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Scale-dependent Newton's constant G in the conformal Newtonian gauge

Herbert W. Hamber[1,*] and Reiko Toriumi[2,†]

[1]*Institut des Hautes Etudes Scientifiques, 35, route de Chartres, 91440 Bures-sur-Yvette, France*
[2]*Department of Physics and Astronomy, University of California, Irvine, California 92697-4575, USA*
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In classical gravity, deviations from the predictions of the Einstein theory are often discussed within the framework of the conformal Newtonian gauge, where scalar perturbations are described by two potentials ϕ and ψ. In this paper we use the above gauge to explore possible cosmological consequences of a running Newton's constant $G(\Box)$, as suggested by the nontrivial ultraviolet fixed point scenario arising from the quantum field-theoretic treatment of Einstein gravity with a cosmological constant term. Here we focus on the effects of a scale-dependent coupling on the so-called gravitational slip functions $\eta = \psi/\phi - 1$, whose classical general relativity value is zero. Starting from a set of manifestly covariant but nonlocal effective field equations derived earlier, we compute the leading corrections in the potentials ϕ and ψ for a nonrelativistic, pressureless fluid. After providing an estimate for the quantity η, we then focus on a comparison with results obtained in a previous paper on matter density perturbations in the synchronous gauge, which gave an estimate for the growth index parameter γ, also in the presence of a running G. Our results indicate that, in the present framework and for a given $G(\Box)$, the corrections tend to be significantly larger in magnitude for the perturbation growth exponents than for the conformal Newtonian gauge slip function.

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I. INTRODUCTION

Recent years have seen the development of a fascinating variety of alternative theories of gravity, in addition to the more traditional alternate frameworks, which used to include just Brans-Dicke, higher derivative, effective quantum gravity, and supergravity theories. Some of the new additions to the by now rather long list include dilaton gravity, $f(R)$ gravity, torsion gravity, loop quantum gravity, holographic modified gravity, and a few others, just to cite here a few representative examples. All of these theories eventually predict some level of deviation from classical gravity, at short- or long-distance scales, which is often parametrized either by a suitable set of post-Newtonian parameters, or more recently, by the introduction of a gravitational slip function [1,2].

In this paper, we will focus on the analysis of departures from general relativity (GR) in the gravitational slip function, obtained in the framework of the conformal Newtonian gauge, and within the rather narrow context of the nontrivial ultraviolet fixed point scenario for Einstein gravity with a cosmological term. Thus, instead of looking at deviations from GR at very short distances, due to new interactions such as the ones suggested by string theories [3], we will be considering here infrared effects, which could manifest themselves at very large distances.

The specific nature of the scenario we will be investigating here is motivated by the field-theoretic treatment of models for quantum gravity, based on the (minimal) Einstein action with a bare cosmological term. The theory's long-distance scaling properties used as the basis for the present work follow from the existence of a nontrivial ultraviolet fixed point of the renormalization group in Newton's constant G. The latter is inaccessible by direct perturbation theory in four dimensions, and can be shown to radically alter the short- and long-distance behavior of the theory when compared to more naive, perturbative expectations. The renormalization-group origin of such fixed points was first discussed in detail by Wilson some time ago for scalar and self-coupled fermion theories [4]. The general field-theoretic methods were later extended and applied to gravity, where they are now referred to as the nontrivial UV fixed point, or asymptotic safety, scenario [5]. It is fair to say that so far this is the only field-theoretic approach known to work consistently in other not perturbatively renormalizable theories, such as the nonlinear sigma model above two dimensions [6]. While perhaps still a bit mundane in the context of gravity, such nontrivial fixed points are well studied and well understood in statistical field theory, where they generally describe phase transitions between ordered and disordered ground states, or between weakly coupled and condensed states.

The paper is organized as follows. First (Sec. II) we recall the effective covariant field equations describing the running of G, and describe briefly the nature of various objects and parameters entering the quantum nonlocal corrections; a more complete description of the basic setup can be found in our previous papers on the subject, and will not be repeated here. We then discuss the zeroth order (in the metric fluctuations) field equations and

*HHamber@uci.edu
†RToriumi@uci.edu

energy-momentum conservation equations for the standard homogeneous isotropic metric, with a running $G(\Box)$. Later (Sec. III) we extend the formalism to deal with small metric and matter perturbations, and list the relevant field and energy conservation equations to first order in the perturbations in the comoving gauge. These above results are then (Sec. IV) reexpressed in two other choices of gauge, the synchronous and the conformal Newtonian gauge. The latter choice of gauge allows us to extract an expression for the gravitational slip function η due to $G(\Box)$ (Sec. V). This quantity is then evaluated within the context of a ΛCDM model, for redshifts corresponding to the present era ($z = 0$). The resulting correction is then compared to current astrophysical observations, as well as to our previous results (and observations) regarding the corrections due to $G(\Box)$ to the matter density perturbation growth exponents. The conclusions provide an interpretation of the theoretical results, and their associated uncertainties, in view of present and future high precision determination of the gravitational slip function and growth exponents.

II. RUNNING NEWTON'S CONSTANT $G(\Box)$

As mentioned in the introduction, it is not the purpose of this paper to provide a satisfactory description, or motivation, for the running of G that arises in the quantum-field-theoretic treatment of Einstein's gravity with a cosmological term. Here we only provide a brief summary, and only the most relevant formulas will be given for later reference; a more complete set of references can be found, for example, in [7].

The running of Newton's constant G has been computed both on the lattice in four dimensions [8,9], and in the continuum within the framework of the background field expansion applied to $d = 2 + \epsilon$ spacetime dimensions [5,10], and later also using truncation methods applied in $d = 4$ [11]. In either case one obtains a momentum-dependent $G(k^2)$, which eventually needs to be reexpressed in a suitable coordinate-independent way, so that it can be consistently applied to more general problems, involving arbitrary background geometries. The first step in analyzing the consequences of a running of G is therefore to rewrite the expression for $G(k^2)$ in a coordinate-independent way, either by the use of a nonlocal Vilkovisky-type effective gravity action [12,13] or by the use of a set of consistent effective field equations. In going from momentum to position space one employs $k^2 \to -\Box$, which then gives for the quantum-mechanical running of the gravitational coupling the replacement $G \to G(\Box)$. Then the running of G is given in the vicinity of the UV fixed point by

$$G(\Box) = G_0\left[1 + c_0\left(\frac{1}{\xi^2 \Box}\right)^{1/2\nu} + \ldots\right], \tag{2.1}$$

where $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the covariant d'Alembertian, and the dots represent higher order terms in an expansion in $1/(\xi^2\Box)$. Current evidence from Euclidean lattice quantum gravity points toward $c_0 > 0$ (implying infrared growth) and $\nu \simeq \frac{1}{3}$ [9]. Within the quantum-field-theoretic renormalization-group treatment, the quantity ξ arises as an integration constant of the Callan-Symanzik renormalization-group equations.

One issue of great relevance to the physical interpretation of the results is therefore a correct identification of the renormalization-group invariant scale ξ. A number of arguments, mostly based on nonperturbative lattice results and scaling considerations involving the gravitational Wilson loop and its relevance for large scale observable curvature [14], can be given in support of the suggestion that the dynamically generated infrared cutoff scale ξ (analogous to the $\Lambda_{\overline{MS}}$ of QCD) can be quite large in the case of gravity (for a recent review, see Ref. [7]). These arguments would then suggest that the new scale ξ is naturally expected to be related to the large scale average curvature, and thus could be of cosmological relevance,

$$\lambda \simeq \frac{3}{\xi^2}. \tag{2.2}$$

These considerations then lead to a more concrete quantitative estimate for the scale in the running $G(\Box)$ of Eq. (2.1), namely, $\xi \sim 1/\sqrt{\lambda/3} \sim 1.51 \times 10^{28}$ cm. Moreover, from these types of arguments one would also infer that the constant G_0 in Eq. (2.1) can, to a very close approximation, be identified with the laboratory value of Newton's constant, $\sqrt{G_0} \sim 1.6 \times 10^{-33}$ cm. The running of G envisioned above would then remain in agreement with laboratory and solar system precision tests of general relativity.

The appearance of the d'Alembertian $\Box$ in the running of G naturally leads to both a nonlocal effective gravitational action and a corresponding set of nonlocal modified field equations. In the simplest scenario, instead of the ordinary Einstein field equations with constant G,

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.3}$$

one is now led to consider the modified effective field equations

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = 8\pi G(\Box) T_{\mu\nu} \tag{2.4}$$

with the nonlocal term due to the $G(\Box)$. By being manifestly covariant, these equations still satisfy some of the basic requirements for a set of consistent field equations incorporating the running of G. Not unexpectedly though, the new nonlocal equations are much harder to solve than the original classical field equations for constant G.

The effective nonlocal field equations of Eq. (2.4) can be recast in a form very similar to the classical field equations, but with a new source term $\tilde{T}_{\mu\nu} = [G(\Box)/G_0]T_{\mu\nu}$ defined as the effective, or gravitationally dressed, energy-momentum tensor [15,16]. Ultimately the consistency of the effective field equations demands that it be exactly conserved, in consideration of the contracted Bianchi identity satisfied by the Ricci tensor. In this picture, therefore, the running of G can be viewed as contributing to a sort of vacuum fluid, introduced in order to account for the new gravitational quantum vacuum-polarization contribution.

Due the appearance of a negative fractional exponent in Eq. (2.1), the covariant operator appearing in the expression for $G(\Box)$ has to be suitably defined by analytic continuation. This can be done, for example, by computing $\Box^n$ for positive integer n, and then analytically continuing to $n \to -1/2\nu$ [15]. Equivalently, $G(\Box)$ can be defined via a regulated parametric integral representation [17], such as

$$\left(\frac{1}{-\Box(g) + \mu^2}\right)^{1/2\nu} = \frac{1}{\Gamma(\frac{1}{2\nu})} \int_0^\infty d\alpha\, \alpha^{1/2\nu - 1} e^{-\alpha(-\Box(g) + \mu^2)}, \tag{2.5}$$

where $\mu \to 0$ is a suitable infrared regulator. As far as the calculations in this paper are concerned, it will not be necessary to commit oneself to an unduly specific form for the running of $G(\Box)$. Thus, for example, although the lattice gravity results only allow for a nondegenerate phase for the case $c_0 > 0$, it will nevertheless be possible later to have either sign for the correction in Eq. (2.1). We note here that a running cosmological constant $\lambda(k) \to \lambda(\Box)$ causes a number of mathematical inconsistencies [15,18] within the manifestly covariant framework, described here by the effective field equations of Eq. (2.4). Indeed if one assumes for the running part of $\lambda(\Box) \sim (\xi^2 \Box)^{-\sigma}$, then the infrared regulated expression in Eq. (2.5) gives no running of λ, after using the identity $\nabla_\lambda g_{\mu\nu} = 0$.[1] This last conclusion is in general agreement with the field-theoretic results of the nontrivial renormalization-group fixed point scenario [7], thereby providing perhaps an independent consistency check. Note that this rather general argument also applies to possible additional contributions from nonzero vacuum expectation values of matter fields, such as the Higgs. As a result, in the present quantum-field-theoretic motivated framework λ is assumed not to run.

A. Zeroth order effective field equations with $G(\Box)$

A scale-dependent Newton's constant is expected to lead to small modifications of the standard cosmological solutions to the Einstein field equations. Here we will summarize what modifications are expected from the effective field equations on the basis of $G(\Box)$. The starting point is the quantum effective nonlocal field equations of Eq. (2.4), with $G(\Box)$ defined in Eq. (2.1). In the Friedmann-Lemaître-Robertson-Walker (FLRW) framework these are applied to the standard homogeneous isotropic metric

$$d\tau^2 = dt^2 - a^2(t)\left\{\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right\}$$
$$k = 0, \pm 1. \tag{2.6}$$

In the following, we will only consider the case $k = 0$ (spatially flat universe). It should be noted that there are in fact *two* related quantum contributions to the effective covariant field equations. The first one arises because of the presence of a nonvanishing cosmological constant $\lambda \simeq 3/\xi^2$, caused by the nonperturbative quantum vacuum condensate $<R> \neq 0$ [14]. As in the case of the standard FLRW cosmology, this is expected to be the dominant contribution at large times t, and gives an exponential (for $\lambda > 0$) or cyclic (for $\lambda < 0$) expansion of the scale factor. The second contribution arises because of the explicit running of $G(\Box)$ in the effective field equations.

The next step therefore is a systematic examination of the nature of the solutions to the full effective field equations, with $G(\Box)$ involving the relevant covariant d'Alembertian operator

$$\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu \tag{2.7}$$

acting on second rank tensors as in the case of $T_{\mu\nu}$. To start the process, we will assume that $T_{\mu\nu}$ is described by the perfect fluid form,

$$T_{\mu\nu} = [p(t) + \rho(t)] u_\mu u_\nu + g_{\mu\nu} p(t), \tag{2.8}$$

for which one needs to compute the action of $\Box^n$ on $T_{\mu\nu}$, and then analytically continue the answer to negative fractional values of $n = -1/2\nu$. The results of [15–18] then show that a nonvanishing pressure contribution is generated in the effective field equations, even if one initially assumes a pressureless fluid, $p(t) = 0$. After a somewhat lengthy derivation one obtains for a universe filled with nonrelativistic matter ($p = 0$) the following set of effective Friedmann equations,

$$\frac{k}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G(t)}{3}\rho(t) + \frac{\lambda}{3}$$
$$= \frac{8\pi G_0}{3}[1 + c_t (t/\xi)^{1/\nu} + \ldots]\rho(t) + \frac{\lambda}{3} \tag{2.9}$$

for the tt field equation, and

[1] To be a bit specific, consider the case of a scale-dependent $\lambda(k)$, which we will write here as $\lambda = \lambda_0 + \delta\lambda(k)$. Let us also assume, for concreteness, that $\delta\lambda(k) \sim c_1 (k^2)^{-\sigma}$, where c_1 and σ are some constants, and then make the transition to coordinate space by using $k^2 \to -\Box$. Thus $\delta\lambda(\Box) \sim (-\Box + \mu^2)^{-\sigma}$, where one should be careful and use the infrared regulated expression in Eq. (2.5). The effective field equations will then contain a term $\frac{1}{2}\delta\lambda(\Box) \cdot g_{\mu\nu} = \frac{1}{2} c_1 \frac{1}{\Gamma(\sigma)} \int_0^\infty d\alpha\, \alpha^{\sigma-1} e^{-\alpha(-\Box(g) + \mu^2)} \cdot g_{\mu\nu} = \frac{1}{2} c_1 (\mu^2)^{-\sigma} \cdot g_{\mu\nu}$, which still gives just a constant multiplying the metric $g_{\mu\nu}$.

$$\frac{k}{a^2(t)}+\frac{\dot{a}^2(t)}{a^2(t)}+\frac{2\ddot{a}(t)}{a(t)}=-\frac{8\pi G_0}{3}[c_t(t/\xi)^{1/\nu}+\ldots]\rho(t)+\lambda \tag{2.10}$$

for the rr field equation. In the above expressions, the running of G appropriate for the Robertson-Walker metric is

$$G(t)\equiv G_0\left(1+\frac{\delta G(t)}{G_0}\right)=G_0\left[1+c_t\left(\frac{t}{\xi}\right)^{1/\nu}+\ldots\right] \tag{2.11}$$

with c_t of the same order as c_0 in Eq. (5.48) [15] (in the quoted reference the estimate $c_t\simeq 0.450\,c_0$ was given for the tensor box operator). From the above form of $\delta G(t)$ one sees that the amplitude of the quantum correction is actually proportional to the combination c_0/ξ^3 for $\nu=1/3$. Note also that the running of G induces an effective pressure term in the second (rr) equation, due to the presence of a relativistic fluid whose origin is in the vacuum-polarization contribution. Another noteworthy general feature of the new field equations is the additional power-law acceleration contribution, on top of the standard one due to the λ term.

B. Introduction of the w_{vac} parameter

It was noted in [15,18] that the field equations with a running G, Eqs. (2.9) and (2.10), can be recast in an equivalent, but slightly more appealing, form by defining a vacuum-polarization pressure p_{vac} and density ρ_{vac}, such that for the FLRW background one has

$$\rho_{\mathrm{vac}}(t)=\frac{\delta G(t)}{G_0}\rho(t),\qquad p_{\mathrm{vac}}(t)=\frac{1}{3}\frac{\delta G(t)}{G_0}\rho(t). \tag{2.12}$$

From this viewpoint, the inclusion of a vacuum-polarization contribution in the FLRW framework seems to amount to a replacement

$$\rho(t)\rightarrow\rho(t)+\rho_{\mathrm{vac}}(t),\qquad p(t)\rightarrow p(t)+p_{\mathrm{vac}}(t) \tag{2.13}$$

in the original field equations. Just as one introduces the parameter w, describing the matter equation of state,

$$p(t)=w\rho(t) \tag{2.14}$$

with $w=0$ for nonrelativistic matter, one can do the same for the remaining contribution by setting

$$p_{\mathrm{vac}}(t)=w_{\mathrm{vac}}\rho_{\mathrm{vac}}(t). \tag{2.15}$$

We should remark here that the original calculations [15], and more recently [18] which included metric perturbations, also indicate that

$$w_{\mathrm{vac}}=\tfrac{1}{3} \tag{2.16}$$

is obtained *generally* for the given class of $G(\Box)$ considered, and is not tied therefore to a specific choice of ν, such as $\nu=\frac{1}{3}$.

The previous, slightly more compact, notation allows one to rewrite the field equations for the FLRW background in an equivalent form, which we will describe next. We note here that, when dealing with density perturbations, we will have to distinguish the background, which will involve a background pressure ($\bar{p}$) and background density ($\bar{\rho}$), from the corresponding perturbations which will be denoted here by δp and $\delta\rho$. With this notation and for constant G_0, the FLRW field equations for the background are written as

$$\begin{aligned} 3\frac{\dot{a}^2(t)}{a^2(t)}&=8\pi G_0\bar{\rho}(t)+\lambda,\\ \frac{\dot{a}^2(t)}{a^2(t)}+2\frac{\ddot{a}(t)}{a(t)}&=-8\pi G_0\bar{p}(t)+\lambda. \end{aligned} \tag{2.17}$$

Then in the presence of a running $G(\Box)$, and in accordance with the results of Eqs. (2.9) and (2.10), the modified FLRW equations for the background read

$$\begin{aligned} 3\frac{\dot{a}^2(t)}{a^2(t)}&=8\pi G_0\left(1+\frac{\delta G(t)}{G_0}\right)\bar{\rho}(t)+\lambda,\\ \frac{\dot{a}^2(t)}{a^2(t)}+2\frac{\ddot{a}(t)}{a(t)}&=-8\pi G_0\left(w+w_{\mathrm{vac}}\frac{\delta G(t)}{G_0}\right)\bar{\rho}(t)+\lambda, \end{aligned} \tag{2.18}$$

using the definitions in Eqs. (2.14) and (2.15), here with $\bar{p}_{\mathrm{vac}}(t)=w_{\mathrm{vac}}\bar{\rho}_{\mathrm{vac}}(t)$.

Of course the procedure of defining a ρ_{vac} and a p_{vac} contribution, arising from quantum vacuum-polarization effects, is not necessarily restricted to the FLRW background metric case. In general one can decompose the full source term in the effective nonlocal field equations of Eq. (2.4), making use of

$$G(\Box)=G_0\left(1+\frac{\delta G(\Box)}{G_0}\right)\quad\text{with}\quad\frac{\delta G(\Box)}{G_0}\equiv c_0\left(\frac{1}{\xi^2}\right)^{1/2\nu}, \tag{2.19}$$

as two contributions,

$$\frac{1}{G_0}G(\Box)T_{\mu\nu}=\left(1+\frac{\delta G(\Box)}{G_0}\right)T_{\mu\nu}=T_{\mu\nu}+T_{\mu\nu}^{\mathrm{vac}}. \tag{2.20}$$

The latter involves the nonlocal part

$$T_{\mu\nu}^{\mathrm{vac}}\equiv\frac{\delta G(\Box)}{G_0}T_{\mu\nu}. \tag{2.21}$$

Consistency of the full nonlocal field equations requires that the sum be conserved,

$$\nabla^\mu(T_{\mu\nu}+T_{\mu\nu}^{\mathrm{vac}})=0. \tag{2.22}$$

In general one cannot expect that the contribution $T_{\mu\nu}^{\mathrm{vac}}$ will always be expressible in the perfect fluid form of Eq. (2.8), even if the original $T_{\mu\nu}$ for matter (or radiation) has such a form. The former will in general contain, for example, nonvanishing shear stress contributions, even if they were originally absent in the matter part.

III. RELATIVISTIC TREATMENT OF MATTER DENSITY PERTURBATIONS

Besides the modified cosmic scale factor evolution just discussed, the running of $G(\Box)$, as given in Eq. (2.1), also affects the nature of matter density perturbations on large scales. In computing these effects, it is customary to introduce a perturbed metric of the form

$$d\tau^2 = dt^2 - a^2(\delta_{ij} + h_{ij})dx^i dx^j, \tag{3.1}$$

with $a(t)$ the unperturbed scale factor and $h_{ij}(\mathbf{x}, t)$ a small metric perturbation, and $h_{00} = h_{i0} = 0$ by choice of coordinates. After decomposing the matter fields into background and fluctuation contribution, $\rho = \bar{\rho} + \delta\rho$, $p = \bar{p} + \delta p$, and $\mathbf{v} = \bar{\mathbf{v}} + \delta\mathbf{v}$, it is customary in these treatments to expand the density, pressure, and metric perturbations in spatial Fourier modes,

$$\begin{aligned} \delta\rho(\mathbf{x}, t) &= \delta\rho_{\mathbf{q}}(t)e^{i\mathbf{q}\cdot\mathbf{x}}, \qquad \delta p(\mathbf{x}, t) = \delta p_{\mathbf{q}}(t)e^{i\mathbf{q}\cdot\mathbf{x}}, \\ \delta\mathbf{v}(\mathbf{x}, t) &= \delta\mathbf{v}_{\mathbf{q}}(t)e^{i\mathbf{q}\cdot\mathbf{x}}, \qquad h_{ij}(\mathbf{x}, t) = h_{\mathbf{q}ij}(t)e^{i\mathbf{q}\cdot\mathbf{x}}, \end{aligned} \tag{3.2}$$

with $\mathbf{q}$ the comoving wave number. Once the Fourier coefficients have been determined, the original perturbations can later be obtained from

$$\delta\rho(\mathbf{x}, t) = \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} e^{-i\mathbf{q}\cdot\mathbf{x}} \delta\rho_{\mathbf{q}}(t) \tag{3.3}$$

and similarly for the other fluctuation components. Then the field equations with a constant G_0 [Eq. (2.3)] are given to zeroth order in the perturbations by Eq. (2.17), which fixes the three background fields $a(t)$, $\bar{\rho}(t)$, and $\bar{p}(t)$. Note that in a comoving frame the four-velocity appearing in Eq. (2.8) has components $u^i = 1$, $u^0 = 0$. Without $G(\Box)$, to first order in the perturbations and in the limit $\mathbf{q} \to 0$ the field equations give

$$\begin{aligned} \frac{\dot{a}(t)}{a(t)}\dot{h}(t) &= 8\pi G_0 \bar{\rho}(t)\delta(t), \\ \ddot{h}(t) + 3\frac{\dot{a}(t)}{a(t)}\dot{h}(t) &= -24\pi G_0 w \bar{\rho}(t)\delta(t) \end{aligned} \tag{3.4}$$

with the matter density contrast defined as $\delta(t) \equiv \delta\rho(t)/\bar{\rho}(t)$, $h(t) \equiv h_{ii}(t)$ the trace part of h_{ij}, and $w = 0$ for nonrelativistic matter. When combined together, these last two equations then yield a single equation for the trace of the metric perturbation,

$$\ddot{h}(t) + 2\frac{\dot{a}(t)}{a(t)}\dot{h}(t) = -8\pi G_0(1 + 3w)\bar{\rho}(t)\delta(t). \tag{3.5}$$

From first order energy conservation, one has $-\frac{1}{2}(1 + w)h(t) = \delta(t)$, which then allows one to eliminate $h(t)$ in favor of $\delta(t)$, which then allows one to obtain a single second order equation for the density contrast $\delta(t)$.

In the case of a running $G(\Box)$, the above equations need to be rederived from the effective covariant field equations of Eq. (2.4), and lead to several additional terms not present at the classical level [18].

A. Zeroth order energy-momentum conservation

As a first step in computing the effects of density matter perturbations, one needs to examine the consequences of energy and momentum conservation, to zeroth and first order in the relevant perturbations. If one takes the covariant divergence of the field equations in Eq. (2.4), the left-hand side has to vanish identically because of the Bianchi identity. The right-hand side then gives $\nabla^\mu(T_{\mu\nu} + T^{\text{vac}}_{\mu\nu}) = 0$, where the fields in $T^{\text{vac}}_{\mu\nu}$ can be expressed, at least to lowest order, in terms of the p_{vac} and ρ_{vac} fields defined in Eqs. (2.12) and (2.15). The first equation one obtains is the zeroth (in the fluctuations) order energy conservation in the presence of $G(\Box)$, which reads

$$\begin{aligned} &3\frac{\dot{a}(t)}{a(t)}\left[(1 + w) + (1 + w_{\text{vac}})\frac{\delta G(t)}{G_0}\right]\bar{\rho}(t) + \frac{\dot{\delta G}(t)}{G_0}\bar{\rho}(t) \\ &\quad + \left(1 + \frac{\delta G(t)}{G_0}\right)\dot{\bar{\rho}}(t) = 0. \end{aligned} \tag{3.6}$$

In the absence of a running G these equations reduce to the ordinary mass conservation equation for $w = 0$,

$$\dot{\bar{\rho}}(t) = -3\frac{\dot{a}(t)}{a(t)}\bar{\rho}(t). \tag{3.7}$$

It is often convenient to solve the energy conservation equation not for $\bar{\rho}(t)$, but instead for $\bar{\rho}(a)$. This requires that, instead of using the expression for $G(t)$ in Eq. (2.11), one uses the equivalent expression for $G(a)$,

$$G(a) = G_0\left(1 + \frac{\delta G(a)}{G_0}\right), \tag{3.8}$$

which is easily obtained once the relationship between t and $a(t)$ is known (see discussion later). Note, for example, that the solution to Eq. (3.6) can be written as

$$\bar{\rho}(a) = \text{const}\exp\left\{-\int \frac{da}{a}\left(3 + \frac{\delta G(a)}{G_0} + a\frac{\delta G'(a)}{G_0}\right)\right\}. \tag{3.9}$$

B. Effective energy-momentum tensor involving ρ_{vac} and p_{vac}

The next step consists in obtaining the equations which govern the effects of small field perturbations. These equations will involve, apart from the metric perturbation h_{ij}, the matter and vacuum-polarization contributions. The latter arise from [see Eq. (2.20)]

$$\left(1 + \frac{\delta G(\Box)}{G_0}\right)T_{\mu\nu} = T_{\mu\nu} + T^{\text{vac}}_{\mu\nu} \tag{3.10}$$

with a nonlocal $T^{\rm vac}_{\mu\nu} \equiv (\delta G(\Box)/G_0) T_{\mu\nu}$. Fortunately to zeroth order in the fluctuations the results of Ref. [15] indicated that the modifications from the nonlocal vacuum-polarization term could simply be accounted for by the substitution

$$\bar\rho(t) \to \bar\rho(t) + \bar\rho_{\rm vac}(t), \qquad \bar p(t) \to \bar p(t) + \bar p_{\rm vac}(t). \tag{3.11}$$

Here we will apply this last result to the small field fluctuations as well, and set

$$\begin{aligned} \delta\rho_{\mathbf q}(t) &\to \delta\rho_{\mathbf q}(t) + \delta\rho_{\mathbf q \rm vac}(t), \\ \delta p_{\mathbf q}(t) &\to \delta p_{\mathbf q}(t) + \delta p_{\mathbf q \rm vac}(t). \end{aligned} \tag{3.12}$$

The underlying assumption is of course that the equation of state for the vacuum fluid still remains roughly correct when a small perturbation is added. Furthermore, just like we had $\bar p(t) = w\bar\rho(t)$ [Eq. (2.14)] and $\bar p_{\rm vac}(t) = w_{\rm vac}\bar\rho_{\rm vac}(t)$ [Eq. (2.15)] with $w_{\rm vac} = \frac{1}{3}$, we now write for the fluctuations

$$\delta p_{\mathbf q}(t) = w\delta\rho_{\mathbf q}(t), \qquad \delta p_{\mathbf q \rm vac}(t) = w_{\rm vac}\delta\rho_{\mathbf q \rm vac}(t), \tag{3.13}$$

at least to leading order in the long wavelength limit, $\mathbf q \to 0$. In this limit we then have simply

$$\begin{aligned} &\delta p(t) = w\delta\rho(t), \\ &\delta p_{\rm vac}(t) = w_{\rm vac}\delta\rho_{\rm vac}(t) \equiv w_{\rm vac}\frac{\delta G(t)}{G_0}\delta\rho(t), \end{aligned} \tag{3.14}$$

with $G(t)$ given in Eq. (2.11), and we have used Eq. (2.12), now applied to the fluctuation $\delta\rho_{\rm vac}(t)$,

$$\delta\rho_{\rm vac}(t) = \frac{\delta G(t)}{G_0}\delta\rho(t) + \ldots, \tag{3.15}$$

where the dots indicate possible additional $O(h)$ contributions. A bit of thought reveals that the above treatment is incomplete, since $G(\Box)$ in the effective field equation of Eq. (2.4) contains, for the perturbed Robertson-Walker metric of Eq. (3.1), terms of order h_{ij}, which need to be accounted for in the effective $T^{\mu\nu}_{\rm vac}$. Consequently the covariant d'Alembertian operator $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ acting here on second rank tensors, such as $T_{\mu\nu}$,

$$\begin{aligned} \nabla_\nu T_{\alpha\beta} &= \partial_\nu T_{\alpha\beta} - \Gamma^\lambda_{\alpha\nu}T_{\lambda\beta} - \Gamma^\lambda_{\beta\nu}T_{\alpha\lambda} \equiv I_{\nu\alpha\beta}, \\ \nabla_\mu(\nabla_\nu T_{\alpha\beta}) &= \partial_\mu I_{\nu\alpha\beta} - \Gamma^\lambda_{\nu\mu}I_{\lambda\alpha\beta} - \Gamma^\lambda_{\alpha\mu}I_{\nu\lambda\beta} \\ &\quad - \Gamma^\lambda_{\beta\mu}I_{\nu\alpha\lambda}, \end{aligned} \tag{3.16}$$

needs to be Taylor expanded in the small field perturbation h_{ij},

$$\Box(g) = \Box^{(0)} + \Box^{(1)}(h) + O(h^2). \tag{3.17}$$

One then obtains for $G(\Box)$ itself

$$G(\Box) = G_0\left[1 + \frac{c_0}{\xi^{1/\nu}}\left(\frac{1}{\Box^{(0)} + \Box^{(1)}(h) + O(h^2)}\right)^{1/2\nu} + \ldots\right], \tag{3.18}$$

which requires the use of the binomial expansion for the operator $(A+B)^{-1} = A^{-1} - A^{-1}BA^{-1} + \ldots$. Thus for sufficiently small perturbations it should be adequate to expand $G(\Box)$ entering the effective field equations in powers of the metric perturbation h_{ij}. Next we turn to a discussion of the above results in different gauges.

IV. GAUGE CHOICES AND CORRESPONDING TRANSFORMATIONS

The previous discussion and summary focused exclusively on the comoving gauge choice for the metric, implicit in the definition of Eq. (2.6). Next we will consider some additional gauges. In this paper we will specifically refer to *three* choices for the metric: the comoving, synchronous, and conformal Newtonian forms. The first two are closely related to each other, and were used to obtain part of the results presented in our previous work [15,18], which was summarized in the previous section. Note that in our previous work [18] we did not include the effects of a stress field s, since it was not necessary for the discussion of density perturbations; new terms arising from such a field are included below. The third form of the metric is the primary focus of the present discussion; the results obtained later on in this paper will either be derived for this metric, or transformed to it by relying on results obtained previously in the other gauges.

A. Comoving, synchronous, and conformal Newtonian gauges

The *comoving* metric has the form

$$g_{\mu\nu} = \bar g_{\mu\nu} + h_{\mu\nu}, \tag{4.1}$$

with background metric

$$\bar g_{\mu\nu} = \mathrm{diag}(-1, a^2, a^2, a^2). \tag{4.2}$$

For the fluctuation one sets

$$h_{0i} = h_{i0} = 0, \tag{4.3}$$

and decomposes the remaining h_{ij} as

$$h_{ij}(\mathbf k, t) = a^2\left[\frac{1}{3}h\delta_{ij} + \left(\frac{1}{3}\delta_{ij} - \frac{k_ik_j}{k^2}\right)s\right] \tag{4.4}$$

so that $\mathrm{Tr}(h_{ij}) = a^2 h$. Besides the scale factor a, the metric is therefore parametrized in terms of the two functions s and h.

On the other hand, in the *synchronous* gauge one sets again $g_{\mu\nu} = \bar g_{\mu\nu} + h_{\mu\nu}$ now with background metric

$$\bar g_{\mu\nu} = a^2\mathrm{diag}(-1, 1, 1, 1). \tag{4.5}$$

For the fluctuation one sets again $h_{0i} = h_{i0} = 0$ and

$$h_{ij}(\mathbf{k}, t) = a^2\left[\frac{k_i k_j}{k^2} h_{\rm sync} + \left(\frac{k_i k_j}{k^2} - \frac{1}{3}\delta_{ij}\right)6\eta\right], \tag{4.6}$$

so that now $\mathrm{Tr}(h_{ij}) = a^2 h_{\rm sync}$. Here, besides the overall scale factor a, the metric is parametrized in terms of the two functions η and $h_{\rm sync}$. From a comparison of the two gauges (comoving and synchronous) one has

$$2\eta = -\tfrac{1}{3}(h + s) \tag{4.7}$$

and

$$h_{\rm sync} + 6\eta = -s. \tag{4.8}$$

Finally the *conformal Newtonian* gauge is in turn described by two scalar potentials ψ and ϕ. In this case the line element is given by

$$\begin{aligned} d\tau^2 &= -g_{\mu\nu}dx^\mu dx^\nu \\ &= a^2\{(1 + 2\psi)dt^2 - (1 - 2\phi)dx_i dx^i\}. \end{aligned} \tag{4.9}$$

Therefore for the metric itself one writes again $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ with $\bar{g}_{\mu\nu} = a^2\mathrm{diag}(-1, 1, 1, 1)$ as for the synchronous case, and furthermore $h_{0i} = h_{i0} = 0$ as before, and now

$$h_{00} = a^2(-2\psi), \tag{4.10}$$

$$h_{ij} = a^2(-2\phi)\delta_{ij}. \tag{4.11}$$

A suitable set of gauge transformations then allows one to go from the synchronous, or comoving, to the conformal Newtonian gauge [19].

B. Tensor box in the comoving gauge

To compute higher order contributions from the h_{ij}'s appearing in the comoving gauge metric, one needs to expand $G(\Box)$ in the various metric perturbations,

$$\begin{aligned} G(\Box) = G_0\bigg[1 + \frac{c_0}{\xi^{1/\nu}}\bigg(\bigg(\frac{1}{\Box^{(0)}}\bigg)^{1/2\nu} \\ - \frac{1}{2\nu}\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h, s)\cdot\bigg(\frac{1}{\Box^{(0)}}\bigg)^{1/2\nu} + \ldots\bigg)\bigg], \end{aligned} \tag{4.12}$$

where the superscripts (0) and (1) refer to zeroth and first order in this expansion, respectively. To get the correction of $O(h, s)$ to the field equations, one therefore needs to consider the relevant term in the expansion of $(1 + \delta G(\Box)/G_0)T_{\mu\nu}$,

$$\begin{aligned} -\frac{1}{2\nu}\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h, s)\cdot\frac{\delta G(\Box^{(0)})}{G_0}\cdot T_{\mu\nu} \\ = -\frac{1}{2\nu}\frac{c_0}{\xi^{1/\nu}}\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h, s)\cdot\bigg(\frac{1}{\Box^{(0)}}\bigg)^{1/2\nu}\cdot T_{\mu\nu}. \end{aligned} \tag{4.13}$$

This last form allows us to use the results obtained previously for the FLRW case, namely,

$$\frac{\delta G(\Box^{(0)})}{G_0} T_{\mu\nu} = T^{\rm vac}_{\mu\nu} \tag{4.14}$$

with here

$$T^{\rm vac}_{\mu\nu} = [p_{\rm vac}(t) + \rho_{\rm vac}(t)]u_\mu u_\nu + g_{\mu\nu}p_{\rm vac}(t) \tag{4.15}$$

to zeroth order in h, and

$$\rho_{\rm vac}(t) = \frac{\delta G(t)}{G_0}\bar{\rho}(t), \qquad p_{\rm vac}(t) = w_{\rm vac}\frac{\delta G(t)}{G_0}\bar{\rho}(t). \tag{4.16}$$

and $w_{\rm vac} = 1/3$. Therefore, in light of the results of Ref. [15], the problem has been reduced to computing the more tractable expression

$$-\frac{1}{2\nu}\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h, s)\cdot T^{\rm vac}_{\mu\nu}. \tag{4.17}$$

To make progress, we will assume a harmonic time dependence for both the perturbations $h(t) = h_0 e^{i\omega t}$ and $s(t) = s_0 e^{i\omega t}$, and for the background quantities $a(t) = a_0 e^{i\Gamma t}$, $\rho(t) = \rho_0 e^{i\Gamma t}$, and $\delta G(t) = \delta G_0 e^{i\Gamma t}$. From now on we shall consider both ω and Γ as slowly varying functions (indeed constants), with the time scale of variations for the perturbation much shorter than the time scale associated with all the background quantities. A more sophisticated treatment will be reserved for future work. Therefore we will take here $\omega \gg \Gamma$ or $\dot{h}/h \gg \dot{a}/a$, which is the same approximation that was used in obtaining the results of Ref. [18].

Let us now list, in sequence, the required matrix elements needed for the present calculation. For the tensor box tt matrix element $(-\frac{1}{2\nu}\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h, s)\cdot T^{\rm vac})_{00}$ one obtains

$$+\frac{1}{2\nu}\frac{11}{3}\frac{\delta G(t)}{G_0}\rho(t)\frac{\Gamma}{\omega}h + \mathcal{O}(k^2). \tag{4.18}$$

For the tensor box ti matrix element $(-\frac{1}{2\nu}\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h, s)\cdot T^{\rm vac})_{0i}$ one obtains

$$-ik_i\frac{1}{2\nu}\frac{2}{9}\frac{\delta G(t)}{G_0}\rho(t)\frac{1}{i\omega}(h - 2s) + \mathcal{O}(k^2). \tag{4.19}$$

For the tensor box ii matrix element, summed over i, $(-\frac{1}{2\nu}\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h, s)\cdot T^{\rm vac})_{ii}$, one obtains

$$3\bigg(+\frac{1}{2\nu}w_{\rm vac}\frac{11}{3}a^2\frac{\delta G(t)}{G_0}\rho(t)\frac{\Gamma}{\omega}h\bigg) + \mathcal{O}(k^2). \tag{4.20}$$

For the tensor box ii matrix element, not summed over i, $(-\frac{1}{2\nu}\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h, s)\cdot T^{\rm vac})_{ii}$, one obtains

$$+\frac{1}{2\nu}a^2\frac{\delta G(t)}{G_0}\rho(t)\left[w_{\rm vac}\frac{11}{3}\frac{\Gamma}{\omega}h+\frac{8}{9}\left(1-3\frac{k_i}{k^2}\right)\frac{\Gamma}{\omega}s\right]+\mathcal{O}(k^2). \tag{4.21}$$

Finally for the tensor box ij matrix element, $(-\frac{1}{2\nu}\frac{1}{\Box^{(0)}}\cdot \Box^{(1)}(h,s)\cdot T^{\rm vac})_{ij}$, one obtains

$$-\frac{k_ik_j}{k^2}\frac{1}{2\nu}a^2\frac{8}{3}\frac{\delta G(t)}{G_0}\rho(t)\frac{\Gamma}{\omega}s+\mathcal{O}(k^2). \tag{4.22}$$

The above expressions are now inserted in the general effective field equations of Eq. (2.4), and will give rise to a set of effective field equations appropriate for this particular gauge, to first order in the field perturbation and with the effects of $G(\Box)$ included.

C. Field equations in the comoving, synchronous, and conformal Newtonian gauges

As a result of the previous manipulations one obtains in the comoving gauge with fields (h, s) the following tt, ti, ii (or $xx+yy+zz$), and ij field equations:

$$\frac{k^2}{3a^2}(h+s)+\frac{\dot a}{a}\dot h=8\pi G_0\left(1+\frac{\delta G}{G_0}\right)\bar\rho\delta+8\pi G_0\frac{\delta G}{G_0}\frac{c_h}{2\nu}h\bar\rho+\mathcal{O}(k^2), \tag{4.23}$$

$$-\frac{1}{3}(\dot h+\dot s)=8\pi G_0\frac{\delta G}{G_0}\left(-\frac{1}{2\nu}\right)\frac{2}{9}\frac{1}{i\omega}(h-2s)\bar\rho+\mathcal{O}(k^2), \tag{4.24}$$

$$-\frac{1}{3}\frac{k^2}{a^2}(h+s)-3\frac{\dot a}{a}\dot h-\ddot h=24\pi G_0\frac{\delta G}{G_0}w_{\rm vac}\bar\rho\delta+24\pi G_0\frac{\delta G}{G_0}w_{\rm vac}\frac{c_h}{2\nu}h\bar\rho+\mathcal{O}(k^2), \tag{4.25}$$

$$\frac{1}{6}\frac{k^2}{a^2}(h+s)-\frac{3}{2}\frac{\dot a}{a}\dot s-\frac{1}{2}\ddot s=-8\pi G_0\frac{\delta G}{G_0}\frac{c_s}{2\nu}s\bar\rho+\mathcal{O}(k^2). \tag{4.26}$$

As in Ref. [18], we have found it convenient here to set in the above expressions

$$c_s\equiv\left(\frac{8}{3}\right)\frac{\Gamma}{\omega} \tag{4.27}$$

and

$$c_h=\equiv(-1)\left(-\frac{11}{3}\right)\frac{\Gamma}{\omega}=\frac{11}{3}\frac{\Gamma}{\omega}. \tag{4.28}$$

In the field equations listed above the terms $\mathcal{O}(k^2)$ arise because of terms $\mathcal{O}(k^2)$ in the expansion of the tensor box operator.

The next step is to convert the left-hand sides of the above field equations, namely, Eqs. (4.23), (4.24), (4.25), and (4.26), which are all expressed in the comoving gauge (h, s), to the synchronous gauge with fields $(h_{\rm sync}, \eta)$. The result of this change of gauge is the sequential replacement

$$\begin{aligned}
\frac{k^2}{3a^2}(h+s)+\frac{\dot a}{a}\dot h&\to-2\frac{k^2}{a^2}\eta+\frac{1}{a^2}\frac{\dot a}{a}\dot h_{\rm sync},\\
-\frac{1}{3}(\dot h+\dot s)&\to 2\dot\eta,\\
-\frac{1}{3}\frac{k^2}{a^2}(h+s)-3\frac{\dot a}{a}\dot h-\ddot h&\to 2\frac{k^2}{a^2}\eta-\frac{1}{a^2}\ddot h_{\rm sync}-2\frac{1}{a^2}\frac{\dot a}{a}\dot h_{\rm sync},\\
\frac{1}{6}\frac{k^2}{a^2}(h+s)-\frac{3}{2}\frac{\dot a}{a}\dot s-\frac{1}{2}\ddot s&\to-\frac{k^2}{a^2}\eta+\frac{1}{2}\frac{1}{a^2}(\ddot h_{\rm sync}+6\ddot\eta)+\frac{1}{a^2}\frac{\dot a}{a}(\dot h_{\rm sync}+6\dot\eta).
\end{aligned} \tag{4.29}$$

The next step involves one more transformation, this time from the synchronous $(h_{\rm sync}, \eta)$ to the desired conformal Newtonian (ϕ, ψ) gauge,

$$\begin{aligned}
\frac{1}{a^2}\left[-2k^2\eta+\frac{\dot a}{a}\dot h_{\rm sync}\right]&\to-\frac{2}{a^2}\left[k^2\phi+3\frac{\dot a}{a}\left(\dot\phi+\frac{\dot a}{a}\psi\right)\right],\\
2\dot\eta&\to 2\left(\dot\phi+\frac{\dot a}{a}\psi\right),\\
\frac{1}{a^2}\left[2k^2\eta-\ddot h_{\rm sync}-2\frac{\dot a}{a}\dot h_{\rm sync}\right]&\to\frac{6}{a^2}\left[\ddot\phi+\frac{\dot a}{a}(\dot\psi+2\dot\phi)+\left(2\frac{\ddot a}{a}-\frac{\dot a^2}{a^2}\right)\psi+\frac{k^2}{3}(\phi-\psi)\right],\\
\frac{1}{a^2}\left[-k^2\eta+\frac{1}{2}(\ddot h_{\rm sync}+6\ddot\eta)+\frac{\dot a}{a}(\dot h_{\rm sync}+6\dot\eta)\right]&\to-\frac{k^2}{a^2}(\phi-\psi).
\end{aligned} \tag{4.30}$$

Equivalently, the above sequence of two transformations can be described by a single transformation, from comoving (h, s) to conformal Newtonian (ϕ, ψ) gauge, which is trivially obtained by combining the previous two. The final outcome of all these manipulations is to achieve a rewrite of the full set of four original field equations, given in Eqs. (4.23), (4.24), (4.25), and (4.26), now with the left-hand side given in the conformal Newtonian gauge and the right-hand side left in the original comoving gauge. One obtains

$$k^2\phi + 3\frac{\dot{a}}{a}\left(\dot{\phi} + \frac{\dot{a}}{a}\psi\right) = -4\pi G_0 a^2\left(1 + \frac{\delta G}{G_0}\right)\bar{\rho}\delta - 4\pi G_0 a^2 \frac{\delta G}{G_0}\frac{c_h}{2\nu}h\bar{\rho} + \mathcal{O}(k^2), \tag{4.31}$$

$$\left(\dot{\phi} + \frac{\dot{a}}{a}\psi\right) = 4\pi G_0 \frac{\delta G}{G_0}\left(-\frac{1}{2\nu}\right)\frac{2}{9}\frac{1}{i\omega}(h - 2s)\bar{\rho} + \mathcal{O}(k^2), \tag{4.32}$$

$$\begin{aligned}\ddot{\phi} + \frac{\dot{a}}{a}(\dot{\psi} + 2\dot{\phi}) + \left(2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)\psi + \frac{k^2}{3}(\phi - \psi) &= 4\pi G_0 a^2\left(w + w_{\text{vac}}\frac{\delta G}{G_0}\right)\bar{\rho}\delta \\ &\quad + 4\pi G_0 a^2 \frac{\delta G}{G_0} w_{\text{vac}}\frac{c_h}{2\nu}h\bar{\rho} + \mathcal{O}(k^2),\end{aligned} \tag{4.33}$$

$$k^2(\phi - \psi) = +8\pi G_0 a^2 \frac{\delta G}{G_0}\frac{c_s}{2\nu}s\bar{\rho} + \mathcal{O}(k^2). \tag{4.34}$$

Note that we have, for convenience, multiplied out the first, third, and fourth equations by a factor of a^2. The last equation involves the quantity

$$\sigma = \frac{2}{3}\frac{\delta G}{G_0}\frac{c_s}{2\nu}\cdot s. \tag{4.35}$$

For the purpose of computing the gravitational slip function $\eta \equiv \psi/\phi - 1$ it will be useful here to record the following relationship between perturbations in the comoving and conformal Newtonian gauge. One has

$$\psi = -\frac{1}{2k^2}a^2\left(\ddot{s} + 2\frac{\dot{a}}{a}\dot{s}\right), \tag{4.36}$$

$$\phi = -\frac{1}{6}(h + s) + \frac{1}{2}\frac{a^2}{k^2}\frac{\dot{a}}{a}\dot{s}. \tag{4.37}$$

Use has been made here of the following relationship between derivatives of an arbitrary function f in the synchronous and comoving gauges:

$$\dot{f}^{\,\text{sync}} = a\dot{f}^{\text{com}} \tag{4.38}$$

and

$$\frac{d}{d\tau_{\text{sync}}} = a\frac{d}{d\tau_{\text{com}}} \tag{4.39}$$

so that

$$\ddot{f}^{\,\text{sync}} = a^2\left(\frac{\dot{a}^{\text{com}}}{a}\dot{f}^{\text{com}} + \ddot{f}^{\text{com}}\right). \tag{4.40}$$

V. GRAVITATIONAL SLIP FUNCTION

The gravitational slip function is commonly defined as

$$\eta \equiv \frac{\psi - \phi}{\phi}. \tag{5.1}$$

In classical GR one has $\phi = \psi$ so that $\eta = 0$, which makes the quantity η a useful parametrization for deviations from classical GR, whatever their origin might be. Using the ij field equation given in Eqs. (4.31), (4.32), (4.33), and (4.34), and the relationship between the conformal Newtonian fluctuation ϕ and the comoving gauge fluctuations h and s, one finally obtains the rather simple result

$$\eta \equiv \frac{\psi - \phi}{\phi} = -16\pi G_0 \frac{\delta G}{G_0}\frac{c_s}{2\nu}\frac{a}{\dot{a}}\frac{s}{\dot{s}}\bar{\rho}. \tag{5.2}$$

The last expression contains the quantity

$$c_s = \left(\frac{8}{3}\right)\frac{i\Gamma}{i\omega_s} \tag{5.3}$$

where ω_s is the frequency associated with the s perturbation, and we have made use of $i\Gamma \to \dot{a}/a$. An equivalent form for the expression in Eq. (5.2) is

$$\begin{aligned}\eta &= -16\pi G_0 \frac{\delta G}{G_0}\frac{1}{2\nu}\frac{8}{3}\frac{1}{i\omega_s}\frac{s}{\dot{s}}\bar{\rho} \\ &= -16\pi G_0 \frac{\delta G}{G_0}\frac{1}{2\nu}\frac{8}{3}\frac{\int s dt}{\dot{s}}\bar{\rho}.\end{aligned} \tag{5.4}$$

In the last expression we now can make use of the equation of motion for the perturbation $s(t)$ to the order we are working, namely,

$$\ddot{s} + 3\frac{\dot{a}}{a}\dot{s} = 0. \tag{5.5}$$

Let us look here first at the very simple limit of $\lambda \simeq 0$; the physically more relevant case of nonzero λ will be discussed a bit later. Note that, in view of Eq. (2.2), this last limit corresponds therefore to a very large ξ. Then for a perfect fluid with equation of state $p = w\rho$ one has simply $a(t) = a_0(t/t_0)^{2/3(1+w)}$ and $\rho(t) = 1/[6\pi G t^2(1 + w)^2]$, and from Eq. (5.4) or (5.17) one obtains for $w = 0$

$$\eta = 4\cdot\frac{8}{3}c_t\left(\frac{t}{\xi}\right)^3\ln\left(\frac{t}{\xi}\right) + \mathcal{O}(t^4) \tag{5.6}$$

whereas for $w \neq 0$ one has

$$\eta = 2 \cdot \frac{8}{3} \frac{c_t}{w(1-w)} \left(\frac{t}{\xi}\right)^3 + \mathcal{O}(t^6). \tag{5.7}$$

Another extreme, but nevertheless equally simple, case is a pure cosmological constant term (no matter of any type), which can be modeled by the choice $w = -1$. In this case t is related to the scale factor by

$$\frac{a(t)}{a_0} = \exp\left\{\sqrt{\frac{\lambda}{3}}(t - t_0)\right\}. \tag{5.8}$$

Then, using the relation in Eq. (2.2), one obtains

$$\frac{t}{\xi} = 1 + \ln\frac{a}{a_\xi}, \tag{5.9}$$

where the quantity a_ξ is therefore related to the time t_0 (''today,'' $a_0 = 1$) and the scale ξ by

$$\frac{t_0}{\xi} = 1 + \ln\frac{1}{a_\xi}. \tag{5.10}$$

Since numerically t_0 is close to, but smaller than, ξ, the scale factor a_ξ will be close to, but slightly larger than, one.

To actually come up with a definite number for η in more realistic cases, one needs (apart from including the effects of $\lambda \neq 0$, which is done below) a value for the coefficient c_t appearing in Eq. (2.11) for $G(t)$, which in turn is related to the original expression for the running Newton's constant $G(\Box)$ in Eq. (2.1). This issue will be discussed in some detail later, but here let us say the following. In Ref. [15] it was estimated that the values of c_t in Eq. (2.11) and c_0 in Eq. (2.11) are of the same order of magnitude, $c_t \approx 0.62 c_0$. The most difficult part has been therefore a reliable estimate of c_0, which is obtained from a lattice computation of invariant curvature correlations at fixed geodesic distance [20], and which, after reexamination of various systematic uncertainties, leads to the recent estimate used in [18] of $c_0 \approx 33.3$. That would give $c_t \approx 20.6$ which, as we will see later, is still very large. Nevertheless it is expected that c_0 (or c_t) enter *all* calculations with $G(\Box)$ with the *same* magnitude and sign.

Let us now go back to the more physical case of $\lambda \neq 0$. The relevant expression for $\eta(t)$ is Eq. (5.4), where we use the equation for $s(t)$, Eq. (5.5), to eliminate the latter. It is also convenient at this stage to change variables from t to $a(t)$, and use the equivalent equation for $s(a)$, namely,

$$s''(a) + \left(\frac{H'(a)}{H(a)} + \frac{4}{a}\right) s'(a) = 0, \tag{5.11}$$

where the prime denotes differentiation with respect to the scale factor a. In the above equation one can use, for nonrelativistic matter with equation of state such that $w = 0$, and to the order needed here, the first Friedmann equation

$$H(a) = \sqrt{\frac{\lambda}{3} + \frac{4}{9a^3}}. \tag{5.12}$$

We have also made use of the unperturbed result for the background matter density valid for $w = 0$ (which follows from energy conservation), namely,

$$\bar{\rho} = \bar{\rho}_0 \frac{1}{a^3}. \tag{5.13}$$

Note that the above expression for $\bar{\rho}$ is valid to zeroth order in δG, which is entirely adequate when substituted into $\eta(a)$, since the rest there is already first order in δG. This finally gives an explicit solution for $s(a)$,

$$s(a) \propto \frac{2}{3a^{3/2}} \sqrt{1 + a^3 \theta}, \tag{5.14}$$

with parameter $\theta \equiv \lambda / 8\pi G_0 \bar{\rho}_0$. The above solution for $s(a)$ can then be substituted directly in Eq. (5.4), provided one changes variables from t to $a(t)$, and in the process uses the following identities:

$$\int s(t) dt = \int s(a) \frac{1}{aH(a)} da, \tag{5.15}$$

as well as

$$\dot{s} = aH(a) \frac{\partial s}{\partial a}, \tag{5.16}$$

with $H(a)$ given a few lines above.

The resulting expression, which still involves an integral over the scale factor $a(t)$, can now be readily evaluated, and leads eventually to a rather simple expression for η. The general result for nonrelativistic matter ($w = 0$) but $\lambda \neq 0$ is

$$\eta(a) = \frac{16}{3\nu} \frac{\delta G(a)}{G_0} \log\left[\frac{a}{a_\xi}\right]. \tag{5.17}$$

This is the main result of the paper. The integration constant a_ξ has been fixed following the requirement that the scale factor $a \to a_\xi$ for $t \to \xi$ [see Eqs. (2.1), (2.11), and (3.8) for the definitions of ξ]. In other words, by switching to the variable $a(t)$ instead of t, the quantity ξ has been traded for a_ξ. In the next section we will show that in practice the quantity a_ξ is generally expected to be slightly larger than the scale factor "today," i.e., for $t = t_0$. As a result the correction in Eq. (5.17) is expected to be negative today.

The next section will be devoted to establishing the general relationship between t and $a(t)$, for nonvanishing cosmological constant λ, so that a quantitative estimate for the slip function η can be obtained from Eq. (5.17) in a realistic cosmological context. Specifically we will be interested in the value of η for a current matter fraction $\Omega \simeq 0.25$, as suggested by current astrophysical measurements.

A. Relating the scale factor a to t, and vice versa

Let us now come back to the general problem of estimating $\eta(a)$, using the expression given in Eq. (5.17), for $\lambda \neq 0$ and a nonrelativistic fluid with $w = 0$. To predict the correct value for the slip function $\eta(a)$ one needs the quantity $\delta G(a)$, which is obtained from the FLRW version of $G(\Box)$, namely, $G(t)$ in Eq. (2.11), via the replacement, in this last quantity, of $t \rightarrow t(a)$. The last step requires therefore that the correct relationship between t and $a(t)$ be established, for any value of λ. In the following we will first relate t to $a(t)$, and vice versa, to zeroth order in the quantum correction δG [we will call them $a^{(0)}(t)$ and $t^{(0)}(a)$], and then compute the first order correction in δG to the above quantities [we will call those $a^{(1)}(t)$ and $t^{(1)}(a)$].

Let us look first at the zeroth order result. The field equations and the energy conservation equation for $a^{(0)}(t)$, without a δG correction, but with the λ term, were already given in Eq. (2.17),

$$3\frac{\dot{a}^{(0)2}(t)}{a^{(0)2}(t)} = 8\pi G_0 \bar{\rho}^{(0)}(t) + \lambda,$$
$$\frac{\dot{a}^{(0)2}(t)}{a^{(0)2}(t)} + 2\frac{\ddot{a}^{(0)}(t)}{a^{(0)}(t)} = -8\pi G_0 w \bar{\rho}^{(0)}(t) + \lambda \tag{5.18}$$

for a spatially flat universe ($k = 0$), and

$$\dot{\bar{\rho}}^{(0)}(t) + 3(1 + w)\frac{\dot{a}^{(0)}(t)}{a^{(0)}(t)}\bar{\rho}^{(0)}(t) = 0. \tag{5.19}$$

From these one can obtain $a^{(0)}(t)$ and then $\bar{\rho}^{(0)}(t)$. As a result the scale factor is found to be related to time by

$$t^{(0)}(a) = \frac{2\,\mathrm{Arcsinh}\,[a^{3/2}\theta^{(1/2)}]}{\sqrt{3\lambda}}, \tag{5.20}$$

where we have defined the parameter

$$\theta \equiv \frac{\lambda}{8\pi G_0 \bar{\rho}_0} = \frac{1 - \Omega}{\Omega} \tag{5.21}$$

with $\bar{\rho}_0$ the current ($t = t_0$) matter density, and Ω the current matter fraction. Note that in practice we will be interested in a matter fraction which today is around 0.25, giving $\theta \simeq 3.0$, a number which is of course quite far from the zero cosmological constant case of $\theta = 0$.

One can express the time today (t_0) in terms of cosmological constant λ, and therefore in terms of θ, as follows:

$$t_0^{(0)} = \frac{2\,\mathrm{Arcsinh}\,(\sqrt{\theta})}{\sqrt{3\lambda}} \tag{5.22}$$

with the normalization for $t^{(0)}(a)$ such that $t^{(0)}(a = 0) = 0$ and $t^{(0)}(a = 1) = t_0$ "today." So here we follow the customary choice of having the scale factor equal to one "today." Then one has

$$\frac{t^{(0)}(a)}{t_0^{(0)}} = \frac{\mathrm{Arcsinh}[\sqrt{a^3\theta}]}{\mathrm{Arcsinh}(\sqrt{\theta})}. \tag{5.23}$$

When expanded out in θ, the above result leads to some perhaps more recognizable terms,

$$\frac{t^{(0)}(a)}{t_0^{(0)}} = a^{3/2}\bigg[1 - \frac{1}{6}(-1 + a^3)\theta + \frac{1}{360}(-17 - 10a^3 + 27a^6)\theta^2 + \cdots\bigg]. \tag{5.24}$$

Conversely, one has for the scale factor as a function of the time

$$a^{(0)}(t) = \left(\frac{\mathrm{Sinh}^2[\frac{\sqrt{3\lambda}}{2}t]}{\theta}\right)^{1/3}, \tag{5.25}$$

which, when expanded out in λ or t, gives the more recognizable result

$$[a^{(0)}(t)]^3 = \frac{3\lambda t^2}{4\theta}\left(1 + \frac{\lambda t^2}{4} + \frac{\lambda^2 t^4}{40} + \cdots\right). \tag{5.26}$$

Similarly for the pressure one obtains

$$\bar{\rho}^{(0)}(t) = \frac{\lambda \mathrm{Csch}^2[\frac{\sqrt{3\lambda}}{2}t]}{8\pi G_0}, \tag{5.27}$$

which when expanded out in λ or t gives the more familiar result

$$\bar{\rho}^{(0)}(t) = \frac{1}{6\pi G_0 t^2(1 + \frac{t^2\lambda}{4} + \frac{t^4\lambda^2}{40} + \frac{3t^6\lambda^3}{2240} + \cdots)}. \tag{5.28}$$

To be more specific, let us set $\theta = 3$, which corresponds to a matter fraction today of $\Omega \sim 0.25$. In addition, we will now make use of Eq. (2.2) and set $\lambda \rightarrow 3/\xi^2$. One then obtains

$$t_0^{(0)}(\theta = 3) = 0.878\xi, \tag{5.29}$$

which shows that t_0 and ξ are rather close to each other (apparently a numerical coincidence).

Then, from the expression for $G(t)$ in Eq. (2.11),

$$\frac{\delta G(t)}{G_0} = c_t\left(\frac{t}{\xi}\right)^{1/\nu}, \tag{5.30}$$

one can obtain $G(a)$ in all generality, by the replacement $t \rightarrow t(a)$ according to the result of Eq. (5.20) or (5.23). For the special case of pure nonrelativistic matter with equation of state $w = 0$ and $\lambda = 0$ one obtains, using Eq. (5.24),

$$\frac{\delta G(a)}{G_0} = c_t\left(\frac{a}{a_\xi}\right)^{\gamma_\nu}, \tag{5.31}$$

with exponent

$$\gamma_\nu = \frac{3}{2\nu}. \tag{5.32}$$

The latter is largely the expression used earlier in the matter density perturbation treatment of our earlier work of Ref. [18].

More generally one can define a_ξ as the value for the scale factor a which corresponds to the scale ξ,

$$a_\xi^{(0)} \equiv \left(\frac{1}{\theta}\right)^{1/3} \mathrm{Sinh}^{2/3}\left[\frac{3}{2}\right] = 1.655\left(\frac{1}{\theta}\right)^{1/3}, \tag{5.33}$$

so that in general $a_\xi \neq a_0$, where $a_0 = 1$ is the scale factor "today." Then for the observationally favored case $\theta \simeq 3$ one obtains

$$a_\xi^{(0)}(\theta = 3) = 1.148, \tag{5.34}$$

which clearly implies $a_\xi^{(0)} > a_0 = 1$.[2] The above expressions will be used in the next section to obtain a quantitative estimate for the slip function $\eta(a)$, evaluated at today's time $t = t_0$.

The discussion above dealt with the case of $\delta G = 0$. Let us now consider briefly the corrections to $a(t)$ and, conversely, $t(a)$ that come about when the running of G is included, in other words when a constant G is replaced by $G(t)$ or $G(a)$ in the effective field equations. In Eq. (2.18) the Friedmann equations were given in the presence of a running G, namely,

$$3\frac{\dot a^2(t)}{a^2(t)} = 8\pi G_0\left(1 + \frac{\delta G(t)}{G_0}\right)\bar\rho(t) + \lambda,$$
$$\frac{\dot a^2(t)}{a^2(t)} + 2\frac{\ddot a(t)}{a(t)} = -8\pi G_0\left(w + w_{\mathrm{vac}}\frac{\delta G(t)}{G_0}\right)\bar\rho(t) + \lambda, \tag{5.35}$$

together with the energy conservation equation

$$3\frac{\dot a(t)}{a(t)}\left[(1 + w) + (1 + w_{\mathrm{vac}})\frac{\delta G(t)}{G_0}\right]\bar\rho(t) + \frac{\delta \dot G(t)}{G_0}\bar\rho(t) + \left(1 + \frac{\delta G(t)}{G_0}\right)\dot{\bar\rho}(t) = 0. \tag{5.36}$$

To solve these equations to first order in δG we set

$$a(t) = a^{(0)}(t)[1 + c_t a^{(1)}(t)], \tag{5.37}$$

$$\bar\rho(t) = \bar\rho^{(0)}(t)[1 + c_t \bar\rho^{(1)}(t)], \tag{5.38}$$

where $a^{(0)}(t)$ and $\bar\rho^{(0)}(t)$ here represent the solutions obtained previously for $\delta G = 0$. One then finds for the correction to the matter density

$$\bar\rho^{(1)}(t) = -\left(\frac{t}{\xi}\right)^{1/\nu}\left(1 + w_{\mathrm{vac}}\frac{\nu}{(1+\nu)}\sqrt{3\lambda}\,t\,\mathrm{Coth}\left[\frac{\sqrt{3\lambda}}{2}t\right]\right) \tag{5.39}$$

and to lowest nontrivial order in t and for $w_{\mathrm{vac}} = 1/3$

$$\bar\rho^{(1)}(t) = -\frac{3 + 5\nu}{3(1+\nu)}\left(\frac{t}{\xi}\right)^{1/\nu} + \ldots. \tag{5.40}$$

For the correction to the scale factor one finds

$$a^{(1)}(t) = -w_{\mathrm{vac}}\frac{\nu}{(1+\nu)}\lambda\int_0^t \frac{t'(\frac{t'}{\xi})^{1/\nu}}{-1 + \mathrm{Cosh}[\sqrt{3\lambda}t']}dt' \tag{5.41}$$

and to lowest nontrivial order in t for $w_{\mathrm{vac}} = 1/3$,

$$a^{(1)}(t) = -\frac{2\nu^2}{9(1+\nu)}\left(\frac{t}{\xi}\right)^{1/\nu} + \ldots. \tag{5.42}$$

After having obtained the relevant formulas for $a(t)$ and $t(a)$ in the general case, i.e., for nonzero λ, we can return to the problem of evaluating the slip function η.

B. Quantitative estimate of the slip function $\eta(z)$

The general expression for the gravitational slip function $\eta(a)$ was given earlier in Eq. (5.17) for $w = 0$ and $\lambda \neq 0$,

$$\eta(a) = \frac{16}{3\nu}\frac{\delta G(a)}{G_0}\log\left[\frac{a}{a_\xi}\right]. \tag{5.43}$$

To obtain $\delta G(a)$ we now use, from Eq. (2.11),

$$\frac{\delta G(t)}{G_0} = c_t\left(\frac{t}{\xi}\right)^{1/\nu} \tag{5.44}$$

and substitute in the above expression for $\delta G(t)$ the correct relationship between t and a, namely, $t(a)$ from Eq. (5.20), which among other things contains the constant defined in Eq. (5.33),

$$a_\xi = \left(\frac{1}{\theta}\right)^{1/3}\mathrm{Sinh}^{2/3}\left[\frac{3}{2}\right]. \tag{5.45}$$

It will be convenient, at this stage, to also make use of the relationship in Eq. (2.2), namely,

$$\lambda \to \frac{3}{\xi^2}. \tag{5.46}$$

The last step left is to make contact with observationally accessible quantities, by expanding in the redshift z, related in the usual way to the scale factor a by $a \equiv 1/(1+z)$. Then for $\nu = 1/3$ and $\theta = 3$ (matter fraction $\Omega = 0.25$) one finally obtains for the gravitational slip function

$$\eta(z) = -1.491c_t - 6.418c_t z + 30.074c_t z^2 + \cdots. \tag{5.47}$$

To obtain an actual number for $\eta(z = 0)$ one needs to address two more issues. They are (i) to provide a bound on the theoretical uncertainties in the above expression and (ii) to give an estimate for the coefficient c_t, which is traced

[2] Let us give here a few more observational numbers for present and future reference. From the present age of the Universe $t_0 \approx 13.75\,G\mathrm{yrs} \simeq 4216$ Mpc, whereas from the observed value of λ (mostly extracted from distant supernovae surveys) one has following Eq. (2.2) $\xi \simeq 4890$ Mpc, which then gives $t_0/\xi \simeq 0.862 = 1/1.160$. This last ratio is similar to the number we used in Eq. (5.29), by setting there $\Omega = 0.25$ exactly.

back to Eq. (2.11) and therefore to the original expression for $G(\Box)$ in Eq. (2.1). The latter contains the coefficient c_0, but in Ref. [15] the estimate was given $c_t = 0.450c_0$ for the tensor box operator; thus c_t and c_0 can safely be assumed to have the same sign, and comparable magnitudes.

To estimate the level of uncertainty in the magnitude of the correction coefficient in Eq. (5.47) we will consider here an infrared regulated version of $G(\Box)$, where an infrared cutoff is supplied so that in Fourier space $k > \xi^{-1}$, and the spurious infrared divergence at small k is removed. This is quite analogous to an infrared regularization used very successfully in phenomenological applications to QCD heavy quark bound states [21,22], and which has recently found some limited justification in the framework of infrared renormalons [23]. As shown already in the first cited reference, it works much better than expected; here a similar prescription will be used just as a means to provide some estimate on the theoretical uncertainty in the result of Eq. (5.47). Therefore, instead of the $G(\Box)$ in Eq. (2.1), which in momentum space corresponds to

$$G(k^2) \simeq G_0\left[1 + c_0\left(\frac{1}{\xi^2 k^2}\right)^{1/2\nu} + \ldots\right], \tag{5.48}$$

we will consider a corresponding infrared regulated version,

$$G(k^2) \simeq G_0\left[1 + c_0\left(\frac{\xi^{-2}}{k^2 + \xi^{-2}}\right)^{1/2\nu} + \ldots\right]. \tag{5.49}$$

Of course the small distance, $k \gg \xi^{-1}$ or $r \ll \xi$, behavior is unchanged, whereas for large distances $r \gg \xi$ the gravitational coupling no longer exhibits the spurious infrared divergence; instead it approaches a finite value $G_\infty \simeq (1 + c_0 + \ldots)G_0$. Now, in momentum space the infrared regulated $\delta G(k)$ reads

$$\frac{\delta G(k^2)}{G_0} = c_0\left(\frac{m^2}{k^2 + m^2}\right)^{1/2\nu}, \tag{5.50}$$

with $m = 1/\xi$, and in position space the corresponding form is

$$\frac{\delta G(\Box)}{G_0} = c_0\left(\frac{1}{-\xi^2\Box + 1}\right)^{1/2\nu}. \tag{5.51}$$

Following the results of Ref. [15], if the above differential operator acts on functions of t only, then one obtains for $\delta G(t)$

$$\frac{\delta G(t)}{G_0} = c_0\left(\frac{1}{(\frac{c_0}{c_t})^{2\nu}(\frac{\xi}{t})^2 + 1}\right)^{1/2\nu} \tag{5.52}$$

with again $c_t/c_0 \approx 0.62$ [15]. Note that the expression in Eq. (5.52) could also have been obtained directly from Eq. (2.11), by a direct regularization.

One can then repeat the whole calculation for $\eta(a)$ with the regulated version of $\delta G(t)$ given in Eq. (5.52). The result is

$$\eta(z) = -0.766c_t - 4.109c_t z + 12.188c_t z^2 + \ldots. \tag{5.53}$$

It seems that the effect of the infrared regularization has been to reduce the magnitude of the effect (at $z = 0$) by about a factor of 2. It is encouraging that, at this stage of the calculation, the negative trend in $\eta(z)$ due to the running of G appears unchanged. Furthermore, in all cases we have looked so far, the value $\eta(z = 0)$ is found to be negative.

C. Slip function $\eta(z)$ for stress perturbation $s = 0$

In Ref. [18] a preliminary estimate of the magnitude of the slip function η was given. The calculation there neglected the stress field s in Eq. (4.4) and only included the metric perturbation h in the comoving gauge. The main reason was that nonrelativistic matter density perturbations, and therefore the growth exponents, are unaffected by the stress field contribution. We will show here that in this case one still obtains a nonvanishing η, whose value we will discuss below. The results will be useful, since now a direct comparison can be done with the full answer (including the stress field) for $\eta(z)$ given in the previous section.

In the absence of stress ($s = 0$) and finite k, the tt and $xx + yy + zz$ field equations read

$$\begin{aligned} &-2\frac{k^2}{a^2}\phi - 8\pi G_0 \frac{c_h}{2\nu}\frac{\delta G}{G_0}\rho\delta\left(-\frac{2}{1+w}\right) \\ &\quad = 8\pi G_0\left(1 + \frac{\delta G}{G_0}\right)\rho\delta, \end{aligned} \tag{5.54}$$

$$\begin{aligned} &2\frac{k^2}{a^2}(\psi - \phi) + 24\pi G_0\frac{c_h}{2\nu}w_{\mathrm{vac}}\frac{\delta G}{G_0}\rho\delta\left(-\frac{2}{1+w}\right) \\ &\quad = -24\pi G_0\left(w + w_{\mathrm{vac}}\frac{\delta G}{G_0}\right)\rho\delta. \end{aligned} \tag{5.55}$$

In both equations we have made use of zeroth order (in $\delta G/G_0$) energy conservation, which leads to $h = -\frac{2}{(1+w)}\delta$, where δ is the matter fraction. One can then take the ratio of the two equations given above, and obtain again an expression for the slip function $\eta = (\psi - \phi)/\phi$. For $w = 0$ (nonrelativistic matter), after expanding in $\delta G/G_0$, one finds the rather simple result

$$\eta = \frac{\psi - \phi}{\phi} = 3w_{\mathrm{vac}}\left(1 - \frac{c_h}{\nu}\right)\frac{\delta G}{G_0}. \tag{5.56}$$

Here the quantity c_h is the same as in Eq. (4.28), and depends on the choices detailed below. In the following we will continue to use $w_{\mathrm{vac}} = 1/3$ [see Eqs. (2.15) and (2.16)] [15,18], which is the correct value associated with $G(\Box)$ in the FLRW background metric.

In Ref. [18] we used the *scalar box* value $c_h = 1/2$, which then gives

$$\eta = \left(1 - \frac{1}{2\nu}\right)\frac{\delta G}{G_0} = \left(1 - \frac{1}{2\nu}\right)c_t\left(\frac{t}{\xi}\right)^{1/\nu} + \ldots. \quad (5.57)$$

In this last case it is then easy to recompute the slip function in terms of the redshift, just as was done in the previous section, and one finds, under the same conditions as before [$\nu = 1/3$, $\theta = 3$, and t_0/ξ as given in Eq. (5.29)], the following result:

$$\eta \simeq -0.338 c_t + O(z). \quad (5.58)$$

For the infrared regulated version of $\delta G/G_0$ given in Eq. (5.52) one obtains instead the slightly smaller value

$$\eta \simeq -0.174 c_t + O(z). \quad (5.59)$$

For the *tensor box* case (also discussed extensively in [18], where it was shown that this is in fact the correct way of doing the calculation) one finds a significantly larger value $c_h \simeq 7.927$, so that in this case the slip function η becomes

$$\eta \simeq \left(1 - \frac{7.927}{\nu}\right)\frac{\delta G}{G_0} = \left(1 - \frac{7.927}{\nu}\right)c_t\left(\frac{t}{\xi}\right)^{1/\nu} + \ldots. \quad (5.60)$$

Also in this case one can recompute the slip function in terms of the redshift, and one finds, under the same conditions as before,

$$\eta \simeq -15.42 c_t + O(z). \quad (5.61)$$

For the infrared regulated $\delta G/G_0$ given in Eq. (5.52) one finds instead

$$\eta \simeq -7.919 c_t + O(z), \quad (5.62)$$

which is again about a factor of 2 smaller than the unregulated value.

We conclude from the above exercise of calculating η with vanishing stress field $s = 0$ three things. The first is that using the scalar box result on the trace of the energy-momentum tensor (which ultimately is not an entirely correct, or at least an incomplete, procedure, given the tensor nature of the matter energy-momentum tensor) underestimates the effects of $G(\Box)$ on the slip function $\eta(z = 0)$ by a factor that can be as large as an order of magnitude.

The second lesson is that the stress field (s) contribution is indeed important, since it reduces the size of the quantum correction significantly [Eqs. (5.47) and (5.53)], compared to the $s = 0$ result [Eqs. (5.60) and (5.61)], again by almost an order of magnitude, which would imply some degree of cancellation between the s and h contributions.

The third observation is that in all cases we have looked at so far the quantum correction to the slip function is *negative* at $z = 0$.

VI. CONCLUSIONS

In the previous sections we computed corrections to the gravitational slip function $\eta = \psi/\phi - 1$ arising from the renormalization-group motivated running $G(\Box)$, as given in Eq. (2.1). The relevant result was presented in Eqs. (5.47) and (5.53), the first expression representing the answer for an unregulated $G(\Box)$, and the second answer found for an infrared regulated version of the same. It should be noted that, so far, in the treatment of metric and matter perturbations we have considered only the $\mathbf{k} \to 0$ limit [see Eq. (3.2)]. Let us focus here for definiteness on the first of the two results [Eq. (5.47)], which is

$$\eta(z) \simeq -1.491 c_t + O(z) \quad (6.1)$$

at $z \simeq 0$. We now come to the last issue, namely, an estimate for the magnitude of the constant c_t. As already discussed previously in Sec. V B, to get an actual number for $\eta(z = 0)$ one needs a number for c_t, whose appearance is traced back to Eq. (2.11), and therefore to the original expression for $G(\Box)$ in Eq. (2.1), with $c_t \approx 0.450 \times c_0$ for the relevant tensor box operator [15].

The value of the constant c_0 has to be extracted from a nonperturbative lattice computation of invariant curvature correlations at fixed geodesic distance [20]; it relates the physical correlation length ξ to the bare lattice coupling G, and is therefore a genuinely nonperturbative amplitude. After a reexamination of various systematic uncertainties, these lead to the recent estimate used in [18] of $c_0 \approx 33.3$. That would give for the amplitude $c_t \approx 20.6$ which still seems rather large. Nevertheless, based on experience with other field-theoretic models which also exhibit nontrivial fixed points such as the nonlinear sigma model, as well as QCD and non-Abelian gauge theories, one would expect this amplitude to be of order unity; very small or very large numbers would seem rather atypical and un-natural.

As far as astrophysical observations are concerned, current estimates for $\eta(z = 0)$ obtained from CMB measurements give values around 0.09 ± 0.7 [24,25], which would then imply an observational bound $c_t \lesssim 0.3$.

Indeed a similar problem of magnitudes for the theoretical amplitudes was found in our recent calculation of matter density perturbations with $G(\Box)$, where again the corrections seemed rather large [18] in view of the above quoted value of c_t. Let us briefly summarize those results here. Specifically, in Ref. [18] a value for the density perturbation growth index γ was obtained in the presence of $G(\Box)$. The quantity γ is in general obtained from the growth index $f(a)$ [26],

$$f(a) \equiv \frac{\partial \ln \delta(a)}{\partial \ln a}, \quad (6.2)$$

where $\delta(a)$ is the matter density contrast. One is mainly interested in the neighborhood of the present era, $a(t) \simeq a_0 \simeq 1$, which leads to the definition of the growth index parameter γ via

$$\gamma \equiv \left.\frac{\ln f}{\ln \Omega}\right|_{a=a_0}. \quad (6.3)$$

The latter has been the subject of increasingly accurate cosmological observations; for some recent references see [27–29].[3]

On the theoretical side, for the tensor box one finds [18], for a matter fraction $\Omega = 0.25$,

$$\gamma = 0.556 - 106.4 c_t + O(c_t^2), \tag{6.4}$$

where the first contribution is the classical GR value from the relativistic treatment of matter density perturbations [26]. The result presented above is in fact a slight improvement over the answer quoted in our earlier work [18], since now the improved relationship between t and a given in Eq. (5.20) has been used, which reduces the magnitude of the correction proportional to c_t. Nevertheless, it should be emphasized that the above result has been obtained in the $\mathbf{k} \to 0$ limit of the perturbation Fourier modes in Eq. (3.2).

Recent observational bounds on x-ray studies of large galactic clusters at distance scales of up to about 1.4 to 8.5 Mpc (comoving radii of $\sim$8.5 Mpc and viral radii of $\sim$1.4 Mpc) [28] favor values for $\gamma = 0.50 \pm 0.08$, and more recently $\gamma = 0.55 + 0.13 - 0.10$ [29]. This would then constrain the amplitude c_t in Eq. (6.4) *at that scale* to $c_t \lesssim 5 \times 10^{-4}$. The latter bound from density perturbations seems a much more stringent bound than the one coming from the observed slip function. Indeed with the bound on c_t coming from the observed density perturbation exponents one would conclude that, according to Eq. (6.1), the correction to the slip function at $z \simeq 0$ must indeed be very small, $\eta \simeq O(10^{-3})$, which is a few orders of magnitude below the observational limit quoted above, $\eta \simeq 0.09 \pm 0.7$.

It is of course possible that the galactic clusters in question are not large enough yet to see the quantum effect of $G(\square)$, since after all the relevant scale in Eq. (2.1) is related to λ and is supposed therefore to be very large, $\xi \simeq 4890$ Mpc.[4] But most likely the theoretical uncertainties in the value of c_t have also been underestimated in [20], and new, high precision lattice calculation will be required to significantly reduce the systematic errors.

Nevertheless it seems clear that the nonperturbative coefficient c_0 (or c_t) enters *all* calculations involving $G(\square)$ with the *same* magnitude and sign. This is simply a consequence of c_0 being part of the renormalization group $G(\square)$ which enters the covariant effective field equations of Eq. (2.4). Consequently, one should be able to relate one set of physical results to another, such as the value of the slip function $\eta(z = 0)$ in Eq. (5.47) to the corrections to the density perturbation growth exponent γ computed in [18], and given here in Eq. (6.4). Then the amplitude c_t can be made to conveniently drop out when computing the ratio of $G(\square)$ corrections to two different physical processes. The resulting predictions are then entirely independent of the theoretical uncertainty in the amplitude c_0, and remain sensitive only to the uncertainties in the two other quantum parameters ξ and ν, which are expected to be significantly smaller. One then obtains for the ratio of the corrections to the growth exponent γ to the slip function $\eta(z = 0)$ at $z \simeq 0$

$$\frac{\delta\gamma}{\delta\eta} \simeq \frac{-106.4 c_t}{-1.491 c_t} \simeq +71.4 \tag{6.5}$$

for the infrared unimproved case. One conclusion that one can draw from the numerical value of the above ratio is that it might be significantly harder to see the $G(\square)$ correction in the slip function than in the matter density growth exponent, by almost 2 orders of magnitude in relative magnitude. Hopefully increasingly accurate astrophysical measurements of the latter will be done in the not too distant future. Of particular interest would be any trend in the growth exponents as a function of the maximum galactic cluster size.

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APPENDIX: SCALAR BOX IN THE COMOVING GAUGE

In this section we will give a short sample calculation of the effects of the covariant d'Alembertian operator $\square \equiv g^{\mu\nu}\nabla_\mu\nabla_n u$ acting on a coordinate scalar, such as the trace of the energy-momentum tensor. The calculation

[3] For a recent detailed review on the many tests of general relativity on astrophysical scales, and a much more complete set of references, see for example [30,31].

[4] One might perhaps think that the running of G envisioned here might lead to small observable consequences on much shorter, galactic length scales. That this is not the case can be seen, for example, from the following argument. For a typical galaxy one has a size $\sim$30 kpc, giving for the quantum correction the estimate, from the potential obtained in the static isotropic metric solution with $G(\square)$ [16] which gives $\delta G(r) \sim (r/\xi)^{1/\nu}$, $(30\ \mathrm{kpc}/4890 \times 10^3\ \mathrm{kpc})^3 \sim 2.31 \times 10^{-16}$ which is tiny given the large size of ξ. It is therefore unlikely that such a correction will be detectable at these length scales, or that it could account for large anomalies in the galactic rotation curves. The above argument also implies a certain sensitivity of the results to the value of the scale ξ; thus an increase in ξ by a factor of 2 tends to reduce the effects of $G(\square)$ by roughly $2^3 = 8$, as can be seen already from Eq. (2.1) with $\nu = 1/3$. More specifically, the amplitude of the quantum correction is proportional, in the noninfrared improved case, to the combination c_0/ξ^3.

presented below will show that the result is unchanged when the stress contribution s is included in the metric for the comoving gauge. Specifically here we will be interested in the correction of order h_{ij} that arises when the operator in Eq. (4.12) acts on the scalar $T_\lambda{}^\lambda = -\bar{\rho}$. Thus, for example, it will give the correction $O(h, s)$ to $\delta\rho_{\rm vac}$, namely, the second term in the expression

$$\delta\rho_{\rm vac}(t) = \frac{\delta G(\Box^{(0)})}{G_0}\delta\rho(t) + \frac{\delta G(\Box)(h, s)}{G_0}\bar{\rho}(t), \qquad \text{(A1)}$$

with the first term being simply given in the FLRW background by $\delta G(t)/G_0 \cdot \delta\rho(t)$. Here the $O(h, s)$ correction is given explicitly by the expression

$$\frac{\delta G(\Box)(h, s)}{G_0}\bar{\rho} = -\frac{1}{2\nu}\frac{c_0}{\xi^{1/\nu}}\frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h, s) \cdot \left(\frac{1}{\Box^{(0)}}\right)^{1/2\nu} \cdot \bar{\rho}. \qquad \text{(A2)}$$

Now the covariant d'Alembertian $\Box$ acting on general scalar functions $S(x)$ simplifies to

$$\Box S(x) \equiv \frac{1}{\sqrt{g}}\partial_\mu g^{\mu\nu}\sqrt{g}\partial_\nu S(x). \qquad \text{(A3)}$$

In the absence of h_{ij} fluctuations this gives for the metric in the comoving gauge

$$\Box^{(0)} S(x) = \frac{1}{a^2}\nabla^2 S - 3\frac{\dot{a}}{a}\dot{S} - \ddot{S}. \qquad \text{(A4)}$$

To first order in the field fluctuation h_{ij} of the comoving gauge one computes

$$\begin{aligned}\Box^{(1)}(h, s)S(x) = {}& \dot{S}\left[-\frac{1}{2}\dot{h}\right] + \partial_x S\left[\frac{1}{6a^2}ik_x(h + 4s)\right] \\ & + \partial_x^2 S\left[-\frac{1}{3a^2}(h + s) + \frac{1}{a^2}\frac{k_x^2}{k^2}s\right] \\ & + \partial_x\partial_y S\left[\frac{2}{a^2}\frac{k_x k_y}{k^2}s\right],\end{aligned} \qquad \text{(A5)}$$

where we have set as usual $h(x) = h(t)e^{i\mathbf{k}\cdot\mathbf{x}}$. But, for a function of time only, one obtains

$$\Box^{(1)}(h)\rho(t) = -\frac{1}{2}\dot{h}(t)\dot{S}(t). \qquad \text{(A6)}$$

Thus to first order in the fluctuations one has

$$\frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h) \cdot (\delta G\bar{\rho}) = \frac{1}{-\partial_t^2 - 3\frac{\dot{a}}{a}\partial_t} \cdot \frac{1}{2}\dot{h}\left(3\frac{\dot{a}}{a}\delta G - \dot{\delta G}\right)\bar{\rho} \qquad \text{(A7)}$$

and there is no change from the result quoted in [18]. There we set $s = 0$, since we were only interested in cosmological density perturbations δ, which couple only to the trace part of the gravitational field fluctuations h_{ij}.

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