configurations. The corresponding universe in this case is $S^d \otimes R$ and for example in spatial dimension $d = 3$ case, in the Robertson-Walker metric, when the $k = +\,1$ case, where the spatial manifold can be closed, and not $k = -\,1,\ 0$ cases where the spatial part of the universe is open.

It is interesting to realize that even though we explicitly considered only the energy constraint (*i.e.,* Hamiltonian constraint (Eq. (2.26)), and not the momentum constraint (Eq. (2.27)) both of which arise from the canonical formulation, our solutions to the energy constraint so far encode the information in the momentum constraint *i.e.,* the spatial part of the diffeomorphism invariance. But of course, this is expected knowing that the energy constraint and the momentum constraint are related by the commutation relations of the position and momentum operators.

2.5.9 Average Area

There are some key quantities associated with phase transitions that one can calculate. Universal critical exponent $\nu \equiv -\left(\partial\, \beta / \partial\, g\right)^{-1}|_{(g=g_c)}$, averages and fluctuations. The universal critical exponent ν is a cutoff independent quantity, therefore allows to compare different theories with different regularization schemes. The averages and fluctuations are important quantities as a divergence of nonanalyticity in partition function, as caused by a phase transition, appear in them. One can see this explicitly in the below because of the log form:

$$\begin{aligned} < V > &\sim \frac{\partial}{\partial\, \lambda_0} \ln Z_{latt} \\ \chi_V &\sim \frac{\partial^2}{\partial\, \lambda_0^2} \ln Z_{latt} \end{aligned} \tag{2.166}$$

where specifically for gravitational case, the lattice partition function is given by

$$Z_{latt}\left(\lambda_0, k\right) = \int \left[dl^2\right] \; e^{-\lambda_0 \sum_h V_h + k \sum_h \delta_h \, A_h} \; . \tag{2.167}$$

In this section, our goal is concentrate on the quantities above to see any signs of phase transitions, and calculate the universal quantity associated with it, *e.g.,* the critical exponent ν as appeared above.

We will now adopt the squared root of the Newton's coupling for simplicity in the notation as defined earlier in Eqs. (2.38) and in units of λ_0, so we have employed $\lambda = \boldsymbol{g}/2$.

In this section we will look at a natural quantum mechanical expectation value, the average total physical area of the lattice simplicial geometry. It is one of many quantities that can be calculated within the lattice quantum gravity formalism, and is clearly both manifestly geometric and diffeomorphism invariant. Here we will use the wave functions given in Eqs. (2.148) and (2.160), originally obtained for the tetrahedron, octahedron and icosahedron, and later extended to any number of triangles N_Δ

$$\Psi(A_{tot}) \;=\; e^{-i\,A_{tot}/\boldsymbol{g}} \; {}_1F_1\left(n + \frac{1}{2} - i\,\beta,\; 2\,n+1,\; 2\,i\,\frac{A_{tot}}{\boldsymbol{g}}\right) \;, \tag{2.168}$$

with $n \equiv \frac{1}{4}\left(N_\Delta - 2\right)$, $\beta \equiv 4\,\pi/\boldsymbol{g}^3$, and again valid up to an overall wave function normalization constant. Due to the structure of the wavefuntion the resulting probability distribution for the area is rather nontrivial, having many peaks associated with the infinitely many minima and maxima of the hypergeometric function. Clearly the most interesting limit is one where one considers an infinite number of triangles, $N_\Delta \to \infty$, which corresponds to $n \to \infty$ in Eq. (2.168). In Figs. 2.8 and 2.9, we display the behavior of the wave function in Eq. (2.168), both with and without the curvature contribution in the Wheeler DeWitt equation. One notices that when the curvature term is included ($\beta \neq 0$), the peak in the wave function shifts away from the origin. This is largely expected, based on the contribution from the repulsive Coulomb term in the wave equation of Eq. (2.147). Note also that if one

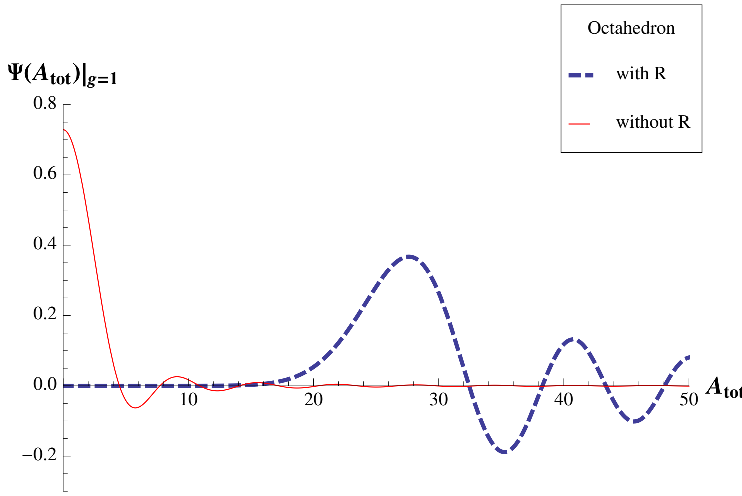

Figure 2.8: Wave Function Ψ versus total area for the octahedron lattice, with and without curvature contribution. The wave function is shown here for $g = \sqrt{G} = 1$, a value chosen here for illustration purposes. The relevant expression for the wave function is given in Eq. (2.168). We refer to the text for further details on how the wave function was obtained, and what its domain of validity is. The wave functions shown here have been properly normalized. Note that with a nonzero curvature term the peak in the wave function moves away from the origin.

compares Figs. 2.8 and 2.9, then one notices that the most probable total area seems to stay at the same value even if one increases the number of triangles N_2. This is in fact the case as can be seen in Figs. 2.10 with the number of triangles taken to be $N_2 = 20, 40, 400$ for illustration. This tells us that physics does not seem to depend on the triangulations in this formalism in a significant way (although most probably A_{tot} seems to shift to a smaller number as the number of triangles N_2 increases, which will be interesting to seek further for future studies), which of course is desired and expected. This also does not ruin our thermodynamic limit that we will take later *i.e.,* $N_2 \to \infty$ and $A_{tot} \to \infty$ at the same time, as we still have the possibility of having $A_{tot} \to \infty$. Furthermore, the limit of $N_2 \to \infty$ and $A_{tot} \to \infty$ at the same time means to take $A_\triangle$ to be finite. Therefore, the thermodynamic limit that we will take later on to obtain the estimate of critical exponent is well defined.

The average total area can then be computed from the above wave function, as the ground

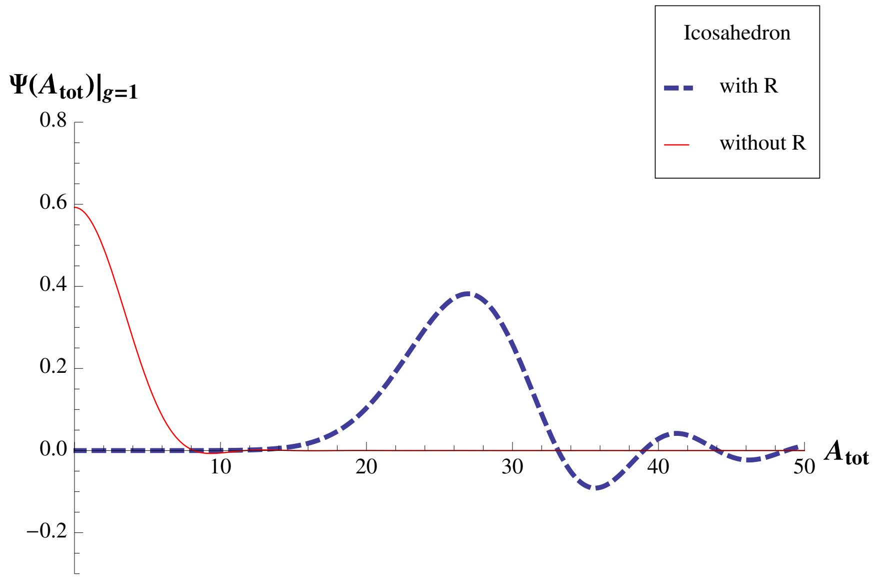

Figure 2.9: Same wave function Ψ as in Fig. 2.8, but now for the icosahedron lattice.

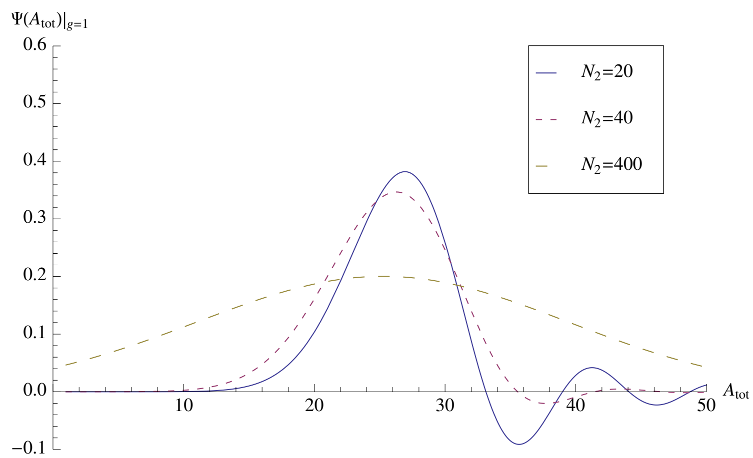

Figure 2.10: ψ (with curvature) vs A_{tot} for different number of triangles N_2. The most probable total area shifts slightly to smaller values as the number of triangles N_2 increases, however, the change is small.

state expectation value

$$< A > = \frac{\langle \Psi | \, A \, | \Psi \rangle}{\langle \Psi | \Psi \rangle} \; = \; \frac{\int \mathcal{D}\mu[g] \; A(g) \; |\Psi(\boldsymbol{g})|^2}{\int \mathcal{D}\mu[g] \; |\Psi(g)|^2} \, , \tag{2.169}$$

where g here is the 3 - metric, and $\mathcal{D}\mu[g]$ denotes a functional integration over all 3 - metrics. In our case we use the measure

$$\int \mathcal{D}\mu[g] \quad \longrightarrow \quad \int_0^\infty dA_{tot} \, , \tag{2.170}$$

which then gives, in terms of the scaled area variable x,

$$< A_{tot} > = \; \boldsymbol{g} \; \frac{\int_0^\infty \, dx \, x \cdot \left| \Psi(x) \right|^2}{\int_0^\infty \, dx \left| \Psi(x) \right|^2} \, . \tag{2.171}$$

In the absence of a curvature term in the Wheeler DeWitt equation ($\beta = 0$) the average area can easily be computed analytically in terms of Bessel function integrals, and the result is

$$< A_{tot} > = \; \boldsymbol{g} \cdot \frac{\pi \, (4n-1) \, \Gamma(4n-2)}{2^{\, 8\, n-5} \, \Gamma(n)^4} \, . \tag{2.172}$$

Note that the average area diverges as $n \to \frac{1}{2}$, which corresponds to the tetrahedron; this entirely spurious divergence prevents us from using the tetrahedron lattice in plotting and numerically extrapolating the remaining two lattices (octahedron and icosahedron) to the infinite lattice limit. For the octahedron one finds $< A_{tot} >= 15 \, \boldsymbol{g}/\pi$, for the icosahedron $< A_{tot} >= 21879 \, \boldsymbol{g}/ \, (3920 \, \pi)$, and in the large n limit $< A_{tot} >= \sqrt{2\, n/\pi} \; \boldsymbol{g} + \mathcal{O}(1/\sqrt{n})$.

One finds that in the presence of a curvature term ($\beta \neq 0$) the resulting integrals are significantly more complicated. We have therefore resorted to a number of tools, which include an analytic expansion in β, the use of known asymptotic expansions for the wave function at large arguments, and an exact numerical integration of the resulting integrals. Let us first discuss here the expansion in β. It is known that the Coulomb wave functions can be expanded in terms of spherical Bessel functions (Neumann expansion) [99, 100, 101], so that one has

$$F_l(\eta, \rho) \;\; = \;\; \frac{2^{l+1}}{\sqrt{\pi}} \;\; \Gamma\left(l + \frac{3}{2}\right) \;\; C_l(\eta) \; \rho \; \sqrt{\frac{\pi}{2\rho}} \; \cdot \left\{ \sum_{k=l}^{\infty} \; b_k(\eta) \; J_{k+\frac{1}{2}}(\rho) \right\} \tag{2.173}$$

with coefficients $b_k(\eta)$ given by a simple recursion relation. When written out explicitly, the expression in curly brackets involves

$$J_{l+\frac{1}{2}}(x) \;+\; \frac{2\,l+3}{l+1}\,\eta\cdot J_{l+\frac{3}{2}}(x) \;+\; \frac{2\,l+5}{l+1}\,\eta^2\cdot J_{l+\frac{5}{2}}(x) \;+\;\cdots\;, \tag{2.174}$$

with additional terms linear in η reappearing at higher orders. That the above expansion is a bit problematic is not entirely surprising, given the modified asymptotic behavior of the Coulomb wave functions for $\eta \neq 0$. In the following, in order to provide initially some insight into the effects of the η (or β) term on the wave function Ψ, we will include the first correction as a perturbation, and drop the rest. Later on, higher order corrections can be included as additional contributions. With this truncation, the Coulomb wave function in Eq. (2.157) becomes

$$F_l(\eta,\rho) \;=\; \frac{2^{l+1}}{\sqrt{\pi}}\;\Gamma\left(l+\frac{3}{2}\right)\,C_l(\eta)\,\rho\,\sqrt{\frac{\pi}{2\,\rho}}\,\left[J_{l+\frac{1}{2}}(\rho)\;+\;\eta\,\frac{2l+3}{l+1}\,J_{l+\frac{3}{2}}(\rho)\;+\;\cdots\right] \tag{2.175}$$

with the last term treated as a perturbation, giving for the wave function itself [see Eq. (2.148)]

$$\begin{aligned}\Psi(x) \;&\simeq\; e^{-i\,x}\;{}_1F_1\left(n+\frac{1}{2}-i\,\beta,\,2\,n+1,\,2\,i\,x\right)\\ &=\; \frac{1}{x^n}\,\left[J_n(x)\;+\;\beta\,\frac{2n+2}{n+\frac{1}{2}}\,J_{n+1}(x)\;+\;\cdots\right]\;,\end{aligned} \tag{2.176}$$

again up to an overall wave function normalization constant $\tilde{\mathcal{N}}$. Note that if m Bessel function terms are kept in Eq. (2.176), beyond the zeroth order, strong coupling, term involving $J_n(x)$, then the resulting expansion in β contains terms up to β^m. One finds to lowest order ($m=1$)

$$\frac{1}{\tilde{\mathcal{N}}^2} \;=\; \frac{\Gamma(n)}{2\,\Gamma(n+\frac{1}{2})\,\Gamma(2n+\frac{1}{2})} \;+\; \frac{4^{1-n}\,(n+1)\,\beta}{(2n+1)\;\left[\Gamma(n+1)\right]^2} \;+\;\cdots \tag{2.177}$$

From the above expressions, the average area can then be computed as some still rather complicated function,

$$\begin{aligned}<\;\; A_{tot}\;\; >&\\ =\;\;\; &\boldsymbol{g}\,\left\{\frac{\pi\,(4n-1)\,\Gamma(4n-2)}{2^{8n-5}\;\left[\Gamma(n)\right]^4}\right.\\ &\left.\;+\;\frac{4(n+1)\beta}{2n+1}\;\left[1-\frac{4^{1-2n}\;\Gamma(n-\frac{1}{2})\,\Gamma(n+\frac{1}{2})\;\left[\Gamma(2n+\frac{1}{2})\right]^2}{n^2\;\left[\Gamma(n)\right]^6}\right]+\cdots\right\}\end{aligned} \tag{2.178}$$

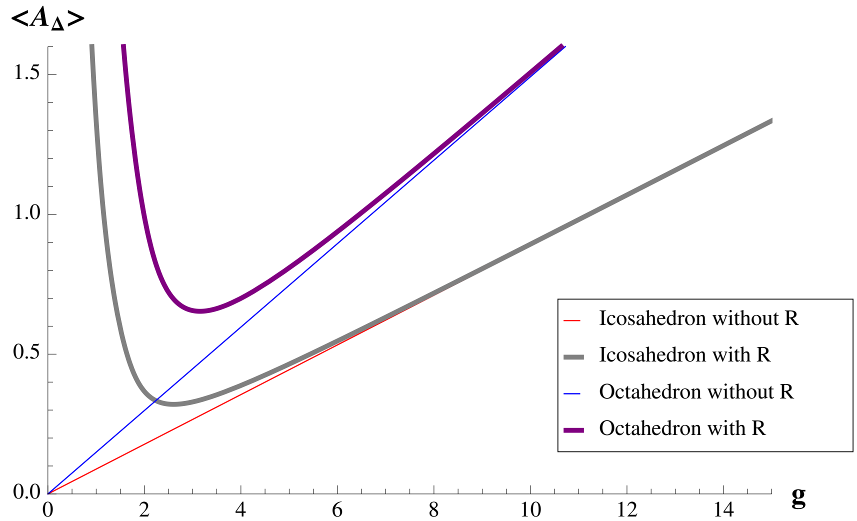

Figure 2.11: Average area of a single triangle vs. $g = \sqrt{G}$ for the octahedron and the icosahedron configurations. The average area was calculated using the expression in Eq. (2.171). Note the qualitative change when one includes the curvature term, with a minimum appearing at $g \sim \mathcal{O}(1)$.

Additional terms can later be included in the Bessel function expansion of Eq. (2.173), so as to obtain more accurate values for the averages; this will be done later.

Fig. 2.11 shows the exact value of the average area for a single triangle $< A_\Delta > = < A_{tot} > / N_\Delta$ as a function of the coupling $\boldsymbol{g}$, obtained by doing the integral in Eq. (2.171) numerically, with the wave function given in Eq. (2.168). One noteworthy aspect is that *a qualitative change seems to occur* when one includes the curvature term: a well defined minimum occurs at $\boldsymbol{g} \sim 1$, which would suggest the appearance of some sort of a phase transition. Doing the integrals numerically one finds a minimum in the average area of a triangle at $\boldsymbol{g}_c \approx 3.1$ for the octahedron, and at $\boldsymbol{g}_c \approx 2.6$ for the icosahedron. On the other hand, using the lowest order Bessel function expansion of Eq. (2.176) for the octahedron ($n = 3/2$) one finds a minimum at $\boldsymbol{g}_c = 2.683$, and for the icosahedron ($n = 9/2$) at $\boldsymbol{g}_c = 2.271$. Adding one more Bessel function correction term then gives $\boldsymbol{g}_c = 3.135$ and $\boldsymbol{g}_c = 2.637$ for the two cases, respectively, which suggests that the expansion is converging.

The limit of a large number of triangles $N_\Delta \to \infty$ corresponds to taking the parameter n in Eq. (2.168) to infinity, since $n \equiv \frac{1}{4}(N_\Delta - 2)$. From the lowest order Bessel function expansion one obtains the following analytic expression for the average total area

$$< A_{tot} > \; = \; \boldsymbol{g} \cdot \sqrt{\frac{2\,n}{\pi}} \left[1 + \frac{3}{16\,n} + \mathcal{O}\left(\frac{1}{n^2}\right) \right] + \frac{2(\pi - 2)\,\boldsymbol{g}}{\pi}\, \beta + \cdots , \tag{2.179}$$

with $\beta \equiv 4\pi/\boldsymbol{g}^3$ [see Eq. (2.43)]. In this limit the resulting function of $\boldsymbol{g}$ has again a well defined minimum at

$$\boldsymbol{g}_c^3 \; = \; \frac{8(\pi - 2)\,\sqrt{2\pi}}{\sqrt{n}} \tag{2.180}$$

or $\boldsymbol{g}_c \simeq 2.839/n^{1/6}$ for large n with one Bessel function correction term. With two Bessel function correction terms in Eq. (2.176) one finds $\boldsymbol{g}_c \simeq 3.276/n^{1/6}$, which again suggests that the expansion is slowly converging. Using the exact wave function to do the integrals numerically one finds for the minimum $\boldsymbol{g}_c \simeq 3.309/n^{1/6}$, which is close to the above answer. Interestingly enough, the above result would suggest that in the limit of infinitely many lattice points the critical point $\boldsymbol{g}_c$ actually moves to the origin, indicating a phase transition located at exactly $\boldsymbol{g} = 0$ ($G = 0$) in the infinite volume ($n \to \infty$) limit (see further discussion later). We note here that the average area for a single triangle is obtained by simply dividing the average total area by the total number of triangles $N_\Delta = 4n + 2$, which then gives in the same limit of large n and strong coupling

$$< A_\Delta > = \frac{\boldsymbol{g}}{2\,\sqrt{2\,\pi\,n}} + \mathcal{O}(\frac{1}{n})\,. \tag{2.181}$$

Quite generally, the average of the area per site in the lattice theory (the spatial volume per site) appears to be well defined mainly due to our wave function normalization choices, and consequently can be explicitly calculated without any leftover ambiguity.

As will be discussed further below in more detail, the estimate for the critical point given in Eq. (2.180) is also in good agreement with a previous variational estimate. In [81] the quantum-mechanical variational (Rayleigh Ritz) method was used to find an approximation for the ground state wave function, using as variational wave function a correlated (Jastrow-Slater) product of single triangle wave functions. There it was found, from the roots of the equation $<\Psi|H|\Psi>=0$, that the variational parameters are almost purely imaginary for strong coupling (large $G>G_c$), whereas for weak enough coupling (small $G<G_c$) they become real. This *abrupt change* in behavior of the wave function at G_c then suggested the presence of a phase transition. With the notation we are using here, the result of [81] reads $\boldsymbol{g}_c^3 \sim 1/N_\Delta$, in qualitative agreement with the result of Eq. (2.180), in the sense that both calculations point to a critical point $G_c=0$ in the infinite volume limit.

Let us now make some additional comments which should help clarify the interpretation of the previous results. It is well known that if there is some sort of continuous phase transition in the lattice theory, the latter is generally associated with a divergent correlation length in the vicinity of the critical point. In our case it is clear that at strong coupling (large $\boldsymbol{g}$) the correlation length is small (of order one) in units of lattice spacing. This can be seen from the fact that (a) the coupling term in the Wheeler DeWitt equation is due mainly to the curvature term, which is small for large $\boldsymbol{g}$, and (b) that the ground state wave function is of the form of a correlated product in the same limit [see Eq. (2.165)]. Then as the effects of the curvature term are included, the correlation length starts to grow due to the additional coupling between edge variables. The previous calculation would then suggest that the point of divergence is located at $\boldsymbol{g}=0$. *It is of course essential that one looks at the limit of infinitely many triangles, $N_\Delta \to \infty$, since no continuous phase transition can occur in a system with a finite number of degrees of freedom.* Each term of the sum in the partition function ($Z=\sum e^{-S[g]}$) is analytic in the coupling constant, and if the sum is only for finite number of configurations, one will never gain the nonanalyticity in partition function [102].

It is also of interest here to discuss how the above (Lorentzian) results relate to what is known about the corresponding *Euclidean* lattice theory in 3 dimensions, which was studied in some detail in [51]. There a phase transition was found between two phases, with the weak coupling phase $G<G_c$ exhibiting a sort of pathological behavior, whereby the lattice collapses into what geometrically could be described as a branched polymer. This is clearly

a nonperturbative phenomenon that cannot be seen from perturbation theory in G. In the Euclidean formulation, average volumes are obtained as suitable derivatives of $\log Z_{latt}$ with respect to the bare cosmological constant λ_0, where Z_{latt} is the lattice path integral

$$Z_{latt} = \int [dl^2] \, e^{-I_{latt}(l^2)} \tag{2.182}$$

with, in 4 dimensions, the action given by

$$I_{latt} = \lambda_0 \sum_h V_h(l^2) - k \sum_h \delta_h(l^2) A_h(l^2) \tag{2.183}$$

and h denoting a hinge [more details can be found in [51]]. Similarly, the average curvature can also be obtained as a derivative of $\log Z$ with respect to $k \equiv 1/(8\pi G)$. More importantly, a non-analyticity in Z, as induced by a phase transition, is expected to show up in local averages as well. From the above expression for Z_{latt} exact sum rules can be derived relating various averages [46, 47]. In the case of the 3 dimensional Euclidean theory the sum rule reads

$$2\,\lambda_0 < \sum_T V_T > - k < \sum_h \delta_h l_h > \; - \, C_0 \;=\; 0 \tag{2.184}$$

where the first term contains a sum over all lattice tetrahedra, and the second term involves as sum over all lattice hinges (just edges in this case). The quantity C_0 here is a constant that solely depends on how the lattice is put together (*i.e.*, on the local coordination number, or incidence matrix).

In [51] it was found that the average curvature goes to zero at some $\boldsymbol{g}_c$ with a characteristic universal exponent δ,

$$< \sum_h \delta_h l_h > = \; - R_0 \; |\boldsymbol{g} - \boldsymbol{g}_c|^{\delta} \tag{2.185}$$

and that the curvature fluctuation diverges in the same limit. From the sum rule in Eq. (2.184) one then deduces that the average volume in the Euclidean theory has a singularity of the type

$$< \sum_T V_T > = \; V_0 \; - \; V_1 \; |\boldsymbol{g} - \boldsymbol{g}_c|^{\delta} \tag{2.186}$$

with the same exponent $\delta \simeq 0.77$. The latter is related by standard universality and scaling arguments [103, 104, 105] (see [25] for details specific to the gravity case) to the correlation length exponent ν by $\nu = (1+\delta)/d$ in d dimensions. To compare to the Lorentzian theory we discussed here, one notes that the 3 dimensional Euclidean theory corresponds to the $2+1$ dimensional Wheeler DeWitt theory, so that the average volume in the above discussion should be taken to correspond to an average area in our case. [4] To conclude, the results for the average area suggest the existence of a phase transition in the Lorentzian theory located at $\boldsymbol{g} = 0$. In the next sections we will present further test of this hypothesis, based on physical observables that can establish directly and un-ambiguously the location of the phase transition point.

2.5.10 Area Fluctuation, Fixed Point and Critical Exponent

Another quantity that can be obtained readily from the wave function Ψ is the fluctuation in the total area

$$\chi_A \;=\; \frac{1}{N_\Delta} \left\{ < (A_{tot})^2 > \;-\; < A_{tot} >^2 \right\} . \tag{2.187}$$

The latter is related to the fluctuations in the individual triangles by

$$\chi_A \;=\; N_\Delta \left\{ < A_\Delta^2 > \;-\; < A_\Delta >^2 \right\} \tag{2.188}$$

with the usual definition of averages, such as the one given in Eq. (2.169).

Generally for a field $\phi(x)$ with renormalized mass m and correlation length $\xi = m^{-1}$, wave function renormalization constant Z, and (Euclidean) propagator

$$< \phi(x) \phi(0) > = \int \frac{d^d p}{(2\pi)^d} \, e^{-ip\cdot x} \, \frac{Z}{p^2 + m^2} \, , \tag{2.189}$$

[4] It should be noted that in the case of the lattice Wheeler DeWitt equation of Eqs. (2.33) and (2.34), and generally in any lattice Hamiltonian continuous time formulation, the lattice continuum limit along the time direction has already been taken. This is due to the fact that one can view the resulting $2+1$ theory as originating from one where there exist initially two lattices spacings, a_t and a. The first one is relevant for the time direction, and the second one for the spatial directions. In the present lattice formulation the limit $a_t \to 0$ has already been taken; the only limit left is $a \to 0$, which requires the existence of an ultraviolet fixed point of the renormalization group.

one has for $\Phi \equiv \int_x \phi(x)$

$$< \Phi^2 > = \int_{x,y} < \phi(x)\phi(y) > = V \int_x < \phi(x)\phi(0) > = V \frac{Z}{m^2} = V Z \xi^2 . \qquad (2.190)$$

Thus the field fluctuation probes the propagator at zero momentum, which in turn is directly related to the renormalized mass (and thus ξ) for the field in question. If the field Φ acquires a nonzero expectation value the above result is modified to

$$\frac{1}{V} \left\{ < \Phi^2 > - < \Phi >^2 \right\} = \frac{Z}{m^2} = Z \xi^2 , \qquad (2.191)$$

involving instead the connected propagator. In the gravity case the quantity A_{tot} plays the role of Φ; if the fluctuation diverges ($\xi \to \infty$) then one has a phase transition, or an ultraviolet fixed point in quantum field theory language [26, 28, 46, 47].

Without the curvature term in the Wheeler DeWitt equation [$\beta = 0$ for the wave function Ψ in Eq. (2.176)] the area fluctuation does not diverge, even when n is large, and is simply proportional to $\boldsymbol{g}^2$. In this case one finds

$$\begin{aligned} \chi_A(\beta = 0) &= \frac{4n - 1}{16} \left[\frac{2n - 1}{2n^2 - n - 1} - \frac{\pi^2 \, (4n - 1) \, [\Gamma(4n - 2)]^2}{2^{16n-13} \, (2n + 1) \, [\Gamma(n)]^8} \right] \boldsymbol{g}^2 \\ &\sim \frac{\pi - 2}{4\pi} \boldsymbol{g}^2 + \mathcal{O}(\frac{1}{n}) . \end{aligned} \qquad (2.192)$$

Note the spurious singularity for the special case of the tetrahedron, $n = 1/2$. When the curvature term is taken into account one finds, from the full wave function Ψ in Eq. (2.176) and in the limit of large n,

$$\chi_A = \left(1 - \frac{2}{\pi} \right) \frac{\boldsymbol{g}^2}{4} + 2 \, (4 - \pi) \sqrt{\frac{2}{n \, \pi}} \, \frac{1}{\boldsymbol{g}} + \cdots \qquad (2.193)$$

Note that *the fluctuations now appear to diverge as* $\boldsymbol{g} \to 0$. Furthermore, χ_A is nonanalytic in the original Newton's coupling $\boldsymbol{G} = \boldsymbol{g}^2$ which suggests that perturbation theory in G is useless. A divergence of the fluctuations as $\boldsymbol{g} \to 0$ implies that in this limit the correlation length diverges in lattice units, signaling the emergence of a massless excitation.

Just as for the case of the average curvature [Eq. (2.184)], an exact sum rule can be derived in the (Euclidean) lattice path integral formulation, relating the local volume fluctuations

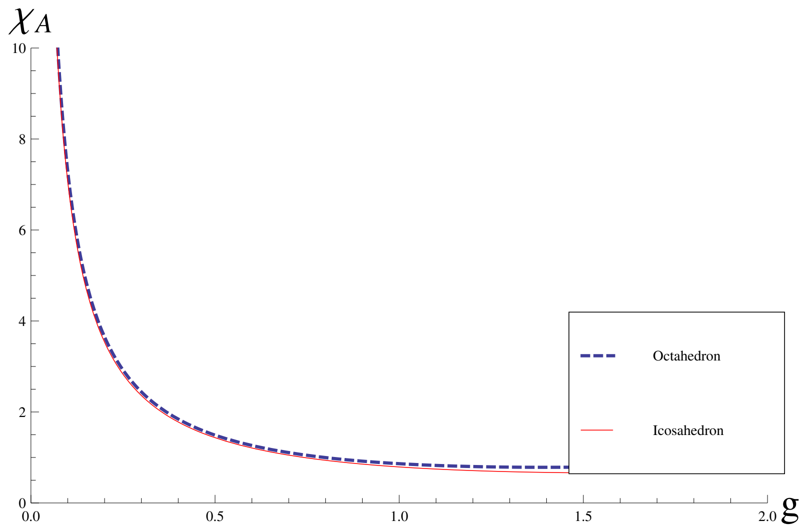

Figure 2.12: Area fluctuation χ_A vs. $g = \sqrt{G}$ for the octahedron and icosahedron, computed from Eq. (2.187). Note the divergence for small g.

to the local curvature fluctuations. In the 3 dimensional Euclidean path integral theory the following exact identity holds for the fluctuations [46, 47]

$$4\lambda_0^2 \left[< (\sum_h V_h)^2 > - < \sum_h V_h >^2 \right] \; - \; k^2 \left[< (\sum_h \delta_h l_h)^2 > - < \sum_h \delta_h l_h >^2 \right] \; - \; 2N_1 \; = \; 0 \, , \tag{2.194}$$

where N_1 is the number of edges on the lattice (further exact sum rules can be derived by considering even higher derivatives of the free energy $\ln Z_L$ with respect to the parameters λ_0 and k). Since the last equation relates the fluctuation in the curvature to fluctuations in the volumes, it also implies a relationship between their singular (divergent) parts. [5]

According to the sum rule of Eq. (2.194) a divergence in the curvature fluctuation

$$\chi_R \; \sim < (\sum_h \delta_h l_h)^2 > - < \sum_h \delta_h l_h >^2 \tag{2.195}$$

[5] We noted previously that in our Hamiltonian formulation the lattice continuum limit along the time direction has already been taken. This results in two lattices spacings, one for the time and one for the space directions, denoted here respectively by a_t and a, with the first lattice spacing already sent to zero. It is then relatively straightforward to relate volumes between the two formulations, such as $V \simeq a_t A$. Relating curvatures (for example, ${}^2R$ in the $2+1$ theory vs. the Ricci scalar R in the original 3 dimensional theory) in the two formulations is obviously less easy, due to the presence of derivatives along the time direction.

for the 3 dimensional (Euclidean) theory generally implies a corresponding divergence in the volume fluctuation

$$\chi_V \;\sim < (\sum_h V_h)^2 > - < \sum_h V_h >^2 \tag{2.196}$$

for the same theory. In our case a divergence is expected in $2+1$ dimensions of the form

$$\chi_A \underset{\boldsymbol{g}\to\boldsymbol{g}_c}{\sim} \left|\boldsymbol{g}-\boldsymbol{g}_c\right|^{-\alpha} \tag{2.197}$$

with exponent $\alpha \equiv 1-\delta = 2-3\nu$, where δ is the universal curvature exponent defined previously in Eq. (2.185), and ν the correlation length exponent. The latter is defined in the usual way [103, 104] [6] through

$$\xi \underset{\boldsymbol{g}\to\boldsymbol{g}_c}{\sim} \left|\boldsymbol{g}-\boldsymbol{g}_c\right|^{-\nu}, \tag{2.198}$$

where ξ is the invariant gravitational correlation length. [7] The scaling relations among various exponents (ν,δ,α) are rather immediate consequences of the scaling assumption for the singular part of the free energy. As one can observe that since the free energy has the form

$$F \;\sim\; \frac{1}{V}\,\ln Z\,, \tag{2.199}$$

a phase transition leads to the appearance of singularity in the free energy. The singular part of the free energy due to a phase transition yields therefore $F_{sing} \sim \xi^{-d}$ in the vicinity of a critical point, and together with Eq. (2.198),

$$F \;\sim\; |g-g_c|^{d\nu}\,. \tag{2.200}$$

[6] Of course from the construction of invariant mass, one can reexpress the renormalization group invariance of mass as $m \sim \Lambda\, e^{-\ln\Lambda} \sim\; \Lambda\, e^{-\int_g d\tilde{g}\;\beta(\tilde{g})^{-1}} \;\sim\; \Lambda\, e^{-\int_g d\tilde{g}\;\left(\beta'(g_c)\,|\tilde{g}-g_c|\right)^{-1}} \;=\; \Lambda\, e^{\,\nu\ln|g-g_c|} = \Lambda\,|g-g_c|^{\nu}\,.$

[7] Note that correlation length is well defined. Physical distance between any two points x and y is given by $d(x,y|g) = \min_\xi \int_{\tau(x)}^{\tau(y)} d\tau \sqrt{g_{\mu\nu}(\xi)\frac{d\xi^\mu}{d\tau}\frac{d\xi^\nu}{d\tau}}$, where τ is some parameter. In the continuum, the geodesic distance between two points can be obtained from solving the geodesic equation, $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\lambda\sigma}\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau} = 0$. On the lattice, the geodesic distance between two vertices x and y requires the determination of the shortest lattice path connecting several lattice vertices given the two vertices as endpoints. This can be done in principle by selecting the shortest path after enumerating all paths connecting the two endpoints. We should define physical correlations at the fixed geodesic distance d and for curvature correlation function is written as $G_R(d) \sim < \sqrt{g}R(x)\sqrt{g}R(y)\delta(|x-y|-d) >_c$ in the continuum, and $G_R(d) \equiv \left\langle \sum_{h\supset x}\delta_h A_h \sum_{h'\supset y}\delta'_h A'_h \delta(|x-y|-d)\right\rangle_c$ on lattice, where $<>_c$ indicates connected average. One expects that the curvature correlation obeys a power law decay for distances sufficiently larger than the lattice spacing l_0, $G_R(d) \overset{d\gg l_0}{\sim} \frac{1}{d^{2n}}$ with an exponent n, and for a very large distances, expects an exponential decay, $G_R(d) \overset{d\gg\xi}{\sim} e^{-d/\xi}$, where ξ is the correlation length. [68]

This then in turn gives

$$
\begin{aligned}
< \mathcal{O} > \sim \frac{1}{V} \frac{\partial}{\partial g} \ln Z \ \sim \ |g - g_c|^{d\nu - 1} \\
\chi \ \sim \ \frac{1}{V} \frac{\partial^2}{\partial g^2} \ln Z \ \sim \ |g - g_c|^{d\nu - 2}
\end{aligned}
\tag{2.201}
$$

Notice that the arguments involving Eq. (2.199), Eq. (2.200), and Eq. (2.201), are general and therefore g is a coupling constant in a theory. Now one can compare Eq. (2.201), and Eq. (2.197), where we defined α; $\alpha = d\nu - 2$ and therefore in $2+1$dimensions, $\alpha = 3\nu - 2$. The preceding argument then implies, via scaling, that a determination of α provides a direct estimate for the correlation length exponent ν defined in Eq. (2.198). Note that based on the results so far one would be inclined to conclude that for $2+1$ gravity the critical point $\boldsymbol{g}_c \to 0$ as $n \to \infty$. Eq. (2.197) can then be re-written either as

$$
\chi_A \underset{\boldsymbol{g} \to \boldsymbol{g}_c}{\sim} \xi^{\alpha/\nu} \tag{2.202}
$$

or, in a finite volume with linear lattice dimensions $L \sim N_0^{1/d} \sim \sqrt{N_\Delta} \sim \sqrt{n}$ (since $N_\Delta = 4n + 2$), as

$$
\chi_A \underset{\boldsymbol{g} \to \boldsymbol{g}_c}{\sim} L^{\alpha/\nu} \ \sim \ n^{1/\nu - 3/2} \, , \tag{2.203}
$$

since, for a very large box and $\boldsymbol{g}$ very close to the critical point $\boldsymbol{g}_c$, the correlation length saturates to its maximum value $\xi \sim L$. Hence the volume- or n-dependence of χ provides a clear and direct way to estimate the critical correlation length exponent ν defined in Eq. (2.198).

2.5.11 Results for Arbitrary Euler Characteristic χ

The results of the previous sections refer to regular triangulations of the sphere ($\chi = 2$) and the torus ($\chi = 0$) in $2+1$ dimensions. It would seem that one has enough information at this point to reconstruct the same type of answers for arbitrary χ. In particular one has for the parameter β [see Eqs. (2.41) and (2.44)]

$$
\beta \ = \ \frac{2\pi\chi}{\boldsymbol{g}^3} \, , \tag{2.204}
$$

relevant for the wave functions in Eqs. (2.148) or (2.160). For the average total area one then finds, using the wave function expansion in Eq. (2.176),

$$
\begin{aligned}
< A_{tot} > \; = \; \boldsymbol{g} \Bigg\{ & \frac{2^{1-2n}\,\Gamma(n-\frac{1}{2})\,\Gamma\left(2n+\frac{1}{2}\right)}{\left[\Gamma(n)\right]^3} \\
& + \frac{8(n+1)\pi\,\chi\left[1-\frac{4^{1-2n}\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\left[\Gamma\left(2n+\frac{1}{2}\right)\right]^2}{n^2\,\left[\Gamma(n)\right]^6}\right]}{\boldsymbol{g}^3\,(2n+1)} + \cdots \Bigg\} .
\end{aligned}
\tag{2.205}
$$

In the large n limit one obtains for the average area of a single triangle

$$
< A_\Delta > = \frac{\boldsymbol{g}}{2\sqrt{2\pi\,n}} \left[1-\frac{5}{16n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right] + \frac{(\pi-2)\,\chi}{\boldsymbol{g}^2\,n}\left[1+\frac{1}{4n(\pi-2)}+\cdots\right] ,
\tag{2.206}
$$

and for the average total area

$$
< A_{tot} > \sim \sqrt{\frac{2\,n}{\pi}}\,\boldsymbol{g} + \frac{4(\pi-2)\,\chi}{\boldsymbol{g}^2} + \cdots .
\tag{2.207}
$$

For the area fluctuation defined in Eq. (2.188) one finds in the same large n limit

$$
\chi_A \; = \; \left(1-\frac{2}{\pi}\right)\frac{\boldsymbol{g}^2}{4} + \mathcal{O}\left(\frac{1}{n}\right) + (4-\pi)\sqrt{\frac{2}{n\,\pi}}\,\frac{\chi}{\boldsymbol{g}} + \cdots .
\tag{2.208}
$$

Again note that the fluctuation appears to diverge as $g \to 0$, which implies that this is the more interesting limit, so from now on we will focus specifically on this limit. It is clear from the analytic expression for $< A_{tot} >$ in Eqs. (2.205) or (2.206) that as $\to \infty$ the gravitational coupling $\boldsymbol{g}(n)$, to this order in the Bessel expansion, has to scale like

$$
\boldsymbol{g}(n) \; \sim \; \frac{1}{\sqrt{n}} ,
\tag{2.209}
$$

so that the expression for $< A_{tot} >$ scales like n or N_Δ, with the expression for $< A_\Delta >$ staying finite.

The result of Eq. (2.208) for χ_A then implies

$$
\chi_A \; \sim \; \frac{1}{\boldsymbol{g}\,\sqrt{n}} \; \sim \; n^0
\tag{2.210}
$$

in the same limit $n \to \infty$. In view of Eqs. (2.205) and (2.203) with $n \sim N_\Delta \sim L^2$, this would imply $2/\nu - 3 = 0$, and thus for the universal critical exponent ν itself $\nu = \frac{2}{3} = 0.666$ to first order ($m = 1$) in the Bessel function expansion of Eq. (2.176) and $\nu = \frac{17}{10} = 0.588$ to the next order ($m = 2$) in the same expansion.

With some additional work one can in fact completely determine the asymptotic behavior of various averages for small β (large $\boldsymbol{g}$) and large n. First one notes that when m Bessel functions are included in the expansion for the wave function given in Eq. (2.176), beyond the leading order one at strong coupling, one obtains a wave function which contains powers of β up to β^m. For a given fixed m one then finds for the average area per triangle the following asymptotic result

$$< A_\Delta > \;\sim\; \frac{1}{\boldsymbol{g}^{3m-1}\, n^{\frac{m+1}{2}}} \,, \tag{2.211}$$

up to terms which contain higher powers of $1/n$ (making these less relevant in the limit $n \to \infty$), and also up to terms which are less singular in $\boldsymbol{g}$ for small $\boldsymbol{g}$. The requirement that the average area per triangle be finite as $n \to \infty$ then requires that the coupling $\boldsymbol{g}$ itself should scale with n according to

$$\boldsymbol{g}(n) \;\sim\; \frac{1}{n^{\frac{m+1}{2(3m-1)}}} \,. \tag{2.212}$$

For the area fluctuation itself one then computes in the same limit

$$\chi_A \;\sim\; \frac{1}{\boldsymbol{g}^{3m-2}\, n^{\frac{m}{2}}} \,, \tag{2.213}$$

again to leading order in $1/n$ and $1/\boldsymbol{g}$. The requirement that $\boldsymbol{g}(n)$ scale according to Eq. (2.212) then implies from Eq. (2.213) that the area fluctuation diverges in the limit $n \to \infty$ as

$$\chi_A(n) \;\sim\; n^{\frac{m-1}{3m-1}} \,. \tag{2.214}$$

By comparing with Eqs. (2.202) and (2.203) one obtains immediately for the exponent

$$\frac{\alpha}{\nu} = \frac{2\,m-2}{3\,m-1} \,, \tag{2.215}$$

and therefore from the scaling relation $\alpha = 2 - 3\,\nu$ finally

$$\nu \, = \, \frac{6\,m - 2}{11\,m - 5} \, . \tag{2.216}$$

One can now take the limit $m \rightarrow \infty$ [infinite number of Bessel functions retained in the expansion of Eq. (2.176)], which in turn leads us to include all the orders of the Bessel functions as one can see from Eq. (2.176)

$$\nu \, = \, \frac{6}{11} \, = \, 0.5454 \ldots \, . \tag{2.217}$$

This therefore is the exact result for the correlation length exponent of $2+1$ dimensional quantum gravity. The derivation shows that *the exponent ν does not seem to depend on the Euler characteristic χ, and therefore on the boundary conditions.* [8] Furthermore one can compare the above value for ν with the (numerically exact) Euclidean 3 dimensional quantum gravity result obtained over twenty years ago in [51], namely $\nu \simeq 0.59(2)$. It would of course be of great interest to repeat the above Euclidean lattice calculation in order to refine the estimate and improve on the statistical and systematic uncertainty. The exponent ν is expected to represent a universal quantity, independent of short-distance regularization details, and therefore characteristic of gravity's universal scaling properties on distances much larger than the lattice cutoff. As such, it should apply equally to both the Lorentzian and the Euclidean formulation, and our results are consistent with this conclusion. Moreover, in $3+1$ dimensions the exponent ν is a key physical quantity as it determines the power for the running of the gravitational constant G [106, 107], and for the Euclidean theory it is known [46, 47] that the universal scaling exponent is consistent with $\nu = 1/3$.

It is perhaps worthwhile at this point to compare with other attempts at determining the critical exponent ν in 3 dimensional gravity. The latest and best results for quantum gravity in the perturbative diagrammatic $2+\epsilon$ continuum expansion using the background field

[8] One might wonder if the value for ν is affected by the choice of normalization in Eqs. (2.69) and (2.170). It is easy to check that at least the inclusion of a weight factor A^m, with m integer, does not change the result given in Eq. (2.217).

method [108, 23, 40, 48, 49] give in $d = 3$ ($\epsilon = 1$ and central charge $c = 1$)

$$\nu^{-1} = 1 + \frac{3}{5} + \ldots \tag{2.218}$$

to 2 - loop order, and therefore $\nu \approx 0.625$, with a substantial uncertainty of about 50% (which can be estimated for example by comparing the one- and 2 - loop results). On the other hand, truncated renormalization group calculations for gravity directly in 3 dimensions [109, 110, 50] give to lowest order in the truncation (*i.e.*, with the inclusion of the cosmological and Einstein-Hilbert terms only) the estimate

$$\nu^{-1} = \frac{2\,D\,(D-2)}{D+2} \tag{2.219}$$

and therefore in $D = 3$ the value $\nu \approx 0.833$. This last result is also affected by a rather substantial uncertainty (again as much as 50%), which can be estimated, for example, by including curvature squared terms in the truncated expansion. Nevertheless, and in light of the uncertainties associated with the various methods, it is very encouraging to note that widely different calculations (on the lattice and in the continuum) give values for the universal scaling exponent ν that are roughly in the same ballpark.

From Eq. (2.217) one obtains the fractal dimension for a gravitational path in 2+1 dimensions

$$\nu^{-1} \; = \; d_F \; = \; \frac{11}{6} \; = \; 1.8333... \tag{2.220}$$

This is slightly smaller than the value for a free scalar field $d_F = 2$, corresponding to the Brownian motion (or Wiener path) value. It is closer to the value expected for a dilute branched polymer in the same dimension [18, 111], and the best match at this point seems to be the $O(n)$ vector model for $n = -1$. The exact value $\nu = 6/11$ for $2 + 1$ gravity would then suggests a connection between the ground state properties of quantum gravity and the geometry of dilute branched polymers in the same dimension.

In light of the results obtained so far it is possible to make a number of additional observations. First note from Eq. (2.212) that as $n \to \infty$ the critical point (or renormalization group ultraviolet fixed point) moves to $\boldsymbol{g} = 0$

$$\boldsymbol{g}(n) \underset{m \to \infty}{\sim} \frac{1}{n^{1/6}} \, . \tag{2.221}$$

For comparison, a variational calculation based on correlated product (Slater-Jastrow) wave functions [81] in $2+1$ dimensions gave

$$\boldsymbol{g}_c^3 \; = \; \frac{4\,\pi\,\chi}{N_\Delta \sqrt{\sigma_0(\sigma_0 - 2)}} \, , \tag{2.222}$$

where $\sigma_0 > 2$ was a parameter associated there with the choice of functional measure over edges. The variational result of Eq. (2.222) can be compared directly with the result of Eqs. (2.180) and (2.221), for $\chi = 2$ and $N_\Delta = 2\,n + 2$. Thus in both treatments the limiting value for the critical point for $\boldsymbol{g}$ in $2+1$ dimensions is zero, $\boldsymbol{g}_c \to 0$ as the number of triangles $N_\Delta \to \infty$.

Physically, this last result implies that there is no weak coupling phase ($\boldsymbol{g} < \boldsymbol{g}_c$, or in terms of Newton's constant $G < G_c$): the only surviving phase for gravity in 3 dimensions is the strongly coupled one ($\boldsymbol{g} > \boldsymbol{g}_c$ or $G > G_c$). Furthermore, the correlation length ξ of Eq. (2.198) is finite for $\boldsymbol{g} > 0$, and diverges at $\boldsymbol{g} = 0$. In particular the weak field expansion, which assumes $\boldsymbol{g}$ small, is expected to have zero radius of convergence. [9] In a sense this is a welcome result, as in the Euclidean theory the weak coupling phase was found to be pathological and thus physically unacceptable in both 3 [51] and 4 dimensions [26, 28, 46, 47]. It would seem therefore that the Euclidean and Lorentzian lattice results are ultimately completely consistent: quantum gravity in $2+1$ dimensions always resides in the strong coupling, *i.e.,* gravitational antiscreening phase; the weak coupling, *i.e.,* gravitational screening phase is physically excluded. In addition, the exact value for ν determines, through standard renormalization group arguments, the scale dependence of the gravitational coupling in the vicinity of the ultraviolet fixed point [106, 107]. [10]

[9] These circumstances are perhaps unfamiliar in the gravity context, but are nevertheless rather similar to what happens in gauge theories, including compact Quantum Electrodynamics in $2+1$ dimensions [112]. There the theory always resides in the strong coupling or disordered phase, with a finite correlation length which eventually diverges at zero charge.

[10] Specifically, the universal exponent ν is related to the behavior of the Callan Symanzik β function for

2.5.12 Discussions

We discussed the form of the gravitational wave function which arises as a solution of the lattice Wheeler DeWitt equation [Eqs. (2.33),(2.34) and (2.71)] for finite lattices. The main result was the wave function Ψ given in Eqs. (2.148), (2.160) and (2.168) with strong coupling limit (curvature term absent) corresponding to the choice of parameter $\beta = 0$.

To summarize, and for the purpose of the following discussion, the wave function Ψ given in Eq. (2.168) can be written in the most general form as

$$\Psi \sim e^{-i\,x}\,{}_1F_1\,(a,\, b,\, 2\,i\,x) \tag{2.223}$$

up to an overall normalization constant $\tilde{N}$, and with parameters related to various geometric invariants

$$\begin{aligned} a &\equiv \frac{1}{4}\,N_\Delta - \frac{\sqrt{2}\,\pi\,i}{\sqrt{\lambda}\,\boldsymbol{G}}\,\chi \;=\; \frac{1}{4}\,N_\Delta - \frac{i}{2\sqrt{2}\,\sqrt{\lambda}\,\boldsymbol{G}}\int d^2y\,\sqrt{g}\,R \\ b &\equiv \frac{1}{2}\,N_\Delta \\ x &\equiv \sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\,A_{tot} \;=\; \sqrt{2}\,\frac{\sqrt{\lambda}}{\boldsymbol{G}}\int d^2y\,\sqrt{g}\,. \end{aligned} \tag{2.224}$$

In the above definitions one can trade, if one so desires, the total number of triangles N_Δ for the total area

$$N_\Delta \;=\; \frac{1}{< A_\Delta >}\,A_{tot} \;=\; \frac{1}{< A_\Delta >}\int d^2y\,\sqrt{g}\,. \tag{2.225}$$

Use has been made of the relationship between various coupling constants $(\boldsymbol{g}, \boldsymbol{G}, \beta, \lambda)$ to re-express the wave function ψ in slightly more general terms, as a function of the original couplings λ and $\boldsymbol{G}$ appearing in the original form of the Wheeler DeWitt equation [see for example Eq. (2.37)]. We did show that an equivalent form for the wave function Ψ can be

Newton's constant G in the vicinity of the ultraviolet fixed point by $\beta'(G)|_{G=G_c} = -1/\nu$. Integration of the renormalization group equations for G then determines the scale dependence of $G(\mu)$ or $G(\Box)$ in the vicinity of the ultraviolet fixed point. Concretely, ν determines the exponent in the running of G. One finds $G(\Box) \sim \left(\xi^2\Box\right)^{-1/2\nu}$, with $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ the covariant d'Alembertian and ξ the renormalization group invariant correlation length.

given in terms of Coulomb wave functions [see Eq. (2.160)], with argument

$$\beta \;\equiv\; \frac{\sqrt{2}\,\pi\,\chi}{\sqrt{\lambda}\,\boldsymbol{G}} \;=\; \frac{1}{2\sqrt{2}\,\sqrt{\lambda}\,\boldsymbol{G}} \int d^2y\,\sqrt{g}\;R \tag{2.226}$$

and x defined as in Eq. (2.224).

The above wave function is exact in the limit of large areas, and completely independent of the weak field expansion. Nevertheless it is only correct to some low order in the same expansion in the limit of small areas. We interpret the situation as follows: For large areas one has a very large number of triangles, and the short distance details of the lattice setup play a vanishingly small role in this limit. One recognizes this limit as being relevant for universal scaling properties, including critical exponents. For small areas on the other hand a certain sensitivity to the short distance properties of the lattice regularization persists, and thus a universal behavior is, not unexpectedly, hard to achieve. In any case this last limit, in the absence of a truly fundamental and explicit microscopic theory, is always expected to be affected by short distance details of the regularization, no matter what its ultimate nature might be (a lattice of some sort, dimensional regularization, or an invariant continuum momentum cutoff etc.)

In principle any well defined diffeomorphism invariant average can be computed using the above wave functions. This will involve at some point the evaluation of a vacuum expectation value of some operator $\tilde{\mathcal{O}}(g)$

$$\langle\Psi\big|\tilde{\mathcal{O}}(g)\big|\Psi\rangle \;=\; \frac{\int \mathcal{D}\mu[g]\;\;\tilde{\mathcal{O}}(g_{ij})\;\;\big|\Psi[g_{ij}]\big|^2}{\int \mathcal{D}\mu[g]\;\;\big|\Psi[g_{ij}]\big|^2} \tag{2.227}$$

where $\mathcal{D}\mu[g]$ is the appropriate functional measure over 3 - metrics g_{ij}. Evaluating such an average is in general nontrivial, as it requires the computation of an (Euclidean) lattice path integral in one dimensionless

$$\langle\Psi\big|\tilde{\mathcal{O}}(g)\big|\Psi\rangle \;=\; \mathcal{N}\int \mathcal{D}\mu[g]\;\tilde{\mathcal{O}}(g_{ij})\;\exp\left\{-S_{eff}[g]\right\} \tag{2.228}$$

with $S_{eff}[g] \equiv -\ln\big|\Psi[g_{ij}]\big|^2$ and $\mathcal{N}$ a normalization constant. The operator $\tilde{\mathcal{O}}(g)$ itself can be local, or nonlocal as in the case of the gravitational Wilson loop discussed in [42, 43]. Note that the statistical weights have many zeros corresponding to the nodes of the wave function Ψ, and that S_{eff} is infinite there.

In the previous subsections we have shown that the wave function allows one to calculate a number of useful and physically relevant averages and fluctuations, which were later extrapolated to the infinite volume limit of infinitely many triangles. It was found that these diffeomorphism invariant observables point in $2+1$ dimensions to the existence of a fixed point at the origin, $G_c = 0$. One concludes therefore that the weak coupling (gravitational screening) phase has completely disappeared in the lattice nonperturbative formulation, and that the theory resides in the strong coupling phase only. By contrast, in the Euclidean theory it was found in [51] that the weak coupling (gravitational screening) phase exists but is pathological, corresponding to a degenerate branched polymer. A similar set of results is found in the 4 dimensional Euclidean theory, where the weak coupling, gravitational screening, phase also describes a branched polymer. [11]

The calculations presented here and in[81] are consistent with the conclusions reached earlier from the Euclidean framework, and no new surprises occur when considering the Lorentzian $2+1$ theory. Furthermore, we have emphasized before that the results obtained point at a nonanalyticity in the coupling at $G = 0$, signaling a strong vacuum instability of quantum gravitation in this dimension. In view of these results it is therefore not surprising that calculations that rely on the weak field, semiclassical or small G expansion run into a serious trouble and uncontrollable divergences very early on. Such an expansion does not seem to exist, if the nonperturbative lattice results presented here are taken seriously. The correct physical vacuum apparently cannot in any way be obtained as a small perturbation of flat or near flat spacetime.

Furthermore, it is expected that the result obtained, in particular the universal critical expo-

[11] Branched polymer phase corresponds to dimensions ~ 2. If the spacetime manifold never reached dimension ~ 4, then it suggests that there is no physical continuum limits available in that phase, and therefore, it suggests that this phase is unphysical.

nent ν should not depend on the topology of the spacetime manifold that we are considering, which we demonstrated in our study with spherical and toric manifolds, and we have shown that the ν does not depend on the Euler characteristics, therefore the topology of the manifold. Usually in Euclidean treatment of lattice calculations, the topology is restricted to a torus, corresponding to periodic boundary conditions. However in general, one can perform calculation with lattices employing different boundary conditions or topology, but one expects that the universal scaling properties of the theory should be determined exclusively by the short distance renormalization effects. In fact the Feynman rules of perturbation theory do not depend in any way on boundary terms.

2.6 Closer to Reality : Discretized Wheeler DeWitt Equation in $3 + 1$ Dimensions

We have argued previously that the correct identification of the true ground state for quantum gravitation necessarily requires the introduction of a consistent nonperturbative cutoff, followed by the construction of the continuum limit in accordance with the methods of the renormalization group. To this day the only known way to introduce such a nonperturbative cutoff reliably in quantum field theory is via the lattice formulation. A wealth of results have been obtained over the years using the Euclidean lattice formulation, which allows the identification of the physical ground state and the accurate calculations of gravitational scaling dimensions, relevant for the scale dependence of Newton's constant in the universal scaling limit.

In this work we will focus instead on the Hamiltonian approach to gravity, which assumes from the beginning a metric with Lorentzian signature. In order to obtain useful insights regarding the nonperturbative ground state, a Hamiltonian lattice formulation was written down based on the Wheeler DeWitt equation, where the quantum gravity Hamiltonian is expressed in the metric space representation. In [81, 113] a general discrete Wheeler DeWitt equation was given for pure gravity, based on the simplicial lattice formulation of gravity developed by Regge and Wheeler. Here, we extend the work initiated in [81, 113] to the physical case of $3+1$ dimensions, and show how nonperturbative vacuum solutions to the lattice Wheeler DeWitt equations can be obtained for arbitrary values of Newton's constant G. The procedure we follow is similar to what was done earlier in $2+1$ dimensions. We solve the lattice equations exactly for several finite regular triangulations of the 3 - sphere, and then extend the result to an arbitrarily large number of tetrahedra. For large enough volumes the exact lattice wave functional is expected to depend on geometric quantities only, such as the total volumes and the total integrated curvature. The regularity condition on the solutions of the wave equation at small areas is then shown to play an essential role in constraining the form of the vacuum wave functional. A key ingredient in the derivation of the results is the local diffeomorphism invariance of the Regge Wheeler lattice formulation.

From the structure of the resulting wave function a number of suggestive physical results can be obtained, the first one of which is that the nonperturbative correlation length is found to be finite for sufficiently large G. At the critical point $G = G_c$, which we determine exactly from the structure of the wave function, fluctuations in the curvature become unbounded, thus signaling a divergence in the nonperturbative gravitational correlation length. Such a result can be viewed as consistent with the existence of a nontrivial ultraviolet fixed point (or a phase transition in statistical field theory language) in G. The behavior of the theory in the vicinity of such a fixed point is then expected to determine, through standard renormalization group arguments, the scale dependence of the gravitational coupling in the vicinity of the ultraviolet fixed point.

A short outline of this Section 2.6 is as follows. In Section 2.6.1, we make explicit set up and point out properties of the lattice Wheeler DeWitt equation by spelling out various quantities appearing in it. This section also emphasizes the important role of continuous lattice diffeomorphism invariance in the Regge framework, as it applies to the present case of $3+1$ dimensional gravity. Section 2.6.2 gives an outline of the general method of solution for the lattice equations, which are later in Section 2.6.3 and Section 2.6.4 -2.6.7 discussed in some detail for a number of regular triangulations of the 3 - sphere. Then a general form of the wave function is given that covers all the previous discrete cases, and subsequently allows a study of the infinite volume limit in Section 2.6.8. Section 2.6.9 discusses the issue of how to define an average volume and thus an average lattice spacing, an essential ingredient in the interpretation of the results given later. Section 2.6.10 discusses modifications of the wave function solution obtained when the explicit curvature term in the Wheeler DeWitt equation is added. Later a partial differential equation for the wave function is derived in the curvature and volume variables. General properties of the solution to this equation are discussed in Section 2.6.11 Section 2.5.11 contains a brief summary of the results obtained so far.

2.6.1 Explicit Setup for the Lattice Wheeler DeWitt Equation in $3+1$ dimensions

In the following we will now focus on a lattice made up of a large number of tetrahedra, with squared edge lengths considered as the fundamental degrees of freedom. For ease of notation, we define $l_{01}^2 = a$, $l_{12}^2 = b$, $l_{02}^2 = c$, $l_{03}^2 = d$, $l_{13}^2 = e$, $l_{23}^2 = f$. For the tetrahedron labeled as in Figure 2.2, we have

$$g_{11} \;=\; a\,, \quad g_{22} \;=\; c\,, \quad g_{33} \;=\; d\,, \tag{2.229}$$

$$g_{12} \;=\; \frac{1}{2}(a\,+\,c\,-\,b)\,, \quad g_{13} \;=\; \frac{1}{2}(a\,+\,d\,-\,e)\,, \quad g_{23} \;=\; \frac{1}{2}(c\,+\,d\,-\,f)\,, \tag{2.230}$$

and its volume V is given by

$$V^2 \;=\; \frac{1}{144}[\,af(-a-f+b+c+d+e)\;+\;bd(-b-d+a+c+e+f)\;+ \\ +\;\;ce(-c-e+a+b+d+f)\;-\;abc\;-\;ade\;-\;bef\;-\;cdf\,]\,. \qquad (2.231)$$

The matrix G^{ij} is then given by

$$G^{ij} \;=\; -\,\frac{1}{24V}\begin{pmatrix} -2f & e+f-b & b+f-e & d+f-c & c+f-d & p \\ e+f-b & -2e & b+e-f & d+e-a & q & a+e-d \\ b+f-e & b+e-f & -2b & r & b+c-a & a+b-c \\ d+f-c & d+e-a & r & -2d & c+d-f & a+d-e \\ c+f-d & q & b+c-a & c+d-f & -2c & a+c-b \\ p & a+e-d & a+b-c & a+d-e & a+c-b & -2a \end{pmatrix}, \qquad (2.232)$$

where the 3 quantities p, q and r are defined as

$$p=-2a-2f+b+c+d+e, \quad q=-2c-2e+a+b+d+f, \quad r=-2b-2d+a+c+e+f\,. \qquad (2.233)$$

To obtain G_{ij} one can then either invert the above expression, or evaluate

$$G_{ij,kl} \;=\; \frac{1}{2\sqrt{g}}(g_{ik}\,g_{jl}\;+\;g_{il}\,g_{jk}\;-\;g_{ij}\,g_{kl}), \qquad (2.234)$$

and later replace derivatives with respect to the metric by derivatives with respect to the squared edge lengths, as in $\frac{\partial}{\partial g_{11}} = \frac{\partial}{\partial a} + \frac{\partial}{\partial b} + \frac{\partial}{\partial e}$ etc. One finds [81] that the matrix representing the coefficients of the derivatives with respect to the squared edge lengths is the same as the inverse of G^{ij}, a result that provides a nontrivial confirmation of the correctness of the Lund Regge result of Eq. (2.31) Then in 3+1 dimensions, the discrete Wheeler DeWitt equation is

$$\left\{-\,\boldsymbol{G}^2\,G_{ij}\frac{\partial^2}{\partial s_i \partial s_j}\;-\;2\,n_{\sigma h}\sum_h \sqrt{s_h}\,\delta_h\;+\;2\lambda V\right\}\Psi[\,s\,]\;=\;0\,, \qquad (2.235)$$

where the sum is over hinges (edges) h in the tetrahedron. Note the mild nonlocality of the equation in that the curvature term, through the deficit angles, involves edge lengths from

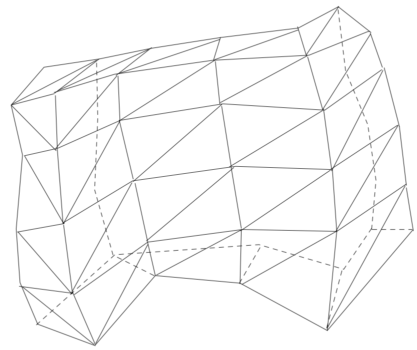

Figure 2.13: A small section of a suitable spatial lattice for quantum gravity in $3+1$ dimensions.

neighboring tetrahedra. In the continuum, the derivatives also give some mild nonlocality. Figure 2.13 gives a pictorial representation of lattices that can be used for numerical studies of quantum gravity in $3+1$ dimensions.

In the following we will be concerned at some point with various discrete, but generally regular, triangulations of the 3 - sphere [114, 115]. These were already studied in some detail within the framework of the Regge theory in [26, 28], where in particular the 5 cell α_4, the 16 cell β_4 and the 600 cell regular polytopes (as well as a few others) were considered in some detail. For a very early application of these regular triangulations to general relativity see [116].

We shall not dwell here on a well-known key aspect of the Regge-Wheeler theory, which is the presence of a continuous, local lattice diffeomorphism invariance, whose main aspects in regards to its relevance for the $3+1$ formulation of gravity were already addressed in some detail in various works, both in the framework of the lattice weak field expansion [73, 74, 81], and beyond it [75, 96] Here we will limit ourselves to some brief remarks on how this local invariance manifests itself in the $3+1$ formulation, and, in particular, in the case of the discrete triangulations of the sphere studied later on. In general, lattice diffeomorphisms in the Regge Wheeler theory correspond to local deformations of the edge lengths about a vertex, which leave the local geometry physically unchanged, the latter being described by the values of local lattice operators corresponding to local volumes and curvatures [73, 74, 75, 96]. The case of flat space (curvature locally equal to zero) or near flat space (curvature locally small) is obviously the simplest to analyze [96]: by moving the location of the vertices around on a smooth manifold one can find different assignments of edge lengths representing locally the same flat, or near-flat, geometry. Then it is easy to show that one obtains a $d \cdot N_0$ parameter family of local transformations for the edge lengths, as expected for lattice diffeomorphisms. For the present case, the relevant lattice diffeomorphisms are the ones that apply to the 3 dimensional, spatial theory. The reader is referred to [51] and, more recently, [81] for their explicit form within the framework of the lattice weak field expansion.

With these observations in mind, we can now turn to a discussion of the solution method for the lattice Wheeler DeWitt equation in $3+1$ dimensions. One item that needs to be brought up at this point is the proper normalization of various terms (kinetic, cosmological and curvature) appearing in the lattice equation of Eqs. (2.34) and (2.235). For the lattice gravity action in d dimensions one has generally the following correspondence

$$\int d^d x \sqrt{g} \quad \longleftrightarrow \quad \sum_\sigma V_\sigma \; , \tag{2.236}$$

where V_σ is the volume of a simplex; in 2 dimensions it is simply the area of a triangle. The curvature term involves deficit angles in the discrete case,

$$\frac{1}{2} \int d^d x \sqrt{g} \, R \quad \longleftrightarrow \quad \sum_h V_h \, \delta_h \; , \tag{2.237}$$

where δ_h is the deficit angle at the hinge h, and V_h the associated "volume of a hinge" [91]. In 4 dimensions the latter is the area of a triangle (usually denoted by V_h), whereas in 3 dimensions it is simply given by the length l_h of the edge labeled by h. In this work we will focus almost exclusively on the case of $3+1$ dimensions; consequently the relevant formulas will be Eqs. (2.236) and (2.237) for dimension $d = 3$.

The continuum Wheeler DeWitt equation is local, as can be seen from Eq. (2.26). One can integrate the Wheeler DeWitt operator over all space and obtain

$$\left\{ - \boldsymbol{G}^2 \int d^3x \, \boldsymbol{\Delta}(g) + 2\lambda \int d^3x \, \sqrt{g} - \int d^3x \, \sqrt{g} \, R \right\} \Psi = 0 \; , \tag{2.238}$$

with the super Laplacian on metrics defined as

$$\boldsymbol{\Delta}(g) \;\equiv\; G_{ij,kl}(\mathbf{x}) \, \frac{\delta^2}{\delta g_{ij}(\mathbf{x}) \, \delta g_{kl}(\mathbf{x})} \; . \tag{2.239}$$

In the discrete case one has one local Wheeler DeWitt equation for *each* tetrahedron [see Eqs. (2.33) and (2.34)], which therefore takes the form

$$\left\{ - \boldsymbol{G}^2 \, \boldsymbol{\Delta}(l^2) - \, \kappa \sum_{h \subset \sigma} \delta_h \, l_h + 2 \, \lambda \, V_\sigma \right\} \Psi = 0 \; , \tag{2.240}$$

where $\boldsymbol{\Delta}(l^2)$ is the lattice version of the super Laplacian, and we have set for convenience $\kappa = 2 \, n_{\sigma \, h}$. As we shall see below, for a regular lattice of fixed coordination number, κ is a constant and does not depend on the location on the lattice. In the above expression $\boldsymbol{\Delta}(l^2)$ is a discretized form of the covariant super Laplacian, acting locally on the space of $s = l^2$ variables,

$$\boldsymbol{\Delta}(l^2) \;\equiv\; G_{ij} \, \frac{\partial^2}{\partial s_i \partial s_j} \; , \tag{2.241}$$

with the matrix G^{ij} given explicitly in Eq. (2.232). Note that the curvature term involves six deficit angles δ_h, associated with the six edges of a tetrahedron. Now, Eq. (2.235) applies to a single given tetrahedron (labeled here by σ), with one equation to be satisfied at each tetrahedron on the lattice. But one can also construct the total Hamiltonian by simply summing over all tetrahedra, which leads to

$$\left\{ - \boldsymbol{G}^2 \sum_\sigma \boldsymbol{\Delta}(l^2) \, + \, 2 \, \lambda \sum_\sigma V_\sigma \, - \, \kappa \, \sum_\sigma \sum_{h \subset \sigma} l_h \, \delta_h \right\} \Psi = 0 \, . \tag{2.242}$$

Summing over all tetrahedra (σ) is different from summing over all hinges (h), and the above equation is equivalent to

$$\left\{ - \boldsymbol{G}^2 \sum_\sigma \boldsymbol{\Delta}(l^2) \, + \, 2 \, \lambda \sum_\sigma V_\sigma \, - \, \kappa \, q \, \sum_h l_h \, \delta_h \right\} \Psi = 0 \; , \tag{2.243}$$

where q here is the lattice coordination number. The latter is determined by how the lattice is put together (which vertices are neighbors to each other, or, equivalently, by the so called incidence matrix). Here, q is the number of neighboring simplexes that share a given hinge (edge). For a flat triangular lattice in 2 dimensional $q = 6$, whereas for the regular triangulations of S^3 we will be considering below one has $q = 3,\, 4,\, 5$; for more general, irregular triangulations q might change locally throughout the lattice. For proper normalization in Eq. (2.242) one requires the 3 dimensional version of Eqs. (2.236) and (2.237), which fixes the overall normalization of the curvature term

$$\kappa \;\equiv\; 2\, n_{\sigma\, h} \;=\; \frac{2}{q} \;, \tag{2.244}$$

thus determining the relative weight of the local volume and curvature terms.

2.6.2 Outline of the General Method of Solution in 3+1 dimensions

Here the general procedure for finding a solution will be rather similar to what was done in $2+1$ dimensions, as the formal issues in obtaining a solution are not dramatically different. First an exact solution is found for *equilateral* edge lengths s. Later this solution is extended to determine whether it is consistent to higher order in the weak field expansion, where one writes for the squared edge lengths the expansion

$$l_{ij}^2 \;=\; s\,(1 \,+\, \epsilon\, h_{ij}) \;\;, \tag{2.245}$$

with ϵ a small expansion parameter. The resulting solution for the wave function can then be obtained as a suitable power series in the h variables, combined with the standard Frobenius method, appropriate for the study of quantum mechanical wave equations for suitably well-behaved potentials. In this method one first determines the correct asymptotic behavior of the solution for small and large arguments, and later constructs a full solution by writing the remainder as a power series or polynomial in the relevant variable. While this last method is rather time consuming, we have found that in some cases (such as the single triangle in

$2+1$ dimensions and the single tetrahedron in $3+1$ dimensions, described in [81] and also below), one is lucky enough to find immediately an exact solution, without having to rely in any way on the weak field expansion.

More importantly, in [113] it was found that already in $2+1$ dimensions this rather laborious weak field expansion of the solution is not really necessary, for the following reason. Diffeomorphism invariance (on the lattice and in the continuum) of the theory severely restricts the form of the Wheeler DeWitt wave function to a function of invariants only, such as total 3 volumes and curvatures, or powers thereof. In other words, the wave function is found to be a function of invariants such as $\int d^d x \sqrt{g}$ or $\int d^d x \sqrt{g}\, R^n$ etc. (These will be listed in more detail below for the specific case of $3+1$ dimensions, where one has $d=3$ in the above expressions).

For concreteness and computational expedience, in the following we will look at a variety of 3 dimensional simplicial lattices, including regular triangulations of the 3 - sphere S^3 constructed as convex 4 polytopes, the latter describing closed and connected figures composed of lower dimensional simplices. Here these will include the 5-cell 4-simplex or hypertetrahedron (Schläfli symbol $\{3,3,3\}$) with 5 vertices, 10 edges and 5 tetrahedra; the 16 cell hyperoctahedron (Schläfli symbol $\{3,3,4\}$) with 8 vertices, 24 edges and 16 tetrahedra; and the 600 cell hypericosahedron (Schläfli symbol $\{3,3,5\}$) with 120 vertices, 720 edges and 600 tetrahedra [114, 115]. Note that the Euler characteristic for all 4 polytopes that are topological 3 - spheres is zero, $\chi = N_0 - N_1 + N_2 - N_3 = 0$, where N_d is the number of simplices of dimension d. We also note here that there are no known regular equilateral triangulations of the flat 3 - torus in 3 dimensions, although very useful slightly irregular (but periodic) triangulations are easily constructed by subdividing cubes on a square lattice into tetrahedra [51].

In the following we will also recognize that there are natural sets of variables for displaying the results. One of them is the scaled total volume x, defined as

$$x \;\equiv\; \frac{4\sqrt{2}\,\sqrt{\lambda}}{q\,\boldsymbol{G}} \sum_{\sigma} V_{\sigma} \;=\; \frac{4\sqrt{2}\,\sqrt{\lambda}}{q\,\boldsymbol{G}} V_{tot}\,. \tag{2.246}$$

Later on we will be interested in making contact with continuum manifolds, by taking the infinite volume (or thermodynamic) limit, defined in the usual way as

$$\begin{aligned} N_{\sigma} &\;\to\; \infty\,, \\ V_{tot} &\;\to\; \infty\,, \\ \frac{V_{tot}}{N_{\sigma}} &\;\to\; \text{const.}\,, \end{aligned} \tag{2.247}$$

with $N_{\sigma} \equiv N_3$ here the total number of tetrahedra. It should be clear that this last ratio can be used to define a fundamental lattice spacing a_0, for example via $V_{tot}/N_{\sigma} \equiv V_{\sigma} = a_0^3/\left(6\sqrt{2}\right)$.

The full solution of the quantum mechanical problem will, in general, require that the wave functions be properly normalized, as in Eq. (2.25). This will introduce at some stage wave function normalization factors $\mathcal{N}$, which will later be fixed by the standard rules of quantum mechanics. If the wave function were to depend on the total volume V_{tot} only (which is the case in $2+1$ dimensions, but not in $3+1$), then the relevant requirement would simply be

$$\|\Psi\|^2 \;\equiv\; \int \mathcal{D}\mu[g] \cdot |\,\Psi[g_{ij}]\,|^2 = \int_0^{\infty} dV_{tot} \cdot V_{tot}^m \cdot |\,\Psi(V_{tot})\,|^2 = \;1\,, \tag{2.248}$$

where $\mathcal{D}\mu[g]$ is the appropriate functional measure over the 3 - metric g_{ij}, and m a positive real number representing the correct entropy weighting. But, not unexpectedly, in 3+1 dimensions the total curvature also plays a role, so the above can only be regarded as roughly correct in the strong coupling limit (large G), where the curvature contribution to the Wheeler DeWitt equation can safely be neglected. As in nonrelativistic quantum mechanics, the normalization condition in Eqs. (2.25) and (2.248) plays a crucial role in selecting out of the two solutions the one that is regular, and therefore satisfies the required wave function normalizability condition.

To proceed further, it will be necessary to discuss each lattice separately in some detail. For each lattice geometry, we will break down the presentation into two separate discussions. The first part will deal with the case of no explicit curvature term in the Wheeler DeWitt equation. Each regular triangulation of the 3 - sphere will be first analyzed separately, and subjected to the required regularity conditions. Here a solution is first obtained in the equilateral case, and later promoted on the basis of lattice diffeomorphism invariance to the case of arbitrary edge lengths, as was done in [113]. Later a single general solution will be written down, involving the parameter q, which covers all previous triangulation cases, and thereby allows a first study of the infinite volume limit. The second part deals with the extension of the previous solutions to the case when the curvature term in the Wheeler DeWitt equation is included. This case is more challenging to treat analytically, and the only results we have obtained so far deal with the large volume limit, for which the solution is nevertheless expected to be exact (as was the case in $2+1$ dimensions [113]).

2.6.3 Nature of Solutions in $3+1$ Dimensions

In this work we will be concerned with the solution of the Wheeler DeWitt equation for discrete triangulations of the 3 - sphere S^3. In general, for an arbitrary triangulation of a smooth closed manifold in 3 dimensions, one can write down the Euler equation

$$N_0 - N_1 + N_2 - N_3 = 0 \tag{2.249}$$

and the Dehn Sommerville relation

$$N_2 = 2\, N_3 \, . \tag{2.250}$$

The latter follows from the fact that each triangle is shared by two tetrahedra and each tetrahedron has 4 triangles, thus $2\, N_2 = 4\, N_3$. In addition, for the regular triangulations of the 3 - sphere we will be considering here, one has the additional identity

$$N_1 = \frac{6}{q}\, N_3 \; , \tag{2.251}$$

where q is the local coordination number, defined as the number of tetrahedra meeting at an edge. For the 3 regular triangulations of the 3 - sphere we will look at one has $q = 3, 4, 5$. The above relations then allow us to relate the number of sites (N_0) to the number of tetrahedra (N_3),

$$N_0 = N_3 \left(\frac{6}{q} - 1 \right) . \tag{2.252}$$

It will also turn out to be convenient to collect here a number of useful definitions, results and identities that apply to the regular triangulations of the 3 - sphere, valid strictly when all edge lengths take on the same identical value $l = \sqrt{s}$. For the total volume

$$V_{tot} \equiv \sum_\sigma V_\sigma \quad \longleftrightarrow \quad \int d^3x \sqrt{g} \tag{2.253}$$

one has

$$V_{tot} \; = \; N_3 \, V_\sigma = \frac{s^{3/2}}{6\sqrt{2}} \, N_3 \; , \tag{2.254}$$

whereas the total curvature

$$R_{tot} \; \equiv \; 2 \sum_h \delta_h \, l_h \quad \longleftrightarrow \quad \int d^3x \sqrt{g} \, R \tag{2.255}$$

is given by

$$R_{tot} = \frac{12 \sqrt{s}}{q} \left[2\,\pi - q \, \cos^{-1} \left(\frac{1}{3} \right) \right] \; N_3 \, . \tag{2.256}$$

The latter relationship can be inverted to give the parameter q as a function of the curvature

$$q \; = \; q_0 \left(1 - \frac{R_{tot}}{R_{tot} + \frac{24\,\pi\,\sqrt{s}}{q_0} \; N_3} \right) \; , \tag{2.257}$$

and its inverse as

$$q^{-1} \; = \; q_0^{-1} + \frac{R_{tot}}{24\,\pi \sqrt{s} \; N_3} \; , \tag{2.258}$$

so that this last quantity is just linear in R_{tot}. A very special value for q corresponds to the choice $q = q_0$ for which $R_{tot} = 0$. For this case one has

$$q_0 \; \equiv \; \frac{2\,\pi}{\cos^{-1}(\frac{1}{3})} = 5.1043 \, . \tag{2.259}$$

Then, summarizing all the previous discussions, the discretized Wheeler DeWitt equation one wants to solve here is

$$\left\{ - \boldsymbol{G}^2 \sum_{i,j \subset \sigma} G_{ij}\,(\sigma) \, \frac{\partial^2}{\partial l_i^2 \, \partial l_j^2} \; - \; \kappa \sum_{h \subset \sigma} l_h \, \delta_h \; + \; 2 \lambda \, V_\sigma \right\} \psi[\, l^2 \,] \; = \; 0 \;\; , \tag{2.260}$$

with parameter κ given by

$$\kappa = \frac{2}{q} \ . \tag{2.261}$$

If the reader is not interested in the details of the solution for each individual lattice, then (s)he can skip the following sections and move on directly to Sec. (2.6.8).

2.6.4 1-cell Complex (Single Tetrahedron)

As a first case we consider here the quantum mechanical problem of a single tetrahedron. One has $N_0 = 4$, $N_1 = 6$, $N_2 = 4$, $N_3 = 1$ and $q = 1$ [note that these do not satisfy the Euler and Dehn Sommerville relations; only the relation between N_1, N_3, and q, Eq. (2.252), is satisfied for a single tetrahedron]. The single tetrahedron problem is relevant for the strong coupling (large G) limit. In this limit one can neglect the curvature term, which couples different tetrahedra to each other, and one is left with the local degrees of freedom, involving a single tetrahedron.

The Wheeler DeWitt equation for a single tetrahedron with a constant curvature density term R reads

$$\left\{ - \boldsymbol{G}^2 \, G_{ij} \frac{\partial^2}{\partial s_i \partial s_j} \; + \; (2\lambda - R) \, V \right\} \Psi[\, s \,] \; = \; 0 \; , \tag{2.262}$$

where now the squared edge lengths $s_1 \dots s_6$ are all part of the same tetrahedron, and G_{ij} is given by a rather complicated, but explicit, 6×6 matrix given earlier.

As in the $2+1$ case previously discussed in [113], here too it is found that, when acting on functions of the tetrahedron volume, the Laplacian term still returns some other function of the volume only, which makes it possible to readily obtain a full solution for the wave function. In terms of the volume of the tetrahedron V_σ one has the equivalent equation for $\Psi[s] = \Psi(V_\sigma)$

$$\psi''(V_\sigma) + \frac{7}{V_\sigma} \psi'(V_\sigma) + \frac{32 \, \lambda}{\boldsymbol{G}^2} \psi(V_\sigma) = 0 \; , \tag{2.263}$$

with primes indicating derivatives with respect to V_σ. From now on we will set the constant curvature density R=0. If one introduces the dimensionless (scaled volume) variable

$$x \;\equiv\; \frac{4\sqrt{2\,\lambda}}{\boldsymbol{G}}\, V_{tot} \;, \tag{2.264}$$

where $V_{tot} \equiv V_\sigma$ is the volume of the tetrahedron, then the differential equation for a single tetrahedron becomes simply

$$\psi''\,(x) + \frac{7}{x}\,\psi'\,(x) + \psi\,(x) \;=\; 0\,. \tag{2.265}$$

Solutions to Eqs. (2.263) or (2.265) are Bessel functions J_m or Y_m with $m = 3$,

$$\psi_R(V_{tot}) \;=\; \text{const.}\; J_3\left(\frac{4\sqrt{2\lambda}}{\boldsymbol{G}} V_{tot}\right) / V_{tot}^3 \;, \tag{2.266}$$

or

$$\psi_S(V_{tot}) \;=\; \text{const.}\; Y_3\left(\frac{4\sqrt{2\lambda}}{\boldsymbol{G}} V_{tot}\right) / V_{tot}^3 \,. \tag{2.267}$$

Only $J_m(x)$ is regular as $x \to 0$, $J_m(x) \sim \Gamma(m+1)^{-1}(x/2)^m$. In terms of the variable x the regular solution is therefore

$$\psi\,(V_{tot}) \;\propto\; \frac{J_3\,(x)}{x^3} \;\propto\; \frac{J_3\left(\frac{4\sqrt{2\lambda}}{\boldsymbol{G}}\, V_{tot}\right)}{{V_{tot}}^3} \;, \tag{2.268}$$

and the only physically acceptable wave function is

$$\Psi(a, b, \ldots f) \;=\; \Psi(V_{tot}) \;=\; \mathcal{N}\; \frac{J_3\left(\frac{4\sqrt{2\lambda}}{\boldsymbol{G}} V_{tot}\right)}{V_{tot}^3} \;, \tag{2.269}$$

with normalization constant

$$\mathcal{N} \;=\; \frac{45\sqrt{77\pi}}{1024\, 2^{3/4}} \; \left(\frac{\boldsymbol{G}}{\sqrt{\lambda}}\right)^{5/2} \,. \tag{2.270}$$

The latter is obtained from the wave function normalization requirement

$$\int_0^\infty dV_{tot} \; | \, \Psi(V_{tot}) \, |^2 \;=\; 1 \,. \tag{2.271}$$

Note that the solution given in Eq. (2.268) is exact, and a function of the volume of the tetrahedron only; its only dependence on the values of the edge lengths of the tetrahedron [or, equivalently, on the metric, see Eq. (2.28)] is through the *total* volume.

One can compute the average volume of the single tetrahedron, and it is given by

$$\langle\, V_{tot}\,\rangle \;\equiv\; \int_0^\infty dV_{tot}\cdot V_{tot}\cdot |\,\Psi(V_{tot})\,|^2 \;=\; \frac{31185\,\pi\,\boldsymbol{G}}{262144\,\sqrt{2\,\lambda}} \;=\; 0.2643\,\frac{\boldsymbol{G}}{\sqrt{\lambda}}\,. \tag{2.272}$$

This last result allows us to define an average lattice spacing, by comparing it to the value for an equilateral tetrahedron for which $V_{tot} = (1/6\sqrt{2})\,a_0^3$. One obtains

$$a_0 \;=\; 1.3089\,\left(\frac{\boldsymbol{G}}{\sqrt{\lambda}}\right)^{1/3}\,. \tag{2.273}$$

With the notation of Eq. (2.38) one has as well $\boldsymbol{G}/\sqrt{\lambda} = \sqrt{2}\,\boldsymbol{G} = \sqrt{2}\,\boldsymbol{g}$. Then for a single tetrahedron one has $\langle\, V_{tot}\,\rangle \;\equiv \langle\, V_\sigma\,\rangle \;=\; 0.3738\,\boldsymbol{g}$.

The single tetrahedron problem is clearly quite relevant for the limit of strong gravitational coupling, $1/G \to 0$. In this limit lattice quantum gravity has a finite correlation length, comparable to one lattice spacing,

$$\xi \;\sim a_0\,. \tag{2.274}$$

This last result is seen here simply as a reflection of the fact that for large G the edge lengths, and therefore the metric, fluctuate more or less independently in different spatial regions, due to the absence of the curvature term in the Wheeler DeWitt equation. This is of course true also in the Euclidean lattice theory, in the same limit [26, 28]. It is the inclusion of the curvature term that later leads to a coupling between fluctuations in different spatial regions, an essential ingredient of the full theory.

2.6.5 5-cell Complex (Configuration of 5 Tetrahedra)

The first regular triangulation of S^3 we will consider is the 5 cell complex, sometimes referred to as the hypertetrahedron. Here one has $N_0 = 5$, $N_1 = 10$, $N_2 = 10$, $N_3 = 5$ and $q = 3$, since there are 3 tetrahedra meeting on each edge. Then for the parameter κ appearing in Eq. (2.260) one has

$$\kappa \;=\; \frac{2}{3}\,. \tag{2.275}$$

First we will consider the case of no curvature term in the lattice Wheeler DeWitt equation of Eq. (2.260). The curvature term will be re-introduced at a later stage [see Sec. (2.6.10)], as its presence considerably complicates the solution of the lattice equations.

Solving the lattice equations directly (by brute force, one might say) in terms of the edge length variables is a rather difficult task, since many edge lengths are involved, increasingly more so for finer triangulations. Nevertheless it can be done, to some extent, in $2+1$ dimensions [113], and possibly even in $3+1$ dimensions, analytically for some special cases, or numerically for more general cases. To obtain a full solution to the lattice equations we rely here instead on a simpler procedure, already employed successfully (and checked explicitly) in $2+1$ dimensions. First, an exact wave function solution to the lattice Wheeler DeWitt equations is obtained for the equilateral case, where all edges in the simplicial complex are assumed to have the same length. Then, in the next step, the diffeomorphism invariance of the simplicial lattice theory is used to promote the previously obtained expression for the wave function to its unique general coordinate invariant form, involving various geometric volume and curvature terms. It is a nontrivial consequence of the invariance properties of the theory that such an invariant expression can be obtained, without any further ambiguity. In a number of instances such a procedure can be checked explicitly and systematically within the framework of the weak field expansion, and used to show that the form of the relevant wave function solution is indeed, as expected, strongly constrained by diffeomorphism invariance [113].

In the case of the 5-cell complex, and for now without an explicit curvature term in the Wheeler DeWitt equation, one obtains the following differential equation

$$\psi''(V_{tot}) + \frac{95}{9\,V_{tot}}\,\psi'(V_{tot}) + \frac{32\,\lambda}{9\,\boldsymbol{G}^2}\,\psi(V_{tot}) \;=\; 0 \tag{2.276}$$

for a wave function that, for now, depends only on the total volume, $\psi = \psi(V_{tot})$. To obtain this result, it is assumed at first that the simplicial complex is built out of equilateral tetrahedra; in accordance with the previous discussion, this constraint will be removed below.

In terms of the dimensionless variable x defined as

$$x \equiv \frac{4\sqrt{2\lambda}}{3\,\boldsymbol{G}}\,V_{tot} \tag{2.277}$$

one has the equivalent form for Eq. (2.276)

$$\psi''(x) + \frac{95}{9\,x}\,\psi'(x) + \psi(x) \,=\, 0\,. \tag{2.278}$$

This last equation can then be solved immediately, and the solution is

$$\psi(V_{tot}) \,\propto\, \frac{J_{\frac{43}{9}}(x)}{x^{\frac{43}{9}}} \,\propto\, \frac{J_{\frac{43}{9}}\left(\frac{4\sqrt{2\,\lambda}}{3\,\boldsymbol{G}}\,V_{tot}\right)}{V_{tot}{}^{\frac{43}{9}}}\,, \tag{2.279}$$

up to an overall wave function normalization constant. As in the previously discussed tetrahedron case, and also as in $2+1$ dimensions, one discards the Bessel function of the second kind (Y) solution, since it is singular at the origin.

2.6.6 16-cell Complex (Configuration of 16 Tetrahedra)

The next regular triangulation of S^3 we will consider is the 16-cell complex, sometimes referred to as the hyperoctahedron. One has in this case $N_0 = 8$, $N_1 = 24$, $N_2 = 32$, $N_3 = 16$ and $q = 4$, since there are 4 tetrahedra meeting on each edge. For the parameter κ in Eq. (2.260) one has

$$\kappa \,=\, \frac{2}{4}\,. \tag{2.280}$$

In the case of the 16-cell complex (again for now without an explicit curvature term in the Wheeler DeWitt equation) one obtains the following differential equation

$$\psi''(V_{tot}) + \frac{47}{2\,V_{tot}}\,\psi'(V_{tot}) + \frac{2\,\lambda}{\boldsymbol{G}^2}\,\psi(V_{tot}) = 0 \tag{2.281}$$

for a wave function that depends only on the total volume, $\psi = \psi(V_{tot})$. In terms of the variable

$$x \,\equiv\, \frac{\sqrt{2\,\lambda}}{\boldsymbol{G}}\,V_{tot} \tag{2.282}$$

one has an equivalent form for Eq. (2.281)

$$\psi''(x) + \frac{47}{2\,x}\,\psi'(x) + \psi(x) \;=\; 0\,. \tag{2.283}$$

The correct wave function solution is now

$$\psi(V_{tot}) \;\propto\; \frac{J_{\frac{45}{4}}(x)}{x^{\frac{45}{4}}} \;\propto\; \frac{J_{\frac{45}{4}}\left(\frac{\sqrt{2\,\lambda}}{\boldsymbol{G}}\,V_{tot}\right)}{V_{tot}^{\;\frac{45}{4}}}\,, \tag{2.284}$$

up to an overall wave function normalization constant. Again, we discarded the Bessel function of the second kind (Y) solution, since it is singular at the origin.

2.6.7 600-cell Complex (Configuration of 600 Tetrahedra)

The last, and densest, regular triangulation of S^3 we will consider here is the 600 cell complex, often called the hypericosahedron. For this lattice one has $N_0 = 120$, $N_1 = 720$, $N_2 = 1200$, $N_3 = 600$ and $q = 5$, since there are now five tetrahedra meeting at each edge. For the parameter κ in Eq. (2.260) one has

$$\kappa \;=\; \frac{2}{5}\,. \tag{2.285}$$

For this 600 cell complex (again for now without an explicit curvature term in the Wheeler DeWitt equation) one obtains the following differential equation

$$\psi''(V_{tot}) + \frac{672}{V_{tot}}\,\psi'(V_{tot}) + \frac{32\,\lambda}{25\,\boldsymbol{G}^2}\,\psi(V_{tot}) \;=\; 0 \tag{2.286}$$

for a wave function that depends only on the total volume, $\psi = \psi(V_{tot})$. In terms of the variable

$$x \;\equiv\; \frac{4\sqrt{2\,\lambda}}{5\,\boldsymbol{G}}\,V_{tot} \tag{2.287}$$

one has an equivalent form for Eq. (2.286)

$$\psi''(x) + \frac{672}{x}\,\psi'(x) + \psi(x) \;=\; 0\,. \tag{2.288}$$

Then the solution of the Wheeler DeWitt equation without a curvature term is

$$\psi\left(V_{tot}\right) \;\propto\; \frac{J_{\frac{671}{2}}\left(x\right)}{x^{\frac{671}{2}}} \;\propto\; \frac{J_{\frac{671}{2}}\left(\frac{4\sqrt{2}\,\lambda}{5\,\boldsymbol{G}}\,V_{tot}\right)}{V_{tot}{}^{\frac{671}{2}}} \;, \tag{2.289}$$

again up to an overall wave function normalization constant. As in previous cases, we discard the Bessel function of the second kind (Y) solution, since it is singular at the origin.

2.6.8 Generalized Solution for Zero Explicit Curvature

In this section we summarize and extend the previous results for the wave functions, obtained so far for the 3 separate cases of the 5 cell, 16 cell and 600 cell triangulation of the 3 - sphere S^3. The single tetrahedron case is somewhat special (it cannot contain a curvature term), and will be left aside for the moment. Also, all the previous results so far apply to the case of no explicit curvature term in the Wheeler DeWitt equation of Eq. (2.260); the inclusion of the curvature term will be discussed later. Consequently the following discussion still focuses on the strong coupling limit, $G \to \infty$.

For the following discussion the relevant Wheeler DeWitt equation is the one in Eq. (2.260),

$$\left\{ -\,\boldsymbol{G}^2 \sum_{i,j \subset \sigma} G_{ij}\left(\sigma\right) \frac{\partial^2}{\partial l_i^2\, \partial l_j^2} \;-\; \kappa \sum_{h \subset \sigma} l_h\, \delta_h \;+\; 2\lambda\, V_\sigma \right\} \psi[\, l^2\,] \;=\; 0 \;\;, \tag{2.290}$$

which depends on the parameter

$$\kappa = \frac{2}{q} \;, \tag{2.291}$$

where q represents the number of tetrahedra meeting at an edge. The above equation is quite general and not approximate in any way. Nevertheless it depends on the local lattice coordination number q (how the edges are connected to each other, or, in other words, on the incidence matrix).

Now, all previous differential equations for the wave function as a function of the total volume

V_{tot} [Eqs. (2.276), (2.281) and (2.286)] can be summarized as a single equation

$$\psi''(V_{tot}) + \frac{(11+9\,q)}{2\,q^2}\,\frac{N_3}{V_{tot}}\,\psi'(V_{tot}) + \frac{32}{q^2}\,\frac{\lambda}{\boldsymbol{G}^2}\,\psi(V_{tot}) = 0 \ . \tag{2.292}$$

Equivalently, in terms of the scaled area variable defined as

$$x \equiv \frac{4\sqrt{2\,\lambda}}{q\,\boldsymbol{G}}\,V_{tot} \ , \tag{2.293}$$

one can summarize the results of Eqs. (2.278), (2.283) and (2.288) through the single equation

$$\psi''(x) + \frac{(11+9\,q)}{2\,q^2}\,\frac{N_3}{x}\,\psi'(x) + \psi(x) \ = \ 0\,. \tag{2.294}$$

It will be convenient here to define the (Bessel function) index n as

$$n \equiv \frac{11+9\,q}{4\,q^2}\,N_3 \ - \ \frac{1}{2}\,, \tag{2.295}$$

so that for the 5-cell, 16-cell and 600-cell one has

$$\begin{aligned} 2\,n+1 \ &= \ \frac{95}{9}\ \ (q=3,\ N_3=5)\,, \\ &= \ \frac{47}{2}\ \ (q=4,\ N_3=16)\,, \\ &= \ 672\,(q=5,\ N_3=600)\ \ , \end{aligned} \tag{2.296}$$

respectively, and in the general case

$$2\,n+1 = \frac{(11+9\,q)}{2\,q^2}\,N_3\ , \tag{2.297}$$

thus reproducing $n = 43/9$, $45/4$ and $671/2$, respectively, in the 3 cases. Then Eq. (2.294) is just

$$\psi''(x) + \frac{2\,n+1}{x}\,\psi'(x) + \psi(x) \ = \ 0\,. \tag{2.298}$$

Consequently the wave function solutions are

$$\psi \ \propto \ \frac{J_n(x)}{x^n} \ \propto \ \frac{J_n\left(\frac{4\sqrt{2\,\lambda}}{q\,\boldsymbol{G}}\,V_{tot}\right)}{\left(\frac{4\sqrt{2\,\lambda}}{q\,\boldsymbol{G}}\,V_{tot}\right)^n}\ , \tag{2.299}$$

up to an overall wave function normalization constant, thus summarizing all the results so far for the individual regular triangulations [Eqs. (2.279), (2.284) and (2.289)]. A more explicit, but less transparent, form for the wave function solution is

$$\psi\left(V_{tot}\right) \;=\; \mathcal{N} \cdot V_{tot}^{\frac{1}{2}-\frac{N_3(11+9q)}{4\,q^2}} \cdot J_{-\,\frac{1}{2}+\frac{N_3(11+9q)}{4q\,q^2}} \left(\frac{4\,\sqrt{2\,\lambda}}{q\;\boldsymbol{G}}\; V_{tot} \right) \,, \tag{2.300}$$

with $\mathcal{N}$ an overall wave function normalization constant. Its large volume behavior is completely determined by the asymptotic expansion of the Bessel J function,

$$\psi(x) \;\simeq\; \frac{J_n(x)}{x^n} \underset{x\,\to\,\infty}{\sim} \; x^{-n}\,\sqrt{\frac{2}{\pi x}}\;\sin\left(x + \frac{\pi}{4} - \frac{n\,\pi}{2}\right) \,+\, \mathcal{O}\left(\frac{1}{x^{n+\frac{3}{2}}}\right) \,. \tag{2.301}$$

It is also easy to see that the argument of the Bessel function solution J in Eqs. (2.299) and (2.300) has the following expansion for large volumes

$$x \;=\; \frac{4\,\sqrt{2\,\lambda}}{q_0\;\boldsymbol{G}}\,V_{tot} \,+\, \frac{a_0^2}{36\,\sqrt{2}\,\pi}\,\frac{\sqrt{2\lambda}}{\boldsymbol{G}}\;R_{tot}\;, \tag{2.302}$$

with a_0 ($a_0^3 \equiv 6\sqrt{2}\,V/N_3$) representing here the average lattice spacing. Thus the second correction is of order $(V/N_3)^{2/3}\,R_{tot}$. Note that nothing particularly interesting is happening in the structure of the wave function so far. Similarly, the index n of the Bessel function solution in Eqs. (2.299) and (2.300) has the following expansion for large volumes and small curvatures,

$$n \;=\; \frac{(11+9\,q_0)}{4\,q_0^2}\,N_3 - \frac{1}{2} \,+\, \frac{(22+9\,q_0)}{96\,\pi\,q_0\,a_0}\;R_{tot} \,+\, \mathcal{O}\left(R^2\right)\;, \tag{2.303}$$

with a_0 again defined as above. Note here that the second correction is of order $(N_3/V)^{1/3}\,R_{tot}$. It follows that the asymptotic behavior for the exponent of the fundamental wave function solutions for large volume and small curvature is given by

$$\begin{aligned} \pm\;\; i\left[\,\frac{4\,\sqrt{2\,\lambda}}{q_0\;\boldsymbol{G}}\,V_{tot} \,+\, \frac{a_0^2}{36\,\sqrt{2}\,\pi}\,\frac{\sqrt{2\,\lambda}}{\boldsymbol{G}}\;R_{tot} \,+\, \mathcal{O}\left(R^2\right)\right] \qquad\qquad \\ -\;\left[\,\frac{11+9\,q_0}{4\,q_0^2}\,N_3 \,+\, \frac{22+9\,q_0}{96\,\pi\,q_0\,a_0}\;R_{tot} \,+\, \mathcal{O}\left(R^2\right)\right]\,\ln V_{tot}\,. \end{aligned} \tag{2.304}$$

Let us make here some additional comments. One might wonder what concrete lattices correspond to values of n greater that $671/2$, which is after all the highest value attained for a regular triangulation of the 3 - sphere, namely the 600-cell complex. For each of the 3 regular triangulations of S^3 with N_0 sites one has for the number of edges $N_1 = \frac{6}{6-q} N_0$, for the number of triangles $N_2 = \frac{2\,q}{6-q} N_0$ and for the number of tetrahedra $N_3 = \frac{q}{6-q} N_0$, where q is the number of tetrahedra meeting at an edge (the local coordination number). In the 3 cases examined previously q was of course an integer between 3 and 5; in 2 dimensions it is possible to have one more integer value of q corresponding to the regularly triangulated torus, but this is not possible here. In any case, one always has for a given triangulation of the 3 - sphere the Euler relation $N_0 - N_1 + N_2 - N_3 = 0$. The interpretation of other, even noninteger, values of q is then clear. Additional triangulations of the 3 - sphere can be constructed by considering irregular triangulations, where the parameter q is now seen as an *average* coordination number. Of course the simplest example is what could be described as a semiregular lattice, with N_a edges with coordination number q_a and N_b edges with coordination number q_b, such that $N_a + N_b = N_1$. Various irregular and random lattices were considered in detail some time ago in [93], and we refer the reader to this work for a clear exposition of the properties of these kind of lattices. In the following we will assume that such constructions are generally possible, so that even non-integer values of q are meaningful and are worth considering.

2.6.9 Average Volume and Average Lattice Spacing

At this stage it will be useful to examine the question of what values are allowed for the average volume. The latter will be needed later on to give meaning to the notion of an average lattice spacing. In general the average volume is defined as

$$\langle V_{tot} \rangle \;\equiv\; \frac{\langle \Psi | V_{tot} | \Psi \rangle}{\langle \Psi | \Psi \rangle} \;=\; \frac{\int \mathcal{D}\mu[g] \cdot V_{tot}[g_{ij}] \cdot |\, \Psi[g_{ij}]\, |^2}{\int \mathcal{D}\mu[g] \cdot |\, \Psi[g_{ij}]\, |^2}\,, \tag{2.305}$$

where $\mathcal{D}\mu[g]$ is the appropriate (DeWitt) functional measure over 3 - metrics g_{ij}.

Now consider the wave function obtained given in Eq. (2.299), with n defined in Eq. (2.295).

This wave function is relevant for the strong coupling limit, where the explicit curvature term in the Wheeler DeWitt equation can be neglected. In this limit one can then compute the average total volume

$$\langle V_{tot} \rangle \;=\; \frac{\int_0^\infty dV_{tot} \cdot V_{tot} \cdot |\psi\;(V_{tot})|^2}{\int_0^\infty \, dV_{tot} \cdot |\psi(V_{tot})|^2} \, . \tag{2.306}$$

One then obtains immediately for the average volume of a tetrahedron

$$\langle V_\sigma \rangle \;=\; \frac{2^{-\frac{3}{2}-2\,n}\;\Gamma\;\left(n-\frac{1}{2}\right)\;\Gamma\left(2\,n+\frac{1}{2}\right)}{\Gamma\left(n\right)^3\;N_3} \cdot \frac{q\,\boldsymbol{G}}{\sqrt{\lambda}} \, . \tag{2.307}$$

If the whole lattice is just a single tetrahedron, then one has $n = 3$, and therefore

$$\langle V_\sigma \rangle \;=\; \frac{31185\,\pi\,\boldsymbol{G}}{262144\,\sqrt{2}\,\sqrt{\lambda}} \;=\; 0.2643\,\frac{\boldsymbol{G}}{\sqrt{\lambda}} \, , \tag{2.308}$$

from which one can define an average lattice spacing a_0 via $\langle V_\sigma \rangle = a_0^3/6\,\sqrt{2}$. For large N_3 one has

$$a_0^3 \;=\; \frac{3\,\sqrt{11+9\,q}}{2\,\sqrt{2\,\pi}\,N_3}\,\frac{\boldsymbol{G}}{\sqrt{\lambda}} \, . \tag{2.309}$$

But in general one cannot assume a trivial entropy factor from the functional measure, and one should evaluate instead

$$\langle V_{tot} \rangle \;=\; \frac{\int_0^\infty dV_{tot} \cdot V_{tot}^m \cdot V_{tot} \cdot |\,\psi\;(V_{tot})\,|^2}{\int_0^\infty dV_{tot} \cdot V_{tot}^m \cdot |\,\psi(V_{tot})\,|^2} \, , \tag{2.310}$$

with some power $m = c_0\,N_3$ and c_0 a real positive constant. One then obtains for the average volume of a single tetrahedron

$$\langle V_\sigma \rangle \;=\; \frac{1}{N_3}\,\langle V_{tot} \rangle \;=\; \sqrt{c_0\,[11+q_0(9-c_0\,q_0)]}\;\frac{\boldsymbol{G}}{8\,\sqrt{2\,\lambda}} \, , \tag{2.311}$$

which is finite as $N_3 \to \infty$. Note that in order for the above expression to make sense one requires $c_0 < (11+9q_0)/q_0^2 \simeq 2.185$. If the exponent in the entropy factor is too large, the integrals diverge. One then finds that the corresponding lattice spacing is given by

$$a_0^3 \;=\; \sqrt{c_0\,[11+q_0(9-c_0\,q_0)]}\;\frac{3\,\boldsymbol{G}}{4\,\sqrt{\lambda}} \, . \tag{2.312}$$

The lesson learned from this exercise is that in gravity the lattice spacing a_0 (the fundamental length scale, or the ultraviolet cutoff if one wishes) is itself dynamical, and thus set by the bare values of G and λ. In a system of units for which $\lambda_0 = 1$ one then has $a_0 \sim g^{1/3}$. Either way, the choice for a_0 has no immediate direct physical meaning, and has to be viewed instead in the context of a subsequent consistent renormalization procedure. In the following it will be safe to assume, based on the results of Eqs. (2.273) and (2.312) that

$$a_0^3 \;=\; f^3 \, \frac{\boldsymbol{G}}{\sqrt{\lambda}} \,, \tag{2.313}$$

in units of the UV cutoff, where f is a numerical constant of order one (for concreteness, in the single tetrahedron case one has $f \approx 1.3089$).

2.6.10 Large Volume Solution for Nonzero Curvature

The next task in line is to determine the form of the wave function when the curvature term in the Wheeler DeWitt equation of Eq. (2.260) is not zero. In particular we will be interested in the changes to the wave function given in Eqs. (2.299) and (2.300), with argument x in Eq. (2.302) and parameter n in Eq. (2.303). We define here the total integrated curvature R_{tot} as in Eq. (2.255), which is of course different from the local curvature appearing in the lattice Wheeler DeWitt equation of Eq. (2.260),

$$R_\sigma \equiv \sum_{h \subset \sigma} \delta_h \, l_h \, . \tag{2.314}$$

In order to establish the structure of the solutions for large volumes V_{tot} we will assume, based in part on the results of the previous sections, and on the analogous calculation in $2+1$ dimensions [113], that the fundamental wave function solutions for large volumes have the form

$$\exp \left\{ \pm \, i \left(\alpha \int d^3x \sqrt{g} \, + \beta \int d^3x \sqrt{g} \, R + \gamma \int d^3x \sqrt{g} \, R^2 + \delta \int d^3x \sqrt{g} \, R_{\mu\nu} R^{\mu\nu} + \cdots \right) \right\} \, . \tag{2.315}$$

Note here that the structure of the above expression, and the nature of the terms that enter into it, are basically dictated by the requirement of diffeomorphism invariance as it applies to the argument of the wave functional. Apart from the cosmological term, allowed terms are all the ones that can be constructed from the Riemann tensor and its covariant derivatives, for a a fixed topology of 3 space. Clearly, at large distances (infrared limit) the most important terms will be the Einstein and cosmological terms, with coefficients β and α, respectively. In 3 dimensions the Riemann and Ricci tensor have the same number of algebraically independent components (6), and are related to each other by

$$R^{\mu\nu}{}_{\lambda\sigma} \;=\; \epsilon^{\mu\nu\kappa}\,\epsilon_{\lambda\sigma\rho}\,\left(R^{\rho}{}_{\kappa} \,-\, \frac{1}{2}\,\delta^{\rho}{}_{\kappa} \right)\,. \tag{2.316}$$

The Weyl tensor vanishes identically, and one has

$$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} - 3R^2 \;=\; 0\,, \qquad C_{\mu\nu\lambda\sigma}C^{\mu\nu\lambda\sigma} = 0\,. \tag{2.317}$$

As a consequence, there is in fact only *one* local curvature squared term one can write down in 3 spatial dimensions. Nevertheless, higher derivative terms will only become relevant at very short distances, comparable or smaller than the Planck length $\sqrt{G}$; in the scaling limit it is expected that these can be safely neglected.

When expressed in lattice language, the above form translates to an ansatz of the form

$$\exp\{\,\pm\, i\,(c_0\,V_{tot} \,+\, c_1\,R_{tot}^{m}\,)\}\,, \tag{2.318}$$

with m assumed to be an integer. In addition, from the studies of lattice gravity $2+1$ dimensions one expects a $\ln V_{tot}$ term as well in the argument of the exponential [113]. This suggests a slightly more general ansatz ,

$$\exp\{\,\pm\, i\,(\,c_0\,V_{tot} + c_1\,R_{tot}^{m}\,) + c_2\,\ln V_{tot} + c_3\,\ln R_{tot}\,\}\;\;. \tag{2.319}$$

The next step is to insert the above expression into the lattice Wheeler DeWitt equation Eq. (2.260) and determine the values of the five constants $c_0 \dots c_3$, m. This can be done consistently just to leading order in the weak field expansion of Eq. (2.245), which is entirely adequate here, as it will provide enough information to uniquely determine the coefficients.

Here we will just give the result of this exercise. For the 5-cell complex ($q = 3$) one obtains

$$\psi \sim \exp\left\{ \pm i \left(\frac{4\sqrt{2}\sqrt{\lambda}}{3\,\boldsymbol{G}} V_{tot} \; - \; \frac{\sqrt{2}}{\boldsymbol{G}\sqrt{\lambda}} R_{tot} \right) - \frac{95}{18} \ln V_{tot} \right\} , \tag{2.320}$$

whereas for 16-cell complex ($q = 4$)one finds

$$\psi \sim \exp\left\{ \pm i \left(\frac{\sqrt{2}\sqrt{\lambda}}{\boldsymbol{G}} V_{tot} \; - \; \frac{3\sqrt{2}}{4\,\boldsymbol{G}\sqrt{\lambda}} R_{tot} \right) - \frac{47}{4} \ln V_{tot} \right\} , \tag{2.321}$$

and finally for 600-cell complex ($q = 5$)

$$\psi \sim \exp\left\{ \pm i \left(\frac{4\sqrt{2}\sqrt{\lambda}}{5\,\boldsymbol{G}} V_{tot} \; - \; \frac{3\sqrt{2}}{5\,\boldsymbol{G}\sqrt{\lambda}} R_{tot} \right) - 336 \ln V_{tot} \right\} . \tag{2.322}$$

These expressions allow us again to identify the answer for general q as

$$\psi \sim \exp\left\{ \pm i \left(\frac{4\sqrt{2\,\lambda}}{q\,\boldsymbol{G}} V_{tot} \; - \; \frac{3\sqrt{2}}{q\,\boldsymbol{G}\sqrt{\lambda}} R_{tot} \right) \; - \; \frac{(11 + 9\,q)\,N_3}{4\,q^2} \ln V_{tot} \right\} . \tag{2.323}$$

Note that in deriving the above results we considered the large volume limit $V \to \infty$, treating the number of tetrahedra N_3 as a fixed parameter. Then from the previous expression we can now read off the values for the various coefficients, namely

$$\begin{aligned}
c_0 &= \frac{4\sqrt{2\,\lambda}}{q\,\boldsymbol{G}} \\
c_1 &= -\frac{3\sqrt{2}}{q\,\boldsymbol{G}\sqrt{\lambda}} \\
c_2 &= -\frac{(11 + 9\,q)\,N_3}{4\,q^2} \\
c_3 &= 0
\end{aligned} \tag{2.324}$$

with the only possible value $m = 1$.

In order to make contact with the strong coupling result for the wave function derived in the previous sections [Eqs. (2.300), (2.302), (2.303) and (2.304)], one needs to again expand the above answer for small curvatures. One obtains for the exponent of the wave function the following expression

$$\begin{aligned}
\pm\, i \left\{ \frac{4\sqrt{2\,\lambda}}{q_0\,\boldsymbol{G}} V_{tot} \; + \; \left(\frac{a_0^2}{36\sqrt{2}\,\pi} \frac{\sqrt{2\,\lambda}}{\boldsymbol{G}} \; - \; \frac{6}{q_0\,\boldsymbol{G}\sqrt{2\,\lambda}} \right) R_{tot} \; + \; \mathcal{O}\left(R^2\right) \right\} \\
- \left\{ \frac{11 + 9\,q_0}{4\,q_0^2} N_3 \; + \; \frac{22 + 9\,q_0}{96\,\pi\,q_0\,a_0} R_{tot} \; + \; \mathcal{O}\left(R^2\right) \right\} \ln V_{tot} \, ,
\end{aligned} \tag{2.325}$$

with a_0 again representing the average lattice spacing, $a_0^3 \equiv 6\sqrt{2}\, V/N_3$. This finally determines uniquely the coefficients α and β appearing in Eq. (2.315),

$$
\begin{aligned}
\alpha &= \frac{4}{q_0} \cdot \frac{\sqrt{2\,\lambda}}{\boldsymbol{G}} \\
\beta &= \frac{a_0^2}{36\,\sqrt{2}\,\pi} \cdot \frac{\sqrt{2\,\lambda}}{\boldsymbol{G}} - \frac{6}{q_0} \cdot \frac{1}{\boldsymbol{G}\,\sqrt{2\,\lambda}} \,.
\end{aligned}
\tag{2.326}
$$

The most important result so far is the appearance of two contributions of opposite sign in β, signaling the appearance of a critical value for G where β vanishes.

This critical point is located at $\lambda_c = 108\,\sqrt{2}\pi/q_0\,a_0^2$ or, in a system of units where $\lambda = \boldsymbol{G}/2$ [see Sec. (2.4.1)],

$$
\boldsymbol{G}_c = \frac{216\,\sqrt{2}\pi}{q_0} \cdot \frac{1}{a_0^2} \,. \tag{2.327}
$$

But since the average lattice spacing a_0 is itself a function of G and λ[see Eqs. (2.273), (2.312) and (2.313)] one obtains in the same system of units

$$
\boldsymbol{G}_c = \frac{36\; 2^{3/8}\, 3^{1/4}\, \pi^{3/4}}{f^{3/2}\; q_0^{3/4}} \simeq 28.512 \,, \tag{2.328}
$$

using the value of f for the single tetrahedron, or equivalently $\boldsymbol{g}_c \simeq 5.340$, a rather large value. Here we keep in mind that we are using a system of units where we set $16\pi G \to \boldsymbol{G}$. In a conventional system of units, one has the more reasonable result $G_c \approx 0.567$ in units of the fundamental UV cutoff. Evidence for a phase transition in lattice gravity in $3+1$ dimensions was also seen earlier from the application of the variational method, using Jastrow-Slater correlated product variational wavefunctions [81]. [12] Note that the results of Eqs. (2.325) and (2.326) imply a dependence of the fundamental wave function on the curvature, of the type

$$
\psi(R) \;\sim\; e^{\pm\, i\, R_{tot}/R_0} \,, \tag{2.329}
$$

[12] One can compare the above value for G_c obtained in the Lorentzian $3+1$ theory with the corresponding value in the Euclidean 4 dimensional theory. There one finds $G_c \approx 0.624$ [46, 47]. The two values are not expected to be the same in the two formulations, due to the different nature of the cutoffs. In particular, in the lattice Hamiltonian formulation the continuum limit has already been taken in the time direction. Nevertheless, it is encouraging that they are quite comparable in magnitude.

with R_0 a characteristic scale for the total, integrated curvature. Thus $R_0 \sim 1/(g - g_c)$ with G_c, and therefore $\boldsymbol{g}_c = \sqrt{\boldsymbol{G}_c}$, given in Eq. (2.327). Therefore at the critical point fluctuations in the curvature become unbounded, just as is the case for the fluctuations in a scalar field when the renormalized mass approaches zero. [13]

At this stage one can start to compare with the results obtained previously without the explicit curvature term in the Wheeler DeWitt equation, Eqs. (2.302) and (2.303). The main change is that here one would be led to identify

$$x \;=\; \frac{4\sqrt{2\,\lambda}}{q_0\,\boldsymbol{G}}\,V_{tot} \,+\, \left(\frac{a_0^2}{36\sqrt{2}\,\pi} \cdot \frac{\sqrt{2\,\lambda}}{\boldsymbol{G}} \;-\; \frac{6}{q_0} \cdot \frac{1}{\boldsymbol{G}\sqrt{2\,\lambda}} \right) R_{tot} \,, \tag{2.330}$$

so that the Bessel function argument x [see Eq. (2.302)] now contains a new contribution, of opposite sign, in the curvature term. Its origin can be traced back to the new curvature contribution c_1 in Eq. (2.324), which in turn arises because of the explicit curvature term now present in the full Wheeler DeWitt equation. On the other hand, as is already clear from the result for c_2 in Eq. (2.324), the index n of the Bessel function solution in Eqs. (2.299) and (2.300) is left unchanged,

$$n \;=\; \frac{11 + 9\,q_0}{4\,q_0^2}\,N_3 - \frac{1}{2} + \frac{22 + 9\,q_0}{96\,\pi\,q_0\,a_0}\,R_{tot} \,+\, \mathcal{O}\left(R_{tot}^2\right) \,, \tag{2.331}$$

with again an average lattice spacing a_0 defined as before.

But there is a better way to derive correctly the modified form of the wave function. From the asymptotic solution for the wave function of Eq. (2.323) it is possible to first obtain a partial differential equation for $\psi(R_{tot}, V_{tot})$. The equation reads (in the following we shall write R_{tot} as R and V_{tot} as V to avoid unnecessary clutter)

$$\frac{\partial^2 \psi}{\partial V^2} + c_V\,\frac{\partial\,\psi}{\partial\,V} + c_R\,\frac{\partial\,\psi}{\partial\,R} + c_{VR}\,\frac{\partial^2\,\psi}{\partial\,V\partial\,R} + c_{RR}\,\frac{\partial^2\,\psi}{\partial\,R^2} + c_\lambda\,\psi + c_{curv}\,\psi \;=\; 0\,. \tag{2.332}$$

[13] It is tempting to try to extract a critical exponent from the result of Eq. (2.329). In analogy to the wave functional for a free scalar field with mass m, and thus correlation length $\xi = 1/m$, one would obtain for the correlation length exponent ν (with ν defined by $\xi \sim |g - g_c|^{-\nu}$) from the above wave function the *semi-classical* estimate $\nu = \frac{1}{2}$. In the $2+\epsilon$ perturbative expansion for pure gravity one finds in the vicinity of the UV fixed point $\nu^{-1} = (d-2) + \frac{3}{5}(d-2)^2 + \mathcal{O}((d-2)^3)$[20, 23, 40, 48, 49, 108]. The above lowest order lattice result would then agree only with the leading, semi-classical term.

The coefficients in the above equation are given by

$$
\begin{aligned}
c_V &= \frac{11+9\,q}{2\,q^2}\cdot\frac{N_3}{V} = \frac{11+9\,q_0}{2\,q_0^2}\cdot\frac{N_3}{V} + \frac{22+9\,q_0}{48\,\sqrt{2}\,3^{1/3}\,\pi\,q_0}\cdot\frac{N_3^{1/3}\,R}{V^{4/3}} + \mathcal{O}(R^2) \\
c_R &= -\frac{2}{9}\,\frac{R}{V^2} + \frac{11+9\,q_0}{6\,q_0^2}\cdot\frac{N_3\,R}{V^2} + \mathcal{O}(R^2) \\
c_{VR} &= \frac{2}{3}\,\frac{R}{V} + \mathcal{O}(R^2) \\
c_{RR} &= \frac{2}{9}\,\frac{R^2}{V^2} \\
c_\lambda &= \frac{32\,\lambda}{q^2\,G^2} = \frac{32}{G^2\,q_0^2} + \frac{4\sqrt{2}\lambda}{3\,3^{1/3}\,\pi\,q_0\,G}\cdot\frac{R}{N_3^{2/3}\,V^{1/3}} + \mathcal{O}(R^2) \\
c_{curv} &= -\frac{16}{G^2\,q^2}\cdot\frac{R}{V} = -\frac{16}{G^2\,q_0^2}\cdot\frac{R}{V} + \mathcal{O}(R^2)\,.
\end{aligned}
\tag{2.333}
$$

Note that in the small curvature, large volume limit [this is the limit in which, after all, Eq. (2.323) was derived] one can safely set the coefficients c_R and c_{RR} to zero. It is then easy to check that the solution in Eq. (2.323) satisfies Eqs. (2.332) and (2.333), up to terms of order $1/V^2$. Also note that here, and in Eqs. (2.320), (2.321), (2.322) and (2.323), we take the large volume limit $V \to \infty$, treating the number of tetrahedra N_3 as a large, fixed parameter. A differential equation in the variable V only can be derived as well (with coefficients that are functions of R), but then one finds that the required coefficients are not real, which makes this approach less appealing.

In the limit $R \to 0$ Eq. (2.332) reduces to

$$
\frac{\partial^2\psi}{\partial V^2} + \frac{11+9\,q_0}{2\,q_0^2}\cdot\frac{N_3}{V}\cdot\frac{\partial\,\psi}{\partial\,V} + \frac{32\,\lambda}{G^2\,q_0^2}\,\psi \;=\; 0\,,
\tag{2.334}
$$

which is essentially Eq. (2.292) in the same limit, with solution given previously in Eq. (2.299).

2.6.11 Nature of the Wave Function Solution ψ

In this section we discuss some basic physical properties that can be extracted from the wave function solution $\psi(V,R)$. So far we have not been able to find a general solution to the fundamental Eq. (2.332), but one might suspect that the solution is still close to a Bessel or hypergeometric function, possibly with arguments “shifted” according to Eqs. (2.330) and (2.331), as was the case in $2+1$ dimensions. As a consequence, some physically motivated approximations will be necessary in the following discussion. Let us discuss here in detail one possible approach. If one sets the troublesome coefficient $c_{VR} = 0$ in Eq. (2.332), and keeps only the leading term in c_V, then the relevant differential equation becomes

$$\frac{\partial^2 \psi}{\partial V^2} + c_V \frac{\partial \psi}{\partial V} + c_\lambda \psi + c_{curv} \psi = 0 \,, \tag{2.335}$$

with coefficients given in Eq. (2.333), except that from now on only the leading term in c_V and c_λ will be retained (otherwise it seems again difficult to find an exact solution). Note that the above equation still contains an explicit curvature term proportional to R, from c_{curv}. Now a complete solution can be found in terms of the confluent hypergeometric function of the first kind, ${}_1F_1(a, b, z)$[99, 100, 101]. Up to an overall wave function normalization constant, it is

$$\begin{aligned}\psi(V,\, R) \quad \simeq \quad & e^{-\frac{4\, i \sqrt{2\lambda}\, V}{q_0\, \boldsymbol{G}}} \cdot \frac{\Gamma\left(\frac{(11+9\, q_0)\, N_3}{4\, q_0^2} + \frac{i \sqrt{2}\, R}{q_0\, \boldsymbol{G} \sqrt{\lambda}}\right)}{\Gamma\left(1 - \frac{(11+9\, q_0)\, N_3}{4\, q_0^2} + \frac{i \sqrt{2}\, R}{q_0\, \boldsymbol{G} \sqrt{\lambda}}\right)} \\ & \times {}_1F_1\left(\frac{(11+9\, q_0) N_3}{4\, q_0^2} - \frac{i\, \sqrt{2}\, R}{q_0\, \sqrt{\lambda}\, \boldsymbol{G}},\, \frac{(11+9\, q_0) N_3}{2\, q_0^2},\, \frac{8\, i \sqrt{2\lambda}\, V}{q_0\, \boldsymbol{G}}\right) . \end{aligned} \tag{2.336}$$

Here again q_0 is just a number, given previously in Eq. (2.259), and N_3 the total number of tetrahedra for a given triangulation of the manifold. Note that this last solution still retains three key properties: it is a function of geometric invariants (V, R) only; it is regular at the origin in the variable V (the irregular solution is discarded due to the normalizability constraint); and finally it agrees, as it should, with the zero curvature solution of Eqs. (2.299) and (2.300) in the limit $R = 0$.

The above wave function exhibits some intriguing similarities with the exact wave function solution found in $2+1$ dimensions; the difference is that the total curvature R here plays the role of the Euler characteristic χ there. Let us be more specific, and discuss each argument separately. For the arguments of the confluent hypergeometric function of the first kind, ${}_1F_1(a, b, z)$, one finds again $b = 2a$ for $R = 0$, with both a and b proportional to the total number of lattice sites, as in $2+1$ dimensions [113]. Specifically, here one has

$$Re(a) \;=\; \frac{11 + 9\,q_0}{4\,q_0^2}\,N_3 \;\approx\; 0.5464\,N_3\,, \tag{2.337}$$

whereas in $2+1$ dimensions the analogous result is

$$Re(a) \;=\; \frac{1}{4}\,N_2\,. \tag{2.338}$$

The curvature contribution in both cases then appears as an additional contribution to the first argument (a), and is purely imaginary. Here one has

$$Im(a) \;=\; -\,\frac{\sqrt{2}}{q_0\,\sqrt{\lambda}\,\boldsymbol{G}}\;\int d^3x\,\sqrt{g}\,R\,, \tag{2.339}$$

whereas in $2+1$ dimensions the corresponding result is

$$Im(a) \;=\; -\,\frac{1}{2\,\sqrt{2\,\lambda}\,\boldsymbol{G}}\;\int d^2x\,\sqrt{g}\,R\,. \tag{2.340}$$

Finally, here again the third argument z is purely imaginary and simply proportional to the total volume. From the above solution

$$z \;=\; i\,\frac{8\,\sqrt{2\lambda}}{q_0\,\boldsymbol{G}}\;\int d^3x\,\sqrt{g}\,, \tag{2.341}$$

whereas in $2+1$ dimensions

$$z \;=\; i\,\frac{2\,\sqrt{2\lambda}}{\boldsymbol{G}}\;\int d^2x\,\sqrt{g}\,. \tag{2.342}$$

Nevertheless we find also some important additional differences with the $2+1$ result, most notably the various gamma function coefficients involving the curvature R, which are entirely absent in the lower dimensional case, as well as the fact that the critical (UV fixed) point is located at some finite G_c here [see Eq. (2.327)], whereas it is exactly at $G_c = 0$ in $2+1$ dimensions [113].

Let us now continue here with a discussion of the main properties of the wave function in Eq. (2.336). First let us introduce some additional notational simplification. By using the coupling $\boldsymbol{g}$ [see Sec. (2.4.1) and Eq. (2.38)] one can make the above expression for ψ slightly more transparent

$$\psi(V,\,R) \;\simeq\; e^{-\frac{4\,i\,V}{q_0\,\boldsymbol{g}}} \cdot \frac{\Gamma\left(\frac{(11+9\,q_0)\,N_3}{4\,q_0^2}+\frac{2\,i\,R}{q_0\,\boldsymbol{g}^3}\right)}{\Gamma\left(1-\frac{(11+9\,q_0)\,N_3}{4\,q_0^2}+\frac{2\,i\,R}{q_0\,\boldsymbol{g}^3}\right)} \times {}_1F_1\left(\frac{(11+9\,q_0)N_3}{4\,q_0^2}-\frac{2\,i\,R}{q_0\,\boldsymbol{g}^3},\;\frac{(11+9\,q_0)N_3}{2\,q_0^2},\;\frac{8\,i\,V}{q_0\,\boldsymbol{g}}\right) . \qquad (2.343)$$

We remind the reader that, by virtue of Eq. (2.259), in all the above expressions q_0 is just a numerical constant, $q_0 \;\equiv\; 2\,\pi/\cos^{-1}(\frac{1}{3}) \;=\; 5.1043$. Note that for weak coupling the curvature terms become more important due to the $1/\boldsymbol{g}^3$ coefficient. The resulting probability distribution $|\psi(V,R)|^2$ is shown, for some illustrative cases, in Figures 2.14, 2.15, and 2.16.

One important proviso should be stated here first. We recall that having obtained an (exact or approximate) expression for the wave function does not lead immediately to a complete solution of the problem. This should be evident, for example, from the general expression for the average of a generic quantum operator $\mathcal{O}(g)$

$$\langle \mathcal{O}(g) \rangle \;\equiv\; \frac{\langle \Psi|\mathcal{O}|\Psi\rangle}{\langle \Psi|\Psi\rangle} \;=\; \frac{\int \mathcal{D}\mu[g] \cdot \mathcal{O}(g_{ij}) \cdot |\,\Psi[g_{ij}]\,|^2}{\int \mathcal{D}\mu[g] \cdot |\,\Psi[g_{ij}]\,|^2} \,, \qquad (2.344)$$

where $\mathcal{D}\mu[g]$ is the appropriate (DeWitt) functional measure over the 3 - metric g_{ij}. Because of the general coordinate invariance of the state functional, the inner products shown above clearly contain an infinte redundancy due to the geometrical indistinguishability of 3-metrics which differ only by a coordinate transformation [31, 32, 33]. Nevertheless this divergence is of no essence here, since it cancels out between the numerator and the denominator.

On the lattice the above average translates into

$$\langle \mathcal{O}(l^2) \rangle \;\equiv\; \frac{\langle \Psi|\mathcal{O}|\Psi\rangle}{\langle \Psi|\Psi\rangle} \;=\; \frac{\int \mathcal{D}\mu[l^2] \cdot \mathcal{O}(l^2) \cdot |\,\Psi[l^2]\,|^2}{\int \mathcal{D}\mu[l^2] \cdot |\,\Psi[l^2]\,|^2} \,, \qquad (2.345)$$

where $\mathcal{D}\mu[l^2]$ is the Regge Wheeler lattice transcription of the DeWitt functional measure [31, 32, 33] in terms of edge length variables, here denoted collectively by l^2. The latter includes an integration over all squared edge lengths, constrained by the triangle inequalities and their higher dimensional analogs [51, 51]. Again, because of the continuous local diffeomorphism invariance of the lattice theory, the individual inner products shown above will contain an infinte redundancy due to the geometrical indistinguishability of 3-metrics which differ only by a lattice coordinate transformation. And, again, this divergence will be of no essence here, as it is expected to cancel between numerator and denominator [75].

It seems clear then that, in general, the full functional measure cannot be decomposed into a simple product of integrations over V and R. It follows that the averages listed above are in general still highly nontrivial to evaluate. In fact, quantum averages can be written again quite generally in terms of an effective (Euclidean) 3 dimensional action

$$\langle\Psi|\tilde{\mathcal{O}}(g)|\Psi\rangle \;=\; \mathcal{N}\int \mathcal{D}\mu[g]\;\tilde{\mathcal{O}}(g_{ij})\;\exp\left\{-S_{eff}[g]\right\}\;, \tag{2.346}$$

with $S_{eff}[g] \equiv -\ln|\Psi[g_{ij}]|^2$ and $\mathcal{N}$ a normalization constant. The operator $\tilde{\mathcal{O}}(g)$ itself can be local, or nonlocal as in the case of correlations such as the gravitational Wilson loop [42, 43]. Note that the statistical weights have zeros corresponding to the nodes of the wave function Ψ, so that S_{eff} is infinite there. [14]

Nevertheless it will make sense here to consider a *semiclassical* expansion for the $3+1$ dimensional theory, where one simply focuses on the clearly identifiable stationary points (maxima) of the probability distribution $|\psi|$, obtained by squaring the solution in Eqs. (2.336) or (2.343). In the following we will therefore focus entirely on the properties of the probability distribution $|\psi(V,R)|^2$ obtained from Eq. (2.336) or (2.343). For illustrative purpose, the reader is referred to Figures 2.14, 2.15 and 2.16 below.

As discussed previously, the asymptotic expansion for the wave function at large volumes implies the existence of a phase transition at some $G = G_c$ [see for example Eqs. (2.326)

[14] In practical terms, the averages in Eqs. (2.344) and (2.345) are difficult to evaluate analytically, even once the complete wave function is known explicitly, due to the nontrivial nature of the gravitational functional measure; in the most general case these averages will have to be evaluated numerically. The presence of infinitely many zeros in the statistical weights complicates this issue considerably, from a numerical point of view.

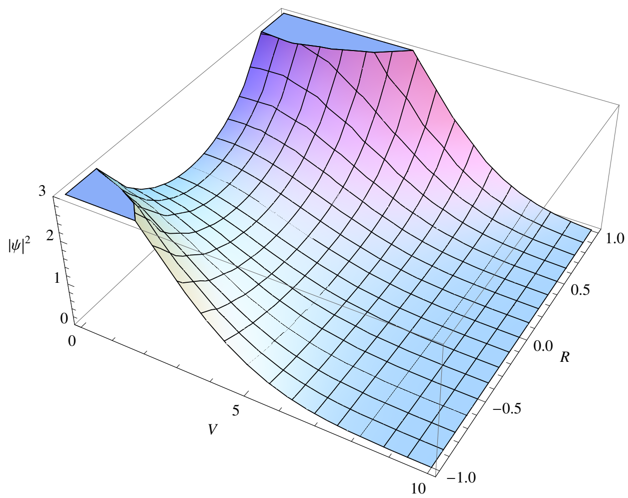

Figure 2.14: Wave function of Eq. (2.343) squared, $|\psi(V,R)|^2$, plotted as a function of the total volume V and the total curvature R, for coupling $g = \sqrt{G} = 1$ and $N_3 = 10$. One notes that for strong enough coupling g the distribution in curvatures is fairly flat around $R = 0$, giving rise to large fluctuations in the curvature. These become more pronounced as one approaches the critical point at g_c.

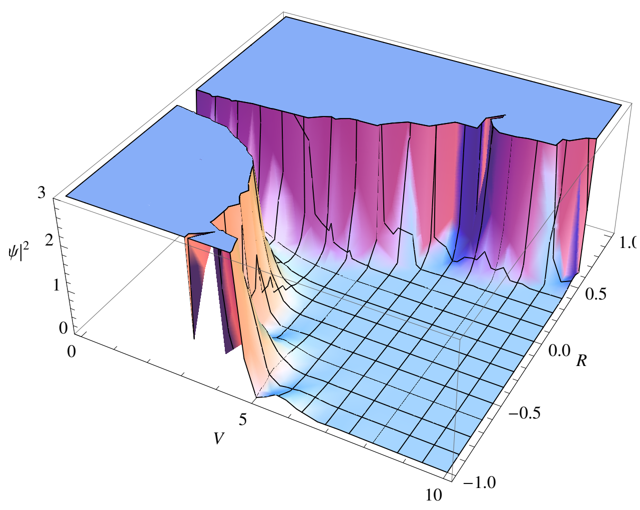

Figure 2.15: Same wave function of Eq. (2.343) squared, $|\psi(V,R)|^2$, plotted as a function of the total volume V and the total curvature R, but now for weaker coupling $g = \sqrt{G} = 0.5$, and still $N_3 = 10$. For weak enough coupling g the distribution in curvature is such that values around $R = 0$ are almost completely excluded, as these are associated with a very small probability. Note that, unless the total volume V is very small, the probability distribution is markedly larger towards positive curvatures.

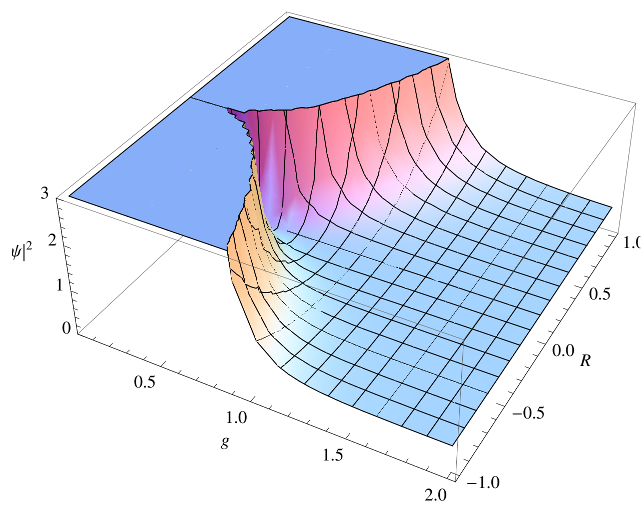

Figure 2.16: Curvature distribution in R as a function of the coupling $g = \sqrt{G}$. The strong coupling relationship between the average volume and the coupling g [Eq. (2.272)] allows one to plot the wave function of Eq. (2.343) squared as a function of the coupling g and the total curvature R only (we use again here $N_3 = 10$ for illustrative purposes). Then, for strong enough coupling $g = \sqrt{G}$, the probability distribution $|\psi|^2$ is again fairly flat around $R = 0$, giving rise to large fluctuations in the curvature. The latter are interpreted here as signaling the presence of a massless particle. On the other hand, for weak enough coupling g one notices that curvatures close to $R = 0$ have essentially vanishing probability. The distribution shown here points therefore toward a pathological ground state for weak enough coupling $g < g_c$ [given in Eq. (2.327)], with no sensible continuum limit.

and (2.327)]. In addition, the explicit solution in Eq. (2.343) allows a more precise nonperturbative characterization of the two phases. In view of the nontrivial and generally complex arguments of both the gamma function and the confluent hypergeometric function, the analytic properties of the wave function, and therefore of the probability distribution, are quite rich in features, at least for the more general and physically relevant case of nonzero curvature.

One first notes that for strong enough coupling $\boldsymbol{g}$ the distribution in curvature is fairly flat around $R = 0$, giving rise to large fluctuation in the latter (see Fig. 2.14). On the other hand, for weak enough coupling $\boldsymbol{g}$ the probability distribution in curvature is such that values around $R = 0$ are almost excluded, since they are associated with a very small probability. Furthermore, unless the volume V is very small, the probability distribution is also generally markedly larger towards positive curvatures (see Fig. 2.15).

In order to explore specifically the curvature (R) dependence of the probability distribution, it would be desirable to factor out or remove the dependence of the wave function $\psi(V, R)$ on the total volume V. To achieve this, one can employ a mean-field-type prescription, and replace the total volume V by its average $\langle V \rangle$. After all, the probability distribution in the volume is well behaved at large G [see Sec. (2.6.9)], and does not exhibit any marked change in behavior for intermediate G [as can be inferred, for example, from the asymptotic form of the wave function in Eq. (2.323)]. Consequently we will now make the replacement in $\psi(V, R)$

$$V \;\longrightarrow\; \langle\, V \,\rangle \;\equiv\; N_3 \,\langle\, V_\sigma \,\rangle \;=\; 0.2643\, \frac{\boldsymbol{G}}{\sqrt{\lambda}} \;=\; 0.3738\, \boldsymbol{g}\;, \tag{2.347}$$

obtained by inserting the result of Eq. (2.272). This replacement then makes it possible to plot the wave function of Eq. (2.343) squared as a function of the coupling $\boldsymbol{g}$ and the total curvature R *only* (in the following we use again $N_3 = 10$ for illustrative purposes); see Figure 2.16. One then notes that for strong enough gravitational coupling $\boldsymbol{g} = \sqrt{\boldsymbol{G}}$ the probability distribution is again fairly flat around $R = 0$, giving rise to large fluctuations in the curvature. On the other hand, for weak enough coupling $\boldsymbol{g}$ one observes that curvatures close to zero have near vanishing probability. The distributions shown suggest therefore a clearly pathological ground state for weak enough coupling $\boldsymbol{g} < \boldsymbol{g}_c$ [or $G < G_c$, see Eq. (2.327)], with no sensible 4 dimensional continuum limit.

At this point some preliminary conclusions, based on the behavior of the wave function discussed previously in Sec. (2.6.10) and the shape shown in Figures 2.14, 2.15 and 2.16, are as follows. For large enough $G > G_c$, but nevertheless close to the critical point, the flatness in the curvature probability distribution implies that different curvature scales are all equally important. The corresponding gravitational correlation length is finite in this region as long as $G > G_c$, and expected to diverge at the critical point, thus presumably signaling the presence of a massless excitation at G_c [see the argument after Eq. (2.329)]. On the other hand for weak enough coupling, $G < G_c$ we observe that the probability distribution appears to change dramatically. The main evidence for this is the shape of the approximate wave function given in Eq. (2.336), which points to a vanishing relative probability for metric field configurations for which the curvature is small $R \approx 0$. This would suggest that the weak coupling phase, for which $G < G_c$, has *no* continuum limit in terms of manifolds that appear smooth, at least at large scales. The geometric character of the manifold is then inevitably dominated by nonuniversal short distance lattice artifacts; no sensible scaling limit exists in this phase.

If this is indeed the case, then the results obtained in the present, Lorentzian, $3 + 1$ theory generally agree with what is found in the Euclidean case, where the weak coupling phase was found to be pathological as well [26, 28, 53] (it bears more resemblance to a branched polymer, and has thus no sensible interpretation in terms of smooth 4 dimensional manifolds). In either case, the only physically acceptable phase, leading to smooth manifolds at large distances, seems to be the one with $G > G_c$. It is a simple consequence of renormalization group arguments that in this phase the gravitational coupling at large distances can only flow to larger values, implying therefore gravitational antiscreening as the only physically possible outcome.

2.6.12 Discussions

In this work [117] we have discussed the nature of gravitational wave functions that arise as solutions of the lattice Wheeler DeWitt equation for finite simplicial lattices. The main results were given in Eqs. (2.332), (2.336) and (2.343). While there are many aspects of this problem that still remain open and unexplored, we showed that the very structure of the wave function is such that it allows one to draw a number of useful and possibly physically relevant conclusions about ground state properties of pure quantum gravity in 3+1 dimensions. These include the observation that the theory exhibits a phase transition at some critical value of Newton's constant G_c [given in Eq. (2.327)].

The structure of the wave function further suggests that the weak coupling phase, for which the coupling $G < G_c$, is pathological and cannot be interpreted in terms of smooth manifolds at *any* distance scale. The correct physical vacuum apparently cannot in any way be obtained as a small perturbation of flat, or near flat, spacetime. On the other hand the strong coupling phase does *not* exhibit any such pathology, and is therefore a good candidate for a physically acceptable ground state for pure quantum gravity. In this phase, renormalization group studies argue that in this phase Newton's constant grows with distance, and thus this phase exhibits gravitational antiscreening as discussed in Chapter 1.

In $3+1$ dimensions, it is much more difficult to solve for entirely analytically keeping all the original degrees of freedom. For future prospect, one intermediate step one may take will be for example to work out the lattice discretized Wheeler DeWitt equations in a similar set up with Belinsky-Khalatnikov-Lifshitz (BKL) solutions to cosmology, therefore in T^3 (three torus) spatial boundary conditions. BKL solutions are classical chaotic solutions to Einstein's field equations, providing us the dynamical evolution of oscillatory universe expanding and contracting around singularity. Initially the spatial part of the metric has 6 degrees of freedom, however, one easily sees that the wavefunction only depends on the 3 degrees of freedom still capturing all the dynamics given [118, 119].

Chapter 3

Cosmological Implications of the running G

As we have seen in the Chapter 1, studying Einstein gravity with a cosmological constant term in the framework of quantum field theory and renormalization group tells us that Newton's constant G to run with a scale. In this Chapter, we explore possible cosmological consequences of a running Newton's constant $G(\Box)$, as suggested by the nontrivial ultraviolet fixed point scenario in the quantum field theoretic treatment of Einstein gravity with a cosmological constant term. In particular we focus here on what possible effects the scale dependent coupling might have on large scale cosmological density perturbations.

Firstly, we will work on a gauge called the comoving Friedmann - Lemaître - Robertson - Walker (FLRW) gauge in Section 3.2. Starting from a set of manifestly covariant effective field equations derived, we systematically develop the linear theory of density perturbations for a nonrelativistic, pressureless fluid. The result is a modified equation for the matter density contrast, which can be solved and thus provides an estimate for the growth index parameter γ in the presence of a running G.

Later, we will work on a gauge called conformal Newtonian gauge by gauge transforming the results obtained in the comoving FLRW gauge by mediating the synchronous gauge Section 3.3. In classical gravity deviations from the predictions of the Einstein theory can be often discussed in the conformal Newtonian gauge, where scalar perturbations are described by two potentials ϕ and ψ. Note that we use the above gauge to explore possible cosmological consequences of a running Newton's constant $G(\Box)$, as suggested by the nontrivial ultraviolet fixed point scenario arising from the quantum field theoretic treatment of Einstein gravity with a cosmological constant term. Here we focus on the effects of a scale dependent coupling on the so called gravitational slip functions $\eta = \psi/\phi - 1$, whose classical general relativity value is zero. Starting from a set of manifestly covariant but nonlocal effective field equations derived earlier, we compute the leading corrections in the potentials ϕ and ψ for a nonrelativistic, pressureless fluid. After providing an estimate for the quantity η, we then focus on a comparison with results obtained on matter density perturbations in the comoving gauge, which gave an estimate for the growth index parameter γ, also in the presence of a running G. Our results indicate that, in the present framework and for a given $G(\Box)$, the corrections tend to be significantly larger in magnitude for the perturbation growth exponents than for the conformal Newtonian gauge slip function.

The Chapter is organized as follows. First in Section 3.1, we recall the effective covariant field equations describing the running of G, and describe briefly the nature of various objects and parameters entering the quantum nonlocal corrections. In Section 3.2We then discuss the zeroth order (in the metric fluctuations) field equations and energy momentum conservation equations for the standard homogeneous isotropic metric, with a running $G(\Box)$. Later we extend the formalism to deal with small metric and matter perturbations, and list the relevant field and energy conservation equations to first order in the perturbations in the comoving gauge.

After showing the overall consistency of the derived equations, we proceed to derive the modified differential equation for the density contrast $\delta(t)$. Later this is rewritten, following customary procedures, as a function of the scale factor as $\delta(a)$. The resulting differential equation for the density contrast is then solved and the results for the growth exponents compared to the standard classical result. We provide an interpretation of the theoretical results and their associated uncertainties vis-à-vis present and future high precision galaxy clustering measurements.

These above results are then Section 3.3 reexpressed in two other choices of gauge, the co-moving FLRW and the conformal Newtonian gauge. The latter choice of gauge allows us to extract an expression for the gravitational slip function η due to $G(\Box)$ (Section 3.4). This quantity is then evaluated within the context of a ΛCDM model, for redshifts corresponding to the present era (z=0). The resulting correction is then compared to current astrophysical observations, as well as to our previous results (and observations) regarding the corrections due to $G(\Box)$ to the matter density perturbation growth exponents. We will discuss an interpretation of the theoretical results, and their associated uncertainties, in view of present and future high precision determination of the gravitational slip function and growth exponents.

3.1 Running Newton's Constant $G(\Box)$

The running of G can be computed on the lattice directly in 4 dimensions [25, 26, 27, 28, 65, 80, 52, 46, 47], or in the continuum within the framework of the background field expansion applied to $2+\epsilon$ spacetime dimensions [6, 23, 40, 48, 49] and later using truncation methods (functional renormalization group equation method) applied in 4 dimensions [50, 120, 121]. One obtains a momentum dependent $G(k^2)$, (where k is a 4 momentum,) which should be reexpressed in a coordinate independent way, so that it can be usefully applied to more general problems involving arbitrary background geometries.

The first step in analyzing the consequences of a running of G is therefore to rewrite the expression for $G(k^2)$ in a coordinate independent way, for example by the use of a nonlocal Vilkovisky type effective gravity action [122, 123], or by the use of a set of consistent effective field equations. In going from momentum to position space one usually employs $k^2 \to -\Box$, which then gives for the quantum mechanical running of the gravitational coupling the replacement $G \;\to\; G(\Box)$. One then finds that the running of G is given in the vicinity of the UV fixed point by

$$G(\Box) \;=\; G_0 \left[\, 1 \,+\, c_0 \left(\frac{1}{\xi^2 \Box} \right)^{1/2\nu} \,+\, \dots \right] \,, \tag{3.1}$$

where $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the covariant d'Alembertian, and the dots represent higher order terms in an expansion in $1/(\xi^2 \Box)$. Current evidence from Euclidean lattice quantum gravity points toward $c_0 > 0$ (implying infrared growth) and $\nu \simeq 1/3$ [52, 46, 47]. Within the quantum field theoretic renormalization group treatment, the quantity ξ arises as an integration constant of the Callan Symanzik renormalization group equations. As we discussed in Chapter 1, one key issue, and of great relevance to the physical interpretation of the results, is a correct identification of the renormalization group invariant scale ξ. Arguments can be given (see below and Sections 1.4.3 and 1.5 of Chapter 1) in support of the suggestion that the infrared scale ξ (very much analogous to the $\Lambda_{\overline{MS}}$ of QCD) can be very large, *i.e.*, cosmological, for gravity. From these arguments one can infer that the constant G_0, to a very close approximation, will be identified with the laboratory value of Newton's constant, $\sqrt{G_0} \sim 1.6 \times 10^{-33}$cm, since the laboratory scale is much much smaller than the cosmological scale of the size of the universe.

The appearance of the d'Alembertian $\Box$ in running G naturally yields us a nonlocal effective gravitational action and a corresponding set of nonlocal modified field equations. Instead of the ordinary Einstein field equations with constant G *i.e.*,

$$R_{\mu\nu} \,-\, \frac{1}{2} \, g_{\mu\nu} \, R \,+\, \lambda \, g_{\mu\nu} \;=\; 8 \pi \, G \, T_{\mu\nu} \;, \tag{3.2}$$

one is now lead to consider the modified effective field equations *i.e.*,

$$R_{\mu\nu} \,-\, \frac{1}{2} \, g_{\mu\nu} \, R \,+\, \lambda \, g_{\mu\nu} \;=\; 8 \pi \, G(\Box) \, T_{\mu\nu} \,, \tag{3.3}$$

with a new nonlocal term arising from $G(\Box)$. By being manifestly covariant they still satisfy some of the basic requirements for a set of consistent field equations incorporating the running of G. [1]

It is instructive to note, as pointed out in [124], that the effective nonlocal field equations of Eq. (3.3) can be recast in a form very similar to the classical field equations, but with a new source term $\tilde{T}_{\mu\nu} = (G(\Box)/G_0)\, T_{\mu\nu}$ defined as the gravitationally dressed, effective energy momentum tensor. The consistency of the effective field equations ultimately demands that it be exactly conserved, in consideration of the contracted Bianchi identity satisfied by the Ricci tensor. The running of G in the given picture above can be viewed as contributing to a vacuum fluid, introduced in order to account for the new gravitational vacuum polarization.

One has to take care of the appearance of a negative fractional exponent in Eq. (3.1), the covariant operator appearing in the expression for $G(\Box)$, which should suitably defined by analytic continuation. This can be done, for example, by computing $\Box^n$ for positive integer n, and then analytically continuing to $n \to -1/2\nu$ [124]. Equivalently, $G(\Box)$ can be defined via a suitable regulated parametric integral representation [125], such as

$$\left(\frac{1}{-\Box(g)+m^2}\right)^{1/2\nu} = \frac{1}{\Gamma(\frac{1}{2\nu})}\int_0^\infty d\alpha\;\alpha^{1/2\nu-1}\;e^{-\alpha\,(-\Box(g)+m^2)}\,. \tag{3.4}$$

As far as our calculations go, it will not be necessary to commit oneself to an excessively specific form for the running of $G(\Box)$. Thus for example, although the lattice gravity results only allow for a nondegenerate phase for the case $c_0 > 0$, it will nevertheless be possible later to have either sign for the correction in Eq. (3.1), in the sense that the very existence of a nontrivial ultraviolet fixed point implies in principle the appearance of two physically distinct phases, each of which may or may not be physically realized due to issues of nonperturbative stability. In principle, observation can be used to constrain one or the other choice. Furthermore, the value of the exponent ν need not to be specified until the very end of the

[1] In this Chapter, we discuss how the renormalization group running of $G(\Box)$ can be incorporated in a set of manifestly covariant effective field equations as given in Eq. (3.3). In Appendix C, we will approach the same problem from the point of view of an effective gravitational action.

calculation, so that most of the results remain general. [2]

3.1.1 (Zeroth Order) Effective Field Equations with $G(\Box)$

A scale dependent Newton's constant is expected to lead to small modifications of the standard cosmological solutions to the Einstein field equations. Here we will summarize what modifications are expected from the effective field equations on the basis of $G(\Box)$, as given in Eq. (3.1), which itself originates in Eq. (3.223). The starting point are the quantum effective field equations of Eq. (3.3), with $G(\Box)$ defined in Eq. (3.1). In the FLRW framework these are applied to the standard homogeneous isotropic metric

$$d\tau^2 \;=\; dt^2 - a^2(t) \left\{ \frac{dr^2}{1 - k\, r^2} + r^2 \, \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \right\} \,, \quad k = 0, \pm 1\,. \tag{3.8}$$

[2] A running cosmological constant $\lambda(k) \to \lambda(\Box)$ causes a number of mathematical inconsistencies [124] within the manifestly covariant framework, described here by the effective field equations of Eq. (3.3). In Chapter 1, we discussed the fact that a running cosmological constant $\lambda(k)$ is both inconsistent with the overall scaling properties of the gravitational functional integral in the continuum (Sec. 1.4.1) and with gauge invariance in the perturbative treatment about two dimensions (Sec. 1.4.2). The expectation is that serious inconsistencies will arise when a running cosmological constant is formulated within a fully covariant effective theory approach. The first step therefore is to promote again an renormalization group running in momentum space to a manifestly covariant form, $\lambda(k) \to \lambda(\Box)$ in the effective field equation of Eq. (3.3). To be more specific, consider the case of a scale dependent $\lambda(k)$, which we will write as $\lambda(k) = \lambda^{(0)} + \delta\lambda(k)$. We will assume for concreteness, that $\delta\lambda(k) \sim c_1 (k^2)^{-\sigma}$, where c_1 and σ are some constants. Then make the transition to coordinate space by replacing $k^2 \to -\Box$. This gives

$$\delta\lambda(\Box) \sim (-\Box(g) + m^2)^{\sigma} \,, \tag{3.5}$$

where we have been careful and used the infrared regulated expression given in Eq. (3.4). The effective field equations in Eq. (3.3) now contain the additional (running cosmological) term

$$\delta\lambda(\Box) \cdot g_{\mu\nu} = c_1 \frac{1}{\Gamma(\sigma)} \int_0^\infty d\alpha \, \alpha^{\sigma-1} e^{\alpha(-\Box(g)+m^2)} \cdot g_{\mu\nu} = c_1 (m^2)^{-\sigma} \cdot g_{\mu\nu} \,. \tag{3.6}$$

The result as shown above is still a numerical constant multiplying the metric $g_{\mu\nu}$. Note that we used the key result that covariant derivatives of the metric tensor vanish identically:

$$\nabla_\lambda g_{\mu\nu} = 0 \,. \tag{3.7}$$

The conclusion of this exercise is that λ cannot run. Note also another key aspect of the derivation that what matters is not just the form of $\lambda(\Box)$, but also the object it acts on. This last aspect is missed completely if one just focuses on $\lambda(k)$. Moreover, the above argument applies to possible additional contributions to the vacuum energy from various condensates and nonzero vacuum expectation values of matter fields, such as the QCD color field condensate, the quark condensate and the Higgs field. One is lead therefore to the conclusion that, quite generally, a running of λ in the effective field equations inevitably ends up in conflict with general covariance, in essence by virtue of Eq. (3.7).

In the following we will mainly consider the case $k = 0$ (spatially flat universe). It should be noted that there are in fact *two* related quantum contributions to the effective covariant field equations. The first one arises because of the presence of a nonvanishing cosmological constant $\lambda \simeq 3/\xi^2$, caused by the nonperturbative quantum vacuum condensate $< R > \neq 0$ [41, 42, 43]. As in the case of standard FLRW cosmology, this is expected to be the dominant contributions at large times t, and gives an exponential (for $\lambda > 0$), or cyclic (for $\lambda < 0$) expansion of the scale factor. The second contribution arises because of the explicit running of $G(\Box)$ in the effective field equations. The next step therefore is a systematic examination of the nature of the solutions to the full effective field equations, with $G(\Box)$ involving the relevant covariant d'Alembertian operator

$$\Box \; = \; g^{\mu\nu} \, \nabla_\mu \nabla_\nu \tag{3.9}$$

acting on second rank tensors as in the case of $T_{\mu\nu}$,

$$\begin{aligned} \nabla_\nu T_{\alpha\beta} \; &= \; \partial_\nu T_{\alpha\beta} - \Gamma_{\alpha\nu}^{\lambda} T_{\lambda\beta} - \Gamma_{\beta\nu}^{\lambda} T_{\alpha\lambda} \; \equiv \; I_{\nu\alpha\beta} \\ \nabla_\mu \left(\nabla_\nu T_{\alpha\beta} \right) &= \; \partial_\mu I_{\nu\alpha\beta} - \Gamma_{\nu\mu}^{\lambda} I_{\lambda\alpha\beta} - \Gamma_{\alpha\mu}^{\lambda} I_{\nu\lambda\beta} - \Gamma_{\beta\mu}^{\lambda} I_{\nu\alpha\lambda} \, . \end{aligned} \tag{3.10}$$

and in general requires the calculation of 1920 terms, of which fortunately many vanish by symmetry due to specific choice of metric.

To start the process, one assumes for example that $T_{\mu\nu}$ has a perfect fluid form,

$$T_{\mu\nu} = \left[\, p(t) + \rho(t) \, \right] u_\mu \, u_\nu + g_{\mu\nu} \, p(t) \tag{3.11}$$

for which one needs to compute the action of $\Box^n$ on $T_{\mu\nu}$, and then analytically continues the answer to negative fractional values of $n = -1/2\nu$. Even in the simplest case, with $G(\Box)$ acting on a *scalar* such as the trace of the energy momentum tensor $T_\lambda{}^\lambda$, one finds for the choice $\rho(t) = \rho_0 \, t^\beta$ and $a(t) = a_0 \, t^\alpha$ the rather unwieldy expression

$$\Box^n \left[-\rho(t) \right] \to 4^n \left(-1 \right)^{n+1} \frac{\Gamma\left(\frac{\beta}{2} + 1 \right) \, \Gamma\left(\frac{\beta + 3\,\alpha + 1}{2} \right)}{\Gamma\left(\frac{\beta}{2} + 1 - n \right) \, \Gamma\left(\frac{\beta + 3\,\alpha + 1}{2} - n \right)} \, \rho_0 \, t^{\beta - 2n} \, , \tag{3.12}$$

with an integer n later analytically continued to $n \to -\frac{1}{2\nu}$, with $\nu = 1/3$.

A more general calculation shows that a non-vanishing pressure contribution is generated in the effective field equations, even if one initially assumes a pressureless fluid, $p(t) = 0$. After a somewhat lengthy derivation one obtains for a universe filled with nonrelativistic matter ($p = 0$) the following set of effective Friedmann equations,

$$\begin{aligned} \frac{k}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)} &= \frac{8\pi\, G(t)}{3}\, \rho(t) + \frac{\lambda}{3} \\ &= \frac{8\pi\, G_0}{3} \left[1 + c_t\, (t/\xi)^{1/\nu} + \ldots \right] \rho(t) + \frac{\lambda}{3} \end{aligned} \tag{3.13}$$

for the tt field equation, and

$$\frac{k}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)} + \frac{2\,\ddot{a}(t)}{a(t)} = -\frac{8\pi\, G_0}{3} \left[c_t\, (t/t_0)^{1/\nu} + \ldots \right] \rho(t) + \lambda \tag{3.14}$$

for the rr field equation. In the above expressions the running of G appropriate for the RW metric is

$$G(t) \equiv G_0 \left(1 + \frac{\delta G(t)}{G_0} \right) = G_0 \left[1 + c_t \left(\frac{t}{t_0} \right)^{1/\nu} + \ldots \right] \tag{3.15}$$

with c_t of the same order as c_0 in Eq. (3.223), and $t_0 = \xi$ [124]; in the quoted reference it was estimated $c_t = 0.450\ c_0$ for the tensor box operator. Note that it is the running of G that induces an effective pressure term in the second (rr) equation, corresponding to the presence of a relativistic fluid due to the vacuum polarization contribution. One important feature of the new equations is an additional power law acceleration contribution, on top of the standard one due to λ.

3.1.2 Introduction of the w_{vac} Parameter

It was noted in [124] that the field equations with a running G, Eqs. (3.13) and (3.14), can be recast in an equivalent, but slightly more appealing, form by defining a vacuum polarization pressure p_{vac} and density ρ_{vac}, such that for the FLRW background one has

$$\rho_{vac}(t) = \frac{\delta G(t)}{G_0}\, \rho(t) \qquad\qquad p_{vac}(t) = \frac{1}{3}\, \frac{\delta G(t)}{G_0}\, \rho(t)\,. \tag{3.16}$$

Consequently the source term in the tt field equation can be regarded as a combination of two density terms

$$\left(1 + \frac{\delta G(t)}{G_0}\right) \rho(t) \equiv \rho(t) + \rho_{vac}(t)\,, \tag{3.17}$$

while the rr equation involves the new vacuum polarization pressure term

$$\frac{1}{3}\,\frac{\delta G(t)}{G_0}\,\rho(t) \equiv p_{vac}(t)\,. \tag{3.18}$$

Form this viewpoint, the inclusion of a vacuum polarization contributions in the FLRW framework seems to amount to a replacement

$$\rho(t) \to \rho(t) + \rho_{vac}(t) \qquad\qquad p(t) \to p(t) + p_{vac}(t) \tag{3.19}$$

in the original field equations. Just as one introduces the parameter w, describing the matter equation of state,

$$p(t) = w\,\rho(t) \tag{3.20}$$

with $w = 0$ for nonrelativistic for matter, one can do the same for the remaining contribution by setting

$$p_{vac}(t) = w_{vac}\;\rho_{vac}(t)\,. \tag{3.21}$$

We should remark here that

$$w_{vac} = \frac{1}{3} \tag{3.22}$$

is obtained *generally* for the given class of $G(\Box)$ considered, and is not tied therefore to a specific choice of ν, such as $\nu = \frac{1}{3}$. Then in terms of the two w parameters

$$\left(w + w_{vac}\frac{\delta G(t)}{G_0}\right) \rho(t) = p(t) + p_{vac}(t) \tag{3.23}$$

with, according to Eqs. (3.13) and (3.14) and following the results of [124], $w_{vac} = 1/3$ in a FLRW background. We should remark here that the calculations of [124] also indicate that $w_{vac} = 1/3$ is obtained *generally* for the given class of $G(\Box)$ considered, and is not tied therefore to a specific choice of ν, such as $\nu = 1/3$.

The previous, slightly more compact, notation allows one to re-write the field equations for the FLRW background in an equivalent form, which we will describe below. First we note though that in the following we will restrict our attention mainly to a spatially flat geometry, $k = 0$. Furthermore, when dealing with density perturbations we will have to distinguish between the background, which will involve a background pressure ($\bar{p}$) and background density ($\bar{\rho}$), from the corresponding perturbations which will be denoted here by δp and $\delta\rho$. Then with this notation and for constant G_0, the FLRW field equations for the background are written as

$$\begin{aligned} 3\,\frac{\dot{a}^2(t)}{a^2(t)} &= 8\pi\, G_0\, \bar{\rho}(t) + \lambda \\ \frac{\dot{a}^2(t)}{a^2(t)} + 2\,\frac{\ddot{a}(t)}{a(t)} &= -8\pi\, G_0\, \bar{p}(t) + \lambda\,. \end{aligned} \tag{3.24}$$

Now in the presence of a running $G(\Box)$, and in accordance with the results of Eqs. (3.13) and (3.14), the modified FLRW equations for the background read

$$\begin{aligned} 3\,\frac{\dot{a}^2(t)}{a^2(t)} &= 8\pi\, G_0 \left(1 + \frac{\delta G(t)}{G_0}\right) \bar{\rho}(t) + \lambda \\ \frac{\dot{a}^2(t)}{a^2(t)} + 2\,\frac{\ddot{a}(t)}{a(t)} &= -8\pi\, G_0 \left(w + w_{vac} \frac{\delta G(t)}{G_0}\right) \bar{\rho}(t) + \lambda\,, \end{aligned} \tag{3.25}$$

using the definitions in Eqs. (3.20) and (3.21), here with $\bar{p}_{vac}(t) = w_{vac}\, \bar{\rho}_{vac}(t)$.

We note here that the procedure of defining a ρ_{vac} and a p_{vac} contribution, arising entirely from quantum vacuum polarization effects, is not necessarily restricted to the FLRW background metric case [124]. In general one can decompose the full source term in the effective nonlocal field equations of Eq. (3.3), making use of

$$G(\Box) = G_0 \left(1 \,+\, \frac{\delta G(\Box)}{G_0}\right) \qquad \text{with} \quad \frac{\delta G(\Box)}{G_0} \equiv c_0 \left(\frac{1}{\xi^2 \Box}\right)^{1/2\nu}, \tag{3.26}$$

as two contributions,

$$\frac{1}{G_0}\, G(\Box)\, T_{\mu\nu} \;=\; \left(1 + \frac{\delta G(\Box)}{G_0}\right) T_{\mu\nu} \;=\; T_{\mu\nu} + T_{\mu\nu}^{vac}\,. \tag{3.27}$$

The latter involves the nonlocal part [3]

$$T^{vac}_{\mu\nu} \equiv \frac{\delta G(\Box)}{G_0} \, T_{\mu\nu} \, . \tag{3.28}$$

In addition, consistency of the full nonlocal field equations requires that the sum be conserved,

$$\nabla^{\mu} \left(T_{\mu\nu} + T^{vac}_{\mu\nu} \right) = 0 \, . \tag{3.29}$$

It is important to note at this stage that the nature of the covariant d'Alembertian $\Box \equiv g^{\mu\nu} \, \nabla_\mu \nabla_\nu$ is such that the result depends on the type of the object it acts on. Here $T_{\mu\nu}$ is a second rank tensor (as in Eq. (3.10)), which causes a reshuffling of components in $T_{\mu\nu}$ due to the matrix nature of both tensor $\Box$ and tensor $G(\Box)$, and eventually accounts for the generation of a nonvanishing induced pressure term. This is clearly seen in the effective field equations of Eqs. (3.13) and (3.14), and in the ensuing definitions of Eq. (3.16).

In general though one cannot expect that the contribution $T^{vac}_{\mu\nu}$ will always be expressible in the perfect fluid form of Eq. (3.11), even if the original $T_{\mu\nu}$ for matter (or radiation) has such a form. The former will in general contain, for example, nonvanishing shear stress contributions, even if they were originally absent in the matter part. Nevertheless the interesting question arises of whether, for example, $w_{vac} = 1/3$ continues to hold beyond the FLRW case treated above. In part this question will be answered affirmatively below, in the case of matter density perturbations.

[3] One normally does not include the *left hand side* field equation contribution $+\lambda g_{\mu\nu}$ as part of the *right hand side* matter part $T^{vac}_{\mu\nu}$, although it might be sensible to do so, given its large radiative (quantum) content [123]. We note here that the former is expected to contain the fundamental length scale ξ as well, in the form $\simeq +(3/\xi^2) \, g_{\mu\nu}$.

3.2 Relativistic Treatment of Matter Density Perturbations

Besides the modified cosmic scale factor evolution just discussed, the running of $G(\Box)$ given in Eq. (3.26) also affects the nature of matter density perturbations on very large scales. In computing these effects, it is customary to introduce a perturbed metric of the form

$$d\tau^2 = dt^2 - a^2\,(\delta_{ij} + h_{ij})\,dx^i dx^j\,, \tag{3.30}$$

with $a(t)$ the unperturbed scale factor and $h_{ij}(\mathbf{x}, t)$ a small metric perturbation, and $h_{00} = h_{i0} = 0$ by choice of coordinates. As will become clear later, we will mostly be concerned here with the trace mode $h_{ii} \equiv h$, which determines the nature of matter density perturbations. After decomposing the matter fields into background and fluctuation contribution, $\rho = \bar{\rho} + \delta\rho$, $p = \bar{p} + \delta p$, and $\mathbf{v} = \bar{\mathbf{v}} + \delta\mathbf{v}$, it is customary in these treatments to expand the density, pressure and metric trace perturbation modes in spatial Fourier modes,

$$\begin{aligned}
\delta\rho(\mathbf{x}, t) &= \delta\rho_{\mathbf{q}}(t)\, e^{i\,\mathbf{q}\cdot\mathbf{x}}\,, \qquad \delta p(\mathbf{x}, t) = \delta p_{\mathbf{q}}(t)\, e^{i\,\mathbf{q}\cdot\mathbf{x}} \\
\delta\mathbf{v}(\mathbf{x}, t) &= \delta\mathbf{v}_{\mathbf{q}}(t)\, e^{i\,\mathbf{q}\cdot\mathbf{x}}\,, \qquad h_{ij}(\mathbf{x}, t) = h_{\mathbf{q}\,ij}(t)\, e^{i\,\mathbf{q}\cdot\mathbf{x}}
\end{aligned} \tag{3.31}$$

with $\mathbf{q}$ the comoving wavenumber. Once the Fourier coefficients have been determined, the original perturbations can later be obtained from

$$\delta\rho(\mathbf{x}, t) \;=\; \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}}\, e^{-i\,\mathbf{q}\cdot\mathbf{x}}\, \delta\rho_{\mathbf{q}}(t) \tag{3.32}$$

and similarly for the other fluctuation components. Then the field equations with a constant G_0 (Eq. (3.2)) are given to zeroth order in the perturbations by Eq. (3.24), which fixes the 3 background fields $a(t)$, $\bar{\rho}(t)$ and $\bar{p}(t)$. Note that in a comoving frame the 4-velocity appearing in Eq. (3.11) has components $u^i = 1,\ u^0 = 0$. To first order in the perturbations and in the limit $\mathbf{q} \to 0$ the field equations give

$$\begin{aligned}
\frac{\dot{a}(t)}{a(t)}\,\dot{h}(t) &= 8\pi\, G_0\, \bar{\rho}(t)\, \delta(t) \\
\ddot{h}(t) + 3\,\frac{\dot{a}(t)}{a(t)}\,\dot{h}(t) &= -\,24\pi\, G_0\, w\, \bar{\rho}(t)\, \delta(t)
\end{aligned} \tag{3.33}$$

with the matter density contrast defined as $\delta(t) \equiv \delta\rho(t)/\bar{\rho}(t)$, $h(t) \equiv h_{ii}(t)$ the trace part of h_{ij}, and $w = 0$ for nonrelativistic matter. When combined together, these last two equations then yield a single equation for the trace of the metric perturbation,

$$\ddot{h}(t) + 2\,\frac{\dot{a}(t)}{a(t)}\,\dot{h}(t) \;=\; -\,8\pi\,G_0(1+3\,w)\,\bar{\rho}(t)\,\delta(t)\,. \tag{3.34}$$

From first order energy conservation one has $-\frac{1}{2}\,(1+w)\;h(t) = \delta(t)$, which then allows one to eliminate $h(t)$ in favor of $\delta(t)$. This finally gives a single second order equation for the density contrast $\delta(t)$,

$$\ddot{\delta}(t) + 2\,\frac{\dot{a}}{a}\,\dot{\delta}(t) - 4\pi\,G\,\bar{\rho}(t)\,\delta(t) = 0\,. \tag{3.35}$$

In the case of a running $G(\Box)$ these equations need to be rederived from the effective covariant field equations of Eq. (3.3), and lead to several additional terms not present at the classical level. Not surprisingly, as we shall see below, the correct field equations with a running G are not given simply by a naive replacement $G \to G(t)$, which would lead to incorrect results, and violate general covariance.

3.2.1 Zeroth Order Energy Momentum Conservation

As a first step in computing the effects of density matter perturbations one needs to examine the consequences of energy and momentum conservation, to zeroth and first order in the relevant perturbations. If one takes the covariant divergence of the field equations in Eq. (3.3), the *left hand side* has to vanish identically because of the Bianchi identity. The *right hand side* then gives $\nabla^\mu\left(T_{\mu\nu} + T^{vac}_{\mu\nu}\right) = 0$, where the fields in $T^{vac}_{\mu\nu}$ can be expressed, at least to lowest order, in terms of the p_{vac} and ρ_{vac} fields defined in Eqs. (3.16) and (3.21). The first equation one obtains is the zeroth (in the fluctuations) order energy conservation in the presence of $G(\Box)$, which reads

$$3\,\frac{\dot{a}(t)}{a(t)}\,\left[(1+w) + (1+w_{vac})\;\frac{\delta G(t)}{G_0}\right]\bar{\rho}(t) + \frac{\delta\dot{G}(t)}{G_0}\,\bar{\rho}(t) + \left(1 + \frac{\delta G(t)}{G_0}\right)\,\dot{\bar{\rho}}(t) = 0\,. \tag{3.36}$$

For $w = 0$ and $w_{vac} = \frac{1}{3}$ this reduces to

$$\left[3\,\frac{\dot{a}(t)}{a(t)} + 4\,\frac{\dot{a}(t)}{a(t)}\,\frac{\delta G(t)}{G_0} + \frac{\delta \dot{G}(t)}{G_0} \right] \bar{\rho}(t) + \left(1 + \frac{\delta G(t)}{G_0} \right) \dot{\bar{\rho}}(t) = 0 \,, \tag{3.37}$$

or equivalently in terms of the variable $a(t)$ only

$$\left[\frac{3}{a} + \frac{4}{a}\frac{\delta G(a)}{G_0} + \frac{\delta G'(a)}{G_0} \right] \bar{\rho}(a) + \left(1 + \frac{\delta G(a)}{G_0} \right) \bar{\rho}'(a) = 0 \,. \tag{3.38}$$

In the absence of a running G these equations reduce to the ordinary mass conservation equation for $w = 0$,

$$\dot{\bar{\rho}}(t) = -3\,\frac{\dot{a}(t)}{a(t)}\,\bar{\rho}(t) \,. \tag{3.39}$$

It will be convenient in the following to solve the energy conservation equation not for $\bar{\rho}(t)$, but instead for $\bar{\rho}(a)$. This requires that, instead of using the expression for $G(t)$ in Eq. (3.15), one uses the equivalent expression for $G(a)$

$$G(a) = G_0 \left(1 + \frac{\delta G(a)}{G_0} \right) , \qquad \text{with} \;\; \frac{\delta G(a)}{G_0} \equiv c_a \left(\frac{a}{a_0} \right)^{\gamma_\nu} + \dots \,. \tag{3.40}$$

In this last expression the power is $\gamma_\nu = 3/2\nu$, since from Eq. (3.15) one has for nonrelativistic matter $a(t)/a_0 \approx (t/t_0)^{2/3}$ in the absence of a running G. In the following we will almost exclusively consider the case $\nu = 1/3$ [52, 46, 47] for which therefore $\gamma_\nu = 9/2$. [4] Then in the above expression $c_a \approx c_t$ if a_0 is identified with a scale factor appropriate for a universe of size ξ; to a good approximation this should correspond to the universe "today", with the relative scale factor customarily normalized at such a time to $a/a_0 = 1$. Consequently, and with the above proviso, the constant c_a in Eq. (3.40) can safely be taken to be of the same order as the constant c_0 appearing in the original expressions for $G(\Box)$ in Eq. (3.26).

Then the solution to Eq. (3.37) can be written as

$$\bar{\rho}(a) = \text{const.}\; \exp \left\{ - \int \frac{da}{a} \, \left(3 + \frac{\delta G(a)}{G_0} + a\,\frac{\delta G'(a)}{G_0} \right) \right\} \,, \tag{3.41}$$

[4] This implicitly assumes that the cosmic evolution is largely matter dominated, if $p = w\rho$ then $a(t)/a_0 = (t/t_0)^{2/3(1+w)}$. In the opposite regime where a cosmological constant can eventually prevail one has instead $a(t)/a_0 = \exp \sqrt{\lambda/3}(t - t_0)$. Then $\frac{t}{t_0} = 1 + \frac{1}{t_0}\sqrt{\frac{3}{\lambda}} \log \frac{a}{a_0}$ and for $t_0 \simeq \xi$ and $\sqrt{\frac{3}{\lambda}} \simeq \xi$ one has simply $\frac{t}{t_0} = 1 + \log \frac{a}{a_0}$.

or, more explicitly, as

$$\bar{\rho}(a) = \bar{\rho}_0 \left(\frac{a_0}{a}\right)^3 \left(\frac{1+c_a}{1+c_a\left(\frac{a}{a_0}\right)^{\gamma_\nu}}\right)^{(1+\gamma_\nu)/\gamma_\nu} \simeq \bar{\rho}_0 \left(\frac{a_0}{a}\right)^3 \frac{1+(1+\gamma_\nu^{-1})\, c_a}{1+(1+\gamma_\nu^{-1})\, c_a \left(\frac{a}{a_0}\right)^{\gamma_\nu}} \tag{3.42}$$

with $\bar{\rho}(a)$ normalized so that $\bar{\rho}(a = a_0) = \bar{\rho}_0$. For $c_a = 0$ the above expression reduces of course to the usual result for nonrelativistic matter,

$$\bar{\rho}(t) = \bar{\rho}_0 \left(\frac{a_0}{a}\right)^3 . \tag{3.43}$$

Furthermore, here one also finds that the zeroth order momentum conservation equation is identically satisfied, just as in the case of constant G.

3.2.2 Zeroth Order Field Equations with Running $G(\Box)$

The zeroth order field equations with the running of G included were already given in Eq. (3.25). One can subtract the two equations from each other to get an equation that does not contain λ,

$$\frac{\dot{a}^2(t)}{a^2(t)} - \frac{\ddot{a}(t)}{a(t)} = 4\pi G_0 \left[(1+w) + (1+w_{vac})\, \frac{\delta G(t)}{G_0}\right] \bar{\rho}(t)\, . \tag{3.44}$$

Alternatively, from Eqs. (3.25) one can obtain a single equation that only involves the acceleration term with $\ddot{a}(t)$,

$$3\, \frac{\ddot{a}(t)}{a(t)} = -4\pi G_0 \left[(1+3\, w) + (1+w_{vac})\, \frac{\delta G(t)}{G_0}\right] \bar{\rho}(t) + \lambda\, . \tag{3.45}$$

It is also rather easy to check the overall consistency of the energy conservation equation, Eq. (3.37), and of the two field equations in Eq. (3.25). This is done by (i) taking the time derivative of the first tt equation in Eq. (3.25), (ii) replacing terms involving $\dot{\bar{\rho}}$ by $\bar{\rho}$ using the energy conservation equation, Eq. (3.37), and (iii) finally by substituting again the result of the first (tt) equation into Eq. (3.25) to obtain the second (rr) equation in Eq. (3.25).

3.2.3 Effective Energy Momentum Tensor ρ_{vac}, p_{vac}

The next step consists in obtaining the equations which govern the effects of small field perturbations. These equations will involve, apart from the metric perturbation h_{ij}, the matter and vacuum polarization contributions. The latter arise from (see Eq. (3.27))

$$\left(1+\frac{\delta G(\Box)}{G_0}\right) T_{\mu\nu} \;=\; T_{\mu\nu}+T^{vac}_{\mu\nu} \tag{3.46}$$

with a nonlocal $T^{vac}_{\mu\nu} \equiv (\delta G(\Box)/G_0)\, T_{\mu\nu}$. Fortunately to zeroth order in the fluctuations the results of Ref. [124] indicated that the modifications from the nonlocal vacuum polarization term could simply be accounted for by the substitution

$$\bar\rho(t) \to \;\bar\rho(t)+\bar\rho_{vac}(t) \qquad\qquad \bar p(t) \to \;\bar p(t)+\bar p_{vac}(t)\,. \tag{3.47}$$

Here we will apply this last result to the small field fluctuations as well, and set

$$\delta\rho_{\mathbf q}(t) \to \;\delta\rho_{\mathbf q}(t)+\delta\rho_{\mathbf q\,vac}(t) \qquad\qquad \delta p_{\mathbf q}(t) \to \;\delta p_{\mathbf q}(t)+\delta p_{\mathbf q\,vac}(t)\,. \tag{3.48}$$

The underlying assumptions is of course that the equation of state for the vacuum fluid still remains roughly correct when a small perturbation is added. Furthermore, just like we had $\bar p(t) = w\,\bar\rho(t)$ (Eq. (3.20)) and $\bar p_{vac}(t) = w_{vac}\,\bar\rho_{vac}(t)$ (Eq. (3.21)) with $w_{vac} = 1/3$, we now write for the fluctuations

$$\delta p_{\mathbf q}(t) = w\,\delta\rho_{\mathbf q}(t) \qquad \delta p_{\mathbf q\,vac}(t) = w_{vac}\,\delta\rho_{\mathbf q\,vac}(t)\,, \tag{3.49}$$

at least to leading order in the long wavelength limit, $\mathbf q \to 0$. In this limit we then have simply

$$\delta p(t) = w\,\delta\rho(t) \qquad \delta p_{vac}(t) = w_{vac}\,\delta\rho_{vac}(t) \equiv w_{vac}\,\frac{\delta G(t)}{G_0}\delta\rho(t)\,, \tag{3.50}$$

with $G(t)$ given in Eq. (3.15), and we have used Eq. (3.16), now applied to the fluctuation $\delta\rho_{vac}(t)$,

$$\delta\rho_{vac}(t) \;=\; \frac{\delta G(t)}{G_0}\,\delta\rho(t)+\ldots \tag{3.51}$$

where the dots indicate possible additional $\mathcal{O}(h)$ contributions.

A bit of thought reveals that the above treatment is incomplete, since $G(\Box)$ in the effective field equation of Eq. (3.3) contains, for the perturbed RW metric of Eq. (3.30), terms of order h_{ij}, which need to be accounted for in the effective $T^{\mu\nu}_{vac}$. Consequently the covariant d'Alembertian has to be Taylor expanded in the small field perturbation h_{ij},

$$\Box(g) \; = \; \Box^{(0)} + \Box^{(1)}(h) + O(h^2) \, , \tag{3.52}$$

and similarly for $G(\Box)$

$$G(\Box) \; = \; G_0 \left[1 + \frac{c_0}{\xi^{1/\nu}} \left(\frac{1}{\Box^{(0)} + \Box^{(1)}(h) + \mathcal{O}(h^2)} \right)^{1/2\nu} + \ldots \right] , \tag{3.53}$$

which requires the use of the binomial expansion for the operator $(A + B)^{-1} \; = \; A^{-1} - A^{-1} B A^{-1} + \ldots$. Thus for sufficiently small perturbations it should be adequate to expand $G(\Box)$ entering the effective field equations in powers of the metric perturbation h_{ij}. Since a number of subtleties arise in this expansion, we shall first consider the simpler case of a trace box, where some of the issues we think can be clearly identified, and addressed. After that, we will consider the more complicated case of the tensor box. This will be followed by a determination of the effects of the running of G on the relevant matter and metric perturbations, by the use of the modified field equations, now expanded to first order in the perturbations.

3.2.4 $\mathcal{O}(h)$ Correction using Trace Box

In this section the term $\mathcal{O}(h)$ in $\delta\rho_{vac}$ of Eq. (3.48) will be determined, using a set of formal manipulations involving the covariant trace box operator. Instead of considering the full field equations with a running $G(\Box)$, as given in Eq. (3.3),

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R + \lambda\, g_{\mu\nu} = 8\pi\, G_0 \left(1 + \frac{\delta G(\Box)}{G_0} \right) T_{\mu\nu} \tag{3.54}$$

we will consider here instead the action of a scalar $G(\Box)$ on the trace of the field equations for $\lambda = 0$,

$$R = -8\pi G_0 \left(1 + \frac{\delta G(\Box)}{G_0} \right) T_\lambda^{\ \lambda} \, , \tag{3.55}$$

or equivalently, by having the operator $G(\Box)$ act on the *left hand side*,

$$\left(1 - \frac{\delta G(\Box)}{G_0} + \dots \right) R = -8\pi \, G_0 \, T_\lambda^{\ \lambda} \, . \tag{3.56}$$

For a perfect fluid one has simply $T_\lambda^{\ \lambda} = -\rho$, which then gives [124]

$$G_0 \left(1 + \frac{\delta G(\Box)}{G_0} \right) T_\lambda^{\ \lambda} \;\; \rightarrow \;\; G_0 \left[1 + c_t \left(\frac{t}{t_0} \right)^{1/\nu} + \dots \right] T_\lambda^{\ \lambda} \equiv G(t) \, T_\lambda^{\ \lambda} \, , \tag{3.57}$$

or equivalently

$$G_0 \left[1 + c_t \left(\frac{t}{t_0} \right)^{1/\nu} + \dots \right] \bar{\rho}(t) \equiv G(t) \, \bar{\rho}(t) \, , \tag{3.58}$$

with $c_t \simeq 0.785 \, c_0$, and $t_0 = \xi$ [124] (in the tensor box case a slightly smaller value was found, $c_t \simeq 0.450 \, c_0$). The two terms in Eq. (3.58) are of course recognized, up to a factor of G_0, as the combination

$$\bar{\rho}(t) + \bar{\rho}_{vac}(t) \tag{3.59}$$

of Eq. (3.47), with $\bar{\rho}_{vac}(t) \equiv \delta G(t)/G_0 \cdot \bar{\rho}(t)$. Thus the zeroth order result obtained by the use of the scalar d'Alembertian acting on the trace of the field equations is consistent with what has been used so far for $G(t)$.

To compute the higher order terms in the h_{ij}'s appearing in the metric of Eq. (3.30) one needs to expand $G(\Box)$ according to Eq. (3.53) giving

$$G(\Box) = G_0 \left[1 + \frac{c_0}{\xi^{1/\nu}} \left(\left(\frac{1}{\Box^{(0)}} \right)^{1/2\nu} - \frac{1}{2\,\nu} \frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h) \cdot \left(\frac{1}{\Box^{(0)}} \right)^{1/2\nu} + \dots \right) \right] . \tag{3.60}$$

Here we are interested in the correction of order h_{ij}, when the above operator acts on the scalar $T_\lambda^{\ \lambda} = -\bar{\rho}$. This would then give the correction $\mathcal{O}(h)$ to $\delta\rho_{vac}$, namely the second term in

$$\delta\rho_{vac}(t) = \frac{\delta G(\Box^{(0)})}{G_0} \, \delta\rho(t) + \frac{\delta G(\Box)(h)}{G_0} \, \bar{\rho}(t) \, , \tag{3.61}$$

with the first term being simply given in the FLRW background by $\delta G(t)/G_0 \cdot \delta\rho(t)$. Here the $\mathcal{O}(h)$ correction is given explicitly by the expression

$$\frac{\delta G(\Box)(h)}{G_0}\,\bar{\rho} \;=\; -\frac{1}{2\,\nu}\,\frac{c_0}{\xi^{1/\nu}}\,\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h)\cdot\left(\frac{1}{\Box^{(0)}}\right)^{1/2\nu}\cdot\bar{\rho}\,. \tag{3.62}$$

The effect of the $(\Box^{(0)})^{-1/2\nu}$ term is essentially to make the coupling time dependent, *i.e.*, to correctly reproduce the required overall time-dependent factor $\delta G(t)/G_0$.

Now the scalar d'Alembertian $\Box = g^{\mu\nu}\,\nabla_\mu\nabla_\nu$ acting on scalar functions $\varphi(x)$ has the form

$$\Box\,\varphi(x) \equiv \frac{1}{\sqrt{g}}\,\partial_\mu\, g^{\mu\nu}\sqrt{g}\,\partial_\nu\,\varphi(x) \tag{3.63}$$

In the absence of h_{ij} fluctuations this gives for the metric in Eq. (3.30)

$$\Box^{(0)}\varphi(x) = \frac{1}{a^2}\nabla^2\varphi - 3\,\frac{\dot{a}}{a}\,\dot{\varphi} - \ddot{\varphi} \;\;\rightarrow\;\; \left(-\partial_t^2 - 3\,\frac{\dot{a}}{a}\partial_t\right)\varphi(t)\,, \tag{3.64}$$

where in the second expression we have used the properties of the RW background metric: we only need to consider functions that are time dependent, so that $\varphi(\mathbf{x},t)\to\varphi(t)$. To first order in the field fluctuation h_{ij} of Eq. (3.30) one computes

$$\Box^{(1)}(h)\,\varphi(x) = -\frac{1}{2}\dot{h}\,\dot{\varphi} - \frac{1}{a^2}h_{xx}\,\partial_x^2\varphi + \frac{1}{a^2}\,(-\partial_x\,h_{xx})\cdot\partial_x\varphi + \frac{1}{2\,a^2}\,\partial_x h\cdot\partial_x\varphi + \dots \tag{3.65}$$

with the trace $h(t) = h_{xx}(t) + h_{yy}(t) + h_{zz}(t)$. But for a function of the time only one obtains

$$\Box^{(1)}(h)\,\rho(t) = -\frac{1}{2}\,\dot{h}(t)\,\dot{\varphi}(t)\,. \tag{3.66}$$

Thus to first order in the fluctuations one obtains the expression

$$\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h)\cdot(\,\delta G\,\bar{\rho}\,) = \frac{1}{-\partial_t^2 - 3\,\frac{\dot{a}}{a}\,\partial_t}\cdot\frac{1}{2}\,\dot{h}\left(3\,\frac{\dot{a}}{a}\,\delta G - \delta\dot{G}\right)\bar{\rho} \tag{3.67}$$

where use has been made of the zeroth order mass conservation equation in Eq. (3.39). Note that this result also correctly incorporates the effect of $G(\Box^{(0)})$ on functions of t, as given for example in Eq. (3.57), which ensures the proper running of $\delta G(t)$.

Now in our treatment we are generally interested in mass density and metric perturbations around a near-static background described by $\dot{a}/a = H(t)$, and $\bar{\rho}(t)$. For these we expect the relevant time variations in $\delta\rho$ and h to be somewhat larger than for the background itself. Thus for sufficiently slowly varying background fields we retain only $h(t)$ and its derivatives, and for a sufficiently slowly varying $h(t)$ only $h(t)$ and the lowest derivatives. Then the factors of $\dot{a}/a$ are seen to cancel out at leading order between numerator and denominator in Eq. (B.61), and one is left simply with

$$\frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h) \cdot \delta G(t) \, \bar{\rho}(t) \; = \; - \, \frac{1}{2} \, \delta G(t) \, h(t) \, \bar{\rho}(t) + \dots \tag{3.68}$$

Putting everything together, one finds for the $\mathcal{O}(h)$ correction

$$\frac{\delta G(\Box)(h)}{G_0} \, \bar{\rho}(t) \simeq + \frac{1}{4\,\nu} \, \frac{\delta G(t)}{G_0} \, h(t) \, \bar{\rho}(t) \, . \tag{3.69}$$

The trace box calculation just described allows one to compute the correction $\mathcal{O}(h)$ to $\delta\rho_{vac}(t)$ in Eq. (B.55), and leads to the following $\mathcal{O}(h)$ modification of Eq. (3.51)

$$\delta\rho_{vac}(t) = \frac{\delta G(t)}{G_0} \, \delta\rho(t) + \frac{1}{2\,\nu} \, c_h \, \frac{\delta G(t)}{G_0} \, h(t) \, \bar{\rho}(t) \tag{3.70}$$

and similarly from $\delta p_{vac}(t) = w_{vac} \delta\rho_{vac}(t)$,

$$\delta p_{vac}(t) = w_{vac} \, \left(\frac{\delta G(t)}{G_0} \, \delta\rho(t) + \frac{1}{2\,\nu} \, c_h \, \frac{\delta G(t)}{G_0} \, h(t) \, \bar{\rho}(t) \right) \tag{3.71}$$

with $w_{vac} = 1/3$. The second $\mathcal{O}(h)$ terms in both expressions account for the feedback of the metric fluctuations h on the vacuum density $\delta\rho_{vac}$ and pressure δp_{vac} fluctuations.

The potential flaw with the preceding argument is that it assumes that certain very specific functions of the background stay constant, or at least very slowly varying. In the case at hand this was $\dot{a}/a \equiv H(a) \approx$ const. and $\rho \approx$ const., which in principle is not the only possibility, and would seem therefore a bit restrictive. A slightly more general approach, and a check, to the evaluation of the expression in Eq. (B.61) goes as follows. One assumes instead a harmonic time dependence for the metric fluctuation $h(t) = h_0 \, e^{i \omega t}$, and similarly for $a(t) = a_0 \, e^{i \Gamma t}$, $\bar{\rho}(t) = \bar{\rho}_0 \, e^{i \Gamma t}$, and $\delta G(t) = \delta G_0 \, e^{i \Gamma t}$; different frequencies for a and $\bar{\rho}$ could be considered as well, but here we will just stick with the simplest possibility. Then from the last expression in Eq. (B.61) one has

$$\frac{1}{-\partial_t^2 - 3\,\frac{\dot a}{a}\,\partial_t} \cdot \frac{1}{2}\,\dot h \left(3\,\frac{\dot a}{a}\,\delta G - \delta \dot G \right) \bar\rho \;=\; \frac{1}{\omega^2 + 7\Gamma\omega + 10\Gamma^2} \cdot (-\,\Gamma\,\omega\,\delta G\,h\,\bar\rho)\;. \tag{3.72}$$

In the limit $\omega \gg \Gamma$, corresponding to $\dot h/h \gg \dot a/a$, one obtains for the above expression

$$-\,\frac{\Gamma}{\omega}\,\delta G(t)\,h(t)\,\bar\rho(t) \;\simeq\; -\,\left(\frac{\dot a}{a}\,\frac{h}{\dot h} \right)\,\delta G(t)\,h(t)\,\bar\rho(t)\,, \tag{3.73}$$

after substituting back $\dot h/h = i\omega$ and $\dot a/a = i\Gamma$ in the last expression. Then $\delta\rho_{vac}(t)$ in Eq. (3.70) now involves the quantity c_h

$$c_h \;=\; \frac{\dot a}{a}\,\frac{h}{\dot h}\,. \tag{3.74}$$

At first this last factor (a function and not a constant) would seem rather hard to evaluate, and perhaps not even close to constant in time. But a bit of thought reveals that, to the order we are working, one can write

$$\frac{\dot h}{h}\,\frac{a}{\dot a} \;=\; \frac{\partial \log h(a)}{\partial \log a} \;=\; \frac{\partial \log \delta(a)}{\partial \log a} \equiv f(a)\,, \tag{3.75}$$

where $\delta(a)$ is the matter density contrast, and $f(a)$ the known density growth index [126]. In the absence of a running G (which is all that is needed, to the order one is working here) an explicit form for $f(a)$ is known in terms of derivatives of a Gauss hypergeometric function, which will be given below. One can then either include the explicit form for $f(a)$ in the above formula for$\delta\rho_{vac}(t)$, or use the fact that for a scale factor referring to "today" $a/a_0 \approx 1$, and for a matter fraction $\Omega \approx 0.25$, one knows that $f(a = a_0) \simeq 0.4625$, and thus in Eq. (3.70) one obtains the improved result $c_h \simeq 2.1621$. This can then be compared to the earlier result, which gave $c_h \simeq 1/2$.

A similar analysis can now be done in the opposite, but in our opinion less physical, $\omega \ll \Gamma$ limit, for which one now obtains for the expression in Eq. (3.72)

$$-\,\frac{1}{10}\,\left(\frac{a}{\dot a}\,\frac{\dot h}{h} \right)\,\delta G(t)\,h(t)\,\bar\rho(t)\,. \tag{3.76}$$

This new limit is less physical because of the fact that now the background is assumed to be varying more rapidly in time than the metric perturbation itself, $\dot{a}/a \gg \dot{h}/h$. For $\delta\rho_{vac}(t)$ one then obtains a similar expression to the one in Eq. (3.70), with a different coefficient

$$c_h \;=\; \frac{1}{10}\,\frac{a}{\dot{a}}\,\frac{\dot{h}}{h} \tag{3.77}$$

still involving the quantity $(a/\dot{a})(\dot{h}/h) \equiv f(a)$. By the same chain of arguments used in the previous paragraph one can now either include the explicit form for $f(a)$ in the formula for $\delta\rho_{vac}(t)$, or use the fact that for a scale factor referring to "today" $a/a_0 \approx 1$ and a matter fraction $\Omega \approx 0.25$ one knows that $f(a = a_0) \simeq 0.4625$. In this case one then has in Eq. (3.70) $c_h \simeq (1/10) \times 0.4625 = 0.0463$. One disturbing, but not entirely surprising, general aspect of the whole calculation in this second $\omega \ll \Gamma$ limit (as opposed to the previous treatment in the opposite limit) is its rather significant sensitivity, in the final result, to the set of assumptions initially made about the time development of the background as specified by the functions $a(t)$ and $\bar{\rho}(t)$. Therefore in the following we shall not consider this limit further.

To summarize, the results for a trace box and a slowly varying background, $\dot{h}/h \gg \dot{a}/a$, give the $\mathcal{O}(h)$ corrected expression for $\delta\rho_{vac}(t)$ in Eq. (3.70) and $\delta p_{vac}(t) = w_{vac}\,\delta\rho_{vac}(t)$, with $c_h \simeq +2.1621$.

3.2.5 $\mathcal{O}(h)$ Correction using Tensor Box

The results of Eqs. (B.55) and (3.70) for the vacuum polarization contribution,

$$\delta\rho_{vac}(t) = \frac{\delta G(t)}{G_0}\,\delta\rho(t) + \frac{1}{2\,\nu}\,c_h\,\frac{\delta G(t)}{G_0}\,h(t)\,\bar{\rho}(t) \tag{3.78}$$

and similarly for $\delta p_{vac}(t) = w_{vac}\,\delta\rho_{vac}(t)$ with $w_{vac} = 1/3$, were obtained using a scalar d'Alembertian to implement $G(\Box)$ by considering the trace of the field equation, Eq. (3.55). In this section we will discuss instead the result for the full tensor d'Alembertian, as it appears originally in the effective field equations of Eqs. (3.3) and (3.54).

Now the d'Alembertian operator $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ acts on the second rank tensor $T_{\mu\nu}$ as in Eq. (3.10), and should therefore be regarded as a 4×4 matrix, transforming $T_{\mu\nu}$ into $[\Box T]_{\mu\nu}$. Indeed it is precisely this matrix nature of $\Box$, and therefore of $G(\Box)$, that accounts for the fact that a vacuum pressure is induced in the first place, leading to a $w_{vac} \neq 0$.

To compute the correction of $\mathcal{O}(h)$ to $\delta\rho_{vac}(t)$ one needs to consider the relevant term in the expansion of $(1+\delta G(\Box)/G_0)\, T_{\mu\nu}$, which we write as

$$-\frac{1}{2\,\nu}\,\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h)\cdot\frac{\delta G(\Box^{(0)})}{G_0}\cdot T_{\mu\nu} \;=\; -\frac{1}{2\,\nu}\,\frac{c_0}{\xi^{1/\nu}}\,\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h)\cdot\left(\frac{1}{\Box^{(0)}}\right)^{1/2\nu}\cdot T_{\mu\nu}\,. \quad (3.79)$$

This last form allows us to use the results obtained previously for the FLRW case in [124], namely

$$\frac{\delta G(\Box^{(0)})}{G_0}\,T_{\mu\nu} \;=\; T^{vac}_{\mu\nu} \quad (3.80)$$

with here

$$T^{vac}_{\mu\nu} \;=\; [p_{vac}(t)+\rho_{vac}(t)]\,u_\mu\,u_\nu + g_{\mu\nu}\,p_{vac}(t) \quad (3.81)$$

and (see Eqs. (3.16) and (3.28)), to zeroth order in h,

$$\rho_{vac}(t) = \frac{\delta G(t)}{G_0}\,\bar\rho(t)\,, \qquad p_{vac}(t) = w_{vac}\,\frac{\delta G(t)}{G_0}\,\bar\rho(t)\,. \quad (3.82)$$

with $w_{vac} = 1/3$. Therefore, in light of the results of Ref. [124], the problem has been dramatically reduced to just computing the much more tractable expression

$$-\frac{1}{2\,\nu}\,\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h)\cdot T^{vac}_{\mu\nu}\,, \quad (3.83)$$

and in fact the only ordering for which the expression $(\delta G(\Box)/G_0)\,T_{\mu\nu}$ is calculable within reasonable effort. Still, in general the resulting expression for $\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h)$ is rather complicated if evaluated for arbitrary functions, although it does have a structure similar to the one found for the trace box in Eq. (B.61).

Here we will resort, for lack of better insights, to a treatment that parallels what was done before for the trace box, where one assumed a harmonic time dependence for the metric trace fluctuation $h(t) = h_0 \, e^{i\omega t}$, and similarly for $a(t) = a_0 \, e^{i\Gamma t}$ and $\rho(t) = \rho_0 \, e^{i\Gamma t}$. In the limit $\omega \gg \Gamma$, corresponding to $\dot{h}/h \gg \dot{a}/a$, one finds for the fluctuation $\delta\rho_{vac}(t)$ in Eq. (3.70)

$$\delta\rho_{vac}(t) = \frac{\delta G(t)}{G_0} \, \delta\rho(t) + \frac{1}{2\,\nu} \, c_h \, \frac{\delta G(t)}{G_0} \, h(t) \, \bar{\rho}(t) \, . \tag{3.84}$$

The $\mathcal{O}(h)$ correction factor c_h for the tensor box is now given by

$$c_h = \frac{11}{3} \, \frac{\dot{a}}{a} \, \frac{h}{\dot{h}} \, , \tag{3.85}$$

with all other off-diagonal matrix elements vanishing. Furthermore one finds to this order, but only for the specific choice $w_{vac} = 1/3$ in the zeroth order $T_{\mu\nu}^{vac}$,

$$\delta p_{vac}(t) \;\; = \;\; \frac{1}{3} \, \delta\rho_{vac}(t) \tag{3.86}$$

i.e., the $\mathcal{O}(h)$ correction preserves the original result $w_{vac} = 1/3$. In other words, the first order result $\mathcal{O}(h)$ just obtained for the tensor box would have been somewhat inconsistent with the zeroth order result, unless one had $w_{vac} = 1/3$ to start with. Now, one would not necessarily expect that the first order correction could be still be cast in the form of the same equation of state $p_{vac} \simeq (1/3)\rho_{vac}$ as the the zeroth order result, but it would nevertheless seem attractive that such a simple relationship can be preserved beyond the lowest order.

As far as the magnitude of the correction c_h in Eq. (3.85) one can argue again, as was done in the trace box case, that from Eq. (3.75) one can relate the combination $(\dot{h}/h)(a/\dot{a})$ to the growth index $f(a)$. Then, in the absence of a running G (which is all that is needed here, to the order one is working), an explicit form for $f(a)$ is known in terms of suitable derivatives of a Gauss hypergeometric function. These can then be inserted into Eq. (3.85). Alternatively, one can make use again of the fact that for a scale factor referring to "today" $a/a_0 \approx 1$, and for a matter fraction $\Omega \approx 0.25$, one knows that $f(a = a_0) \simeq 0.4625$, and thus in Eq. (3.70) $c_h \simeq (11/3) \times 2.1621 = +7.927$. This last result can then be compared to the earlier scalar result which gave $c_h \simeq +2.162$ using the same set of approximations (slowly varying background fields). It is encouraging that the new correction is a bit larger but not too different from what was found before in the trace box case. Note that so far the sign of the $\mathcal{O}(h)$ correction is the same in all physically relevant cases examined.

Next, as in the trace box case, one can do the same analysis in the opposite, but less physical, limit $\omega \ll \Gamma$ or $\dot{h}/h \ll \dot{a}/a$. One now obtains from the tt matrix element the $\mathcal{O}(h)$ correction in the expression for $\delta\rho_{vac}$ given in Eq. (3.70), namely

$$\frac{1}{2\,\nu}\, c_h \,\frac{\delta G(t)}{G_0}\; h(t)\,\bar{\rho}(t)\,. \tag{3.87}$$

with a coefficient

$$c_h = -\frac{121}{60}\,\frac{\omega^2}{\Gamma^2} = \simeq -\frac{121}{60}\,\left(\frac{a}{\dot{a}}\right)^2\,\frac{\ddot{h}}{h} = \mathcal{O}(\ddot{h}/h)\,. \tag{3.88}$$

Similarly for the ii matrix element of the $\mathcal{O}(h)$ correction one finds

$$\frac{1}{2\,\nu}\, a^2(t)\, c_h' \,\frac{\delta G(t)}{G_0}\; h(t)\,\bar{\rho}(t)\,. \tag{3.89}$$

with

$$c_h' = -\frac{5}{18} \tag{3.90}$$

giving now the $\delta p_{vac}(h)$ correction. Again all off-diagonal matrix elements are equal to zero. It seems therefore that this limit, $\omega \ll \Gamma$ or $\dot{h}/h \ll \dot{a}/a$, leads to rather different results compared to what had been obtained before: the only surviving contribution to $\mathcal{O}(h)$ is a rather large pressure contribution, with a sign that is opposite to all other cases encountered previously. Furthermore here the relationship $w_{vac} = 1/3$ is no longer preserved to $\mathcal{O}(h)$. But, as emphasized in the previous discussion of the trace box case, this second limit is in our opinion less physical, because of the fact that now the background is assumed to be varying more rapidly in time than the metric perturbation itself, $\dot{a}/a \gg \dot{h}/h$. Furthermore, as in the trace box calculation, one disturbing but not entirely surprising general aspect of the whole calculation in this second $\omega \ll \Gamma$ limit, is its extreme sensitivity as far as magnitudes and signs of the results are concerned, to the set of assumptions initially made about the time development of the background. As a final sample calculation let us mention here the case, similar to what was done originally for the trace box, where one assumes instead

$\dot{a}/a \equiv H(a) \approx$ const. and $\bar{\rho} \approx$ const., which, as we mentioned previously, seems now a bit restrictive. Nevertheless we find it instructive to show how sensitive the calculations are to the nature of the background, and in particular its assumed time dependence. In the notation of Eqs. (3.87), (3.88) and (3.90) one finds in this case

$$c_h = + \frac{625}{192} \frac{\omega^2}{H^2} = - \frac{625}{192} \frac{1}{H^2} \frac{\ddot{h}}{h} , \qquad c_h' = - \frac{4}{9} . \tag{3.91}$$

Again here the pressure contribution $\delta p_{vac}(h)$ is the dominant contribution, the $\delta \rho_{vac}(h)$ part being negligible, $O(\ddot{h})$. For the reasons mentioned, in the following we will no longer consider this limit of rapid background fluctuations any further.

To summarize, the results for a trace box and for a very slowly varying background, $\dot{h}/h \gg \dot{a}/a$, give the $\mathcal{O}(h)$ corrected expression for $\delta \rho_{vac}(t)$ in Eq. (3.70) and $\delta p_{vac}(t) = w_{vac} \, \delta \rho_{vac}(t)$ with $c_h \simeq +2.162$, while the tensor box calculation, under essentially the same assumptions, gives the somewhat larger result $c_h \simeq +7.927$. From now on, these will be the only two choices we shall consider here.

3.2.6 First Order Energy Momentum Conservation

The next step in the analysis involves the derivation of the energy momentum conservation to first order in the fluctuations, and a derivation of the relevant field equations to the same order. After that, energy conservation will be used to eliminate the h field entirely, and thus obtain a single equation for the matter density fluctuation δ.

The results so far can be summarized as follows. For the metric in Eq. (3.30), and in the limit $\mathbf{q} \to 0$, the field equations in Eq. (3.3) can now be written as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda \, g_{\mu\nu} = 8\pi \, G_0 \left(T_{\mu\nu} + T_{\mu\nu}^{vac} \right) , \tag{3.92}$$

with $T_{\mu\nu}^{vac} \equiv (\delta G(\Box)/G_0) \, T_{\mu\nu}$. Here $T_{\mu\nu}$ describes the ordinary matter contribution, in the form of a perfect fluid as given in Eq. (3.11), here with $p = w\rho$ and $w \simeq 0$, while $T_{\mu\nu}^{vac}$

describes the additional vacuum polarization contribution

$$T_{\mu\nu}^{vac} = [\, p_{vac}(t) + \rho_{vac}(t)\,]\, u_\mu\, u_\nu + g_{\mu\nu}\, p_{vac}(t) \tag{3.93}$$

with $p_{vac} = w_{vac}\, \rho_{vac}$ and $w_{vac} = 1/3$, as in Eq. (3.21). Furthermore, each field now contains both a background and a perturbation contribution,

$$\rho(t) = \bar{\rho}(t) + \delta\rho(t)\,, \qquad p(t) = w\, \rho(t) \tag{3.94}$$

and similarly

$$\rho_{vac}(t) = \bar{\rho}_{vac}(t) + \delta\rho_{vac}(t)\,, \qquad p_{vac}(t) = w_{vac}\, \rho_{vac}(t)\,. \tag{3.95}$$

From Eq. (3.16) one has

$$\bar{\rho}_{vac}(t) = \frac{\delta G(t)}{G_0}\, \rho(t)\,, \tag{3.96}$$

while from Eq. (3.70) one has

$$\delta\rho_{vac}(t) = \frac{\delta G(t)}{G_0}\, \delta\rho(t) + \frac{1}{2\,\nu}\, c_h\, \frac{\delta G(t)}{G_0}\, h(t)\, \bar{\rho}(t) \tag{3.97}$$

and similarly $\delta p_{vac}(t) = w_{vac}\, \delta\rho_{vac}(t)$. The second $\mathcal{O}(h)$ terms in both expressions physically account for the feedback of the metric fluctuations h on the vacuum density $\delta\rho_{vac}$ and pressure δp_{vac} fluctuations. In light of the discussion of the previous section, we will limit our derivations below to the case of constant c_h; the case of a non-constant c_h as in Eq. (3.75) can be dealt with as well, but the resulting equations are found to be quite a bit more complicated to write down.

Consequently all quantities in the effective field equations of Eq. (3.92) have been specified to the required order in the field perturbation expansion. First we will look here at the implications of energy momentum conservation, $\nabla^\mu \left(T_{\mu\nu} + T_{\mu\nu}^{vac} \right) = 0$, to first order in the fluctuations. The zeroth order energy conservation equation was already obtained in Eq. (3.36), and its explicit solution for $\bar{\rho}(a)$ given in Eq. (3.42). After defining the matter density contrast $\delta(t)$ as the ratio $\delta(t) \equiv \delta\rho(t)/\bar{\rho}(t)$, the energy conservation equation to first order in the perturbations is found to be

$$\left[-\frac{1}{2} \left((1+w) + (1+w_{vac}) \frac{\delta G(t)}{G_0} \right) - \frac{1}{2\nu} c_h \frac{\delta G(t)}{G_0} \right] \quad \dot{h}(t)$$
$$+ \left[\frac{1}{2\nu} \, c_h \, \left(3\,(w - w_{vac}) \, \frac{\dot{a}(t)}{a(t)} \frac{\delta G(t)}{G_0} - \frac{\delta \dot{G}(t)}{G_0} \right) \right] \quad h(t) \quad = \left[1 + \frac{\delta G(t)}{G_0} \right] \dot{\delta}(t). \quad (3.98)$$

In the absence of a running G ($\delta G(t) = 0$) this reduces simply to $-\frac{1}{2}\,(1+w)\;\dot{h}(t) = \dot{\delta}(t)$, and thus to the standard result for the metric trace perturbation in terms of the density contrast

$$-\,\frac{1}{2}\;(1+w)\;h(t) = \delta(t)\,. \quad (3.99)$$

This last result then allows us to solve explicitly, at the given order, *i.e.*, to first order in the fluctuations and to first order in δG, for the metric perturbation $\dot{h}(t)$ in terms of the matter density fluctuation $\delta(t)$ and $\dot{\delta}(t)$,

$$\dot{h}(t) = \; - \; \frac{2}{1+w} \left[1 + \frac{1}{1+w} \left((w - w_{vac}) - 2\,c_h \, \frac{1}{2\,\nu} \right) \frac{\delta G(t)}{G_0} \right] \dot{\delta}(t)$$
$$- \; \frac{1}{2\,\nu} \, \frac{4\,c_h}{(1+w)^2} \left[3 \; (w - w_{vac}) \; \frac{\dot{a}(t)}{a(t)} \, \frac{\delta G(t)}{G_0} \; - \; \frac{\delta \dot{G}(t)}{G_0} \right] \delta(t)\,. \quad (3.100)$$

Similarly, by differentiating the above relationship, an expression for $\ddot{h}(t)$ in terms of δ and its derivatives can be obtained as well.

3.2.7 First Order Field Equations

To first order in the perturbations, the tt and ii effective field equations become, respectively,

$$\frac{\dot{a}(t)}{a(t)} \, \dot{h}(t) - 8\pi \, G_0 \, \frac{1}{2\nu} \, c_h \, \frac{\delta G(t)}{G_0} \, \bar{\rho}(t) \, h(t) = 8\pi \, G_0 \left(1 + \frac{\delta G(t)}{G_0} \right) \bar{\rho}(t) \, \delta(t) \quad (3.101)$$

and

$$\ddot{h}(t) + 3 \, \frac{\dot{a}(t)}{a(t)} \, \dot{h}(t) + 24\pi \, G_0 \, \frac{1}{2\nu} \, c_h \, w_{vac} \, \frac{\delta G(t)}{G_0} \, \bar{\rho}(t) \, h(t) = -\,24\pi \, G_0 \left(w + w_{vac} \, \frac{\delta G(t)}{G_0} \right) \bar{\rho}(t) \, \delta(t) \quad (3.102)$$

In the second ii equation, the zeroth order ii field equation of Eq. (3.25) has been used to achieve some simplification.

As a final exercise, it is easy to check the overall consistency of the first order energy conservation equation of Eq. (3.98), and of the two field equations given in Eqs. (3.101) and (3.102). To do so, one needs to (i) take the time derivative of the tt equation in Eq. (3.101); (ii) get rid of $\dot{\bar{\rho}}$ consistently by using energy conservation to zeroth order in δG and in the fluctuations from Eq. (3.98) for terms of order δG times a fluctuation, combined with the use of energy conservation to first order in δG but without fluctuations as in Eq. (3.37) for the terms that are already of first order in the fluctuations; (iii) eliminate the $\dot{\delta}$ terms using the energy conservation equation to first order in δG without field fluctuations (Eq. (3.37)) for terms proportional to δG times a fluctuation, and using the energy conservation equation to first order in δG and in the fluctuation (again Eq. (3.98)) for terms of zeroth order in the fluctuations; (iv) use the combination of Eqs. (3.25) that does not contain λ, Eq. (3.44), to get rid of $\ddot{a}/a$ terms; (v) Finally use the tt equation for the fluctuation, Eq. (3.101), to eliminate some terms proportional to $\bar{\rho}$ times a fluctuation so as to finally obtain the second ii field equation Eq. (3.102).

3.2.8 Matter Density Contrast Equation in Time t

To obtain an equation for the matter density contrast $\delta(t) = \delta\rho(t)/\bar{\rho}(t)$ one needs to eliminate the metric trace field $h(t)$ from the field equations. This is first done by taking a suitable linear combination of the two field equations in Eqs. (3.101) and (3.102), to get the equivalent equation

$$\begin{aligned} \ddot{h}(t) + 2\,\frac{\dot{a}(t)}{a(t)}\,\dot{h}(t) \quad &+ \quad 8\pi\, G_0\, \frac{1}{2\nu}\, c_h\, \left(1 + 3\, w_{vac}\right) \frac{\delta G(t)}{G_0}\, \bar{\rho}(t)\, h(t) \\ &= \quad -\, 8\pi\, G_0 \left[(1 + 3\, w) + (1 + 3\, w_{vac})\ \frac{\delta G(t)}{G_0} \right] \bar{\rho}(t)\, \delta(t)\,. \end{aligned} \tag{3.103}$$

Then the first order energy conservation equations to zeroth (Eq. (3.99)) and first (Eq. (3.100)) order in δG allow one to completely eliminate the h, $\dot{h}$ and $\ddot{h}$ field in terms of the matter density perturbation $\delta(t)$ and its derivatives. The resulting equation reads, for $w = 0$ and

$w_{vac} = 1/3$,

$$\begin{aligned}\ddot{\delta}(t) \;\; &+ \left[\left(2\,\frac{\dot{a}(t)}{a(t)} - \frac{1}{3}\,\frac{\delta\dot{G}(t)}{G_0} \right) - \frac{1}{2\nu}\cdot 2\,c_h \cdot \left(\frac{\dot{a}(t)}{a(t)}\,\frac{\delta G(t)}{G_0} + 2\,\frac{\delta\dot{G}(t)}{G_0} \right) \right] \dot{\delta}(t) \\ &+ \Bigg[-4\pi\,G_0 \left(1 + \frac{7}{3}\,\frac{\delta G(t)}{G_0} - \frac{1}{2\nu}\cdot 2\,c_h \cdot \frac{\delta G(t)}{G_0} \right) \bar{\rho}(t) \\ &\quad - \frac{1}{2\nu}\cdot 2\,c_h \cdot \left(\frac{\dot{a}^2(t)}{a^2(t)}\,\frac{\delta G(t)}{G_0} + 3\,\frac{\dot{a}(t)}{a(t)}\,\frac{\delta\dot{G}(t)}{G_0} + \frac{\ddot{a}(t)}{a(t)}\,\frac{\delta G(t)}{G_0} + \frac{\delta\ddot{G}(t)}{G_0} \right) \Bigg] \delta(t) = 0\,. \end{aligned} \tag{3.104}$$

This last equation then describes matter density perturbations to linear order, taking into account the running of $G(\Box)$, and is therefore the main result. The terms proportional to c_h, which can be clearly identified in the above equation, describe the feedback of the metric fluctuations h on the vacuum density $\delta\rho_{vac}$ and pressure δp_{vac} fluctuations. The equation given above can now be compared with the corresponding, much simpler, equation obtained for constant G, *i.e.,* for $G \to G_0$ and still $w = 0$ (see for example [127] and [126])

$$\ddot{\delta}(t) + 2\,\frac{\dot{a}}{a}\,\dot{\delta}(t) - 4\pi\,G_0\,\bar{\rho}(t)\,\delta(t) = 0 \tag{3.105}$$

from which one obtains for the growing mode

$$\delta_{\mathbf{q}}(t) = \delta_{\mathbf{q}}(t_0)\,\left(\frac{t}{t_0}\right)^{2/3}, \tag{3.106}$$

which is the standard result in the matter dominated era.

3.2.9 Matter Density Contrast Equation in $a(t)$

It is common practice at this point to write an equation for the density contrast $\delta(a)$ as a function not of t, but of the scale factor $a(t)$. This is done by utilizing the following simple derivative identities

$$\dot{f}(t) = a\,H(a)\,\frac{\partial f(a)}{\partial a} \tag{3.107}$$

$$\ddot{f}(t) = a^2\,H^2(a)\left(\frac{\partial \ln H(a)}{\partial a} + \frac{1}{a}\right)\frac{\partial f(a)}{\partial a} + a^2\,H^2(a)\,\frac{\partial^2 f(a)}{\partial a^2} \tag{3.108}$$

where f is any function of t, and $H \equiv \dot{a}(t)/a(t)$ the Hubble constant. This last quantity can be obtained from the zeroth order tt field equation

$$H^2(a) \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\,\pi\,G_0}{3}\,\bar{\rho} + \frac{\lambda}{3}\,. \tag{3.109}$$

Often this last equation is written in terms of current density fractions,

$$H^2(a) \equiv \left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{\dot{z}}{1+z}\right)^2 = H_0^2\left[\Omega\,\left(1+z\right)^3 + \Omega_R\,\left(1+z\right)^2 + \Omega_\lambda\right] \tag{3.110}$$

with $a/a_0 = 1/(1+z)$ where z is the red shift, and a_0 the scale factor "today". Then H_0 is the Hubble constant evaluated today, Ω the (baryonic and dark) matter density, Ω_R the space curvature contribution corresponding to a curvature k term, and Ω_λ the dark energy or cosmological constant part, all again measured *today*. In the absence of spatial curvature $k=0$ one has today

$$\Omega_\lambda \equiv \frac{\lambda}{3\,H_0^2}\,, \qquad \Omega \equiv \frac{8\,\pi\,G_0\,\bar{\rho}_0}{3\,H_0^2}\,, \qquad \Omega + \Omega_\lambda = 1\,. \tag{3.111}$$

In terms of the scale factor $a(t)$ the equation for matter density perturbations for constant $G = G_0$, Eq. (3.105), becomes

$$\frac{\partial^2 \delta(a)}{\partial a^2} + \left[\frac{\partial \ln H(a)}{\partial a} + \frac{3}{a}\right]\,\frac{\partial\,\delta(a)}{\partial a} - 4\pi\,G_0\,\frac{1}{a^2 H(a)^2}\,\bar{\rho}(a)\,\delta(a) = 0\,. \tag{3.112}$$

The quantity $H(a)$ is most simply obtained from the FLRW field equations

$$H(a) = \sqrt{\frac{8\pi}{3}\,G_0\,\bar{\rho}(a) + \frac{\lambda}{3}}\,, \tag{3.113}$$

with the matter density given in Eq. (3.43), which can in principle be solved for $a(t)$,

$$t - t_0 = \int \frac{da}{a\,\sqrt{\frac{8\pi}{3}\,G_0\,\bar{\rho}_0\,\left(\frac{a_0}{a}\right)^3 + \frac{\lambda}{3}}}\,. \tag{3.114}$$

It is convenient at this stage to introduce a parameter θ describing the cosmological constant fraction as measured today,

$$\theta \equiv \frac{\lambda}{8\,\pi\,G_0\,\bar{\rho}_0} = \frac{\Omega_\lambda}{\Omega} = \frac{1-\Omega}{\Omega}\,. \tag{3.115}$$

While the following discussion will continue with some level of generality, in practice one is mostly interested in the observationally favored case of a current matter fraction $\Omega \approx 0.25$, for which $\theta \approx 3$. In terms of the parameter θ the equation for the density contrast $\delta(a)$ for constant G can then be recast in the slightly simpler form

$$\frac{\partial^2 \delta(a)}{\partial a^2} + \frac{3\,(1 + 2\,a^3\,\theta)}{2\,a\,(1 + a^3\,\theta)}\,\frac{\partial\,\delta(a)}{\partial a} - \frac{3}{2\,a^2\,(1 + a^3\,\theta)}\,\delta(a) = 0\,. \tag{3.116}$$

A general solution of the above equation is given by a linear combination of the two solutions

$$\delta_0(a) = c_1 \cdot \sqrt{1 + a^3\,\theta}\,a^{-3/2} + c_2 \cdot a \cdot {}_2F_1\left(\frac{1}{3}, 1; \frac{11}{6}; -a^3\,\theta\right) \tag{3.117}$$

where c_1 and c_2 are arbitrary constants, and ${}_2F_1$ is the Gauss hypergeometric function. The subscript 0 in $\delta_0(a)$ is to remind us that this solution is appropriate for the case of constant $G = G_0$. Since one is only interested in the growing solution, the constant $c_1 = 0$.

To evaluate the correction to $\delta_0(a)$ coming from the terms proportional to c_a one sets

$$\delta(a) \propto \delta_0(a)\,\left[\,1 + c_a\,\mathcal{F}(a)\,\right]\,, \tag{3.118}$$

and inserts the resulting expression in Eq. (3.104), written now as a differential equation in $a(t)$, after using Eqs. (3.107) and (3.108) to replace

$$\begin{aligned} \dot{a}(t) &= a\,H \\ \ddot{a}(t) &= a^2\,H^2\left(\frac{\partial \ln H}{\partial a} + \frac{1}{a}\right)\,. \end{aligned} \tag{3.119}$$

One only needs to determine the differential equations for density perturbations δ up to first order in the fluctuations, so it will be sufficient to obtain an expression for Hubble constant H from the tt component of the effective field equation to zeroth order in the fluctuations, namely the first of Eqs. (3.25). One has

$$H(a) = \sqrt{\frac{8\pi}{3}\,G_0\left(1 + \frac{\delta G(a)}{G_0}\right)\,\bar{\rho}(a) + \frac{\lambda}{3}} \tag{3.120}$$

with $G(a)$ given in Eq. (3.40) and $\bar{\rho}(a)$ given in Eq. (3.41). [5] In this last expression the exponent is $\gamma_\nu = 3/2\nu \simeq 9/2$ for a matter dominated background universe, although more

[5] We have noted before that Eq. (3.120) is suggestive of a small additional matter contribution, $\Omega_{vac} \simeq (8\pi/3)\delta G(a)\bar{\rho}_0/H_0^2$, to the overall balance in Eq. (3.111).

general choices, such as $\gamma_\nu = 3(1+w)/2\nu$ or even the use of Eq. (3.114), are possible and should be explored (see discussion later). Also, $c_a \approx c_t$ if a_0 is identified with a scale factor corresponding to a universe of size ξ; to a good approximation this corresponds to the universe "today", with the relative scale factor customarily normalized at that time to $a/a_0 = 1$. In [124] it was found that in Eq. (3.15) $c_t \simeq 0.785\, c_0$ in the trace box case, and $c_t \simeq 0.450\, c_0$ in the tensor box case; in the following we will use the average of the two values.

After the various substitutions and insertions have been performed, one obtains, after expanding to linear order in a_0, a second order linear differential equation for the correction $\mathcal{F}(a)$ to $\delta(a)$, as defined in Eq. (3.118). Since this equation looks rather complicated for general $\delta G(a)$ it won't be recorded here, but it is easily obtained from Eq. (3.104) by a sequence of straightforward substitutions and expansions. The resulting equation can then be solved for $\mathcal{F}(a)$, giving the desired density contrast $\delta(a)$ as a function of the parameter Ω.

Nevertheless with the specific choice for $G(a)$ given in Eq. (3.40) an explicit form for the equation for $\delta(a)$ reads:

$$\frac{\partial^2 \delta(a)}{\partial a^2} + A(a)\,\frac{\partial \delta(a)}{\partial a} + B(a)\,\delta(a) = 0\,. \tag{3.121}$$

with the two coefficients given by

$$A(a) = \frac{3(1+2a^3\theta)}{2a(1+a^3\theta)} - \frac{c_a\left(9a^3(1+\gamma_\nu)\,\theta\nu + a^{\gamma_\nu}\left(6c_h\gamma_\nu\,(1+2\gamma_\nu)\left(1+a^3\theta\right)^2 + \left(-9a^3\theta + \gamma_\nu\,\left(1+a^3\theta\right)\left(3+2\gamma_\nu\,\left(1+a^3\theta\right)\right)\right)\nu\right)\right)}{6\,a\,\nu\,\gamma_\nu\,\left(1+a^3\theta\right)^2} \tag{3.122}$$

and

$$B(a) = -\frac{3}{2a^2(1+a^3\theta)} - \frac{c_a\left(3a^3(1+\gamma_\nu)\,\theta\nu + a^{\gamma_\nu}\left(c_h\gamma_\nu(2+\gamma_\nu)\left(1+a^3\theta\right)\left(-1+2\gamma_\nu+2a^3(1+\gamma_\nu)\,\theta\right) + \left(4\gamma_\nu + a^3(-3+4\gamma_\nu)\,\theta\right)\nu\right)\right)}{2\,\nu\,\gamma_\nu\,a^2\,\left(1+a^3\theta\right)^2} \tag{3.123}$$

and the variable a considered just as a stand-in for what should really be the variable a/a_0. To obtain an explicit solution to the above equation one needs to know the coefficient c_a and the exponent γ_ν in Eq. (3.40), whose likely values are discussed above and right after the quoted expression for $G(a)$. For the exponent ν one has $\nu \simeq 1/3$, whereas for the value for c_h one finds, according to the discussion in the previous sections, $c_h \simeq 7.927$ for the tensor box case. Furthermore one needs at some point to insert a value for the matter density fraction parameter θ as given in Eq. (3.115), which based on current observation is close to $\theta = (1 - \Omega)/\Omega \simeq 3$.

3.2.10 Relativistic Growth Index γ with $G(\Box)$ in the Comoving Gauge

The solution of the above differential equation for the matter density contrast in the presence of a running Newton's constant $G(\Box)$ leads to an explicit form for the function $\delta(a) = \delta_0(a)[1 + c_a \mathcal{F}(a)]$. From it, an estimate of the size of the corrections coming from the new terms due to the running of G can be obtained. It is clear from the previous discussion, and the form of $G(\Box)$, that such corrections will become increasingly important in the present era $t \approx t_0$ or $a \approx a_0$. When discussing the growth of density perturbations in classical General Relativity it is customary at this point to introduce a scale-factor-dependent *growth index* $f(a)$ defined as

$$f(a) \equiv \frac{\partial \ln \delta(a)}{\partial \ln a} \, , \tag{3.124}$$

which is in principle obtained from the differential equation for any scale factor $a(t)$. Nevertheless, here one is mainly interested in the neighborhood of the present era, $a(t) \approx a_0$. One therefore introduces today's *growth index parameter* γ via

$$f(a = a_0) \equiv \left. \frac{\partial \ln \delta(a)}{\partial \ln a} \right|_{a=a_0} \equiv \, \Omega^\gamma \, , \tag{3.125}$$

so that the exponent γ itself is obtained via

$$\gamma \equiv \left. \frac{\ln f}{\ln \Omega} \right|_{a=a_0} . \tag{3.126}$$

The solution of the above differential equation for $\delta(a)$ then determines an explicit value for the growth index γ parameter, for any value of the current matter fraction Ω. In the end, because of observational constraints, one is mostly interested in the range $\Omega \approx 0.25$, so the following discussion will be limited to this case only, although from the original differential equation for $\delta(a)$ one can in principle obtain a solution for any sensible Ω. Numerically the differential equation for $\delta(a)$ can in principle be solved for any value of the parameters. In practice we have found it convenient, and adequate, to obtain the solution as a power series in either Ω or $1 - \Omega$. In the first case the resulting series is asymptotic and only slowly convergent around $\Omega \approx 0.25$, while in the latter case the convergence is much more rapid. In this last case we have carried therefore the expansion up to eighth order, which gives the answers given below (see also Figures 3.1, 3.2, 3.3, and 3.4) to an accuracy of several decimals.

It is known that in the absence of a running Newton's constant G ($G \to G_0$, thus $c_a = 0$) one has $f(a_0) = 0.4625$ and $\gamma = 0.5562$ for the standard ΛCDM scenario with $\Omega = 0.25$ [126]. On the other hand, when the running of $G(\Box)$ is taken into account, one finds from the solution to Eq. (3.104) for the growth index parameter γ at $\Omega = 0.25$ the following set of results.

Note that the $c_h \neq 0$, scalar and tensor box results can be summarized into the slightly more general formula

$$\gamma = 0.5562 - (0.703 + 25.04\, c_h)\, c_a + \mathcal{O}(c_a^2) . \tag{3.127}$$

showing again the overall importance of the c_h contribution to $\delta\rho_{vac}$ in Eq. (3.70). This last term is responsible for the feedback of the metric fluctuations h on the vacuum density $\delta\rho_{vac}$ and pressure δp_{vac} fluctuations. The above Eq. (3.127) reduces to the cases [1] and [2] below:

[1] For the tensor box case discussed in Sec. (3.2.5) one has the value $c_h = (11/3) \times 2.1621 = 7.927$ in Eqs. (3.70) and (3.97), which gives

$$\gamma = 0.5562 - 199.2\, c_a + \mathcal{O}(c_a^2)\,. \tag{3.128}$$

[2] For the trace box case discussed in Sec. (3.2.4) one has instead $c_h = 2.1621$ and in this case one finds

$$\gamma = 0.5562 - 54.8\, c_a + \mathcal{O}(c_a^2)\,. \tag{3.129}$$

As a comparison, we have also computed the exponent γ for the case $c_h = 0$ in Eqs. (3.70) and (3.97).

[3] Finally for the Newtonian (nonrelativistic) treatment, described in Appendix A, one finds the much smaller correction

$$\gamma = 0.5562 - 0.0142\, c_a + \mathcal{O}(c_a^2)\,. \tag{3.130}$$

Among these last expressions, the tensor box case is supposed to give ultimately the correct answer; the trace box case only serves as a qualitative comparison, and the $c_h = 0$ case is done to estimate independently the size of the correction coming from the ubiquitous $\mathcal{O}(h)$ or $\frac{1}{2\nu}\, c_h$ terms (see for example the differential equation for the density perturbations $\delta(t)$ in Eq. (3.104)).

It should be emphasized here once again that all of the above results have been obtained by solving the differential equation for $\delta(a)$, Eq. (3.121), with $G(a)$ given in Eq. (3.40), and exponent $\gamma_\nu = 3/2\nu \simeq 9/2$ relevant for a matter dominated background universe. It is this last choice that needs to be critically analyzed, as it might give rise to a definite bias. Our value for γ_ν so far reflects our choice of a matter dominated background. More general choices, such as an "effective" $\gamma_\nu = 3(1+w)/2\nu$ with and "effective" w, or even the use of Eq. (3.114), are in principle possible. Then, although Eq. (3.104) for $\delta(t)$ remains unchanged, Eq. (3.121) for $\delta(a)$ would have to be solved with new parameters. In the next section we will

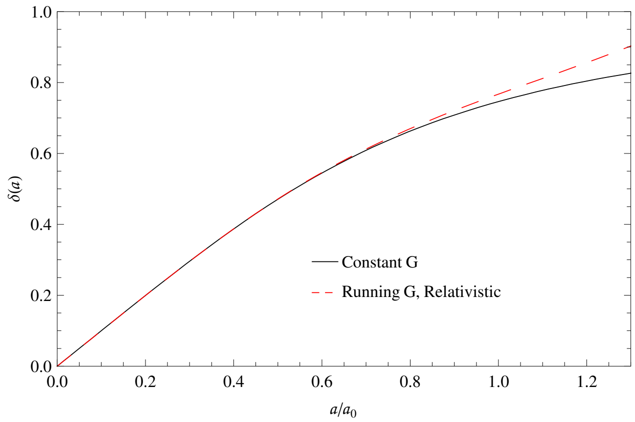

Figure 3.1: Illustration of the matter density contrast $\delta(a)$ as a function of the scale factor $a(t)$, in the fully relativistic treatment (tensor box) and for a given matter fraction $\Omega = 0.25$, obtained from the solution of the density contrast equation of Eq. (3.105), with $G(a)$ given in Eq. (3.40) with $\gamma_\nu = 9/2$ and for $c_a = 0.001$. In the case of a running $G(\Box)$, one generally observes a slightly faster growth rate for later times, as compared to the solution for the case of constant G and with the same choice of Ω, described by Eq. (3.116).

discuss a number of options which should allow one to increase on the accuracy of the above result, and in particular correct the possible shortcomings coming so far from the specific choice of the exponent γ_ν.

3.2.11 Further Elaborations of the Results of Growth Index γ

Looking at these last results (see also Figs. 3.1,3.2,3.3 and 3.4), they seem to indicate that (a) the correction due to the h (or $1/2\nu$) terms in Eq. (3.70) and in the differential equation, Eq. (3.104), for $\delta(a)$ is rather large, and that (b) it is more than twice as large in the tensor box case than it is in the trace box case. Furthermore they seem to suggest that (c) the Newtonian (nonrelativistic) result, which does not contain a ρ_{vac} contribution, substantially underestimates the size of the quantum correction. To quantitatively estimate the actual size of the correction in the above expressions for the growth index parameter γ, and make some preliminary comparison to astrophysical observations, some additional information is

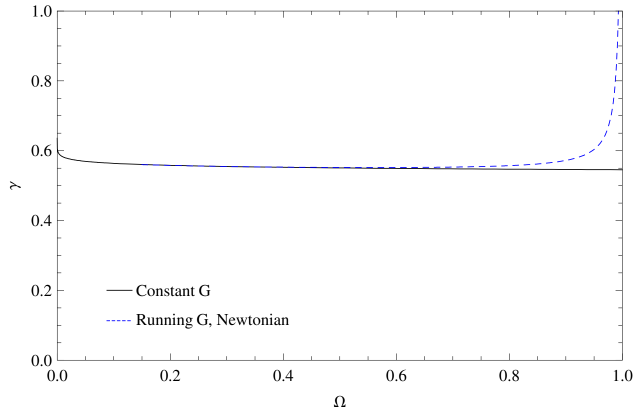

Figure 3.2: Illustration of the growth index parameter γ of Eq. (3.240) as a function of the matter density fraction Ω, computed in the Newtonian (nonrelativistic) theory with a running $G(a)$ given in Eq. (3.40), and obtained by solving Eq. (A.51), here with $\gamma_\nu = 9/2$ and $c_a = 0.01$. For the specific choice of matter fraction $\Omega = 0.25$, suggested by ΛCDM models, one then obtains the estimates for the growth index parameter given in Eq. (3.130).

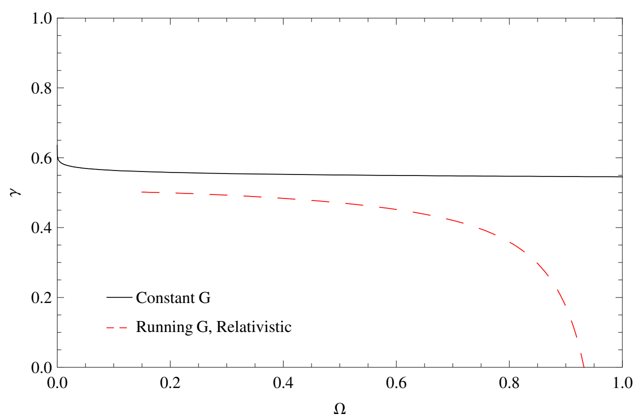

Figure 3.3: Illustration of the growth index parameter γ of Eq. (3.240) as a function of the matter density fraction Ω, computed in the fully relativistic (tensor box) theory with a running $G(a)$ as given in Eq. (3.40), and obtained by solving Eq. (3.104) with $\gamma_\nu = 9/2$ and $c_a = 0.0003$. For the specific choice of matter fraction $\Omega = 0.25$ one then obtains the estimates given for the tensor box in Eq. (3.128). Not surprisingly the deviations from the standard result for γ become more visible for larger values of Ω.

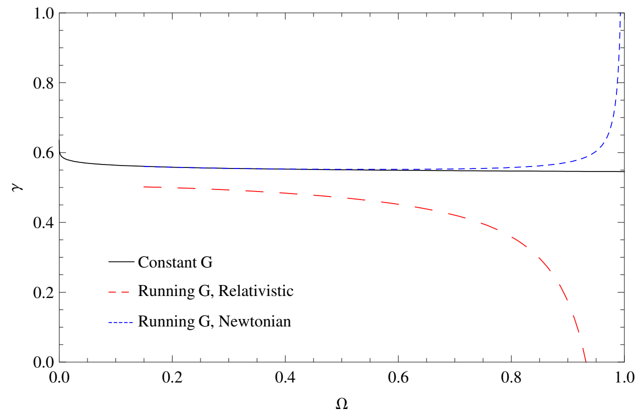

Figure 3.4: Qualitative comparison of the growth index parameters γ of Eq. (3.240) as a function of the matter density fraction Ω, computed first in the relativistic (tensor box) theory with a running $G(a)$ and $c_a = 0.0003$, then in the Newtonian (nonrelativistic) treatment also with a running $G(a)$ and $c_a = 0.01$, both with $\gamma_\nu = 9/2$, and finally compared to the usual treatment with constant G. In both cases the deviations from the standard result for γ are most visible for larger values of Ω, corresponding to a greater matter fraction.

needed.

The first item is the coefficient $c_0 \approx 8$ in Eq. (3.26) as obtained from recent computation (2013) [personal communication with Herbert Hamber]. This number is the result of reanalysis of [128] which involved rather large uncertainties for this particular quantity. The value of the constant c_0 has to be extracted from a nonperturbative lattice computation of invariant curvature correlations at fixed geodesic distance [128]; it relates the physical correlation length ξ to the bare lattice coupling G, and is therefore a genuinely nonperturbative amplitude. Based on experience with other field theoretic models which also exhibit nontrivial fixed points such as the nonlinear sigma model, as well as QCD and non Abelian gauge theories, one would expect this amplitude to be of order unity; very small or very large numbers would appear rather atypical and unnatural.

The next item that is needed here is a quantitative estimate for the magnitude of the coefficient c_a in Eq. (3.40) in terms of c_t in Eq. (3.15), and therefore in terms of c_0 in the original Eq. (3.26). First of all one has $c_a \approx c_t$, if a_0 is identified with a scale factor corresponding to a universe of size ξ; to a good approximation this corresponds to the universe "today", with the relative scale factor customarily normalized at that time to $a/a_0 = 1$, although some large conversion factor might be hidden in this perhaps naive identification (see below).

Regarding the numerical value of the coefficient c_t itself, it was found in [124] that in Eq. (3.15) $c_t \simeq 0.785\, c_0$ in the trace box case, and $c_t \simeq 0.450\, c_0$ in the tensor box case. In both cases these estimates refer to values obtained from the zeroth order covariant effective field equations. In the following we will take for concreteness the average of the two values, thus $c_t \approx 0.618\, c_0$. Then for all 3 covariant calculations recorded above $c_a \approx 0.618 \times 8 \approx 4.9$ ($c_a \approx c_t \approx 0.618\, c_0$ with $c_0 \approx 8$).

From all of these considerations one would tend to get estimates for the growth parameter γ with rather large corrections! For example, in the tensor box case the corrections would add up to $-199.\, c_a = -199. \times 0.618 \times 8 = -984$. In the Newtonian (nonrelativistic) case, where the correction is found to be the smallest, $c_a \approx c_t \approx 2.7\, c_0$ (see Appendix A), so the correction to the index γ becomes $-0.0142 \times 2.7 \times 8 = -0.31$.

Correcting for relative scales. It would seem though that one should account somewhere for the fact that the largest galaxy clusters and superclusters studied today up to redshifts $z \simeq 1$ extend for only about, at the very most, $1/20$ the overall size of the visible universe. This would suggest then that the corresponding scale for the running coupling $G(t)$ or $G(a)$ in Eqs. (3.15) and (3.40) respectively, should be reduced by a suitable ratio of the two relevant length scales, one for the largest observed galaxy clusters or superclusters, and the second for the very large, cosmological scale $\xi \sim 1/\sqrt{\lambda/3} \sim 1.51 \times 10^{28}$cm entering the expression for $\delta G(\Box)$ in Eqs. (3.3) and (3.26). This would dramatically reduce the magnitude of the quantum correction by as much as a factor of the order of $(1/20)^{\gamma_\nu} = (1/20)^{4.5} \approx 1.398 \times 10^{-6}$. When this correction factor is roughly taken into account, one obtains the more reasonable (and perhaps observationally more compatible) estimates for the tensor box case, and Eq. (3.128) is corrected to

$$\gamma = 0.5562 - 0.0057\, c_a + \mathcal{O}(c_a^2)\,. \tag{3.131}$$

and for the trace box case, Eq. (3.129) is corrected

$$\gamma = 0.5562 - 0.0016\, c_a + \mathcal{O}(c_a^2)\,. \tag{3.132}$$

while in the nonrelativistic *i.e.,* Newtonian case one finds Eq. (3.130) corrected to

$$\gamma = 0.5562 - 4.08 \times 10^{-7} c_a + \mathcal{O}(c_a^2)\,. \tag{3.133}$$

In the tensor box case this would then amount to a slightly reduced value for the growth index γ *at these scales* as compared to the constant G case, by as much as a few percent, which could perhaps be observable in the not too distant future. Of course, on larger scales the effects would be more significant, and somewhat bigger for larger values of Ω.

Adjusting further on $a \leftrightarrow t$. A second possibility we will pursue here briefly is to consider a shortcoming, mentioned previously, in the use of $a(t) \sim a_0(t/t_0)^{2/3}$ in relating $G(a)$ in Eq. (3.40) to $G(t)$ in Eq. (3.15). In general, if w is not small, one should use instead Eq. (3.114) to relate the variable t to $a(t)$. The problem here is that, loosely speaking, for $w \neq 0$ at least two w's are involved, $w = 0$ (matter) and $w = -1$ (λ term). Unfortunately, this issue complicates considerably the problem of relating $\delta G(t)$ to $\delta G(a)$, and therefore the solution to the resulting differential equation for $\delta(a)$. As a tractable approximation, one sets instead $a(t) \sim a_0(t/t_0)^{2/3(1+w)}$, and use an "effective" value of $w \approx -7/9$, which would appears appropriate for the final target value of $\Omega \approx 0.25$. For this choice one then obtains a significantly reduced power in Eq. (3.40), namely $\gamma_\nu = 3(1+w)/2\nu = 1$. Furthermore, the resulting differential equation for $\delta(a)$, Eq. (3.121), is still relatively easy to solve, by the same methods used in the previous section. One finds Eq. (3.127) to be corrected to

$$\gamma = 0.5562 - (0.92 + 7.70\, c_h)\, c_a + \mathcal{O}(c_a^2)\,. \tag{3.134}$$

In particular for the tensor box case one has again $c_h = 7.927$, which can be used to compare to the previous result of Eq. (3.128). Thus by reducing the value of γ_ν by about a factor of 4, the c_a coefficient in the above expression has been reduced by about a factor of 3, a significant change.

(Adjusting further on $a \leftrightarrow t$ and) Correcting for relative scales. After using the above improved value for the power γ_ν, the problem of correcting for relative scales needs to be addressed again, in light of the corrected estimate for the growth exponent parameter of Eq. (3.134). Given this new choice for $\gamma_\nu = 1$, on can now consider, for example, the types of galaxy clusters studied recently in [129, 130, 131, 132], which typically involve comoving radii of $\sim 8.5Mpc$ and viral radii of $\sim 1.4Mpc$. For these one would obtain an approximate overall scale reduction factor of $(1.4/4890)^{1/\nu} \approx 2.3 \times 10^{-11}$ with $\nu = 1/3$. This correction seems hidden from our computations as in Eq. (3.49), we approximated our fluctuations to be in the limit of large wave length, $\mathbf{q} \to 0$, which corresponds to the largest scale possible, *i.e.,* at the scale of $\xi \simeq 4890Mpc$. Therefore, implicitly, the reference scale appearing in $G(\Box)$ is of the order of $\xi \simeq 4890Mpc$. This would give for the tensor box ($c_h = 7.927$) correction to the growth index γ in Eq. (3.134) the more reasonable order of magnitude estimate $-62. \times 4.9 \times 2.3 \times 10^{-11} \approx -7.0 \times 10^{-11}$ (where we used $c_h = 7.927$ and $c_a \approx 0.618 \times 8 \approx 4.94$), and

$$\gamma = 0.5562 - 7.0 \times 10^{-11} + \mathcal{O}(c_a^2) \ \sim \ 0.5562\,, \tag{3.135}$$

being unobservable at this scale, and one needs to go larger scales to see any observable corrections. Clearly at this point these should only be considered as rough order of magnitude estimates. [6]

However, this last case is suggestive of a trend, quite independently of the specific value of c_h and therefore of the overall numerical coefficient of the correction in Eq. (3.134): namely

[6] One might perhaps think that the running of G envisioned here might lead to small observable consequences on much shorter, galactic length scales. That this is not the case can be seen, for example, from the following argument. For a typical galaxy one has a size $\sim 30kpc$, giving for the quantum correction the estimate, from Eq. (1.61) for the correction to the static potential, $(30kpc/4890 \times 10^3 kpc)^3 \sim 2.31 \times 10^{-16}$ which is tiny given the large size of ξ. It is therefore unlikely that such a correction will be detectable at these length scales, or that it could account for large anomalies in the galactic rotation curves.

that the correction to the growth index parameter will increase close to linearly (for γ_ν close to one, as we have argued) in the size of the cluster. Consequently one expects that the deviations will increase tenfold in going from a cluster size of $1Mpc$ to one of $10Mpc$, and a hundredfold in going from $1Mpc$ to $100Mpc$.

Finally another possible, and ultimately much more conservative, approach would be to take - at least for the time being - with some caution the rather large value for c_0 obtained from nonperturbative lattice quantum gravity calculations. One could then use instead the observational bounds on x-ray studies of large galactic clusters at distance scales of up to about $1.4 - 8.5Mpc$ [130, 131], namely $\gamma = 0.50 \pm 0.08$, to constrain the value of the constant c_a *at that scale*, giving for example from Eq. (3.134) the bound $c_a \lesssim 8 \times 10^{-4}$ in the case of tensor box, and the much less stringent bound $c_a \lesssim \mathcal{O}(1)$ for the Newtonian (nonrelativistic) case of Eq. (3.130).

3.2.12 Discussions

We have attempted to systematically analyze the effects on matter density perturbations of a running $G(\Box)$ appearing in the original effective, nonlocal covariant field equations of Eq. (3.3). The specific form of $G(\Box)$ in Eq. (3.1) is inspired by the nonperturbative treatment of covariant path integral quantum gravity, and follows from the existence of a nontrivial fixed point in G of the renormalization group in 4 dimensions. The resulting effective field equations are manifestly covariant, and in principle besides the genuinely nonperturbative scale ξ there are no adjustable parameters, since the coefficients (c_0) and scaling dimensions (ν) entering $G(\Box)$ are, again in principle, calculable by systematic field theory and lattice methods ([25] and references therein).

The present work can be viewed in broad terms as consisting of two parts. In the first part we have systematically developed the general formalism necessary to deal with small matter density fluctuations in the presence of a running gravitational coupling $G(\Box)$. Most, if not all, of the results in the first part have been formulated in a way that assumes as little as possible about specific aspects related to how exactly G does run with scale. Indeed many of the equations we have obtained are not restricted to $\nu = 1/3$, and are found to be valid for a wide range of powers ν and coefficients c_0 appearing for example in the original expression for $G(\Box)$ as given in Eq. (3.26). Furthermore, the zeroth order (in the fluctuations) results of [124], on which the present work builds up, do not rely on any specific value for these parameters either, since the expressions obtained there follow from general properties of the covariant d'Alembertian and its powers, as they appear in $G(\Box)$. In particular the flow in the vicinity of the ultraviolet fixed point could in principle allow for c_0 being either negative (gravitational screening) or positive (gravitational antiscreening), and both cases could in principle be described by the results obtained above, for example for the growth index f and the growth index parameter γ. It is only the latter option though that is favored by studies of nonperturbative Euclidean lattice gravity (the weak coupling phase is unstable and found to describe a collapsed degenerate 2 - dimensional spacetime), hence the choice here to discuss primarily this last case. But in principle the fact remains that the sign of c_0 will ultimately determine the direction of the corrections given above, which could eventually become constrained by observation. In the end the only result that is extensively used in the first part is the result of [124] that $w_{vac} = 1/3$, apart from the fact that we choose to restrict our attention from the very beginning primarily to the nonrelativistic matter case $w = 0$, and to the large wavelength limit $\mathbf{q} \to 0$. Later on it was found that for sufficiently slowly varying backgrounds the result $w_{vac} = 1/3$ is preserved also to first order in the perturbations, which seems to suggest some level of consistency in the treatment of the field perturbations.

In spite of the nonlocality of the original effective field equations in Eq. (3.3), one finds quite in general that small perturbations can be treated, in a first approximations, in terms of local terms, described by quantities ρ_{vac} and p_{vac} as they appear in the effective description of $T_{\mu\nu}^{vac}$ in terms of a perfect fluid. The latter should then be regarded as the leading term in a derivative expansion of the nonlocal contribution to the effective field equations, as they apply here to the rather specific case of the $FLRW$ background. Under the physically motivated assumption of a comparatively slowly varying (both in space and time) background, it is then possible to obtain a complete and consistent set of effective field equations, describing small perturbations for the metric trace and matter modes (Eqs. (3.98), (3.101), (3.102) and (3.103)). From these a single equation for the matter density contrast is eventually obtained, Eq. (3.104), which is the main result of this work. The only input needed in this last equation is $\delta G(t)$, the zeroth order (in the fluctuations) running of G as written in Eq. (3.15), with given more or less known parameters ν and c_t. The corresponding result in the Newtonian (nonrelativistic) treatment is obtained in Appendix A, leading to Eq. (A.45).

We remind here that for a translation of the equation for the density contrast $\delta(t)$ into the corresponding equation for $\delta(a)$, we assumed quantum correction in $G(a)$ to be written as a power, with an exponent γ_ν. Subsequently a solution for the differential equation for $\delta(a)$ was obtained, leading to expressions for the growth index $f(a)$ and for the growth index parameter γ. A number of general features can be observed, the first one being the fact that generally the correction to the growth index parameter γ is found to be negative, indicating a less steep rise of f with Ω.

Then we addressed a number of attempts to provide a semi quantitative estimate for the corrections obtained, in order to see if these corrections could be related in some way to current astrophysical observations. For this, one needed to adapt the theoretical calculation for the growth index parameter γ to the kind of observational data available from the study of large galactic clusters. It requires a careful consideration of the relative length scales that come into play. One length scale is given by the size of the largest clusters reached by observation, typically of the order of a few Mpcs. On the other hand it should involve the absolute reference scale given by $\xi = \sqrt{3/\lambda} \simeq 4890 Mpc$. The comparison between theory and observation would then seem straightforward, were it not for the fact that this ratio generally comes in to a certain power, whose detailed knowledge is necessary in order to eventually reduce the quantitative uncertainties. In addition, there is still perhaps a certain level of uncertainty in the actual coefficients c_0 and c_t entering the theoretical predictions, which we have also described above in some detail. The latter could be reduced further by improved nonperturbative lattice computations. Nevertheless, the value of the present calculations lies in our opinion in the fact that so far a discernible trend seems to emerge from the results. The trend we have found seems to suggest that the correction to the growth exponent γ is initially rather small for small clusters, negative in sign, and then slowly increasing in magnitude with scale.

It is clear that the effects discussed in this paper are only relevant for very large scales, much bigger than those usually considered, and well constrained, by laboratory, solar or galactic dynamics tests [133, 134, 135, 136] Future more accurate astrophysical observations might make it possible to see the difference in the predictions of various models [137, 138, 139, 140, 141].

3.3 To Another Gauge: Gauge Choices and Transformations

The previous discussion and summary focused exclusively on the comoving gauge choice for the metric, implicit in the definition of Eq. (3.8). Next we will consider some additional gauges. In this paper we will specifically refer to *three* choices for the metric: the comoving, synchronous and conformal Newtonian forms. The first two are closely related to each other, and were used to obtain part of the results presented in our previous work [107].

3.3.1 Comoving, Synchronous and Conformal Newtonian Gauges

The *comoving* metric has the form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \ , \tag{3.136}$$

with background metric

$$\bar{g}_{\mu\nu} = \mathrm{diag}\left(-1, a^2, a^2, a^2\right) \ . \tag{3.137}$$

For the fluctuation one sets

$$h_{0i} = h_{i0} = 0 \ , \tag{3.138}$$

and decomposes the remaining h_{ij} as

$$h_{ij}(\mathbf{q}, t) \ = \ a^2 \left[\frac{1}{3} \, h \, \delta_{ij} + \left(\frac{1}{3} \, \delta_{ij} - \frac{k_i \, k_j}{k^2} \right) \, s \right] \tag{3.139}$$

so that $Tr(h_{ij}) = a^2 \, h$. Besides the scale factor a, the metric is therefore parameterized in terms of the two functions s and h.

On the other hand, in the *synchronous* gauge one sets again $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ now with background metric

$$\bar{g}_{\mu\nu} = a^2 \, \mathrm{diag}\left(-1, 1, 1, 1\right) \ . \tag{3.140}$$

For the fluctuation one sets again $h_{0i} = h_{i0} = 0$ and

$$h_{ij}(\mathbf{q}, t) \ = \ a^2 \left[\frac{k_i \, k_j}{k^2} \, \widetilde{h} + \left(\frac{k_i \, k_j}{k^2} - \frac{1}{3} \, \delta_{ij} \right) \, 6 \, \widetilde{\eta} \right] \ , \tag{3.141}$$

so that now $Tr(h_{ij}) = a^2 \, \widetilde{h}$. Here, besides the overall scale factor a, the metric is parameterized in terms of the two functions $\widetilde{\eta}$ and $\widetilde{h}$. From a comparison of the two gauges (comoving and synchronous) one has

$$2 \, \widetilde{\eta} \ = \ -\frac{1}{3} \, (h + s) \tag{3.142}$$

and

$$\widetilde{h} + 6 \, \widetilde{\eta} \ = \ -s \ . \tag{3.143}$$

Finally the *conformal Newtonian* gauge is in turn described by two scalar potentials ψ and ϕ. In this case the line element is given by

$$d\tau^2 = -g_{\mu\,\nu} dx^\mu \, dx^\nu = a^2 \left\{ (1 + 2\,\psi)\; dt^2 - (1 - 2\,\phi)\; dx_i \, dx^i \right\} . \tag{3.144}$$

Therefore for the metric itself one writes again $g_{\mu\,\nu} = \bar{g}_{\mu\,\nu} + h_{\mu\,\nu}$ with $\bar{g}_{\mu\,\nu} = a^2 \, diag\,(-1, 1, 1, 1)$ as for the synchronous case, and furthermore $h_{0i} = h_{i0} = 0$ as before, and now

$$h_{00} = a^2 \; (-\,2\,\psi) \tag{3.145}$$

$$h_{ij} = a^2 \; (-\,2\phi)\; \delta_{ij} \; . \tag{3.146}$$

A suitable set of gauge transformations then allows one to go from the synchronous, or comoving, to the conformal Newtonian gauge [142].

3.3.2 Tensor Box in the Comoving Gauge

To compute higher order contributions from the h_{ij} 's appearing in the comoving gauge metric, one needs to expand $G(\Box)$ in the various metric perturbations,

$$G(\Box) = G_0 \left[1 + \frac{c_0}{\xi^{1/\nu}} \left(\left(\frac{1}{\Box^{(0)}} \right)^{1/2\nu} - \frac{1}{2\,\nu} \frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h,s) \cdot \left(\frac{1}{\Box^{(0)}} \right)^{1/2\nu} + \ldots \right) \right] , \tag{3.147}$$

where the superscripts (0) and (1) refer to zeroth and first order in this expansion, respectively. To get the correction of $\mathcal{O}(h,s)$ to the field equations, one therefore needs to consider the relevant term in the expansion of $(1 + \delta G(\Box)/G_0)\, T_{\mu\nu}$,

$$-\frac{1}{2\,\nu} \frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h,s) \cdot \frac{\delta G(\Box^{(0)})}{G_0} \cdot T_{\mu\nu} \;\; = \;\; -\frac{1}{2\,\nu} \frac{c_0}{\xi^{1/\nu}} \frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h,s) \cdot \left(\frac{1}{\Box^{(0)}} \right)^{1/2\nu} \cdot T_{\mu\nu} \; . \tag{3.148}$$

This last form allows us to use the results obtained previously for the FLRW case, namely

$$\frac{\delta G(\Box^{(0)})}{G_0} \, T_{\mu\nu} \;\; = \;\; T^{vac}_{\mu\nu} \tag{3.149}$$

with here

$$T^{vac}_{\mu\nu} \;\; = \;\; [p_{vac}(t) + \rho_{vac}(t)]\, u_\mu \, u_\nu + g_{\mu\nu} \, p_{vac}(t) \tag{3.150}$$

to zeroth order in h, and

$$\rho_{vac}(t) = \frac{\delta G(t)}{G_0}\,\bar{\rho}(t) \qquad\qquad p_{vac}(t) = w_{vac}\,\frac{\delta G(t)}{G_0}\,\bar{\rho}(t)\;. \tag{3.151}$$

and $w_{vac} = 1/3$. Therefore, in light of the results of Ref. [124], the problem has been reduced to computing the more tractable expression

$$-\,\frac{1}{2\,\nu}\,\frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h,s) \cdot T^{vac}_{\mu\nu}\;. \tag{3.152}$$

To make progress, we will assume a harmonic time dependence for both the perturbations $h(t) = h_0\,e^{i\omega t}$ and $s(t) = s_0\,e^{i\omega t}$, and for the background quantities $a(t) = a_0\,e^{i\Gamma t}$, $\rho(t) = \rho_0\,e^{i\Gamma t}$, and $\delta G(t) = \delta G_0\,e^{i\Gamma t}$. From now on we shall consider both ω and Γ as slowly varying functions (indeed constants), with the time scale of variations for the perturbation much shorter than the time scale associated with all the background quantities. A more sophisticated treatment will be reserved for future work. Therefore we will take here $\omega \gg \Gamma$ or $\dot{h}/h \gg \dot{a}/a$, which is the same approximation that was used in obtaining the results of Ref. [106].

Let us now list, in sequence, the required matrix elements needed for the present calculation. For the tensor box tt matrix element $(-\frac{1}{2\,\nu}\,\frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h,s) \cdot T^{vac})_{00}$ one obtains

$$+\;\frac{1}{2\nu}\,\frac{11}{3}\,\frac{\delta G(t)}{G_0}\,\rho(t)\,\frac{\Gamma}{\omega}\,h + \mathcal{O}(k^2)\;. \tag{3.153}$$

For the tensor box ti matrix element $(-\frac{1}{2\,\nu}\,\frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h,s) \cdot T^{vac})_{0i}$ one obtains

$$-\;i\,k_i\,\frac{1}{2\nu}\,\frac{2}{9}\,\frac{\delta G(t)}{G_0}\,\rho(t)\,\frac{1}{i\,\omega}\;(h - 2\,s) + \mathcal{O}(k^2)\;. \tag{3.154}$$

For the tensor box ii matrix element, summed over i, $(-\frac{1}{2\,\nu}\,\frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h,s) \cdot T^{vac})_{ii}$, one obtains

$$3\,\left(+\,\frac{1}{2\nu}\,w_{vac}\,\frac{11}{3}\,a^2\,\frac{\delta G(t)}{G_0}\,\rho(t)\,\frac{\Gamma}{\omega}\,h\right) + \mathcal{O}(k^2)\;. \tag{3.155}$$

For the tensor box ii matrix element, not summed over i, $(-\frac{1}{2\,\nu}\,\frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h,s) \cdot T^{vac})_{ii}$, one obtains

$$+\;\frac{1}{2\nu}\,a^2\,\frac{\delta G(t)}{G_0}\,\rho(t)\,\left[w_{vac}\,\frac{11}{3}\,\frac{\Gamma}{\omega}\,h + \frac{8}{9}\,\left(1 - 3\,\frac{k_i\,k_i}{k^2}\right)\,\frac{\Gamma}{\omega}\,s\right] + \mathcal{O}(k^2)\;. \tag{3.156}$$

Finally for the tensor box ij matrix element, $(-\frac{1}{2\,\nu}\,\frac{1}{\square^{(0)}}\cdot\square^{(1)}(h,s)\cdot T^{vac})_{ij}$, one obtains

$$-\;\frac{k_i\,k_j}{k^2}\,\frac{1}{2\nu}\,a^2\,\frac{8}{3}\,\frac{\delta G(t)}{G_0}\,\rho(t)\,\frac{\Gamma}{\omega}\,s+\mathcal{O}(k^2)\;. \tag{3.157}$$

The above expressions are now inserted in the general effective field equations of Eq. (3.3), and will give rise to a set of effective field equations appropriate for this particular gauge, to first order in the field perturbation and with the effects of $G(\square)$ included.

3.3.3 Field Equations in the Comoving, Synchronous and Conformal Newtonian Gauges

As a result of the previous manipulations one obtains in the comoving gauge with fields (h,s) the following tt, ti, ii (or $xx+yy+zz$), and ij field equations

$$\frac{k^2}{3\,a^2}\,(h+s)+\frac{\dot a}{a}\,\dot h=8\pi G_0\left(1+\frac{\delta G}{G_0}\right)\,\bar\rho\,\delta+8\pi G_0\,\frac{\delta G}{G_0}\,\frac{c_h}{2\nu}\,h\,\bar\rho+\mathcal{O}(k^2) \tag{3.158}$$

$$-\,\frac{1}{3}\left(\dot h+\dot s\right)=8\pi G_0\,\frac{\delta G}{G_0}\left(-\frac{1}{2\nu}\right)\,\frac{2}{9}\,\frac{1}{i\omega}\,(h-2s)\;\bar\rho+\mathcal{O}(k^2) \tag{3.159}$$

$$-\;\frac{1}{3}\,\frac{k^2}{a^2}\,(h+s)-3\frac{\dot a}{a}\,\dot h-\ddot h=24\pi G_0\,\frac{\delta G}{G_0}\,w_{vac}\,\bar\rho\,\delta+24\pi G_0\,\frac{\delta G}{G_0}\,w_{vac}\,\frac{c_h}{2\nu}\,h\bar\rho+\mathcal{O}(k^2) \tag{3.160}$$

$$\frac{1}{6}\,\frac{k^2}{a^2}\,(h+s)-\frac{3}{2}\,\frac{\dot a}{a}\dot s-\frac{1}{2}\,\ddot s=-\,8\pi G_0\,\frac{\delta G}{G_0}\,\frac{c_s}{2\nu}\,s\,\bar\rho+\mathcal{O}(k^2)\;. \tag{3.161}$$

As in Ref. [106], we have found it convenient to here to set in the above expressions

$$c_s\;\equiv\;\left(\frac{8}{3}\right)\,\frac{\Gamma}{\omega} \tag{3.162}$$

and

$$c_h\;=\equiv(-1)\,\left(-\frac{11}{3}\right)\,\frac{\Gamma}{\omega}\;=\;\frac{11}{3}\,\frac{\Gamma}{\omega}\;. \tag{3.163}$$

In the field equations listed above the terms $\mathcal{O}(k^2)$ arise because of terms $\mathcal{O}(k^2)$ in the expansion of the tensor box operator.

The next step is to convert the *left hand sides* of the above field equations, namely Eqs. (3.158), (3.159), (3.160) and (3.161), which are all expressed in the comoving gauge (h,s), to the synchronous gauge with fields $(\widetilde{h},\widetilde{\eta})$. The result of this change of gauge is the sequential replacement

$$
\begin{aligned}
\frac{k^2}{3\,a^2}\,(h+s)+\frac{\dot a}{a}\,\dot h \quad &\longrightarrow \quad -2\,\frac{k^2}{a^2}\widetilde{\eta}+\frac{1}{a^2}\,\frac{\dot a}{a}\,\dot{\widetilde{h}} \\
-\frac{1}{3}\left(\dot h+\dot s\right) \quad &\longrightarrow \quad 2\,\dot{\widetilde{\eta}} \\
-\frac{1}{3}\frac{k^2}{a^2}\,(h+s)-3\frac{\dot a}{a}\,\dot h-\ddot h \quad &\longrightarrow \quad 2\,\frac{k^2}{a^2}\,\widetilde{\eta}-\frac{1}{a^2}\,\ddot{\widetilde{h}}-2\,\frac{1}{a^2}\,\frac{\dot a}{a}\,\dot{\widetilde{h}} \\
\frac{1}{6}\,\frac{k^2}{a^2}\,(h+s)-\frac{3}{2}\,\frac{\dot a}{a}\,\dot s-\frac{1}{2}\,\ddot s \quad &\longrightarrow \quad -\frac{k^2}{a^2}\widetilde{\eta}+\frac{1}{2}\,\frac{1}{a^2}\left(\ddot{\widetilde{h}}+6\,\ddot{\widetilde{\eta}}\right)+\frac{1}{a^2}\,\frac{\dot a}{a}\left(\dot{\widetilde{h}}+6\,\dot{\widetilde{\eta}}\right). \qquad (3.164)
\end{aligned}
$$

The next step involves one more transformation, this time from the synchronous $(\widetilde{h},\widetilde{\eta})$ to the desired conformal Newtonian (ϕ,ψ) gauge,

$$
\begin{aligned}
\frac{1}{a^2}\left[-2\,k^2\,\widetilde{\eta}+\frac{\dot a}{a}\,\dot{\widetilde{h}}\right] \quad &\longrightarrow \quad -\frac{2}{a^2}\left[k^2\,\phi+3\,\frac{\dot a}{a}\,\left(\dot\phi+\frac{\dot a}{a}\,\psi\right)\right] \\
2\,\dot{\widetilde{\eta}} \quad &\longrightarrow \quad 2\,\left(\dot\phi+\frac{\dot a}{a}\,\psi\right) \\
\frac{1}{a^2}\left[2\,k^2\,\widetilde{\eta}-\ddot{\widetilde{h}}-2\,\frac{\dot a}{a}\,\dot{\widetilde{h}}\right] \quad &\longrightarrow \quad \frac{6}{a^2}\left[\ddot\phi+\frac{\dot a}{a}\left(\dot\psi+2\dot\phi\right)+\left(2\frac{\ddot a}{a}-\frac{\dot a^2}{a^2}\right)\psi+\frac{k^2}{3}\,(\phi-\psi)\right] \\
\frac{1}{a^2}\left[-k^2\widetilde{\eta}+\frac{1}{2}\left(\ddot{\widetilde{h}}+6\ddot{\widetilde{\eta}}\right)+\frac{\dot a}{a}\left(\dot{\widetilde{h}}+6\dot{\widetilde{\eta}}\right)\right] \quad &\longrightarrow \quad -\frac{k^2}{a^2}\,(\phi-\psi)\;. \qquad (3.165)
\end{aligned}
$$

Equivalently, the above sequence of two transformations can be described by a single transformation, from comoving (h,s) to conformal Newtonian (ϕ,ψ) gauge, which is trivially obtained by combining the previous two. The final outcome of all these manipulations is to achieve a rewrite of the full set of four original field equations, given in Eqs. (3.158), (3.159), (3.160) and (3.161), now with the *left hand side* given in the conformal Newtonian gauge and the *right hand side* left in the original comoving gauge. One obtains

$$
k^2\,\phi+3\,\frac{\dot a}{a}\,\left(\dot\phi+\frac{\dot a}{a}\,\psi\right)=-\,4\pi G_0\,a^2\,\left(1+\frac{\delta G}{G_0}\right)\,\bar\rho\,\delta-4\pi G_0\,a^2\,\frac{\delta G}{G_0}\,\frac{c_h}{2\nu}\,h\,\bar\rho\;+\;\mathcal{O}(k^2) \qquad (3.166)
$$

$$
\left(\dot\phi+\frac{\dot a}{a}\,\psi\right)=4\pi G_0\,\frac{\delta G}{G_0}\,\left(-\frac{1}{2\nu}\right)\,\frac{2}{9}\,\frac{1}{i\omega}\,(h-2\,s)\;\bar\rho+\;\mathcal{O}(k^2) \qquad (3.167)
$$

$$
\begin{aligned}
\ddot\phi+\frac{\dot a}{a}\,\left(\dot\psi+2\,\dot\phi\right)+\left(2\,\frac{\ddot a}{a}-\frac{\dot a^2}{a^2}\right)\,\psi+\frac{k^2}{3}\,(\phi-\psi) \quad &= \quad 4\pi G_0\,a^2\,\left(w+w_{vac}\,\frac{\delta G}{G_0}\right)\,\bar\rho\,\delta \\
&+ \quad 4\pi G_0\,a^2\,\frac{\delta G}{G_0}\,w_{vac}\,\frac{c_h}{2\nu}\,h\,\bar\rho \\
&+ \quad \mathcal{O}(k^2) \qquad (3.168)
\end{aligned}
$$

$$k^2\,(\phi-\psi) = +8\pi G_0\, a^2\, \frac{\delta G}{G_0}\, \frac{c_s}{2\nu}\, s\, \bar{\rho} \;+\; \mathcal{O}(k^2)\,. \tag{3.169}$$

Note that we have, for convenience, multiplied out the first, third and fourth equations by a factor of a^2. The last equation involves the quantity

$$\sigma \;=\; \frac{2}{3}\, \frac{\delta G}{G_0}\, \frac{c_s}{2\nu} \cdot s\,. \tag{3.170}$$

For the purpose of computing the gravitational slip function $\eta \equiv \psi/\phi - 1$ it will be useful here to record the following relationship between perturbations in the comoving and conformal Newtonian gauge. One has

$$\psi = -\,\frac{1}{2k^2}\, a^2 \left(\ddot{s} + 2\, \frac{\dot{a}}{a}\, \dot{s} \right) \tag{3.171}$$

$$\phi = -\,\frac{1}{6}\, (h+s) + \frac{1}{2}\, \frac{a^2}{k^2}\, \frac{\dot{a}}{a}\, \dot{s} \tag{3.172}$$

Use has been made here of the following relationship between derivatives of an arbitrary function f in the synchronous and comoving gauges where we denote $\tilde{}$ for synchronous.

$$\dot{\tilde{f}} = a\dot{f} \tag{3.173}$$

and

$$\frac{d}{d\tilde{\tau}} = a\, \frac{d}{d\tau} \tag{3.174}$$

so that

$$\ddot{\tilde{f}} = a^2 \left(\frac{\dot{a}}{a}\, \dot{f} + \ddot{\tilde{f}} \right)\,. \tag{3.175}$$

3.4 Gravitational Slip Function η with $G(\Box)$ in the Conformal Newtonian Gauge

The gravitational slip function is commonly defined as

$$\eta \;\equiv\; \frac{\psi - \phi}{\phi}\,. \tag{3.176}$$

In classical GR one has $\phi = \psi$ so that $\eta = 0$, which makes the quantity η a useful parametrization for deviations from classical GR, whatever their origin might be. Using the ij field equation given in Eqs. (3.166), (3.167), (3.168) and (3.169), and the relationship between the conformal Newtonian fluctuation ϕ and the comoving gauge fluctuations h and s, one finally obtains the rather simple result

$$\eta \equiv \frac{\psi - \phi}{\phi} \;=\; -\,16\pi G_0 \, \frac{\delta G}{G_0} \, \frac{c_s}{2\nu} \, \frac{a}{\dot a} \, \frac{s}{\dot s} \, \bar\rho \, . \tag{3.177}$$

The last expression contains the quantity

$$c_s \;=\; \left(\frac{8}{3}\right) \frac{i\Gamma}{i\omega_s} \tag{3.178}$$

where ω_s is the frequency associated with the s perturbation, and we have made use of $i\Gamma \to \dot a / a$. An equivalent form for the expression in Eq. (3.177) is

$$\eta \;=\; -\,16\pi \, G_0 \, \frac{\delta G}{G_0} \, \frac{1}{2\nu} \, \frac{8}{3} \, \frac{1}{i\omega_s} \, \frac{s}{\dot s} \, \bar\rho \;=\; -\,16\pi \, G_0 \, \frac{\delta G}{G_0} \, \frac{1}{2\nu} \, \frac{8}{3} \, \frac{\int s \, \mathrm{d}t}{\dot s} \, \bar\rho \, . \tag{3.179}$$

In the last expression we now can make use of the equation of motion for the perturbation $s(t)$ to the order we are working, namely

$$\ddot s \, + \, 3 \, \frac{\dot a}{a} \, \dot s \;=\; 0 \, . \tag{3.180}$$

Let us look here first at the very simple limit of $\lambda \simeq 0$; the physically more relevant case of nonzero λ will be discussed a bit later. Note that, in view of Eq. (1.60), this last limit corresponds therefore to a very large ξ. Then for a perfect fluid with equation of state $p = w\rho$ one has simply $a(t) = a_0 (t/t_0)^{2/3(1+w)}$ and $\rho(t) = 1/[6\pi G t^2 (1+w)^2]$, and from Eqs. (3.179) or (3.192) one obtains for $w = 0$

$$\eta \;=\; 4 \cdot \frac{8}{3} \, c_t \left(\frac{t}{\xi}\right)^3 \ln\left(\frac{t}{\xi}\right) + \mathcal{O}(t^4) \tag{3.181}$$

whereas for $w \neq 0$ one has

$$\eta \;=\; 2 \cdot \frac{8}{3} \, \frac{c_t}{w\,(1-w)} \left(\frac{t}{\xi}\right)^3 + \mathcal{O}(t^6) \, . \tag{3.182}$$

Another extreme, but nevertheless equally simple, case is a pure cosmological constant term (no matter of any type), which can be modeled by the choice $w = -1$. In this case t is related to the scale factor by

$$\frac{a(t)}{a_0} \; = \; \exp \left\{ \sqrt{\frac{\lambda}{3}} \, (t - t_0) \right\} \; . \tag{3.183}$$

Then, using the relation in Eq. (1.60), one obtains

$$\frac{t}{\xi} \; = \; 1 + \ln \frac{a}{a_\xi} \; , \tag{3.184}$$

where the quantity a_ξ is therefore related to the time t_0 (”today”, $a_0 = 1$) and the scale ξ by

$$\frac{t_0}{\xi} \; = \; 1 + \ln \frac{1}{a_\xi} \; . \tag{3.185}$$

Since numerically t_0 is close to, but smaller than, ξ, the scale factor a_ξ will be close to, but slightly larger than, one.

Let us now go back to the more physical case of $\lambda \neq 0$. The relevant expression for $\eta(t)$ is Eq. (3.179), where we use the equation for $s(t)$, Eq. (3.180), to eliminate the latter. It is also convenient at this stage to change variables from t to $a(t)$, and use the equivalent equation for $s(a)$, namely

$$s''(a) \; + \; \left(\frac{H'(a)}{H(a)} + \frac{4}{a} \right) \, s'(a) \; = \; 0 \; , \tag{3.186}$$

where the prime denotes differentiation with respect to the scale factor a. In the above equation one can use, for nonrelativistic matter with equation of state such that $w = 0$, and to the order needed here, the first Friedmann equation

$$H(a) \; = \; \sqrt{\frac{\lambda}{3} + \frac{4}{9\, a^3}} \; . \tag{3.187}$$

We have also made use of the unperturbed result for the background matter density valid for $w = 0$ (which follows from energy conservation), namely

$$\bar{\rho} = \bar{\rho}_0 \, \frac{1}{a^3} \; . \tag{3.188}$$

Note that the above expression for $\bar{\rho}$ is valid to zeroth order in δG, which is entirely adequate when substituted into $\eta(a)$, since the rest there is already first order in δG. This finally gives an explicit solution for $s(a)$

$$s(a) \; \propto \; \frac{2}{3a^{3/2}} \sqrt{1 + a^3 \, \theta} \; , \tag{3.189}$$

with parameter $\theta \equiv \lambda / 8 \pi G_0 \, \bar{\rho}_0$. The above solution for $s(a)$ can then be substituted directly in Eq. (3.179), provided one changes variables from t to $a(t)$, and in the process uses the following identities

$$\int s(t) \, \mathrm{d}t = \int s(a) \, \frac{1}{a \, H(a)} \, \mathrm{d}a \; , \tag{3.190}$$

as well as

$$\dot{s} = a \, H(a) \, \frac{\partial s}{\partial a} \; , \tag{3.191}$$

with $H(a)$ given a few lines above.

The resulting expression, which still involves an integral over the scale factor $a(t)$, can now be readily evaluated, and leads eventually to a rather simple expression for η. The general result for nonrelativistic matter ($w = 0$) but $\lambda \neq 0$ is

$$\eta(a) \; = \; \frac{16}{3 \, \nu} \, \frac{\delta G(a)}{G_0} \, \log \left[\frac{a}{a_\xi} \right] \; . \tag{3.192}$$

This is the main result of the paper. The integration constant a_ξ has been fixed following the requirement that the scale factor $a \to a_\xi$ for $t \to \xi$ [see Eqs. (3.26), (3.15) and (3.40) for the definitions of ξ]. In other words, by switching to the variable $a(t)$ instead of t, the quantity ξ has been traded for a_ξ. In the next section we will show that in practice the quantity a_ξ is generally expected to be slightly larger than the scale factor "today", i.e. for $t = t_0$. As a result the correction in Eq. (3.192) is expected to be negative today.

The next section will be devoted to establishing the general relationship between t and $a(t)$, for nonvanishing cosmological constant λ, so that a quantitative estimate for the slip function η can be obtained from Eq. (3.192) in a realistic cosmological context. Specifically we will be interested in the value of η for a current matter fraction $\Omega \simeq 0.25$, as suggested by current astrophysical measurements.

3.4.1 Relating the Scale Factor a to Time t, and vice versa

Let us now come back to the general problem of estimating $\eta(a)$, using the expression given in Eq. (3.192), for $\lambda \neq 0$ and a nonrelativistic fluid with $w = 0$. To predict the correct value for the slip function $\eta(a)$ one needs the quantity $\delta G(a)$, which is obtained from the FLRW version of $G(\Box)$, namely $G(t)$ in Eq. (3.15), via the replacement, in this last quantity, of $t \to t(a)$. The last step requires therefore that the correct relationship between t and $a(t)$ be established, for any value of λ. In the following we will first relate t to $a(t)$, and vice versa, to zeroth order in the quantum correction δG [we will call them $a^{(0)}(t)$ and $t^{(0)}(a)$], and then compute the first order correction in δG to the above quantities [we will call those $a^{(1)}(t)$ and $t^{(1)}(a)$].

Let us look first at the zeroth order result. The field equations and the energy conservation equation for $a^{(0)}(t)$, without a δG correction, but with the λ term, were already given in Eq. (3.24),

$$\begin{aligned} 3\,\frac{\dot{a}^{(0)\,2}(t)}{a^{(0)\,2}(t)} &= 8\pi\, G_0\, \bar{\rho}^{(0)}(t) + \lambda \\ \frac{\dot{a}^{(0)\,2}(t)}{{a^{(0)}}^2(t)} + 2\,\frac{\ddot{a}^{(0)}(t)}{a^{(0)}(t)} &= -8\pi\, G_0\, w\, \bar{\rho}^{(0)}(t) + \lambda \end{aligned} \tag{3.193}$$

for a spatially flat universe ($k = 0$), and

$$\dot{\bar{\rho}}^{(0)}(t) + 3\,(1+w)\,\frac{\dot{a}^{(0)}(t)}{a^{(0)}(t)}\,\bar{\rho}^{(0)}(t) = 0\;. \tag{3.194}$$

From these one can obtain $a^{(0)}(t)$ and then $\bar{\rho}^{(0)}(t)$. As a result the scale factor is found to be related to time by

$$t^{(0)}(a) = \frac{2\,\mathrm{Arcsinh}\left[\, a^{3/2}\,\theta^{\frac{1}{2}} \right]}{\sqrt{3\,\lambda}} \tag{3.195}$$

where we have defined the parameter

$$\theta \;\equiv\; \frac{\lambda}{8\pi G_0 \bar\rho_0} \;=\; \frac{1-\Omega}{\Omega} \tag{3.196}$$

with $\bar\rho_0$ the current ($t=t_0$) matter density, and Ω the current matter fraction. Note that in practice we will be interested in a matter fraction which today is around 0.25, giving $\theta \simeq 3.0$, a number which is of course quite far from the zero cosmological constant case of $\theta = 0$.

One can express the time today (t_0) in terms of cosmological constant λ, and therefore in terms of θ, as follows

$$t_0^{(0)} = \frac{2\,\mathrm{Arcsinh}(\sqrt{\theta})}{\sqrt{3\lambda}} \tag{3.197}$$

with the normalization for $t^{(0)}(a)$ such that $t^{(0)}(a=0)=0$ and $t^{(0)}(a=1)=t_0$ "today". So here we follow the customary choice of having the scale factor equal to one "today". Then one has

$$\frac{t^{(0)}(a)}{t_0^{(0)}} = \frac{\mathrm{Arcsinh}\left[\sqrt{a^3\,\theta}\right]}{\mathrm{Arcsinh}(\sqrt{\theta})} \; . \tag{3.198}$$

When expanded out in θ, the above result leads to some perhaps more recognizable terms,

$$\frac{t^{(0)}(a)}{t_0^{(0)}} = a^{\frac{3}{2}} \left[1 - \frac{1}{6}\,\left(-1+a^3\right)\,\theta + \frac{1}{360}\,\left(-17 - 10\,a^3 + 27\,a^6\right)\,\theta^2 + \cdots \right] \; . \tag{3.199}$$

Conversely, one has for the scale factor as a function of the time

$$a^{(0)}(t) = \left(\frac{\mathrm{Sinh}^2\left[\frac{\sqrt{3\,\lambda}}{2}\,t\right]}{\theta} \right)^{\frac{1}{3}} \; , \tag{3.200}$$

which, when expanded out in λ or t, gives the more recognizable result

$$\left[a^{(0)}(t)\right]^3 \;=\; \frac{3\,\lambda\,t^2}{4\theta} \left(1 + \frac{\lambda\,t^2}{4} + \frac{\lambda^2\,t^4}{40} + \cdots \right) \; . \tag{3.201}$$

Similarly for the pressure one obtains

$$\bar\rho^{(0)}(t) = \frac{\lambda\,\mathrm{Csch}^2\left[\frac{\sqrt{3\,\lambda}}{2}\,t\right]}{8\pi G_0} \; , \tag{3.202}$$

which when expanded out in λ or t gives the more familiar result

$$\bar\rho^{(0)}(t) = \frac{1}{6\pi G_0\,t^2\,\left(1 + \frac{t^2\,\lambda}{4} + \frac{t^4\,\lambda^2}{40} + \frac{3\,t^6\,\lambda^3}{2240} + \cdots\right)} \; . \tag{3.203}$$

To be more specific, let us set $\theta = 3$, which corresponds to a matter fraction today of $\Omega \sim 0.25$. In addition, we will now make use of Eq. (1.60) and set $\lambda \to 3/\xi^2$. One then obtains

$$t_0^{(0)}\,(\theta = 3) = 0.878\;\xi\;, \tag{3.204}$$

which shows that t_0 and ξ are rather close to each other (apparently a numerical coincidence).

Then, from the expression for $G(t)$ in Eq. (3.15),

$$\frac{\delta G(t)}{G_0} = c_t \left(\frac{t}{\xi}\right)^{\frac{1}{\nu}}\;, \tag{3.205}$$

one can obtain $G(a)$ in all generality, by the replacement $t \to t(a)$ according to the result of Eqs. (3.195) or (3.198). For the special case of pure nonrelativistic matter with equation of state $w = 0$ and $\lambda = 0$ one obtains, using Eq. (3.199),

$$\frac{\delta G(a)}{G_0} \;=\; c_a \left(\frac{a}{a_\xi}\right)^{\gamma_\nu}\;, \tag{3.206}$$

with exponent

$$\gamma_\nu = \frac{3}{2\nu}\;. \tag{3.207}$$

The latter is largely the expression used earlier in the matter density perturbation treatment of our earlier work of Ref. [106].

More generally one can define a_ξ as the value for the scale factor a which corresponds to the scale ξ,

$$a_\xi^{(0)} \;\equiv\; \left(\frac{1}{\theta}\right)^{\frac{1}{3}} \mathrm{Sinh}^{\frac{2}{3}}\left[\frac{3}{2}\right] = 1.655\,\left(\frac{1}{\theta}\right)^{\frac{1}{3}}\;, \tag{3.208}$$

so that in general $a_\xi \neq a_0$, where $a_0 = 1$ is the scale factor "today". Then for the observationally favored case $\theta \simeq 3$ one obtains

$$a_\xi^{(0)}(\theta = 3) = 1.148\;, \tag{3.209}$$

which clearly implies $a_\xi^{(0)} > a_0 = 1$. [7] The above expressions will be used in the next section to obtain a quantitative estimate for the slip function $\eta(a)$, evaluated at today's time $t = t_0$.

[7] Let us give here a few more observational numbers for present and future reference. From the present age of the Universe $t_0 \approx 13.75\,Gyrs \simeq 4216\,Mpc$, whereas from the observed value of λ (mostly extracted from distant supernovae surveys) one has following Eq. (1.60) $\xi \simeq 4890\,Mpc$, which then gives $t_0/\xi \simeq 0.862 = 1/1.160$. This last ratio is similar to the number we used in Eq. (3.204), by setting there $\Omega = 0.25$ exactly.

The discussion above dealt with the case of $\delta G = 0$. Let us now consider briefly the corrections to $a(t)$ and, conversely, $t(a)$ that come about when the running of G is included, in other words when a constant G is replaced by $G(t)$ or $G(a)$ in the effective field equations. In Eq. (3.25) the Friedmann equations were given in the presence of a running G, namely

$$\begin{aligned} 3\,\frac{\dot{a}^2(t)}{a^2(t)} &= 8\pi\, G_0 \left(1 + \frac{\delta G(t)}{G_0}\right) \bar{\rho}(t) + \lambda \\ \frac{\dot{a}^2(t)}{a^2(t)} + 2\,\frac{\ddot{a}(t)}{a(t)} &= -8\pi\, G_0 \left(w + w_{vac}\frac{\delta G(t)}{G_0}\right) \bar{\rho}(t) + \lambda\;, \end{aligned} \tag{3.210}$$

together with the energy conservation equation

$$3\,\frac{\dot{a}(t)}{a(t)} \left[(1+w) + (1+w_{vac})\,\frac{\delta G(t)}{G_0}\right] \bar{\rho}(t) + \frac{\delta \dot{G}(t)}{G_0}\,\bar{\rho}(t) + \left(1 + \frac{\delta G(t)}{G_0}\right) \dot{\bar{\rho}}(t) = 0\;. \tag{3.211}$$

To solve these equations to first order in δG we set

$$a(t) = a^{(0)}(t)\left[1 + c_t\, a^{(1)}(t)\right] \tag{3.212}$$

$$\bar{\rho}(t) = \bar{\rho}^{(0)}(t)\left[1 + c_t\, \bar{\rho}^{(1)}(t)\right] \tag{3.213}$$

where $a^{(0)}(t)$ and $\bar{\rho}^{(0)}(t)$ here represent the solutions obtained previously for $\delta G = 0$. One then finds for the correction to the matter density

$$\bar{\rho}^{(1)}(t) = -\left(\frac{t}{\xi}\right)^{\frac{1}{\nu}} \left(1 + w_{vac}\,\frac{\nu}{(1+\nu)}\,\sqrt{3\,\lambda}\,t\;\;\mathrm{Coth}\left[\frac{\sqrt{3\,\lambda}}{2}\,t\right]\right) \tag{3.214}$$

and to lowest nontrivial order in t and for $w_{vac} = 1/3$

$$\bar{\rho}^{(1)}(t) = -\,\frac{3+5\nu}{3(1+\nu)}\left(\frac{t}{\xi}\right)^{\frac{1}{\nu}} + \dots\;. \tag{3.215}$$

For the correction to the scale factor one finds

$$a^{(1)}(t) = -\,w_{vac}\,\frac{\nu}{(1+\nu)}\,\lambda \int_0^t \frac{t'\left(\frac{t'}{\xi}\right)^{\frac{1}{\nu}}}{-1 + \mathrm{Cosh}\left[\sqrt{3\,\lambda}\,t'\right]}\,\mathrm{d}t' \tag{3.216}$$

and to lowest nontrivial order in t for $w_{vac} = 1/3$,

$$a^{(1)}(t) = -\,\frac{2\nu^2}{9\,(1+\nu)}\left(\frac{t}{\xi}\right)^{\frac{1}{\nu}} + \dots\;. \tag{3.217}$$

After having obtained the relevant formulas for $a(t)$ and $t(a)$ in the general case, i.e. for nonzero λ, we can return to the problem of evaluating the slip function η.

3.4.2 Quantitative Estimate of the Slip Function η

The general expression for the gravitational slip function $\eta(a)$ was given earlier in Eq. (3.192) for $w = 0$ and $\lambda \neq 0$,

$$\eta(a) \;=\; \frac{16}{3\nu}\,\frac{\delta G(a)}{G_0}\,\log\left[\frac{a}{a_\xi}\right] \;. \tag{3.218}$$

To obtain $\delta G(a)$ we now use, from Eq. (3.15),

$$\frac{\delta G(t)}{G_0} = c_t\,\left(\frac{t}{\xi}\right)^{\frac{1}{\nu}} \tag{3.219}$$

and substitute in the above expression for $\delta G(t)$ the correct relationship between t and a, namely $t(a)$ from Eq. (3.195), which among other things contains the constant defined in Eq. (3.208),

$$a_\xi \;=\; \left(\frac{1}{\theta}\right)^{\frac{1}{3}}\,\mathrm{Sinh}^{\frac{2}{3}}\left[\frac{3}{2}\right] \;. \tag{3.220}$$

It will be convenient, at this stage, to also make use of the relationship in Eq. (1.60), namely

$$\lambda \;\to\; \frac{3}{\xi^2} \;. \tag{3.221}$$

The last step left is to make contact with observationally accessible quantities, by expanding in the redshift z, related in the usual way to the scale factor a by $a \equiv 1/(1+z)$. Then for $\nu = 1/3$ and $\theta = 3$ (matter fraction $\Omega = 0.25$) one finally obtains for the gravitational slip function

$$\eta(z) \;=\; -\,1.491\,c_t - 6.418\,c_t\,z + 30.074\,c_t\,z^2 + \cdots \tag{3.222}$$

To obtain an actual number for $\eta(z = 0)$ one needs to address two more issues. They are (i) to provide a bound on the theoretical uncertainties in the above expression, and (ii) to give an estimate for the coefficient c_t, which is traced back to Eq. (3.15) and therefore to the original expression for $G(\Box)$ in Eq. (3.26). The latter contains the coefficient c_0, but in Ref. [124] the estimate was given $c_t = 0.450\ c_0$ for the tensor box operator, $c_t = 0.785\ c_0$ for the trace box operator ; thus c_t and c_0 can safely be assumed to have the same sign, and comparable magnitudes.

To estimate the level of uncertainty in the magnitude of the correction coefficient in Eq. (3.222) we will consider here an infrared regulated version of $G(\Box)$, where an infrared cutoff is supplied so that in Fourier space $k > \xi^{-1}$, and the spurious infrared divergence at small k is removed. This is quite analogous to an infrared regularization used very successfully in phenomenological applications to QCD heavy quark bound states [143, 144, 145, 146], and which has recently found some limited justification in the framework of infrared renormalons [147, 148]. As shown already in the first cited reference, it works much better than expected; here a similar prescription will be used just as a means to provide some estimate on the theoretical uncertainty in the result of Eq. (3.222). Therefore, instead of the $G(\Box)$ in Eq. (3.26), which in momentum space corresponds to

$$G(k^2) \;\simeq\; G_0 \left[1 \,+\, c_0 \left(\frac{1}{\xi^2 \, k^2} \right)^{1/2\nu} \,+\, \dots \right] \, , \tag{3.223}$$

we will consider a corresponding infrared regulated version,

$$G(k^2) \;\simeq\; G_0 \left[1 \,+\, c_0 \left(\frac{\xi^{-2}}{k^2 \,+\, \xi^{-2}} \right)^{1/2\nu} \,+\, \dots \right] \, . \tag{3.224}$$

Of course the small distance, $k \gg \xi^{-1}$ or $r \ll \xi$, behavior is unchanged, whereas for large distances $r \gg \xi$ the gravitational coupling no longer exhibits the spurious infrared divergence; instead it approaches a finite value $G_\infty \simeq (1 + c_0 + \dots) \, G_0$. Now, in momentum space the infrared regulated $\delta G(k)$ reads

$$\frac{\delta G(k^2)}{G_0} = c_0 \left(\frac{m^2}{k^2 + m^2} \right)^{1/2\,\nu} \, , \tag{3.225}$$

with $m = 1/\xi$, and in position space the corresponding form is

$$\frac{\delta G(\Box)}{G_0} = c_0 \left(\frac{1}{-\,\xi^2 \Box + 1} \right)^{1/2\,\nu} \, . \tag{3.226}$$

Following the results of Ref. [124], if the above differential operator acts on functions of t only, then one obtains for $\delta G(t)$

$$\frac{\delta G(t)}{G_0} \;=\; c_0 \left(\frac{1}{\left(\frac{c_0}{c_t}\right)^{2\,\nu} \, \left(\frac{\xi}{t}\right)^2 + 1} \right)^{\frac{1}{2\,\nu}} \tag{3.227}$$

with again $c_t/c_0 \approx 0.62$ [124]. Note that the expression in Eq. (3.227) could also have been obtained directly from Eq. (3.15), by a direct regularization.

One can then repeat the whole calculation for $\eta(a)$ with the regulated version of $\delta G(t)$ given in Eq. (3.227). The result is

$$\eta(z) = -0.766\, c_t - 4.109\, c_t\, z + 12.188\, c_t\, z^2 + \cdots \; . \tag{3.228}$$

It seems that the effect of the infrared regularization has been to reduce the magnitude of the effect (at $z = 0$) by about a factor of 2. It is encouraging that, at this stage of the calculation, the negative trend in $\eta(z)$ due to the running of G appears unchanged. Furthermore, in all cases we have looked so far, the value $\eta(z = 0)$ is found to be negative.

We note here again what contains in the computation of c_0. $c_0 \approx 8$ was obtained from recent computation (2013) [personal communication with Herbert Hamber]. This number is the result of reanalysis of [128] which involved rather large uncertainties for this particular quantity. The value of the constant c_0 has to be extracted from a nonperturbative lattice computation of invariant curvature correlations at fixed geodesic distance [128]; it relates the physical correlation length ξ to the bare lattice coupling G, and is therefore a genuinely nonperturbative amplitude. Based on experience with other field theoretic models which also exhibit nontrivial fixed points such as the nonlinear sigma model, as well as QCD and non Abelian gauge theories, one would expect this amplitude to be of order unity; very small or very large numbers would appear rather atypical and unnatural. It appears therefore that this value of c_0 may be still large for our expectation.

3.4.3 Slip Function η for Stress Perturbation $s = 0$

We will show here that for the zero stress field s, one still obtains a nonvanishing η, whose value we will discuss below. The results will be useful, since now a direct comparison can be done with the full answer (including the stress field) for $\eta(z)$ given in the previous section.

In the absence of stress ($s = 0$) and finite k, the tt and $xx + yy + zz$ field equations read

$$-\,2\,\frac{k^2}{a^2}\phi - 8\pi G_0 \frac{c_h}{2\nu}\,\frac{\delta G}{G_0}\,\rho\,\delta\,\left(-\frac{2}{1+w}\right) = 8\pi G_0\,\left(1+\frac{\delta G}{G_0}\right)\,\rho\,\delta \tag{3.229}$$

$$2\frac{k^2}{a^2}\,(\psi-\phi) + 24\pi G_0 \frac{c_h}{2\nu}\,w_{vac}\,\frac{\delta G}{G_0}\,\rho\,\delta\,\left(-\frac{2}{1+w}\right) = -\,24\pi G_0\,\left(w + w_{vac}\,\frac{\delta G}{G_0}\right)\,\rho\,\delta\,. \tag{3.230}$$

In both equations we have made use of zeroth order (in $\delta G/G_0$) energy conservation, which leads to $h = -\frac{2}{(1+w)}\,\delta$, where δ is the matter fraction. One can then take the ratio of the two equations given above, and obtain again an expression for the slip function $\eta = (\psi - \phi)\,/\phi$. For $w = 0$ (nonrelativistic matter), after expanding in $\delta G/G_0$, one finds the rather simple result

$$\eta \;=\; \frac{\psi-\phi}{\phi} \;=\; 3\,w_{vac}\,\left(1-\frac{c_h}{\nu}\right)\,\frac{\delta G}{G_0}\;. \tag{3.231}$$

Here the quantity c_h is the same as in Eq. (3.163), and depends on the choices detailed below. In the following we will continue to use $w_{vac} = 1/3$ [see Eqs. (3.21) and (3.22)] [124, 106], which is the correct value associated with $G(\Box)$ in the FLRW background metric.

[1] We used the *trace box* value $c_h = 1/2$ and $c_t \approx 0.785 c_0$, which then gives

$$\eta \;=\; \left(1-\frac{1}{2\,\nu}\right)\,\frac{\delta G}{G_0} \;=\; \left(1-\frac{1}{2\,\nu}\right)\,c_t\,\left(\frac{t}{\xi}\right)^{\frac{1}{\nu}} + \cdots \tag{3.232}$$

In this last case it is then easy to recompute the slip function in terms of the redshift, just as was done in the previous section, and one finds, under the same conditions as before [$\nu = 1/3$, $\theta = 3$, and t_0/ξ as given in Eq. (3.204)] the following result

$$\eta \simeq -\,0.338\,c_t \;+\; \mathcal{O}(z)\,, \tag{3.233}$$

and gives $\eta \simeq -2.1$ with $c_t \approx 0.785 c_0 \approx 6.3$ with $c_0 \approx 8$ for trace box operator. For the infrared regulated version of $\delta G/G_0$ given in Eq. (3.227) one obtains instead the slightly smaller value

$$\eta \simeq -\,0.174\,c_t \;+\; \mathcal{O}(z)\,, \tag{3.234}$$

and gives $\eta \simeq -1.1$ with $c_t \approx 6.3$.

[2] For the *tensor box* case one finds a significantly larger value $c_h \simeq 7.927$ and $c_t \approx 0.450 c_0$, so that in this case the slip function η becomes

$$\eta \simeq \left(1 - \frac{7.927}{\nu}\right) \frac{\delta G}{G_0} = \left(1 - \frac{7.927}{\nu}\right) c_t \left(\frac{t}{\xi}\right)^{\frac{1}{\nu}} + \cdots \tag{3.235}$$

Also in this case one can recompute the slip function in terms of the redshift, and one finds, under the same conditions as before,

$$\eta \simeq -15.42\, c_t \;+\; \mathcal{O}(z)\,, \tag{3.236}$$

yielding $\eta \simeq -55.5$ with $c_t \approx 0.450 c_0 \approx 3.6$ with $c_0 \approx 8$ for tensor box operator. For the infrared regulated $\delta G/G_0$ given in Eq. (3.227) one finds instead

$$\eta \simeq -7.919\, c_t \;+\; \mathcal{O}(z)\,, \tag{3.237}$$

which is about a factor of 2 smaller than the unregulated value, and yields numerically $\eta \simeq -28.5$ with $c_t \approx 3.6$.

We conclude from the above exercise of calculating η with vanishing stress field $s = 0$ three things.

(1) The first is that using the trace box result on the trace of the energy momentum tensor (which ultimately is not an entirely correct, or at least an incomplete, procedure, given the tensor nature of the matter energy momentum tensor) underestimates the effects of $G(\Box)$ on the slip function $\eta(z = 0)$ by a factor that can be as large as an order of magnitude.

(2) The second lesson is that the stress field (s) contribution is indeed important, since it reduces the size of the quantum correction significantly [Eqs. (3.222) and (3.228)], compared to the $s = 0$ result [Eqs. (3.235) and (3.236)], again by almost an order of magnitude, which would imply some degree of cancellation between the s and h contributions.

(3) The third observation is that in all cases we have looked at so far the quantum correction to the slip function is *negative* at $z = 0$.

3.4.4 Discussions

In the previous sections we computed corrections to the gravitational slip function $\eta = \psi/\phi - 1$ arising from the renormalization-group motivated running $G(\Box)$, as given in Eq. (3.26). The relevant result was presented in Eqs. (3.222) and (3.228), the first expression representing the answer for an unregulated $G(\Box)$, and the second answer found for an infrared regulated version of the same. It should be noted that, so far, in the treatment of metric and matter perturbations we have considered only the $\mathbf{q} \to 0$ limit [see Eq. (3.31)]. Let us focus here for definiteness on the first of the two results [Eq. (3.222)], which is

$$\eta(z) \;\simeq\; -1.491\, c_t \;+\; \mathcal{O}(z) \tag{3.238}$$

at $z \simeq 0$. With the value $c_t \approx 0.450 c_0$ for tensor box and $c_0 \approx 8$, the above gives $\eta \approx -5.4$. This is in fact appear too large to fit for the current observations. As discussed before, the value of the constant c_0 has to be extracted from a nonperturbative lattice computation of invariant curvature correlations at fixed geodesic distance [128] and involves nontrivial considerations and possibly still estimated to be too large. Indeed with other field theoretic models which also exhibit nontrivial fixed points such as the nonlinear sigma model, QCD and non Abelian gauge theories, one would expect this amplitude to be of order unity.

As far as astrophysical observations are concerned, current estimates for $\eta(z = 0)$ obtained from CMB measurements give values around 0.09 ± 0.7 [137, 138], which would then imply an observational bound $c_t \lesssim 0.3$, which in fact should be an expected order of values if $c_0 \sim 1$.

Indeed a similar problem of magnitudes for the theoretical amplitudes was found in our recent calculation of matter density perturbations with $G(\Box)$, where again the corrections seemed rather large [106] in view of the above quoted value of c_t. Let us briefly summarize those results here. Specifically, in Ref. [106] a value for the density perturbation growth index γ was obtained in the presence of $G(\Box)$. The quantity γ is in general obtained from the growth index $f(a)$ [126]

$$f(a) \equiv \frac{\partial \ln \delta(a)}{\partial \ln a} \ , \tag{3.239}$$

where $\delta(a)$ is the matter density contrast. One is mainly interested in the neighborhood of the present era, $a(t) \simeq a_0 \simeq 1$, which leads to the definition of the growth index parameter γ via

$$\gamma \equiv \left. \frac{\ln f}{\ln \Omega} \right|_{a=a_0} \ . \tag{3.240}$$

The latter has been the subject of increasingly accurate cosmological observations, for some recent references see [129, 130, 131, 149, 132]. [8]

On the theoretical side, for the tensor box one finds [106], for a matter fraction $\Omega = 0.25$,

$$\gamma = 0.556 - 106.4\, c_t + \mathcal{O}(c_t^2) \ . \tag{3.241}$$

where the first contribution is the classical GR value from the relativistic treatment of matter density perturbations [126]. The result presented above is in fact a slight improvement over the answer quoted in our earlier work [106], since now the improved relationship between t and a given in Eq. (3.195) has been used, which reduces the magnitude of the correction proportional to c_t. Nevertheless, it should be emphasized that the above result has been obtained in the $\mathbf{q} \to 0$ limit of the perturbation Fourier modes in Eq. (3.31).

Recent observational bounds on x-ray studies of large galactic clusters at distance scales of up to about 1.4 to $8.5 Mpc$ (comoving radii of $\sim 8.5 Mpc$ and viral radii of $\sim 1.4 Mpc$) [130, 131] favor values for $\gamma = 0.50 \pm 0.08$, and more recently $\gamma = 0.55 + 0.13 - 0.10$ [149, 132]. This would then constrain the amplitude c_t in Eq. (3.241) *at that scale* to $c_t \lesssim 5 \times 10^{-4}$. The latter bound from density perturbations seems a much more stringent bound than the one coming from the observed slip function. Indeed with the bound on c_t coming from the observed density

[8] For a recent detailed review on the many tests of general relativity on astrophysical scales, and a much more complete set of references, see for example [134, 135].

perturbation exponents one would conclude that, according to Eq. (3.238), the correction to the slip function at $z \simeq 0$ must indeed be very small, $\eta \simeq \mathcal{O}(10^{-3})$, which is a few orders of magnitude below the observational limit quoted above, $\eta \simeq 0.09 \pm 0.7$.

It is of course possible that the galactic clusters in question are not large enough yet to see the quantum effect of $G(\Box)$, since after all the relevant scale in Eq. (3.26) is related to λ and is supposed therefore to be very large, $\xi \simeq 4890 Mpc$. [9] But most likely the theoretical uncertainties in the value of c_t have also been underestimated in [128], and new, high precision lattice calculation will be required to significantly reduce the systematic errors.

Nevertheless it seems clear that the nonperturbative coefficient c_0 (or c_t) enters *all* calculations involving $G(\Box)$ with the *same* magnitude and sign. This is simply a consequence of c_0 being part of the renormalization group $G(\Box)$ which enters the covariant effective field equations of Eq. (3.3). Consequently, one should be able to relate one set of physical results to another, such as the value of the slip function $\eta(z=0)$ in Eq. (3.222) to the corrections to the density perturbation growth exponent γ computed in [106], and given here in Eq. (3.241). Then the amplitude c_t can be made to conveniently drop out when computing the ratio of $G(\Box)$ corrections to two different physical processes. The resulting predictions are then entirely independent of the theoretical uncertainty in the amplitude c_0, and remain sensitive only to the uncertainties in the two other quantum parameters ξ and ν, which are expected to be significantly smaller. One then obtains for the ratio of the corrections to the growth exponent γ to the slip function $\eta(z=0)$ at $z \simeq 0$

$$\frac{\delta\,\gamma}{\delta\,\eta} \;\simeq\; \frac{-106.4\,c_t}{-1.491\,c_t} \;\simeq\; +71.4 \tag{3.242}$$

[9] One might perhaps think that the running of G envisioned here might lead to small observable consequences on much shorter, galactic length scales. That this is not the case can be seen, for example, from the following argument. For a typical galaxy one has a size $\sim 30\,kpc$, giving for the quantum correction the estimate, from the potential obtained in the static isotropic metric solution with $G(\Box)$ [66, 67] which gives $\delta G(r) \sim (r/\xi)^{1/\nu}$, $(30\,kpc/4890 \times 10^3\,kpc)^3 \sim 2.31 \times 10^{-16}$ which is tiny given the large size of ξ. It is therefore unlikely that such a correction will be detectable at these length scales, or that it could account for large anomalies in the galactic rotation curves. The above argument also implies a certain sensitivity of the results to the value of the scale ξ; thus an increase in ξ by a factor of two tends to reduce the effects of $G(\Box)$ by roughly $2^3 = 8$, as can be seen already from Eq. (3.26) with $\nu = 1/3$. More specifically, the amplitude of the quantum correction is proportional, in the non-infrared improved case, to the combination c_0/ξ^3.

for the infrared unimproved case. One conclusion that one can draw from the numerical value of the above ratio is that it might be significantly harder to see the $G(\Box)$ correction in the slip function than in the matter density growth exponent, by almost 2 orders of magnitude in relative magnitude. Hopefully increasingly accurate astrophysical measurements of the latter will be done in the not too distant future. Of particular interest would be any trend in the growth exponents as a function of the maximum galactic cluster size.

Chapter 4

Conclusions

We presented in this thesis the studies of quantum gravity derived from the related calculations and ideas in quantum field theories. The main key concepts were that

(1) the quantum gravity above 2 dimensions should exhibit a nontrivial ultraviolet fixed point, therefore should posses a phase transition. This was researched from the quantum gravity in $2+\epsilon$ dimensions using perturbative quantum field theory treatment in double expansion in G and ϵ, in conjunction with the Euclidean lattice computations based on quantum Regge calculus in 4 dimensional studies.

(2) Furthermore, the physical phase of the quantum gravity should be the antiscreening phase which corresponds to the phase with renormalization group β function negative with a nontrivial UV fixed point where gravitational coupling G should increase with distance.

We have studied further on these above points and presented the results from the studies of

(i) the lattice discretized Hamiltonian formulation of quantum gravity and its analyses on statistical field theories to obtain universal quantities associated with critical phenomena, and

(ii) cosmological implications of the covariant formulation of the running G which increases with the distance scale as suggested by the existence of the nontrivial ultraviolet fixed point in the quantum gravitational theory.

We found that the elaboration of the results support the claims put forward above in (1) and (2).

More specifically, we summarize the results of the studies of (i) and (ii) below:

(i) We discussed the results from the studies on *Regge discretized Wheeler DeWitt equation and its explicit solutions* in $2+1$ and $3+1$ dimensions. The Wheeler DeWitt equation is a position representation of the Hamiltonian of quantum gravity. We took discrete formulation as an advantage to have better control on nonperturbative results. We carefully formulated such that the it retains the diffeomorphism invariance and successfully reduces to the continuum result. We were able to derive the solutions and further, the average quantities such as average areas and area fluctuations were obtained. The fluctuations diverged as G approaches zero, indicating a phase transition and further implying that there is only a strong coupling regime present. Furthermore we determined the universal critical exponent $\nu = 6/11 \simeq 0.545$. This result is complementary to the 3-dimensional Euclidean lattice version of the same theory, which gave $\nu \simeq 0.59$, giving support in the earlier claims. In $3+1$ dimensions, interestingly, even the simple solution in the limit of large volume indicates that there is a critical value for the coupling G. Investigations of the vacuum wave functional further suggest that for weak enough coupling, $G < G_c$, a pathological ground state with no continuum limit appears, where configurations with small curvature have vanishingly small probability. This results suggest that the regime below a certain value of G (weak coupling regime) seems unphysical as our universe should have almost zero curvature.

(ii) We presented the results on *possible consequences on large scale cosmology of a scale dependent Newton's constant G*. If the physical phase is the strong coupling regime, then one finds that the value of the coupling G increases as the distance scale increases. Having started from a set of manifestly covariant but nonlocal effective field equations, we systematically developed the linear theory of density perturbations for a non-relativistic, pressureless fluid. We calculated two cosmological parameters in the presence of manifestly covariant scale dependent G, the growth index γ and the slip function η in two different representations of coordinate systems, namely, the comoving gauge and the conformal Newtonian gauge. Mainly due to the large uncertainties associated with the nonperturbative constant which appears in the expression for the renormalization group equations for the scale dependent gravitational coupling G, the precise predictions of the values of the cosmological parameters, the growth index γ and the slip function η, are not possible at this point, however we determined the general trend in the corrections to the traditional GR due to $G(\Box)$. These results are important as it is possible that in the foreseeable future, cosmological and astrophysical observations (WMAP, weak lensing, supernovae and CMB data) might give more precise insight into these numbers, therefore either serve as a support for this running $G(\Box)$ scenario or otherwise.

We will conclude with some final remarks for each Chapters.

We note here some final comments on the results of the work in Chapter 2 on 2+1 dimensions. Let us add here a few final comments aimed at placing the present work in the context of previous calculations for the same theory. A number of attempts have been made over the years to obtain an estimate for the gravitational wave functional $\Psi[g]$ in the absence of sources. These generally have relied on the weak field expansion in the continuum, as originally done in [86, 87]. Thus for example one finds in $3+1$ dimensions

$$\Psi[h^{TT}] \; = \; \mathcal{N} \; \exp \left\{ - \, \frac{1}{4} \int d^3\mathbf{k} \;\; k \;\; h_{ik}^{TT}(\mathbf{k}) \;\; h_{ik}^{TT*}(\mathbf{k}) \right\} \, , \tag{4.1}$$

where $h_{ik}^{TT}(\mathbf{k})$ is the Fourier amplitude of transverse-traceless modes for the linearized gravitational field in 4 dimensions. The above wave functional describes a collection of harmonic oscillator wave functions, one for each of the infinitely many physical modes of the linearized gravitational field. As in the case of the electromagnetic field, the ground state wave functional can be expressed equivalently in terms of first derivatives of the field potentials (the corresponding $\mathbf{B}$ field for gravity), without having to mention Fourier amplitudes, as

$$\Psi[h^{TT}] \; = \; \mathcal{N} \; \exp \left\{ - \, \frac{1}{8\,\pi^2} \int d^3\mathbf{x} \int d^3\mathbf{y} \; \frac{h_{ik,l}^{TT}(\mathbf{x}) \;\; h_{ik,l}^{TT*}(\mathbf{y})}{|\mathbf{x} \, - \, \mathbf{y}|^2} \right\} \, . \tag{4.2}$$

Clearly both of the above expressions represent only the leading term in an expansion involving infinitely many terms in the metric fluctuation h_{ij} (and since they apply to an expansion about flat space, the cosmological constant term does not appear either). Now, in $2+1$ dimensions the above expressions become meaningless, since there cannot be any transverse-traceless modes, whose naive counting comes from $\frac{D\,(D+1)}{2} - D - D = \frac{D\,(D-3)}{2}$, where the first term is the independent degrees of freedom in a metric, the second term is from Bianchi identity, and the third term is from gauge fixing. The only expectations that remains true is that the wave functional should still depend on physical degrees of freedom only: it should be a function of the intrinsic geometry of 3-space, and should not change under a general coordinate change.

If one restricts oneself to local terms a number of invariants are possible in $2+1$ dimensions. In principle, the wave function could depend on, besides the total area

$$A_{tot} \; = \; \int d^2x \, \sqrt{g} \tag{4.3}$$

and curvature

$$4\pi \, \chi \; = \; \int d^2x \, \sqrt{g} \; R \, , \tag{4.4}$$

other invariants such as

$$r_n \;=\; \int d^2x \,\sqrt{g}\; R^n \tag{4.5}$$

with n an integer. The latter result follows from the fact that in $d = 2$ both the Riemann and Ricci tensors only have one component, related to the scalar curvature,

$$R_{\mu\nu\rho\sigma} \;=\; \frac{1}{2}\, R \, (g_{\mu\sigma}\, g_{\nu\rho} \,-\, g_{\mu\rho}\, g_{\nu\sigma}) \,, \qquad R_{\mu\nu} \;=\; \frac{1}{2}\, R \, g_{\mu\nu} \,. \tag{4.6}$$

Nonlocal terms are possible as well involving inverse powers of the covariant d'Alembertian $\Box$, but these do not seem to play a significant role in the lattice theory.

Now, the relevant Euclidean theory for the present work is of course gravity in 3 (2 space +1 time) dimensions. But in 3 dimensions the Riemann and Ricci tensor have the same number of algebraically independent components (6), and are related to each other by

$$R^{\mu\nu}{}_{\lambda\sigma} \;=\; \epsilon^{\mu\nu\kappa}\, \epsilon_{\lambda\sigma\rho} \left(R^{\rho}{}_{\kappa} \,-\, \frac{1}{2}\, \delta^{\rho}{}_{\kappa} \right) \tag{4.7}$$

The field equations then imply, using Eq. (4.7), that the Riemann tensor is completely determined by the matter distribution implicit in $T_{\mu\nu}$,

$$R_{\mu\nu\rho\sigma} = 8\pi G \, \left[g_{\mu\rho}\, T_{\nu\sigma} + g_{\nu\sigma}\, T_{\mu\rho} + g_{\mu\sigma}\, T_{\nu\rho} - g_{\nu\rho}\, T_{\mu\sigma} + T \, (g_{\mu\sigma}\, g_{\nu\rho} - g_{\mu\rho}\, g_{\nu\sigma}) \right] \tag{4.8}$$

In empty space $T_{\mu\nu} = 0$, which then implies for zero cosmological constant the vanishing of Riemann there

$$R_{\mu\nu\rho\sigma} \;=\; 0 \,. \tag{4.9}$$

As a result in 3 dimensions classical spacetime is locally flat everywhere outside a source, gravitational fields do not propagate outside matter, and two bodies cannot experience any gravitational force: they move uniformly on straight lines. There cannot be any gravitational waves either: the Weyl tensor, which carries information about gravitational fields not determined locally by matter, vanishes identically in 3 dimensions.

What seems rather puzzling at first is that the Newtonian theory seems to make perfect sense in $D = 3$. This can only mean that the Newtonian theory is *not* recovered in the weak field limit of the relativistic theory. To see this explicitly, it is sufficient to consider the trace-reversed form of the field equations,

$$R_{\mu\nu} \;=\; 8\pi G \left(T_{\mu\nu} \;-\; \frac{1}{D-2}\, g_{\mu\nu}\, T \right) \tag{4.10}$$

with $T = T^{\lambda}{}_{\lambda}$, in the weak field limit. In the linearized theory, with $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, and in the gauge $\nabla_\lambda h^{\lambda}{}_{\mu} - \frac{1}{2} \nabla_\mu h^{\lambda}{}_{\lambda} = 0$ one obtains the wave equation

$$\Box\, h_{\mu\nu} \;=\; -16\pi G \left(\tau_{\mu\nu} \;-\; \frac{1}{D-2}\, \eta_{\mu\nu}\, \tau \right) \tag{4.11}$$

with $\tau_{\mu\nu}$ the linearized stress tensor. After neglecting the spatial components of $\tau_{\mu\nu}$ in comparison to the mass density τ_{00}, and assuming that the fields are quasi-static, one obtains a Poisson equation for h_{00},

$$\nabla^2\, h_{00} \;=\; -16\pi G\, \frac{D-3}{D-2}\, \tau_{00} \tag{4.12}$$

In 4 dimensions this is equivalent to Poisson's equation for the Newtonian theory when one identifies the metric with the Newtonian field ϕ in the usual way via $h_{00} \;=\; -2\;\phi$. But in 3 dimensions such a correspondence is obstructed by the fact that, from Eq. (4.12), the nonrelativistic Newtonian coupling appearing in Poisson's equation is given by

$$G_{\mathrm{Newton}} \;=\; \frac{2\,(D-3)}{D-2}\, G \tag{4.13}$$

and the mass density τ_{00} completely decouples from the gravitational field h_{00}. As a result, the linearized theory in 3 dimensions fails to reproduce the Newtonian theory.

In a complementary way one can show that gravitational waves are not possible either in the linearized theory in 3 dimensions. Indeed the counting of physical degrees of freedom for the D-dimensional theory goes as follows. The metric $g_{\mu\nu}$ has $\frac{1}{2}D(D+1)$ independent components; the Bianchi identity and the coordinate conditions reduce this number to $\frac{1}{2}D(D+1) - D - D = \frac{1}{2}D(D-3)$, which gives indeed the correct number of physical degrees of freedom (2) corresponding to a massless spin 2 particle in $D = 4$, and no physical degrees of freedom in $D = 3$ (and minus one degree of freedom in $D = 2$ which is in fact incorrect).

However, investigations of quantum 2 dimensional gravity have uncovered the fact that there can be surviving degrees of freedom in the quantum theory, at least in 2 dimensions. The usual treatment of 2 dimensional gravity [150, 151] starts from the metric in the conformal gauge $g_{\mu\nu}(x) = e^{\phi}(x)\tilde{g}_{\mu\nu}$, where $\tilde{g}_{\mu\nu}$ is a reference metric, usually taken to be the flat one. The conformal gauge fixing then implies a nontrivial Faddeev-Popov determinant, which, when exponentiated, results in an effective Liouville action, with a potential term coming from the cosmological constant contribution. One would therefore conclude from this example that gravity in the functional integral representation needs a careful treatment of the conformal degree of freedom, since in general its dynamics cannot be assumed to be trivial. The calculations presented show that this is indeed the case in $2+1$ dimensions as well.

Finally for the work on Wheeler DeWitt equation in $2+1$ dimensions, it will be interesting to explore further in trying to obtain the exact solution with the explicit curvature term for *arbitrary* edge lengths, as this will give us further insights into in what form the diffeomorphism invariance on lattice in 3 (2 space + 1 time) dimensions is present. For the diffeomorphism invariant argument, we still expect the analytic solution for *arbitrary* edge lengths to depend only on geometrical quantities such as total area, topology of the manifold, and number of the simplexes, however, until one confirms this by solving it, one is not certain. In fact, it is possible that one needs to be careful about the ordering of the kinetic term in the Hamiltonian. For example, more symmetric ordering, Laplacian ordering may be sought instead of the one we took, *i.e.*, we replace such that

$$G_{ij}\frac{\partial^2}{\partial l_i \partial l_j} \;\longrightarrow\; \sqrt{det(G_{ij})}\frac{\partial}{\partial l_i}\left(\frac{1}{\sqrt{det(G_{ij})}}G_{ij}\frac{\partial}{\partial l_j}\right) \tag{4.14}$$

[personal communication with Sergey Cherkas]. One possibility will be that the ordering may be responsible *if* one fails to solve for the analytic solution for *arbitrary* edge lengths in diffeomorphim invariant way.

We add some contemplations for the work in Chapter 2 on $3+1$ dimensions. In the Euclidean lattice theory of gravity in 4 dimensions it was also found early on [26, 28, 53] [see

[46, 47] for more recent numerical investigations of 4 dimensional lattice gravity, including the determination of the critical point and scaling exponents] that the weak coupling (gravitational screening) phase is pathological with no sensible continuum limit, corresponding to a degenerate lower dimensional branched polymer. The calculations presented here can be regarded, therefore, as consistent with the conclusions reached earlier from the Euclidean lattice framework. No new surprises have arisen so far when considering the Lorentzian $3+1$ theory, using an entirely different set of tools.

It is also worthwhile at this point to compare with other attempts at determining the phase structure of quantum gravity in 4 dimensions. Besides the Regge lattice approach, there have been other attempts at searching for a nontrivial renormalization group ultraviolet fixed point in 4 dimensions using continuum methods. In one popular field theoretic approach one develops a perturbative diagrammatic $2+\epsilon$ continuum expansion using the background field method to 2 - loop order [20, 23, 40, 48, 49, 108]. This then leads to a nontrivial UV fixed point $G_c = \mathcal{O}(\epsilon)$ close to 2 dimensions. Two phases emerge, one implying again gravitational screening, and the other antiscreening. In the truncated renormalization group calculations for gravity in 4 dimensions [50, 109, 110], where one retains the cosmological and Einstein Hilbert terms, and possibly later some higher derivative terms, one also finds evidence for a nontrivial UV fixed point scenario. As in the case of gauge theories, both of these methods are ultimately based on renormalization group flows and the weak field expansion, and are therefore unable to characterize the nonperturbative features of either one of the two ground states. Indeed, within the framework of the weak field expansion inherent in these methods, only the weak coupling phase has a chance to start with. It is nevertheless encouraging that such widely different methods tend to point in the same direction, namely a nontrivial phase structure for gravity in 4 dimensions.

We elaborate final points for the work in Chapter 3 on cosmological implications. We conclude that the value of the present calculations lies in the fact that so far a discernible trend seems to emerge from the results. The trend we have found seems to suggest that the correction to the growth exponent γ is negative in sign, and is initially rather small for small clusters, then slowly increasing in magnitude with scale. Therefore our present calculations shows that the observed growth index parameter in a large scale should be smaller than the conventional value (due to ΛCDM) $\gamma = 0.556$. The trend for the slip function also is negative in sign, therefore its observed value should be negative as the conventional value of $\eta = 0$.

Note that since we only performed calculations for $\mathbf{q} \to 0$ which inherently fixes in the calculations the scale to be the largest *i.e.,* the size of the universe, we note that we are missing the explicit expression of $(x\, Mpc/4890\, Mpc)^{1/\nu}$, where x is the scale where we measure cosmological parameters. Therefore, our calculations with $\mathbf{q} \to 0$ implies having $x \to 4890$ and thus $(x\, Mpc/4890\, Mpc)^{1/\nu} \to 1$. If one includes this correction to include the scale dependent effect coming from the higher order corrections in $\mathbf{q}$, the correction to the growth index parameter γ becomes negligible at the scale of clusters of galaxies (Eq. (3.135)). This is expected given the discussion that I will provide below. In fact as obvious from the form of the $G(\Box)$ or $G(\mu^2)$,

$$G(\mu^2) \;\simeq\; G_0 \left[1 \,+\, c_0 \left(\frac{\xi^{-2}}{\mu^2 \,+\, \xi^{-2}} \right)^{1/2\nu} \,+\, \dots \right] . \tag{4.15}$$

the correction to G_0 which is the laboratory value of the Newton's gravitational coupling, becomes apparent only at a scale very close to the size of the universe, as one can see in Fig. 4.1. In fact if one uses $c_0 \approx 8$ [1] as given by the recent recomputation on computer provided by lattice calculations, one reaches $G(l = \xi \sim 4820 Mpc) = 3.8 G_0$, $G(l = 0.58\xi \sim 4820 Mpc) = 2.0 G_0$, and $G(l = 0.24\xi \sim 1170 Mpc) = 1.1 G_0$ where $\xi \sim 4890 Mpc$ is the size of the universe. Recalling that the clusters of galaxy data only reaches about $100 Mpc$ maximum, the scale dependent correction of G may be very difficult to observe unless one can reach the size of the universe. Therefore, most effectively, one may want to find cosmoligical

[1] If one uses $c_0 \sim 1$ as expected from the experiences of the same type of constant for quantum correction in nonlinear sigma models, QCD, etc., the correction becomes even smaller and one notices that it is only $G(l = \xi \sim 4820 Mpc) = 1.4 G_0$ even if one reaches the size of the whole universe.

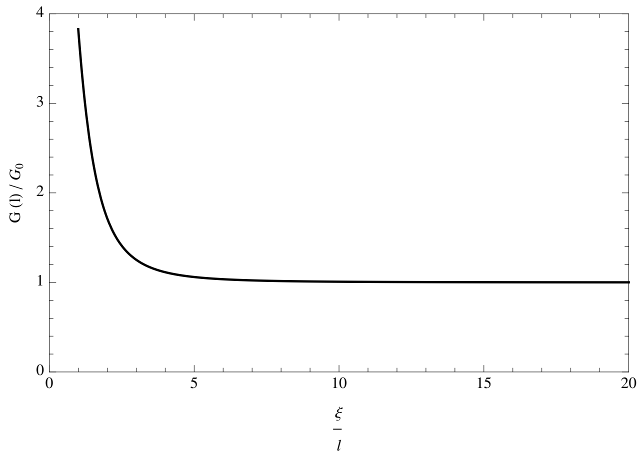

Figure 4.1: Scale dependent G vs ξ/l the length scale normalized by the size of the whole universe $\xi \sim 4890 Mpc$. Here, we have taken $c_0 \sim 8$ in Eq. (4.15).

implications in Cosmic Microwave Background (CMB) data which can reach up to the size of the whole universe. Specifically, for the future prospect, one should look into the effects due to this scale dependent $G(\mu)$ as given above in Eq. (4.15) in the angular power spectrum of CMB data.

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Appendices

A Nonrelativistic (Newtonian) Treatment of Matter Density Perturbations

In this section we discuss the Newtonian theory of small matter fluctuations, first by recalling the relevant equations in the usual treatment, and then by presenting what changes need to be implemented in order to account for the running of G. Later these equations will be solved, so that a comparison can be made with the results in the absence of a running G.

When discussing a nonrelativistic Hubble flow it is customary to define coordinates in the following way

$$\mathbf{x} = \frac{\mathbf{r}}{a(t)}\,, \qquad \mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{\dot{a}}{a}\,\mathbf{r} \tag{A.16}$$

where $\mathbf{x}$ is attached to the comoving frame, while $\mathbf{r}$ is the flat Minkowski space coordinate, such that in the comoving frame $\mathbf{x}$ one has, by construction, $d\mathbf{x}/dt = 0$.

In the following some simplification will arise due to the fact that we shall consider a nonrelativistic fluid with the negligible pressure, $p \simeq 0$ or $w = 0$. The relevant equations are then the continuity equation, the Euler equation and the gravitational field equations. These will be listed below to zeroth and first order in the matter density (ρ), pressure (p), velocity field ($\mathbf{v}$) and gravitational field $\mathbf{g}$.

A.1 Newtonian Treatment without the Running of G

After decomposing the fields into a background and a fluctuation contribution, $\rho = \bar{\rho} + \delta\rho$, $p = \bar{p} + \delta p$, and $\mathbf{v} = \bar{\mathbf{v}} + \delta\mathbf{v}$, one obtains from the continuity equation, to zeroth and first order respectively,

$$\dot{\bar{\rho}} + \nabla \cdot (\bar{\rho}\, \mathbf{v}) = 0 \,, \qquad \dot{\delta\rho} + 3\, \frac{\dot{a}}{a}\, \delta\rho + \frac{\dot{a}}{a}\, (\mathbf{r} \cdot \nabla)\, \delta\rho + \bar{\rho}\, \nabla \cdot \delta\mathbf{v} = 0 \,. \tag{A.17}$$

When the effect of the Hubble flow is included, *i.e.,* Eq. (A.16), the above zeroth order equation reduces to

$$\dot{\bar{\rho}}(t) + 3\, \frac{\dot{a}(t)}{a(t)}\, \bar{\rho}(t) \;= 0 \,, \tag{A.18}$$

with solution $\bar{\rho}(t) = \bar{\rho}_0\, (a_0/a(t))^3$, where $\bar{\rho}_0$ and a_0 are the two integration constants corresponding to the present matter density and to the present scale factor (usually taken to be $a_0 = 1$). We note here that Eq. (A.18), and hence Eq. (3.43), will continue to hold for a running G, as these equations are derived from the kinematics and the continuity equations in the RW background metric given in Eq. (A.17), whose is not affected by the running of $G \to G(\Box)$.

To zeroth and first order in the fluctuations the Euler equations for a fluid in the RW background are given respectively by

$$\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla)\; \mathbf{v} = \mathbf{g} \,, \qquad \dot{\delta\mathbf{v}} + \frac{\dot{a}}{a}\, \delta\mathbf{v} + \frac{\dot{a}}{a}\, (\mathbf{r} \cdot \nabla)\; \delta\mathbf{v} = -\frac{1}{\bar{\rho}}\, \nabla\, \delta p + \delta\mathbf{g} \,. \tag{A.19}$$

Finally the gravitational field equations are given to zeroth and first order in the fluctuations by

$$\nabla \times \mathbf{g} = 0 \,, \qquad \nabla \cdot \mathbf{g} = -\, 4\pi\, G_0\, \bar{\rho} \tag{A.20}$$

$$\nabla \times \delta\mathbf{g} = 0 \,, \qquad \nabla \cdot \delta\mathbf{g} = -\, 4\pi\, G_0\, \delta\rho \,, \tag{A.21}$$

incorporating Gauss' law and the constraint that the gravitational fields are longitudinal. Only the last set of equations contain the gravitational constant G. Hence, in the framework of the Newtonian treatment, the modification of a running $G \rightarrow G(\Box)$ only affects the gravitational Poisson equation.

It is customary at this stage to introduce Fourier components of the fluctuations, and write

$$\delta\rho(\mathbf{r},t) = \delta\rho_{\mathbf{q}}(t)\,\exp\left[\frac{i\,\mathbf{r}\cdot\mathbf{q}}{a(t)}\right] \tag{A.22}$$

and similarly for $\delta\mathbf{v}$, $\delta\mathbf{g}$, and possibly δp. For an adiabatic fluctuation one can also set $\delta p = v_s^2\,\delta\rho$, with v_s the speed of sound.

Then to first order in the fluctuations the continuity equation, Euler equation and the gravitational field equations take on the form, for each mode $\mathbf{q}$,

$$\dot{\delta\rho}_{\mathbf{q}}(t) + 3\,\frac{\dot{a}(t)}{a(t)}\,\delta\rho_{\mathbf{q}}(t) + \frac{i\,\mathbf{q}\cdot\delta\mathbf{v}_{\mathbf{q}}(t)}{a(t)}\,\bar{\rho}(t) = 0 \tag{A.23}$$

$$\dot{\delta\mathbf{v}}_{\mathbf{q}}(t) + \frac{\dot{a}(t)}{a(t)}\,\delta\mathbf{v}_{\mathbf{q}}(t) = -\,\frac{i\,\mathbf{q}}{a(t)}\,\frac{v_s^2}{\bar{\rho}(t)}\,\delta\rho_{\mathbf{q}}(t) + \delta\mathbf{g}_{\mathbf{q}}(t) \tag{A.24}$$

$$\delta\mathbf{g}_{\mathbf{q}}(t) = \frac{4\pi i\,\mathbf{q}}{\mathbf{q}^2}\,a(t)\,G_0\,\delta\rho_{\mathbf{q}}(t)\,. \tag{A.25}$$

Subsequent elimination of the gravitational and velocity fields then leads to a single second order differential equation for the matter density contrast $\delta_{\mathbf{q}}(t) \equiv \delta\rho_{\mathbf{q}}(t)/\bar{\rho}(t)$ describing the physics of compressional modes:

$$\ddot{\delta}_{\mathbf{q}}(t) + 2\,\frac{\dot{a}(t)}{a(t)}\,\dot{\delta}_{\mathbf{q}}(t) + \left(\frac{v_s^2\,\mathbf{q}^2}{a(t)^2} - 4\pi\,G_0\,\bar{\rho}(t)\right)\,\delta_{\mathbf{q}}(t) = 0\,. \tag{A.26}$$

In the limit of very long wavelength fluctuations, $\mathbf{q} \rightarrow 0$, the above equation simplifies to

$$\ddot{\delta}(t) + 2\,\frac{\dot{a}(t)}{a(t)}\,\dot{\delta}(t) - 4\pi\,G_0\,\bar{\rho}(t)\,\delta(t) = 0\,. \tag{A.27}$$

A solution can then be found, using $\bar{\rho}(t) = 1/6\pi G t^2$ and $\dot{a}(t)/a(t) \equiv H(t) = 2/3t$, such that the general form for $\delta(t)$ is given by a linear combination of either $\sim t^{2/3}$ or $\sim t^{-1}$. The latter corresponds to a decaying (as opposed to growing) solution and is usually discarded, giving finally the standard Newtonian result $\delta(a) \propto a$. We note here that the above nonrelativistic equation and solution applies to the case of nonrelativistic matter only; in particular it excludes the presence of a cosmological constant.

A.2 Newtonian Treatment with Running $G(\Box)$

The next step is a modification of the nonrelativistic equations in Eqs. (A.17), (A.19), (A.20) and (A.21) to incorporate a suitable running of G. Since only the latter set of equations, Eqs. (A.20) and (A.21), contain G it is only these that need to be suitably modified. In the presence of a scale dependent coupling one has

$$\delta \mathbf{g} = -\nabla\, \delta\phi \tag{A.28}$$

with the perturbing potential $\delta\phi$ given by a solution to Poisson's equation

$$\nabla^2 \delta\phi(\mathbf{r}, t) = -\nabla \cdot \delta\mathbf{g}(\mathbf{r}, t) = 4\pi G(\Box)\, \delta\rho(\mathbf{r}, t) \tag{A.29}$$

and $G(\Box)$ given in Eq. (3.26). Following Eq. (A.22), as it applies here to $\delta\mathbf{g}$ and $\delta\rho$, we Fourier transform the spatial components of the above Poisson equation, which requires the Fourier transform of $G(\Box)$ as obtained from Eq. (3.26), namely

$$G(\mathbf{q}^2, \partial_t^2) = G_0 \left\{ 1 + c_0 \, \frac{\xi^{-1/\nu}}{\left[-\partial_t^2 - \mathbf{q}^2/a^2(t)\right]^{1/2\nu}} + \dots \right\} . \tag{A.30}$$

As a result the gravitational field perturbation is of the form

$$\delta\mathbf{g}_{\mathbf{q}}(t) = \frac{4\pi\, i\, \mathbf{q}}{\mathbf{q}^2}\, a(t) \, \cdot \, \exp\left[\frac{-\, i\, \mathbf{r} \cdot \mathbf{q}}{a(t)}\right] \, G(\mathbf{q}^2, \partial_t^2) \, \left(\delta\rho_{\mathbf{q}}(t) \, \exp\left[\frac{i\, \mathbf{r} \cdot \mathbf{q}}{a(t)}\right] \right) . \tag{A.31}$$

Since we are mainly interested in the long wavelength limit, it suffices here to evaluate the above expression in the limit $\mathbf{q} \to 0$,

$$
\begin{aligned}
\delta \mathbf{g_q}(t) &= \frac{4\pi\, i\, \mathbf{q}}{\mathbf{q}^2}\, a(t) \left[1 - \frac{i\, \mathbf{r}\cdot\mathbf{q}}{a(t)} + \dots \right] G(\mathbf{q}^2, \partial_t^2) \left(\delta\rho_{\mathbf{q}}(t) \left[1 + \frac{i\, \mathbf{r}\cdot\mathbf{q}}{a(t)} + \dots \right] \right) \\
&\simeq \frac{4\pi\, i\, \mathbf{q}}{\mathbf{q}^2}\, a(t) \Bigg[G(\mathbf{q}^2, \partial_t^2)\, \delta\rho_{\mathbf{q}}(t) \\
&\qquad\qquad - \frac{i\, \mathbf{r}\cdot\mathbf{q}}{a(t)} G(\mathbf{q}^2, \partial_t^2)\, \delta\rho_{\mathbf{q}}(t) + G(\mathbf{q}^2, \partial_t^2)\, \delta\rho_{\mathbf{q}}(t)\, \frac{i\, \mathbf{r}\cdot\mathbf{q}}{a(t)} + \dots \Bigg] ,
\end{aligned}
\tag{A.32}
$$

and for $\mathbf{q} = 0$ only the first term survives. Furthermore, when $G(\Box) = G(\mathbf{q}^2, \partial_t^2)$ acts on a function of t which we will assume here is of the form of a power (*e.g.*, t^α, with the power α a number of order one) one obtains

$$
G(\mathbf{q}^2, \partial_t^2) \cdot t^\alpha \;\to\; G(t) \cdot t^\alpha . \tag{A.33}
$$

Here the running coupling $G(t)$ is given by the expression in Eq. (3.15), with $t_0 \equiv \xi$, and the coefficient

$$
c_t \;=\; \left| \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+1/\nu)} \right| c_0 . \tag{A.34}
$$

Thus for example for $\alpha = -4/3$ (the standard Newtonian result for matter density perturbations) one has $c_t = (27/10)\, c_0$; in the following it will be safe to assume that the coefficient c_t in Eq. (3.15) is a number of the same order of magnitude as the original c_0 in Eq. (3.26).

Consequently, when acting on a density perturbation $\delta\rho_{\mathbf{q}}(t)$ in the form of a power law in t, to leading order in $\mathbf{q}$ one obtains simply

$$
\delta \mathbf{g_q}(t) = \frac{4\pi\, i\, \mathbf{q}}{\mathbf{q}^2}\, a(t)\, G_0 \left[1 + c_t \left(\frac{t}{t_0}\right)^{1/\nu} + \dots \right] \delta\rho_{\mathbf{q}}(t) . \tag{A.35}
$$

This last result can be compared with Eq. (A.25) for the case of a constant G.

As stated previously, the continuity equation for the fluctuations, Eq. (A.23), and the corresponding Euler equation for the fluctuations, Eq. (A.24), are not modified by the presence of a running $G(\Box)$, as given in Eqs. (A.31) and (A.35). To solve the resulting equations of motion for the fluctuations, it is now customary to decompose the velocity perturbation $\delta\mathbf{v}$ into parts perpendicular and parallel to $\mathbf{q}$

$$\delta\mathbf{v}_{\mathbf{q}}(t) = \delta\mathbf{v}_{\mathbf{q}\perp}(t) + i\,\mathbf{q}\,\epsilon_{\mathbf{q}}(t) \tag{A.36}$$

with

$$\mathbf{q}\cdot\delta\mathbf{v}_{\mathbf{q}\perp} \;=\; 0\,, \qquad \epsilon_{\mathbf{q}} \;\equiv\; -\,\frac{i\,\mathbf{q}\cdot\delta\mathbf{v}_{\mathbf{q}}}{\mathbf{q}^2}\,. \tag{A.37}$$

The fractional change in the matter density δ is then defined as

$$\delta_{\mathbf{q}}(t) \equiv \frac{\delta\rho_{\mathbf{q}}(t)}{\bar{\rho}(t)}\,. \tag{A.38}$$

With the above decomposition of the velocity field $\delta\mathbf{v}$ and the expression for the density contrast δ inserted into the first order continuity equation, Eq. (A.23), one obtains the unmodified result

$$\dot{\delta}_{\mathbf{q}}(t) = \frac{\mathbf{q}^2}{a(t)}\,\epsilon_{\mathbf{q}}(t)\,, \tag{A.39}$$

so that there is no change in the relationship between δ and ϵ when $G \to G(\Box)$. In turn the Euler equation for the fluctuation, Eq. (A.24), now becomes the two sets of equations

$$\begin{aligned} Re\,: &\quad \dot{\delta\mathbf{v}}_{\mathbf{q}\perp}(t) + \frac{\dot{a}}{a}\,\delta\mathbf{v}_{\mathbf{q}\perp}(t) = 0 \\ Im\,: &\quad i\,\mathbf{q}\,\dot{\epsilon}_{\mathbf{q}}(t) + \frac{\dot{a}}{a}\,i\,\mathbf{q}\,\epsilon_{\mathbf{q}}(t) = -\,\frac{i\,\mathbf{q}}{a}\,v_s^2\,\delta_{\mathbf{q}}(t) + \delta\mathbf{g}_{\mathbf{q}} \end{aligned} \tag{A.40}$$

with the gravitational field fluctuation $\delta\mathbf{g}_{\mathbf{q}}$ now given by the expression in Eq. (A.31). From the real part (corresponding to rotational modes) one concludes

$$\delta\mathbf{v}_{\mathbf{q}\perp} \propto a^{-1}(t)\,, \tag{A.41}$$

which is of the same form as in the case of a constant G. From the imaginary part (corresponding to compressional modes) in Eq. (A.40) one obtains, using Eq. (A.39),

$$\ddot{\delta}_{\mathbf{q}}(t) + 2\,\frac{\dot{a}}{a}\,\dot{\delta}_{\mathbf{q}}(t) + \frac{\mathbf{q}^2}{a^2}\,v_s^2\,\delta_{\mathbf{q}}(t) - 4\pi\,\exp\left[\frac{-\,i\mathbf{r}\cdot\mathbf{q}}{a(t)}\right]\,G(\mathbf{q}^2,\partial_t^2)\,\left(\exp\left[\frac{i\mathbf{r}\cdot\mathbf{q}}{a(t)}\right]\,\bar{\rho}(t)\,\delta_{\mathbf{q}}(t)\right) = 0\,. \tag{A.42}$$

The latter can be recast into the slightly simpler form

$$\ddot{\delta}_{\mathbf{q}}(t) + 2\,\frac{\dot{a}}{a}\,\dot{\delta}_{\mathbf{q}}(t) + \left(\frac{\mathbf{q}^2}{a^2}\,v_s^2 - 4\pi\,\mathcal{G}\left(\mathbf{q}^2, \partial_t^2\right)\right)\,\delta_{\mathbf{q}}(t) = 0 \tag{A.43}$$

by defining a modified source term

$$\mathcal{G}(\mathbf{q}^2, \partial_t^2) \equiv \frac{1}{\delta_{\mathbf{q}}(t)}\,\left\{\exp\left[\frac{-\,i\mathbf{r}\cdot\mathbf{q}}{a(t)}\right]\,G(\mathbf{q}^2, \partial_t^2)\,\left(\exp\left[\frac{i\mathbf{r}\cdot\mathbf{q}}{a(t)}\right]\,\bar{\rho}(t)\,\delta_{\mathbf{q}}(t)\right)\right\}\,. \tag{A.44}$$

In the limit $\mathbf{q}\to 0$ one obtains immediately

$$\ddot{\delta}(t) + 2\,\frac{\dot{a}}{a}\,\dot{\delta}(t) - 4\pi\,G(t)\,\bar{\rho}(t)\,\delta(t) = 0\,. \tag{A.45}$$

The last two equations can now be compared with the corresponding results for a constant G, given in Eqs. (A.26) and (A.27).

A.3 Computation of the Nonrelativistic (Newtonian) Growth Index with $G(\Box)$

The next step requires a solution of the differential equation for the density perturbations $\delta_{\mathbf{q}}(t)$, in the Newtonian approximation and in the limit $\mathbf{q}\to 0$, as in Eq. (A.45). It is convenient and customary at this point to change variables from t to the scale factor $a(t)$, so that $\delta_{\mathbf{q}}(t)\to\delta_{\mathbf{q}}(a) = \tilde{\delta}_{\mathbf{q}}\cdot\delta(a)$. From Eq. (3.119) one has

$$\begin{aligned}
\dot{\delta}(t) &= a\,H(a)\,\frac{\partial\,\delta(a)}{\partial a} \\
\ddot{\delta}(t) &= a^2\,H^2(a)\left[\frac{\partial\ln H(a)}{\partial a} + \frac{1}{a}\right]\,\frac{\partial\,\delta(a)}{\partial a} + a^2\,H^2(a)\,\frac{\partial^2\delta(a)}{\partial a^2}\,.
\end{aligned} \tag{A.46}$$

Here $H(a)$ is defined as the Hubble "constant" $H(a)\equiv\dot{a}(t)/a(t)$, as it appears in the equations of motion for a background FLRW geometry

$$H(a) = \sqrt{\frac{8\pi}{3}\,G(a)\,\bar{\rho}(a) + \frac{\lambda}{3}}\,, \tag{A.47}$$

but with a running Newton's constant $G(a)$ (see Eq. (3.15))

$$G(a) = G_0\left[1 + \frac{\delta G(a)}{G_0}\right] = G_0\left[1 + c_a\left(\frac{a}{a_0}\right)^{\gamma_\nu} + \dots\right]\,. \tag{A.48}$$

Here the index is $\gamma_\nu = 3/2\nu$, since from Eq. (3.15) one has for non relativistic matter $a(t)/a_0 \approx (t/t_0)^{2/3}$. In the above expression $c_a \approx c_t$ if a_0 is identified with a scale factor corresponding to a universe of size ξ; to a good approximation this corresponds to the universe "today", with the relative scale factor customarily normalized to $a/a_0 = 1$. As a consequence, the constant c_a in Eq. (A.48) can be taken to be of the same order as the constant c_0 appearing in the original expressions for $G(\Box)$ in Eqs. (3.223) and (3.26). Note also that, by the use of Eq. (A.47) for the scale factor, we have allowed for a non-vanishing cosmological constant in our otherwise Newtonian (nonrelativistic) treatment.

After these substitutions one finally obtains the differential equation for the matter density contrast, Eq. (A.45), in the variable $a(t)$

$$\frac{d^2\delta(a)}{da^2} + \left(\frac{d \ln H(a)}{da} + \frac{3}{a} \right) \frac{d\delta(a)}{da} - \frac{4\pi G(a)\, \bar{\rho}(a)}{a^2\, H^2(a)}\, \delta(a) = 0\,. \tag{A.49}$$

Note that in order to compute the leading, in $\delta G(a)/G_0$, correction to the density contrast $\delta(a)$, one only needs $\bar{\rho}(a)$ to lowest order as given in Eq. (3.43), and $H(a)$ as given in Eq. (A.47).

With the aid of the parameter θ (see Eq. (3.115))

$$\theta \equiv \frac{1-\Omega}{\Omega} \tag{A.50}$$

where Ω is the matter density fraction and $1-\Omega$ the cosmological constant fraction as measured today, one obtains the following differential equation for the density contrast $\delta(a)$

$$\frac{\partial^2\delta}{\partial a^2} + \frac{3\,(1+2\,a^3\,\theta)}{2\,a\,(1+a^3\,\theta)} \left(1 + c_a \frac{\gamma_\nu\, a^{\gamma_\nu} + \left(\frac{1}{3}\gamma_\nu - 1\right) a^{3+\gamma_\nu}\theta}{(1+a^3\,\theta)\;(1+2a^3\,\theta)} \right) \frac{\partial\delta}{\partial a} - \frac{3}{2\,a^2\,(1+a^3\,\theta)} \left(1 + c_a \frac{a^{3+\gamma_\nu}\,\theta}{1+a^3\,\theta} \right) \delta = 0 \tag{A.51}$$

for a reference scale $a_0 = 1$; the latter can always be re-introduced later by the trivial replacement $a \to a/a_0$.

Without a scale dependent G ($c_a = 0$ in Eq. (A.48)), the growing solution to the above equation is given by

$$\delta_0(a) \propto a \cdot {}_2F_1\left(\frac{1}{3}, 1; \frac{11}{6}; -a^3\,\theta\right) \tag{A.52}$$

where ${}_2F_1$ is the Gauss hypergeometric function. To evaluate the correction to $\delta_0(a)$ coming from the terms proportional to c_a one sets

$$\delta(a) \propto a \cdot {}_2F_1\left(\frac{1}{3}, 1; \frac{11}{6}; -a^3\,\theta\right)\,[\,1 + c_a\,\mathcal{F}(a)\,]\;, \tag{A.53}$$

then inserts the resulting expression in Eq. (A.51), and finally expands the resulting expression to lowest order in c_a to find the correction $\mathcal{F}(a)$. The resulting differential equation can then be solved for $\mathcal{F}(a)$, giving the density contrast $\delta(a)$ as a function of the two parameters (γ_ν and Ω or $\theta \equiv (1-\Omega)/\Omega$) appearing in Eq. (A.51). In the following we will focus on the specific choice $\nu = 1/3$ obtained from the lattice theory of gravity [52, 46, 47], which leads to the $G(a)$ exponent $\gamma_\nu = \frac{3}{2\,\nu} = 9/2$. It is customary at this point to define the growth index $f(a) \equiv \frac{\partial \ln \delta(a)}{\partial \ln a}$ and the related growth index parameter γ via $\gamma \equiv \frac{\ln f}{\ln \Omega}\big|_{a=a_0}$. Then the solution to Eq. (A.51) gives an explicit expression for the growth index γ parameter, as a function of the matter fraction Ω.

Based on observational constraints, one is mostly interested in the case $\Omega \approx 0.25$, therefore in the following we will limit our discussion to this choice only. In the absence of a running G ($G \to G_0$, thus $c_a = 0$) one has $f(a_0) = 0.4625$ and $\gamma = 0.5562$ for $\Omega = 0.25$ [126]. On the other hand when the running of G is taken into account one finds from the solution to Eq. (A.51) for the growth index parameter γ at $\Omega = 0.25$

$$\gamma = 0.5562 - 0.0142\; c_a + \mathcal{O}(c_a^2)\,. \tag{A.54}$$

In the end it would seem therefore that at least in the Newtonian treatment the correction comes out rather small. Note that both the Newtonian and the relativistic treatment, described in the main body, give a negative sign for the correction arising from the running of G.

To estimate quantitatively the actual size of the correction in Eq. (A.54) one needs an estimate for the coefficient $c_0 \approx 33.3$ in Eq. (3.26), as obtained from the lattice gravity calculations of invariant correlation functions at fixed geodesic distance [128]. In addition one uses the fact that $c_a \approx c_t \approx 2.7\, c_0$ (see the previous discussion related to Eq. (A.48)). From this one would then get the estimate $\gamma = 0.5562 - 1.28$ on the largest scales, which looks like a significant $\mathcal{O}(1)$ correction to γ.

B Trace Box in the Comoving Gauge

In this section we will give a short sample calculation of the effects of the covariant d'Alembertian operator $\Box \equiv g^{\mu\nu}\, \nabla_\mu \nabla_\nu$ acting on a coordinate scalar, such as the trace of the energy momentum tensor. The calculation presented below will show that the result is unchanged when the stress contribution s is included in the metric for the comoving gauge. Specifically here we will be interested in the correction of order h_{ij} that arises when the operator in Eq. (3.147)acts on the scalar $T_\lambda{}^\lambda = -\bar{\rho}$. Thus, for example, it will give the correction $\mathcal{O}(h, s)$ to $\delta\rho_{vac}$, namely the second term in the expression

$$\delta\rho_{vac}(t) = \frac{\delta G(\Box^{(0)})}{G_0}\, \delta\rho(t) + \frac{\delta G(\Box)(h, s)}{G_0}\, \bar{\rho}(t)\,, \tag{B.55}$$

with the first term being simply given in the FLRW background by $\delta G(t)/G_0 \cdot \delta\rho(t)$. Here the $\mathcal{O}(h, s)$ correction is given explicitly by the expression

$$\frac{\delta G(\Box)(h, s)}{G_0}\, \bar{\rho} \;=\; -\frac{1}{2\,\nu}\, \frac{c_0}{\xi^{1/\nu}}\, \frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h, s) \cdot \left(\frac{1}{\Box^{(0)}}\right)^{1/2\nu} \cdot \bar{\rho}\,. \tag{B.56}$$

Now the covariant d'Alembertian $\Box$ acting on general scalar functions $S(x)$ simplifies to

$$\Box\, S(x) \equiv \frac{1}{\sqrt{g}}\, \partial_\mu\, g^{\mu\nu} \sqrt{g}\, \partial_\nu\, S(x)\,. \tag{B.57}$$

In the absence of h_{ij} fluctuations this gives for the metric in the comoving gauge

$$\Box^{(0)} S(x) = \frac{1}{a^2} \nabla^2 S - 3\, \frac{\dot{a}}{a}\, \dot{S} - \ddot{S}\,. \tag{B.58}$$

To first order in the field fluctuation h_{ij} of the comoving gauge one computes

$$\Box^{(1)}(h,s)\, S(x) \;=\; \dot{S}\,\left[-\frac{1}{2}\dot{h}\right] + \partial_x S\,\left[\frac{1}{6a^2}\, i\, k_x\;(h+4\,s)\right] + \partial_x^2 S\,\left[-\frac{1}{3a^2}\;(h+s) + \frac{1}{a^2}\,\frac{k_x^2}{k^2}\,s\right] + \partial_x\partial_y S\,\left[\frac{2}{a^2}\,\frac{k_x\,k_y}{k^2}\,s\right] \tag{B.59}$$

where we have set as usual $h(x) = h(t)\, e^{i\,\mathbf{k}\cdot\mathbf{x}}$. But, for a function of time only, one obtains

$$\Box^{(1)}(h)\,\rho(t) = -\frac{1}{2}\,\dot{h}(t)\,\dot{S}(t)\,. \tag{B.60}$$

Thus to first order in the fluctuations one has

$$\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h)\cdot(\,\delta G\,\bar{\rho}\,) = \frac{1}{-\partial_t^2 - 3\,\frac{\dot{a}}{a}\,\partial_t}\cdot\frac{1}{2}\,\dot{h}\left(3\,\frac{\dot{a}}{a}\,\delta G - \delta\dot{G}\right)\bar{\rho} \tag{B.61}$$

and there is no change from the result quoted in [106]. There we set $s = 0$, since we were only interested in cosmological density perturbations δ, which couple only to the trace part of the gravitational field fluctuations h_{ij}.

C Effective Action with $G(\Box)$

Chapter 3 discussed how the renormalization group running of $G(\Box)$ can be incorporated in a set of manifestly covariant effective field equations. In addition, it was discussed that a running of the cosmological constant in the same equations is essentially ruled out by the requirement of general covariance. One main advantage of the modified Einstein field equations due to the running $G(\Box)$ in the form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = 8\pi G(\Box)T_{\mu\nu} \tag{C.62}$$

is that it is actually tractable, and leads to a number of reasonably unambiguous predictions for homogenous isotropic and static isotropic background metrics as explored in Chapter 3.

In this Appendix, we will approach the same problem from a slightly different perspective, namely from the point of view of an effective gravitational action. In view of the discussion presented in Chapter 1, it should be clear that such an effective action will depend on the two renormalized, dimensionful parameters G and ξ. Note that we will focus on the case of pure gravity, as the addition of matter will leave most of the main conclusions unchanged. Within the framework of an effective action approach, the running of the coupling constants can be implemented by the use of a manifestly covariant effective gravitational action [122, 152, 123, 153, 124]. First consider the cosmological term, for which we write

$$\lambda_0 \to \lambda_0(k^2) \to \lambda_0(\Box)\,. \tag{C.63}$$

It is then easy to see that

$$\lambda_0 \int d^4x \sqrt{g} \to \int d^4x \sqrt{g} \lambda_0(\Box) \cdot 1 \tag{C.64}$$

is meaningless, as $\lambda_0(\Box)$ has nothing to act on [154]. Therefore the λ_0 term in the gravitational action cannot be made to run effectively. The implication here also in this effective form is that if λ_0 is somehow made to run, this can only be achieved by an explicit breaking of general covariance.

One further notices that this is clearly not the case for the rest of the gravitational action, and particularly for the running of G as in given in Eq. (3.1). Indeed, consider here for concreteness the following nonlocal effective gravitational action which is symmetrized (which is a standard procedure: see for example [155])

$$S = -\frac{1}{16\pi G} \int d^4x \sqrt{g} \sqrt{R} (1 - A(\Box)) \sqrt{R}\,, \tag{C.65}$$

with

$$A(\Box) \equiv c_0 \left(-\xi^2 \Box\right)^n \tag{C.66}$$

and G a true constant. In the last expression, n is taken to be an integer with $n \to -\frac{1}{2\nu}$ at the end of the calculation. The next step is to calculate the variation of the above effective action. Note that another possibility is to take the $G(\Box)$ act on the matter term, $\frac{1}{2}\sqrt{g} G(\Box) g^{\mu\nu} T_{\mu\nu}$, but we will not pursue this possibility here. The expression inside the integral requires the evaluation of the four separate terms,

$$-\frac{1}{2}\sqrt{g}(\delta g^{\mu\nu})g_{\mu\nu}\sqrt{R}(1-A(\Box))\sqrt{R}+\sqrt{g}(\delta\sqrt{R})(1-A(\Box))\sqrt{R}$$
$$-n\sqrt{g}\sqrt{R}\frac{A(\Box)}{\Box}(\delta\Box)\sqrt{R}+\sqrt{g}\sqrt{R}(1-A(\Box))(\delta\sqrt{R})\,. \tag{C.67}$$

Further evaluation of these terms require the following elementary variations

$$\delta\sqrt{g}=-\frac{1}{2}\sqrt{g}g_{\mu\nu}(\delta g^{\mu\nu}) \tag{C.68}$$

and

$$\delta R=g^{\mu\nu}(\delta R_{\mu\nu})+R_{\mu\nu}(\delta g^{\mu\nu}) \tag{C.69}$$

with

$$\delta R_{\mu\nu}=\nabla_\alpha(\delta\Gamma^\alpha_{\mu\nu})-\nabla_\mu(\delta\Gamma^\alpha_{\alpha\nu})\,, \tag{C.70}$$

for which one needs

$$\delta\Gamma^\alpha_{\mu\nu}=\frac{1}{2}g^{\alpha\beta}\left(\nabla_\mu(\delta g_{\beta\nu})+\nabla_\nu(\delta g_{\beta\mu})-\nabla_\beta(\delta g_{\mu\nu})\right)\,. \tag{C.71}$$

It then follows that in Eq. (C.69),

$$g^{\mu\nu}(\delta R_{\mu\nu})=\nabla_\mu\nabla_\nu\left(-\delta g^{\mu\nu}+g^{\mu\nu}g_{\alpha\beta}(\delta g^{\alpha\beta})\right)=g_{\alpha\beta}\Box(\delta g^{\alpha\beta})-\nabla_{(\mu}\nabla_{\nu)}(\delta g^{\mu\nu})\,, \tag{C.72}$$

which gives the second and the forth terms in Eq. (C.67). Use has been made of $\delta g_{\mu\nu}=-g_{\mu\alpha}g_{\nu\beta}(\delta g^{\alpha\beta})$. Note that in general $\Box\nabla_\mu\neq\nabla_\mu\Box$, and that $\Box g_{\mu\nu}=0$, but $\Box(\delta g_{\mu\nu})\neq 0$. For the variation of the covariant d'Alembertian

$$\delta\Box=(\delta g^{\mu\nu})\nabla_\mu\nabla_\nu-g^{\mu\nu}(\delta\Gamma^\sigma_{\mu\nu})\nabla_\sigma\,, \tag{C.73}$$

which together with the variation of $\Gamma^\sigma_{\mu\nu}$ in Eq. (C.71) leads to

$$\delta\Box=(\delta g^{\mu\nu})\nabla_\mu\nabla_\nu+\left(\nabla_\mu(\delta g^{\mu\nu})\right)\nabla_\nu-\frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\left(\nabla_\mu(\delta g^{\alpha\beta})\right)\nabla_\nu\,. \tag{C.74}$$

Generally, one encounters an expression which desires to be properly symmetrized, as in the case of

$$\delta(\Box^n)\rightarrow\sum_{k=1}^{n}\Box^{k-1}(\delta\Box)\Box^{n-k}\,. \tag{C.75}$$

Eventually this leads to rather lengthy and complicated expressions: they can be worked out in detail, but we will consider, for illustration, only one such ordering, $\delta(\Box^n) \to n\Box^{n-1}(\delta\Box)$, which gives simply $\delta\left(A(\Box)\right) = n\frac{A(\Box)}{\Box}(\delta\Box)$. Several integrations by parts have been performed, involving both $\Box^n$ with integer n and $g_{\mu\nu}\Box - \nabla_{(\mu}\nabla_{\nu)}$, to isolate the $\delta g^{\mu\nu}$ term. In general, one needs to be careful about the ordering of covariant derivatives, whose commutator is nonvanishing,

$$[\nabla_\mu, \nabla_\nu] T^{\alpha_1\alpha_2\cdots}{}_{\beta_1\beta_2\cdots} = -\sum_i R^{\alpha_i}_{\mu\nu\sigma} T^{\alpha_1\cdots\sigma\cdots}{}_{\beta_1\cdots} - \sum_j R^{\sigma}_{\mu\nu\beta_j} T^{\alpha_1\cdots}{}_{\beta_1\cdots\sigma\cdots} \tag{C.76}$$

with the σ index in $T^{\cdots}{}_{\cdots}$ in the i^{th} position in the first term, and in the j^{th} position in the second term. As a consequence, the $\mathcal{O}(R)$ commutator terms generally give rise to higher derivative terms in the effective field equations, due to the fact that the zeroth order terms in the action are already $\mathcal{O}(R)$. After the manipulations described above, the effective field equations for zero cosmological constant take the form

$$\begin{aligned}
& R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R \\
& - \frac{1}{2}\, R_{\mu\nu} \left(\frac{1}{\sqrt{R}}\, A(\Box) \sqrt{R} + \sqrt{R}\, A(\Box) \frac{1}{\sqrt{R}} \right) \\
& + \frac{1}{2}\, g_{\mu\nu} \, \left(\sqrt{R}\, A(\Box) \sqrt{R} \right) \\
& + \frac{1}{2}\, \left(g_{\mu\nu}\Box - \nabla_{(\mu}\nabla_{\nu)} \right) \left(\frac{1}{\sqrt{R}}\, A(\Box) \sqrt{R} + \sqrt{R}\, A(\Box) \frac{1}{\sqrt{R}} \right) \\
& + n \, \left(\nabla_\mu \sqrt{R} \right) \, \left(\nabla_\nu \frac{A(\Box)}{\Box} \sqrt{R} \right) \\
& - \frac{1}{2}\, n\, g_{\mu\nu} \, \left(\nabla_\alpha \sqrt{R} \right) \, g^{\alpha\beta} \left(\nabla_\beta \frac{A(\Box)}{\Box} \sqrt{R} \right) \\
& - \frac{1}{2}\, n\, g_{\mu\nu}\, \sqrt{R}\, A(\Box) \sqrt{R} \\
& = 8\pi G T_{\mu\nu}\,.
\end{aligned} \tag{C.77}$$

Note that the above effective field equations are not completely symmetric in $\mu \leftrightarrow \nu$ due to our specfic choice of operator ordering. Note also that taking the covariant divergence of the *left hand side* is expected to give zero, which is required for consistency of the field equations (for some terms it is clear that they give zero by inspection). The above effective field equations are still rather complicated. Note that generally any terms of $\mathcal{O}(R^2)$ can safely be dropped, if one is interested in the long distance, small curvature limit.

One more possibility is to generalize the effective action in Eq. (C.65) to the form

$$S = -\frac{1}{16\pi G}\left(\frac{1}{2}\int d^4x\sqrt{g}R^{1-\alpha}(1-A(\Box))R^{\alpha} + \frac{1}{2}\int d^4x\sqrt{g}R^{\alpha}(1-A(\Box))R^{1-\alpha}\right)\,, \quad \text{(C.78)}$$

which now depends on a parameter α taking values between zero and one; the previous case then corresponds to the symmetric choice $\alpha = \frac{1}{2}$. Following the same procedure which yields Eq. (C.77), we obtain for the field equations with zero cosmological constant the following expression

$$\begin{aligned}
& R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{1}{2}g_{\mu\nu}R^{1-\alpha}A\left(\Box\right)R^{\alpha} \\
& - R_{\mu\nu}\left[\left(1-\alpha\right)R^{-\alpha}A(\Box)R^{\alpha} + \alpha R^{\alpha-1}A(\Box)R^{1-\alpha}\right] \\
& + \left(g_{\mu\nu}\Box - \nabla_{(\mu}\nabla_{\nu)}\right)\left[\left(1-\alpha\right)R^{-\alpha}A(\Box)R^{\alpha} + \alpha R^{\alpha-1}A(\Box)R^{1-\alpha}\right] \\
& + \frac{1}{2}n\left(\nabla_{\mu}R^{1-\alpha}\right)\left(\nabla_{\nu}\frac{A(\Box)}{\Box}R^{\alpha}\right) + \frac{1}{2}n\left(\nabla_{\nu}R^{\alpha}\right)\left(\nabla_{\mu}\frac{A(\Box)}{\Box}R^{1-\alpha}\right) \\
& - \frac{1}{4}ng_{\mu\nu}\left(\nabla_{\rho}R^{1-\alpha}\right)g^{\rho\sigma}\left(\nabla_{\sigma}\frac{A(\Box)}{\Box}R^{\alpha}\right) - \frac{1}{4}ng_{\mu\nu}\left(\nabla_{\sigma}R^{\alpha}\right)g^{\rho\sigma}\left(\nabla_{\rho}\frac{A(\Box)}{\Box}R^{1-\alpha}\right) \\
& - \frac{1}{4}ng_{\mu\nu}R^{1-\alpha}A(\Box)R^{\alpha} - \frac{1}{4}ng_{\mu\nu}R^{\alpha}A(\Box)R^{1-\alpha} \\
& = 8\pi G T_{\mu\nu}\,, \qquad \text{(C.79)}
\end{aligned}$$

which incidentally shows that the choice of either $\alpha = 1$ or $\alpha = 0$ is problematic. Then one final technical question remains, namely what is the relationship between the above effective ield equations Eq. (C.77) and (C.79) and the clearly more economical field equations given earlier in Chapter 3, *i.e.,* Eq. (C.62) (with zero cosmological constant to compare with Eqs. (C.77) and (C.79)). Obviously, the equations obtained here significantly from a variational principle appear more complicated. They contain a number of nontrivial terms, some of which are reminiscent of the $1+A(\Box)$ term in Eq. (C.62), and others with a different structure (such as the $g_{\mu\nu} - \nabla_{(\mu}\nabla_{\nu)}$ term). Note that many of the new additional curvature terms which appear on the *left hand side* of the effective field equations in Eqs. (C.77) and

(C.79) can be moved, equivalently, to the *right hand side.* To do so, one makes the use of the fact that to zeroth order in the quantum correction proportional to $A(\Box)$

$$R = 4\lambda - 8\pi G T_\sigma^\sigma \ , \tag{C.80}$$

which then allows a distinction of the source term in pure matter ($T_{\mu\nu}$) and vacuum polarization ($T_{\mu\nu}^{\text{vac}}$) contributions as was done in Chapter 3. Generally, some of the issues that come up in comparing effective field equations reflect an ambiguity of where and at what stage, the replacement $G \to G(\Box)$ is performed. However, we expect that when asking the right physical questions, the answers would largely be unambiguous. It is of course possible that given a specific metric, the two sets of effective field equations will ultimately give similar results, but in general this remains a largely open question. One possibility is that both sets of field equations describe the same running of the gravitational coupling, up to curvature squared (higher derivative) terms, which then become irrelevant at very large distances. In any case, the main purpose of our exercise in this Appendix was to show that in either case (via Eq. (C.62) or Eqs. (C.77) and (C.79)) the running of $G(\Box)$ clearly leads to nonvanishing nontrivial effects.