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Phases of simplicial quantum gravity in four dimensions
Estimates for the critical exponents *

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A numerical evaluation of the discrete path integral for pure lattice gravity, with and without higher derivative terms, and using the lattice analog of the DeWitt gravitational measure, shows the existence of a well-behaved ground state for sufficiently strong gravity ($G \geqslant G_C$). Close to the continuous critical point separating the smooth from the rough phase of gravity, the critical exponents are estimated using a variety of methods on lattices with up to $15 \times 16^4 = 1\,572\,864$ simplices. With periodic boundary conditions (four-torus) the average curvature approaches zero at the critical point. Curvature fluctuations diverge at this point, while the fluctuations in the local volumes remain bounded. The value of the curvature critical exponent is estimated to be $\delta = 0.626(25)$, when the critical point is approached from the smooth phase. In this phase, as well as at the critical point, the fractal dimension is consistent with four, the euclidean value. In the (physically unacceptable) rough, collapsed phase the fractal dimensions is closer to two, in agreement with earlier results which suggested a discontinuity in the fractal dimensions at the critical point. For sufficiently small higher derivative coupling, and in particular for the pure Regge-Einstein action, the transition between the smooth and rough phase becomes first order, suggesting the existence of a multicritical point separating the continuous from the discontinuous phase transition line.

1. Introduction

Recently there has been a renewed interest in discrete models for quantum gravity. In this paper we shall concentrate on the simplicial formulation of quantum gravity, also known as the Regge Calculus approach, and refine and extend some of the results presented in refs. [1,2]. It is well known that at the classical level the theory is completely equivalent to general relativity, and the correspondence is particularly transparent in the usual weak-field expansion [3], with the invariant edge lengths playing the role of infinitesimal geodesics in the continuum. The correspondence between lattice and continuum quantities is straightforward, and so is the interpretation of the terms in the action, as well as the identification and separation of the measure contribution.

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In the limit of smooth manifolds with small curvatures, one can show that the full continuous diffeomorphism invariance is recovered. In addition the lattice model has a trivial piecewise diffeomorphism invariance, corresponding to coordinate transformations within the flat simplices, as well as relabeling of the vertices. Away from almost flat manifolds the continuous diffeomorphism invariance is lost, and different configurations of edge lengths will in general correspond to different manifolds. In this sense there is no gauge invariance in simplicial gravity except in the smooth limit. On the other hand the theory is formulated entirely in terms of coordinate invariant quantities, the edge lengths, which form the elementary degrees of freedom in the theory [2,4,5].

Since quantum gravity is not well defined in the continuum, a number of technical and conceptual difficulties, related to the gravitational measure factor [6], the unboundedness of the Euclidean gravitational action [7], and the lack of perturbative renormalizability above two dimensions [8], persist in the lattice formulation as well (in fact in all known lattice formulations).

It seems clear that non-perturbative methods are necessary in trying to understate the nature of the ground state in quantum gravity. A description of the construction of the action and measure for simplicial lattice gravity, inspired by Regge's original work [5], can be found in ref. [2]. It was shown some time ago that, under suitable conditions, the gravitational measure and the cosmological term stabilize the gravitational action, and lead to a convergent path integral, even in the case of the pure Regge action. As the bare Newton's constant is varied, a continuous phase transition is found, separating a "smooth" from a "rough" phase of gravity [2]. In the first phase the curvature is small and negative, and the fractal dimension is consistent with four. In the second phase the simplices are collapsed, the curvature is large and positive, and the fractal dimension is much less than four, indicating the presence of finger-like structures, reminiscent of the unbounded fluctuations in the conformal mode in the continuum. Approaching the critical point from the only physically acceptable phase, the smooth one, it was found more recently that the curvature vanishes with an exponent $\delta = 0.62(5)$ [1]. At the critical point the curvature fluctuations diverge, leading to the possibility of defining a non-trivial lattice continuum. This paper will be devoted to extending and refining the results presented in refs. [1]. In particular more accurate results for the exponents will be obtained on rather large lattices (up to 1 572 864 simplices, or 3 276 800 hinges), a finite-size scaling analysis will be described, and new results for the fractal dimension will be discussed. At the end of the paper we will touch on some issues related to the nature of correlations in simplicial quantum gravity.

On the lattice there is a lot of freedom in how one chooses to define the action, the measure, the underlying lattice structure, and the correlation functions, just as there are many ways of finite differencing a derivative. Given reasonable geometric and positivity properties, universality is expected to lead to the same results for quantities like physical observables, exponents, anomalous dimensions etc., in some continuum limit. In two dimension this has been verified explicitly in the case of two-dimensional gravity [9], where good agreement is found between the lattice gravity results and the conformal field theory predictions.

It is possible to formulate simplicial quantum gravity on any type of sensible lattice, including a random one. But a great simplification occurs if one adopts a "regular" lattice (in the sense that the coordination number is fixed at each vertex, but leaving of course the edge lengths arbitrary), since it is somewhat easier to work with both from an analytical as well as a computational point of view. The continuous lattice diffeomorphism invariance mentioned previously is of course not lost by going to such a lattice. As in refs. [1,2], we will discuss in the following results for a simplicial complex topologically equivalent to a torus in four dimensions. One could adopt a different set of boundary conditions, but in the end one expects short distance renormalization effects and critical behavior (and therefore the lattice continuum limit) to be independent of the boundary conditions and the topology.

Recently there also has been some work on three- and four-dimensional generalizations of the original dynamical triangulation model [10–13], using equilateral tetrahedra and simplices, and performing the sum over triangulations using Alexander moves [14]. This development represents an alternative and complementary approach to what is being discussed here.

2. Gravitational action and measure

Following ref. [2], the four-dimensional pure gravity action on the lattice will be chosen to be

$$I[l] = \sum_{\text{hinges } h} \left[\lambda\, V_h - k\, A_h \delta_h + a\, \frac{A_h^2 \delta_h^2}{V_h} \right], \tag{2.1}$$

where V_h is the volume per hinge (triangle), A_h is the area of the hinge and δ_h the corresponding deficit angle, proportional to the curvature at h. Classically the lattice action is bounded from below if $4a\lambda - k^2 > 0$. The lattice curvature squared term proportional to δ_h^2 vanishes if and only if the curvature is zero everywhere. In the following we will take the fundamental "lattice spacing" to be equal to one; the appropriate power of the lattice spacing can always be restored at the end by invoking dimensional arguments. Since the lattice is dynamical, the average physical separation between sites will then be equal to this fundamental "lattice spacing" times the average edge length (with our choice of coupling constants and measure, there is typically a factor of two between the two). In the classical continuum limit the above action is equivalent to the continuum action

$$I = \int \mathrm{d}^4x \sqrt{g} \left[\lambda - \frac{k}{2} R + \frac{a}{4} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \cdots \right], \tag{2.2}$$

with a cosmological constant term (proportional to λ), an Einstein–Hilbert term ($k = 1/8\pi G$), and a higher-derivative term [15]. Here the dots indicate higher-order lattice corrections.

One could consider the Regge action by itself ($a = 0$), but then the euclidean action would be unbounded from below, and problems might arise, depending on the choice of measure (this point will be discussed further below). It should be stressed that the Regge action is equivalent to the Einstein action only in the continuum limit. As in any lattice theory, for finite lattice spacing there are higher-order corrections $O(k^4)$, which can be arbitrarily modified by adding extra terms (or by, say, replacing δ by $\sin\delta$) without affecting the continuum limit. In this respect the Regge action is not unique, and we expect many different but similar actions to lead to the same continuum limit, in the region of parameter space where one can be defined (we will argue that the same should be true for a wide class of measures over the invariant edge lengths). Renormalization group arguments then suggest that in general the continuum limit should be explored in this enlarged multi-parameter space. The higher-derivative lattice term introduced here can therefore be regarded also as a way of parameterizing the higher-order lattice corrections in the simplest way.

Different measures in the continuum give rise to different measures on the lattice. The difference between various measures seems to be in the power of $\sqrt{g}$ in the pre-factor, which corresponds to some product of volume factors on the lattice. On the lattice these volume factors do not give rise to coupling terms and are strictly local. DeWitt has argued that the gravitational measure should be [6]

$$\int \mathrm{d}\mu[g] \; = \; \int \prod_x g^{(d-4)(d+1)/8} \prod_{\mu \geqslant \nu} \mathrm{d}g_{\mu\nu}. \tag{2.3}$$

On the simplicial lattice the invariant edge lengths represent the elementary degrees of freedom, which uniquely specify the geometry for a given incidence matrix. Since the induced metric at a simplex is linearly related to the edge lengths squared within that simplex, one would expect the lattice analog of the DeWitt metric to simply correspond to $\mathrm{d}l^2$. But it should be stressed that while the metric tensor is coordinate dependent, all the edge lengths are manifestly coordinate invariant (they should be thought of as lattice versions of elementary infinitesimal geodesics). We will therefore write the lattice measure in general as

$$\int \mathrm{d}\mu_\epsilon[l] = \prod_{\text{edges } ij} \int_0^\infty V_{ij}^{2\sigma} \, \mathrm{d}l_{ij}^2 \, F_\epsilon[l], \tag{2.4}$$

where V_{ij} is the "volume per edge", $F_\epsilon[l]$ is a function of the edge lengths which enforces the higher-dimensional analogs of the triangle inequalities, and $\sigma = (d-4)/4d$ for the lattice analog of the DeWitt measure. The parameter ϵ is introduced as an ultraviolet cutoff at small edge lengths: the function $F_\epsilon[l]$ is zero if any of the edges are equal to or less than ϵ; in the following we will set $\sigma = 0$ and $\epsilon = 0$ (DeWitt measure with no cutoff).

If a D-component scalar field is coupled to gravity the power σ has to be changed, due to an additional factor of $\prod_x g^{D/2}$ in the continuum gravitational measure,

$$\sigma = \frac{D}{2d(d+1)} - \frac{4-d}{4d} \underset{d=4}{\rightarrow} \frac{D}{40}. \tag{2.5}$$

Eventually it would be of interest to explore the sensitivity of the results to the type of gravitational measure employed. On the basis of universality of critical behavior [16], one would expect though that different invariant lattice measures should lead to the same lattice continuum limit. In two [9] and three [17] space-time dimensions numerical studies seem to indicate that different measures, within a certain universality class, will give the same results for infrared sensitive quantities, like critical exponents. On the other hand the lattice path integral might not be meaningful for certain values of σ for which the measure becomes singular, and for which the simplicial lattice tends to degenerate into a lower-dimensional manifold.

In general a simple scaling argument gives the following estimate of the average volume per edge

$$\langle V_l \rangle \sim \frac{2(1+\sigma d)}{\lambda d} \underset{d=4,\,\sigma=0}{\sim} \frac{1}{2\lambda}, \tag{2.6}$$

if curvature terms in the action are neglected, and shows that the volume tends to zero for a singular measure such as the scale-invariant one ($\sigma d = -1$), unless a cutoff is imposed for short edge lengths. (In four dimensions the numerical simulations with $\sigma = 0$ agree to within a fraction of a percent with the above formula; see discussion below).

Some useful identities can be obtained by examining the scaling properties of the action and the measure. The couplings k and λ in the above gravitational action are dimensionful in four dimensions, but one can define the dimensionless coupling k^2/λ, and rescale the edge lengths so as to eliminate the overall length scale $\sqrt{k/\lambda}$. As a consequence the path integral for pure gravity,

$$Z(\lambda, k, a) = \int d\mu_\epsilon[l]\, e^{-I[l]}, \tag{2.7}$$

obeys an equation of the type

$$Z(\lambda, k, a) = \left(\frac{k}{\lambda}\right)^{N_1} Z\left(\frac{k^2}{\lambda}, \frac{k^2}{\lambda}, a\right), \tag{2.8}$$

where N_1 represents the number of edges in the lattice, and we have selected here the dl^2 measure ($\sigma = 0$). This equation implies, in turn, a sum rule for local averages, which (for the dl^2 measure) reads

$$4\lambda\langle V\rangle - 2k\langle\delta A\rangle = 2N_1/N_0. \tag{2.9}$$

Here N_0 represents the number of sites in the lattice, and the averages are defined per site. For the hypercubic lattice we will use, $N_1 = 15N_0$, $N_2 = 50N_0$, $N_3 = 36N_0$ and $N_4 = 24N_0$ [2]. The coefficients on the l.h.s. of the equation reflect the scaling dimensions of the various terms, while the r.h.s. term arises from the scaling of the measure (in d dimensions the coefficients become d, $(d-2)$ and $(d-4)$, respectively). This last formula can be useful in checking the accuracy of numerical calculations, since each term can be estimated independently.

3. Results in four dimensions

In order to explore the ground state of four-dimensional lattice gravity beyond perturbation theory one has to resort to numerical methods. General aspects of the method as applied to simplicial quantum gravity are discussed in refs. [1,2], and will not be repeated here. In the numerical simulations presented below the simple hypercubic lattice was employed, with six face diagonals, four body diagonals and a hyperbody diagonal introduced to make the cube rigid. Lattices of size between $4 \times 4 \times 4 \times 4$ (with 3840 edges) and $16 \times 16 \times 16 \times 16$ (with 983 040 edges) were considered. In all cases the measure was dl^2 ($\sigma = 0$, see eq. (2.4)). Periodic boundary conditions (four-torus) were used, since it is expected that for this choice boundary effects will be minimized. One could perform the numerical studies with lattices of different topologies, but one expects that universal infrared scaling properties of the theory should be determined by short-distance renormalization effects, and should therefore in general be independent of the specific choice of boundary conditions, so the four-torus should be as good as any other choice of topology as far as critical properties are concerned.

The edge lengths are updated by a standard Metropolis algorithm, generating eventually an ensemble of configurations distributed according to the action of eq. (2.1), with the inclusion of the appropriate generalized triangle inequality constraints arising from the nontrivial measure. A stringent test on the program is that it correctly reproduces the analytic weak-field expansion to second order [18]. The lengths of the runs typically varied between $10-40k$ Monte Carlo iterations on the 4^4 lattice, $2-6k$ on the 8^4 lattice, and $0.5k$ on the 16^4 lattice. On the larger lattices duplicated copies of the smaller lattices are used as starting configurations for each k, allowing for additional equilibration sweeps after duplicating the lattice in all four directions. This allows for a substantial savings in run time, since the initial edge length configuration on the larger lattice is already close to a representative configuration. In some special cases (close to k_c

for $a = 0$, and at k_c for $a = 0.005$) much longer runs were performed to increase the accuracy of the results and have good control over the statistical errors. One notices that in all runs the scaling relation of eq. (2.9) is very well verified (to one percent or better), as one would expect if the edge probability distribution is sampled correctly. Furthermore the average volume associated with an edge agrees with the estimate of eq. (2.6) to better than one percent, for all values of the couplings that we have investigated, and suggests that the average volume is only very weakly dependent on k and a, being influenced mostly by the measure and the cosmological term.

One should emphasize that at this point the nature of the results is still rather preliminary, even though some effort has been made to control the systematic errors by computing the critical exponents for four-dimensional gravity for different values of the (naively irrelevant) coupling a, and by a variety of different methods, which presumably have different (and hopefully small) systematic biases.

3.1. LOCAL AVERAGES

Quantities of physical interest which have been computed include the scaled average curvature $\mathcal{R}$

$$\mathcal{R}(\lambda, k, a) \equiv \langle l^2 \rangle \frac{\langle 2 \sum_h \delta_h A_h \rangle}{\langle \sum_h V_h \rangle} \sim \frac{\langle \int \sqrt{g}\, R \rangle}{\langle \int \sqrt{g} \rangle} \tag{3.1}$$

and the average scaled curvature squared $\mathcal{R}^2$

$$\mathcal{R}^2(\lambda, k, a) \equiv \langle l^2 \rangle^2 \frac{\langle 4 \sum_h \delta_h^2 A_h^2 / V_h \rangle}{\langle \sum_h V_h \rangle} \sim \frac{\langle \int \sqrt{g}\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \rangle}{\langle \int \sqrt{g} \rangle}, \tag{3.2}$$

and as usual the sum over hinges h represents a sum over the triangles in the simplicial lattice. In four dimensions it is known that for sufficiently large higher-derivative coupling there is a continuous transition between the "smooth" (small negative average curvature) and "rough" (very large positive average curvature) phase of space-time [1,2]. The average curvature can be used (at least on the four-torus) as an order parameter to distinguish between the two phases; other possible order parameters will be discussed later. In addition one can define the average scaled volume per site

$$\mathcal{V}(\lambda, k, a) \equiv [\langle l^2 \rangle]^{-2} \frac{\langle \sum_h V_h \rangle}{N_0}. \tag{3.3}$$

If all the lattice simplices collapse into lower-dimensional objects, then $\mathcal{V}$ will become very small.

Besides $\mathcal{R}$ and $\mathcal{R}^2$, it is useful to consider the lattice analog of the fluctuation in the local curvature

$$\chi_{\mathcal{R}}(\lambda,k,a) \equiv \frac{1}{\langle \sum_h V_h \rangle} \left[\langle (2\sum_h \delta_h A_h)^2 \rangle - \langle 2\sum_h \delta_h A_h \rangle^2 \right] \tag{3.4}$$

and of the fluctuation in the local volume

$$\chi_V(\lambda,k,a) \equiv \frac{1}{\langle \sum_h V_h \rangle} \left[\langle (\sum_h V_h)^2 \rangle - \langle \sum_h V_h \rangle^2 \right]. \tag{3.5}$$

As discussed in ref. [1], the divergence in the fluctuation is indicative of long-range correlations (a massless particle), since the fluctuations are as usual related to the zero-momentum component of the propagator. In particular fluctuations in the curvature are sensitive to the presence of a spin-two massless particle, while fluctuations in the volume probe only the correlations in the scalar channel. Thus in the case of gravity a dramatic difference is expected in the two type of correlations.

The derivative of the average curvature is then simply related to the fluctuations in the curvature, since one has, from the definition of $\mathcal{R}$ in eq. (3.1),

$$\frac{\partial \mathcal{R}}{\partial k} \sim \frac{2\langle l^2 \rangle}{\langle V \rangle} \frac{\partial \langle \delta\, A \rangle}{\partial k}. \tag{3.6}$$

Only the second term on the r.h.s. is divergent as k approaches k_c, and this is due to a divergence in $\partial\langle \delta A \rangle / \partial k$ (the other derivatives can be shown to remain quite small in comparison). Similarly, the fluctuations in the total average action

$$\langle I \rangle = \lambda \langle V \rangle - k \langle \delta\, l \rangle + a \langle \delta^2\, l^2 / V_l^2 \rangle \tag{3.7}$$

are dominated by the fluctuations in the curvature as well.

Figs. 1 to 4 show the local distribution of edge lengths, volumes and curvatures throughout the lattice, as obtained for a system of size 16^4 (in this particular case with action parameters $\lambda = 1$, $k = 0.244 \approx k_c$ and $a = 0.005$). As can be seen, the distributions are rather smooth and well behaved, at least up to and including the critical point at k_c, to be discussed below. It is a legitimate question to ask to what extent the distributions deviate from a simple form such as a gaussian. In the case of the edge length probability distribution (see fig. 1), one notices that it is essentially zero for small edge lengths, a reflection of the presence of the triangle inequalities. Not unexpectedly, the measure controls the behavior at small edge lengths, while the cosmological constant term provides an exponential cutoff at large edge lengths. The shape of the distribution is well approximated by the function

$$P(l) \sim A\, l^{\alpha} \exp(-b\, |l - l_0|^{\beta}). \tag{3.8}$$

For small l the power dominates and one finds a steep rise, $\alpha \approx 6.5$, while for large l the decay is exponential with $\beta \approx 3.2$ (which is close to the naive

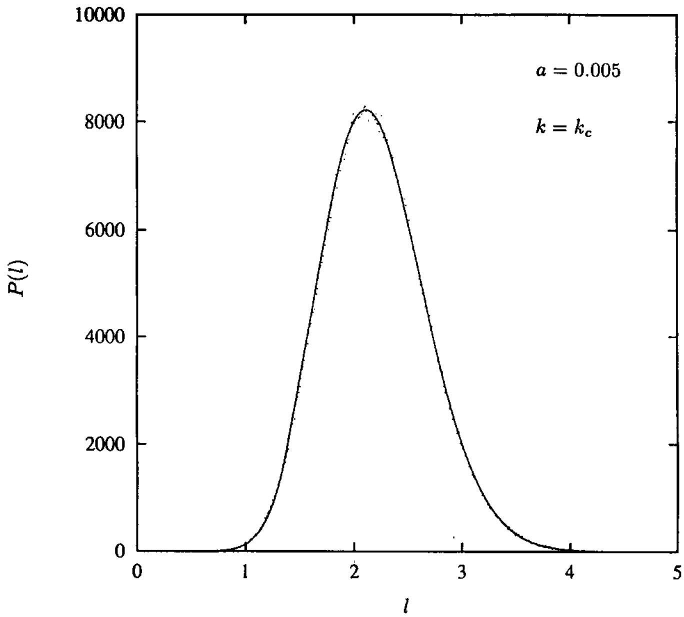

Fig. 1. Histogram of the distribution of edge lengths, $P(l)$, on a 16^4 lattice, for $\lambda = 1$, $k = 0.244 \approx k_c$ and $a = 0.005$ (DeWitt dl^2 measure).

expectation of 4). Similar results are found for the average volume per edge (see fig. 2), $V_l \sim \sqrt{g}(x)$,

$$P(V_l) \sim A\ (V_l)^{\alpha} \exp\left(-b\ |V_l - V_0|^{\beta}\right) \tag{3.9}$$

and one finds $\alpha \approx 3.6$, while for large l the decay is exponential with $\beta \approx 2.5$ (which is somewhat larger than, but still consistent with, the naive expectation of 1). In both cases the location of the peak in the distribution is only weakly dependent on k ($\leqslant k_c$) and a ($> a_0$), a reflection of the fact that the shape of the distribution is mostly affected by the measure and the cosmological constant term (which here and in the following will not be changed), and are quite insensitive for example to the bare Newton's constant. It is clear that the former two entities, and not the latter, that set the scale for the fundamental length scale in the problem, the average edge length $l_0 = \sqrt{\langle l^2 \rangle}$.

In the case of the curvature distribution (figs. 3 and 4), there are clear deviations from gaussian fluctuations, and one has for $\delta_h A_h \sim \sqrt{g}R(x)$

$$P(\delta_h A_) \sim A\ |\delta_h A_h|^{\alpha} \exp\left(-b\ |\delta A_h - r_0|^{\beta}\right) \tag{3.10}$$

with a negligible power contribution ($\alpha \approx 0$), and $\beta \approx 1.4$. In the case of the curvature, the constant r_0 vanishes at the critical point ($k = k_c$). The deviations from a gaussian distributions are seen even more clearly when one considers the log of the probability distribution (see fig. 4). Since the power is close to one,

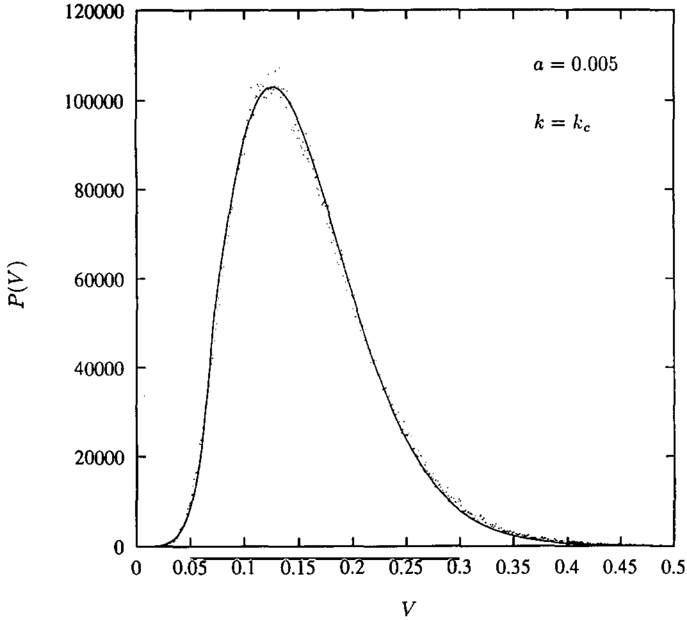

Fig. 2. Histogram of the distribution of local volumes per hinge, $P(V_h)$, for the same parameters as in fig. 1.

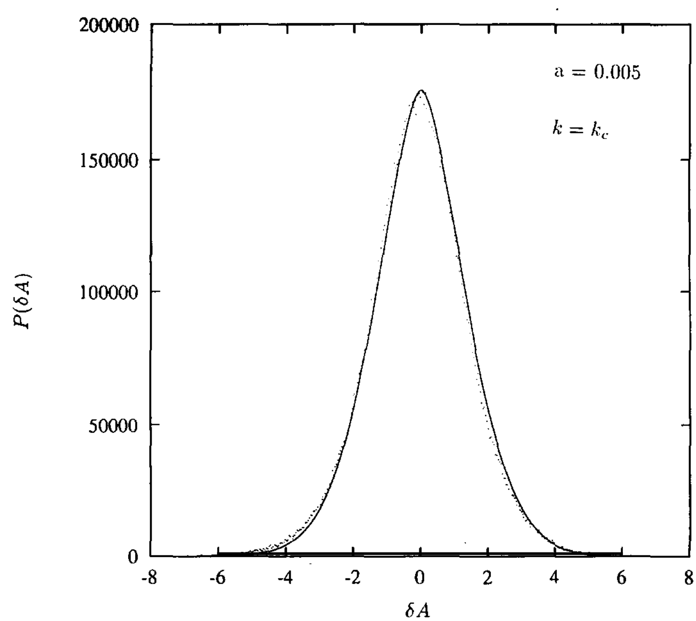

Fig. 3. Histogram of the distribution of local curvatures, $P(A_h \delta_h)$, for the same parameters as in fig. 1.

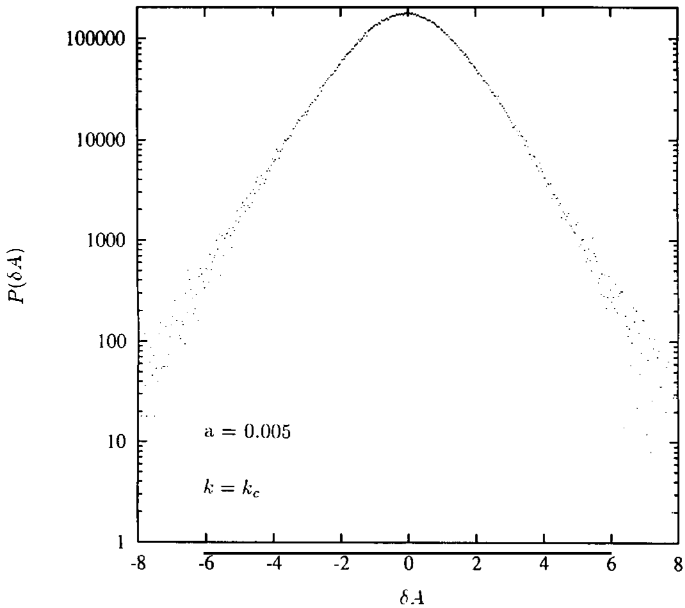

Fig. 4. Same as fig. 3, but using a vertical logarithmic scale in order to show the deviations from gaussian behavior.

$\ln P(\delta A)$ exhibits an almost linear behavior for large $|\delta A|$. (Had one chosen to look at the distribution of $\delta_h A_h / V_h \sim R(x)$, the results would be rather similar. One finds in this case $\beta \approx 1.2$, compatible within errors with the exponent for the $\delta_h A_h$ distribution). Our results at this point are not accurate enough to suggest whether there might actually be a non-analyticity in the curvature distribution at zero curvature, in the sense that the above exponents are expected to describe more accurately the asymptotic falloff of the distribution when the curvatures are large in magnitude. In any case such a singularity would be rather unusual.

Close to a critical point it is possible that some of the local averages, as well as their fluctuations, will develop a singularity in k in the infinite-volume limit. This is certainly true close to two space-time dimensions. In the $2 + \epsilon$ expansion of Einstein's gravity one sets $k^{-1} = 8\pi G$ and performs a double expansion in G and ϵ [8]. The dimensionful bare coupling is written as $G_0 = \Lambda^{2-d} G$, where Λ is an ultraviolet cutoff, for example of the order of the inverse average lattice spacing, $\Lambda \sim \pi / \langle l^2 \rangle^{1/2}$, and G a dimensionless bare coupling constant. Close to two dimensions one finds for the beta function

$$\beta(G) \equiv \frac{\partial G}{\partial \log \Lambda} = \epsilon\, G - \beta_0 G^2 + \mathrm{O}(G^3, G^2\epsilon, G\epsilon^2), \tag{3.11}$$

with $\beta_0 = \frac{2}{3}(25 - D)$, where D is the number of massless scalar fields (for pure gravity $D = 0$). To lowest order the ultraviolet fixed point is at

$$G^* = \frac{\epsilon}{\beta_0} + \mathrm{O}(\epsilon^2) \tag{3.12}$$

Integrating eq. (3.11) close to the non-trivial fixed point in $2+\epsilon$ dimensions one obtains

$$\mu_0 = \Lambda \exp\left(-\int^G \frac{\mathrm{d}G'}{\beta(G')}\right) \underset{G\to G^*}{\sim} \Lambda \mid G - G^* \mid^{-1/\beta'(G^*)} \sim \Lambda \mid G - G^* \mid^{1/\epsilon}, \tag{3.13}$$

where μ_0 is an arbitrary integration constant with the dimension of a mass, and the derivative of the beta function at the fixed point is

$$\beta'(G^*) = -\epsilon = -1/\nu. \tag{3.14}$$

The possibility of algebraic singularities in some of the vacuum expectation values (like the average curvature and its derivatives) close to the fixed point is then a natural one, at least from the point of view of the $2+\epsilon$ expansion.

This result also illustrates how in principle the lattice continuum limit is to be taken: it corresponds to $\Lambda \to \infty$, $G \to G^*$ with μ_0 held constant; for fixed lattice cutoff the continuum limit is approached by tuning G to G^*. Away from G^* one will in general expect to encounter some lattice artifacts, which reflect the non-uniqueness of the lattice transcription of the continuum action and measure, as well as its reduced symmetry properties. In the present model things are complicated further by the fact that we also have a second, higher derivative, coupling a, which is presumably asymptotically free, and should lead to non-trivial scaling properties for large a [15].

3.2. CURVATURE CRITICAL EXPONENT

The results obtained for the average curvature $\mathcal{R}(k)$, defined in eq. (3.1), are shown in figs. 5–11. Four values of a, 0, 0.005, 0.02 and 0.1, have been studied, and $\lambda = 1$ was held fixed (since λ only sets the overall scale in the action). Due to the long runs and the large lattices employed the results have relatively small uncertainties. Since different edge length starting configurations are used for different k's, one has that for different values of k the results for the curvature are completely statistically uncorrelated. In spite of the fact that the lattice volume is not very large, the statistical errors are quite small since there is a rather large number of hinges (triangles) per site, namely $50L^4$, where L is the number of sites in each lattice "direction". The statistical errors in $\mathcal{R}(k)$ are estimated by the usual binning procedure, and represent one standard deviation.

In refs. [1,2] it was found that as k is varied, the curvature is negative for sufficiently small k ("smooth" phase), and appears to go to zero continuously at some finite value k_c. For $k \geqslant k_c$ the curvature becomes very large, and

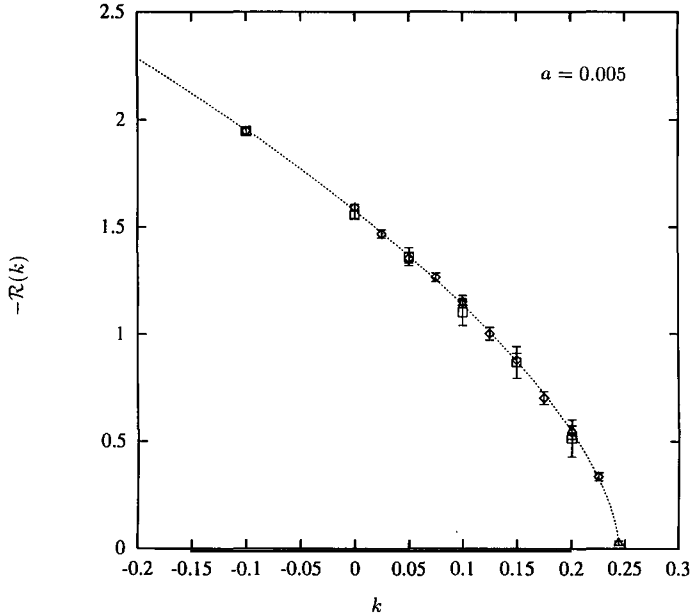

Fig. 5. Average curvature $\mathcal{R}$ as a function of k, for $\lambda = 1$ and $a = 0.005$ (dl^2 measure). The squares refer to $L = 4$, the diamonds to $L = 8$, and the triangles to $L = 16$. The dotted line represents the fit to an algebraic singularity.

the simplices tend to collapse into degenerate configurations with very small volumes ($\langle V\rangle/\langle l^2\rangle^2 \sim 0$). This "rough" or "collapsed" phase is the region of the usual weak-field expansion ($G \to 0$); in the continuum it is characterized by the unbounded fluctuations in the conformal mode (see discussion below). For k close to, but less than, k_c one writes for the average curvature

$$\mathcal{R}(k,a) \underset{k\to k_c(a)}{\sim} -A_{\mathcal{R}}(a)\ (k_c(a)-k)^{\delta} \tag{3.15}$$

and average curvature fluctuation

$$\chi_{\mathcal{R}}(k,a) \underset{k\to k_c(a)}{\sim} A_{\chi}(a)\ (k_c(a)-k)^{\delta-1}, \tag{3.16}$$

where δ is the universal curvature critical exponent (introduced in ref. [1]), characteristic of the gravitational transition. In the case of higher-derivative coupling $a = 0.005$ and $a = 0.02$ (see figs. 5 and 7) the interpretation of the data is quite straightforward: the assumption of an algebraic singularity is well supported by the results. After performing a simultaneous fit to $\mathcal{R}(k)$ in $A_{\mathcal{R}}$, k_c and the exponent δ, and using close to k_c the data on the largest lattice available, one finds the results summarized in table 1. It is quite remarkable that the two estimates for δ agree very well with each other, in spite of the fact that the curvature changes by about a factor of five between the two values of a.

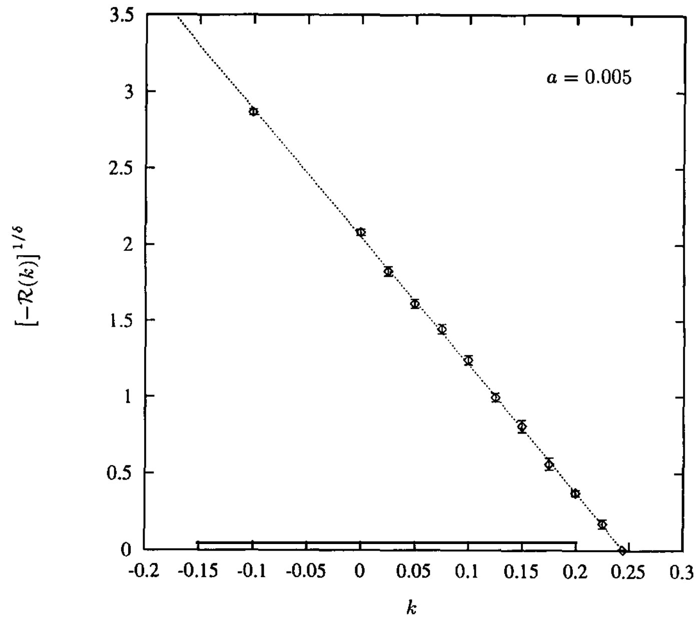

Fig. 6. Average curvature $\mathcal{R}$ raised to the power $1/\delta \approx 1/0.626$, using the data on the largest lattice available ($L = 8$ and 16) for $a = 0.005$; other parameters are the same as in fig. 5. The linearity is quite striking.

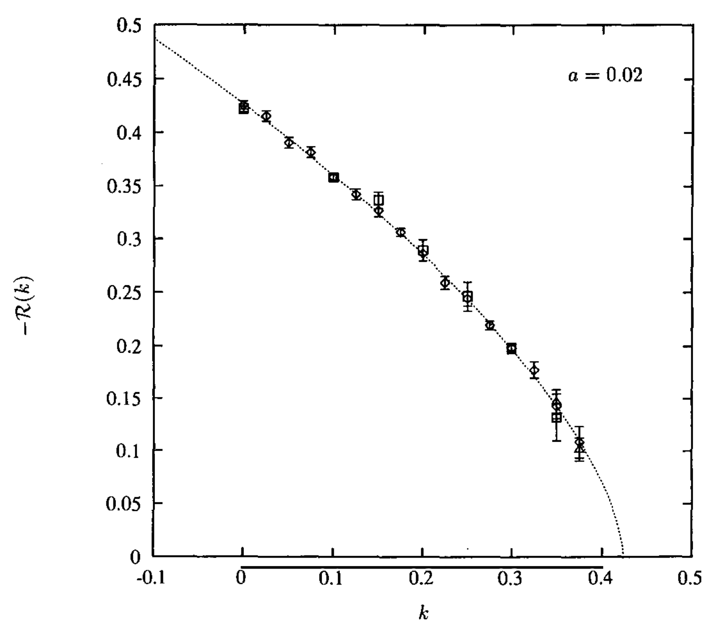

Fig. 7. Same as fig. 5, but for $a = 0.02$.

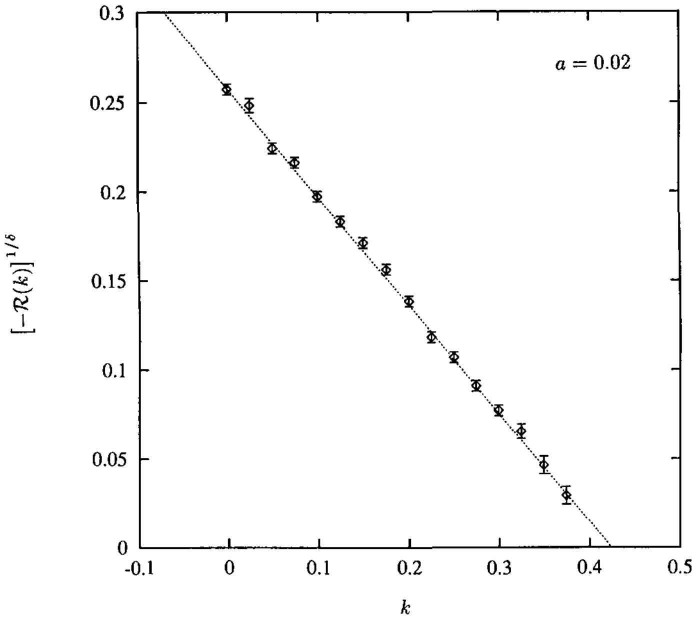

Fig. 8. Same as fig. 6, but for $a = 0.02$.

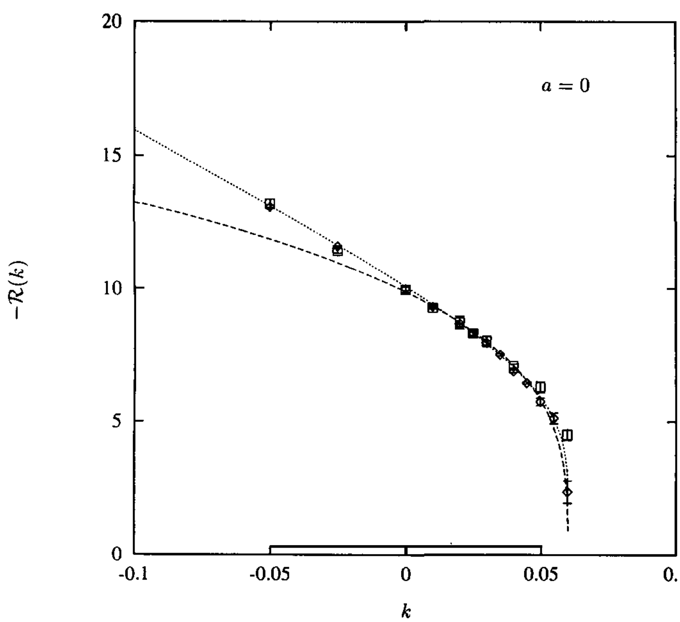

Fig. 9. Same as fig. 5, but for $a = 0$. The dashed line represents a fit to an algebraic singularity with a small power ($\delta \approx 0.30$), while the dotted line corresponds to a purely logarithmic singularity (see eq. (3.23)).

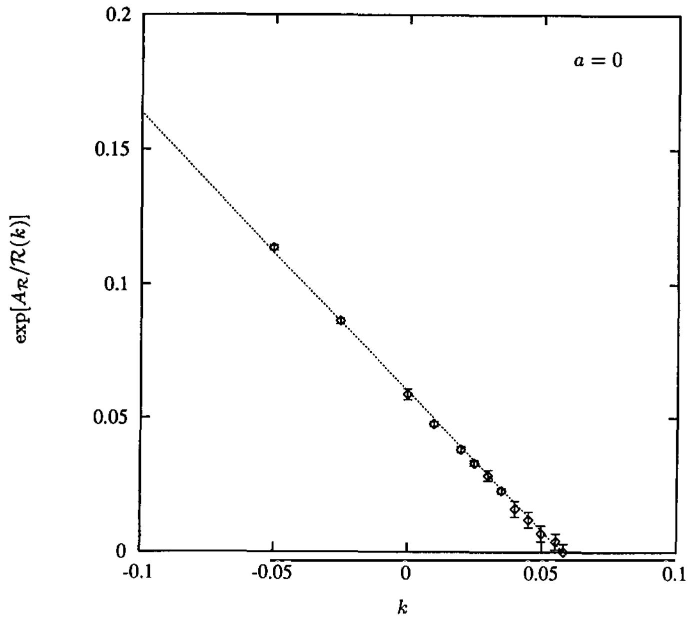

Fig. 10. The quantity $\exp[A_{\mathcal{R}}/\mathcal{R}(k)]$, using the data of fig. 9 on the largest lattice available ($L = 8$) for $a = 0$. Behaviour is close to linear for this particular combination, in agreement with the assumption of a logarithmic singularity in k for $a = 0$.

TABLE 1

Estimates of the critical amplitude $A_{\mathcal{R}}$, the critical point k_c and the critical exponent δ, for different values of a.

a	L	$A_{\mathcal{R}}$	k_c	δ	χ^2/d.o.f
0.000	4 – 8	22.93(16)	0.060(2)	0.30(3)	2.40
0.005	4 – 16	3.794(41)	0.2443(11)	0.624(6)	0.34
0.020	4 – 16	0.732(10)	0.4244(36)	0.628(20)	0.29
0.100	4 – 8	0.065(6)	1.17(4)	0.76(12)	1.32

Furthermore one would expect that the data for $[-\mathcal{R}(k)]^{1/\delta}$ should lie close to a straight line. This appears indeed to be the case, as shown in figs. 7 and 9, and the data is strikingly close to a straight line over a wide range of k values, providing further support for the assumption of an algebraic singularity.

For $a = 0.1$ the statistical accuracy is much lower than in the previous two cases, and the results are presented only for comparison. In this case the average curvature is quite small, even away from k_c, and finite-volume effects are starting to become important. In general one would expect that in order to obtain results relevant for the continuum limit (in the sense that the finite volume corrections

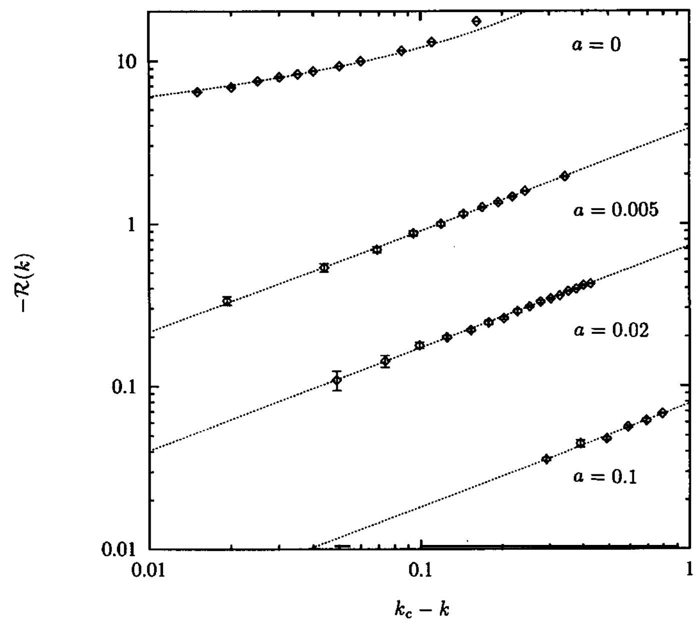

Fig. 11. Average curvature $\mathcal{R}$ as a function of $k_c - k$ on a log–log scale, from $a = 0$ (top) to $a = 0.005$, $a = 0.002$ and $a = 0.1$ (bottom). The data is the same as in figs. 5–10, but presented on a different scale. The dotted line at the top corresponds to a logarithmic singularity, while the remaining three straight lines correspond to $\delta = 0.626$. As before, $\lambda = 1$ and the measure is dl^2.

should be small), the scale of the curvature should be much smaller than the average lattice spacing, but much larger than the size of the system,

$$\frac{1}{\langle l^2 \rangle} \gg \left| \frac{\mathcal{R}(k,a)}{\langle l^2 \rangle} \right| \gg \frac{1}{\langle l^2 \rangle L^2} \tag{3.17}$$

(here L is a characteristic linear size of the system, in units of the average edge length, $L = V^{1/4}$). Stated equivalently, in momentum space the physical scales should be much smaller that the ultraviolet cutoff, but much larger than the infrared one.

Finally in fig. 10 the average curvatures are shown on a log–log scale. Even though the curvature is changing by as much as an order of magnitude, the results at $a = 0.005$, $a = 0.02$ and, to some extent, even at $a = 0.1$, are consistent with one universal critical exponents characterizing the transition. An average of the lattice results so far then gives the estimate

$$\delta = 0.626(25), \tag{3.18}$$

in good agreement with previous estimates on smaller lattices (up to 8^4) $\delta = 0.62(5)$ [1]. In table 1 we summarize the results obtained so far under the assumption of an algebraic singularity in $\mathcal{R}(k)$.

The results for the average curvature $\mathcal{R}$ are not inconsistent with known results within the weak-field expansion in the continuum (at least for small a). Substituting $k^{-1} = 8\pi G_0$, where G_0 is the dimensionful bare Newton's constant, setting $k_c = c\Lambda^2$, where c is a constant independent of k, and Λ the ultraviolet cutoff (here of the order of the average inverse lattice spacing, $\sim \langle l^2 \rangle^{-1/2}$), one obtains from eq. (3.15)

$$\begin{aligned} \mathcal{R}(G_0) &\sim -A_{\mathcal{R}} \left(\frac{-1}{8\pi G_0}\right)^{\delta} \left[1 - c\Lambda^2 8\pi G_0\right]^{\delta} \\ &\sim -A_{\mathcal{R}} \left(\frac{-1}{8\pi G_0}\right)^{\delta} \left[1 + \delta\, c\Lambda^2(-8\pi G_0) \right. \\ &\qquad \left. + \frac{\delta(\delta-1)}{2} (c\Lambda^2)^2 (-8\pi G_0)^2 + \cdots \right], \end{aligned} \tag{3.19}$$

so perhaps $\mathcal{R}(G_0)$ is possibly not even analytic at $G_0 = 0$. Furthermore an expansion in powers of G_0 involves increasingly higher powers of the ultraviolet cutoff Λ, as expected from a theory which is not perturbatively renormalizable in G_0 [8].

We note that the vacuum expectation value of the curvature can be used as a possible definition of the effective, long-distance cosmological constant (or equivalently as the definition for a length scale R_0 associated with some large average curvature radius),

$$\mathcal{R}(\lambda, k, a) \sim \frac{\langle \int \sqrt{g}\, R \rangle}{\langle \int \sqrt{g} \rangle} \sim \left(\frac{4\lambda}{k}\right)_{\text{eff}}. \tag{3.20}$$

As one approaches the fixed point at k_c, $(\lambda/k)_{\text{eff}} \to 0$ and this length scale becomes very large. If the system is of finite extent, with linear dimensions $L = V^{1/4}$, then the scaling laws for $\mathcal{R}$ give the volume dependence of the effective cosmological constant at the fixed point,

$$\left(\frac{k}{\lambda}\right)_{\text{eff}}(L) - \left(\frac{k}{\lambda}\right)(l_0) \underset{L \gg l_0}{\sim} \left(\frac{L}{l_0}\right)^{\delta/\nu}. \tag{3.21}$$

Here $\delta/\nu \approx 1.54$, and $(k/\lambda)(l_0)$ is a ratio of bare coupling constant, at the scale of the average lattice spacing $l_0 = \sqrt{\langle l^2 \rangle}$. (The volume dependence of the results is a topic by itself, and will be discussed more in detail below). Therefore at the fixed point the effective cosmological constant, which is of order one at scales of the order of the cutoff, relaxes to zero as the overall volume is increased. Phrased differently, the curvature is large in magnitude at short distances, and fluctuates wildly, but its average becomes very small if large regions of space-time are considered.

For $a = 0$, corresponding to the pure Regge action with no explicit higher-derivative lattice contribution, however the situation appears not to be as clear. First of all the path integral is still well defined (at least for sufficiently small

$|k|$), since the deficit angles are bounded, and the edge lengths fluctuate around some average value, which is determined by the interplay of the measure and the cosmological constant term. Alternatively, one can think of the fluctuations in the conformal mode as becoming bounded (again at least for sufficiently small $|k|$) when a momentum cutoff of order $\pi/\sqrt{\langle l^2 \rangle}$ is dynamically generated. But in any case the assumption of an algebraic singularity for the average curvature leads to a value for the curvature exponent which is much smaller than the preceding estimates, $\delta \approx 0.30(4)$. Since it is difficult to distinguish a small power from a logarithm, this suggests the ansatz

$$\mathcal{R}(k,a) \underset{k \to k_c(a)}{\sim} -A_{\mathcal{R}}(a)\ (k_c(a)-k)^{\delta}\ \left[-\ln(k_c(a)-k)\right]^{-\bar{\delta}}, \qquad (3.22)$$

and one then finds $\delta = 0.01(2)$ and $\bar{\delta} = 1.05(6)$, with a lower chi-squared. The smallness of the new δ would seem to suggest that the singularity in the average curvature for $a = 0$ is in fact purely logarithmic,

$$\mathcal{R}(k,a) \underset{k \to k_c(a)}{\sim} -A_{\mathcal{R}}(a)\ \left[-\ln(k_c(a)-k)\right]^{-1}, \qquad (3.23)$$

corresponding to $\delta = 0$ and $\bar{\delta} = 1$ in eq. (3.22). One finds in this case $A_{\mathcal{R}} = 28.3(5)$ and $k_c = 0.059(2)$, which is close to the algebraic singularity result of table 1. Fig. 9 shows the curvature data with both the pure power ($\delta = 0.30$) and the pure logarithmic fit; the latter one is significantly better. Fig. 10 displays the combination $\exp[A_{\mathcal{R}}/\mathcal{R}(k)]$, which shows indeed close to linear behavior, in agreement with the suggestion of a purely logarithmic singularity. A more conservative conclusion about the $a = 0$ case would be that the exponents are certainly different from the previous cases (by several standard deviations), and that the results are at least suggestive of a logarithmic singularity ($\delta \approx 0$).

From the analysis of the curvature fluctuation $\chi_{\mathcal{R}}(k)$ (defined in eq. (3.4)) one obtains similar values for δ and k_c, but with somewhat larger errors, since the fluctuations are more difficult to compute accurately than the averages. (Alternatively, one can calculate the fluctuations as derivatives of the averages, but this procedure would lead to estimates similar to the ones obtained before from the curvature, since it is the same data that is being analyzed in a different way; the errors are larger since a derivative of numerical data has to be taken first). The results for the fluctuations computed directly are shown in figs. 12–14. If one tries to fit the curvature fluctuation to an algebraic singularity, as in eq. (3.16), the results for the exponents turn out to be completely consistent with the previous estimate for δ, but have a larger error. Therefore we only show in the figures the fits obtained if we assume the values for δ obtained previously from the average curvature ($\delta \approx 0.626$ for $a = 0.005$ and 0.02). On the other hand for $a = 0$ the previous discussion suggests that the singularity in $\mathcal{R}(k)$ is

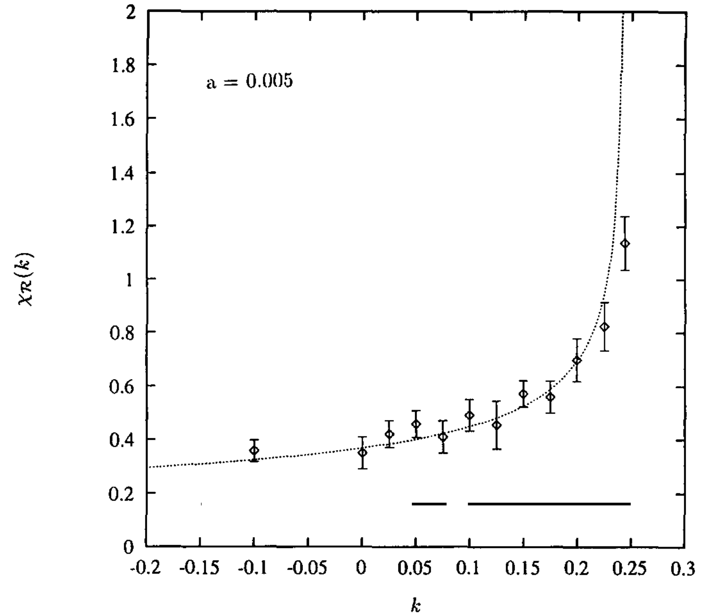

Fig. 12. Curvature fluctuation $\chi_{\mathcal{R}}$ measured directly, as a function of k and for $a = 0.005$. The line is a fit to the data assuming an algebraic singularity, with the exponent determined from $\mathcal{R}(k)$, namely $1 - \delta \approx 0.374$.

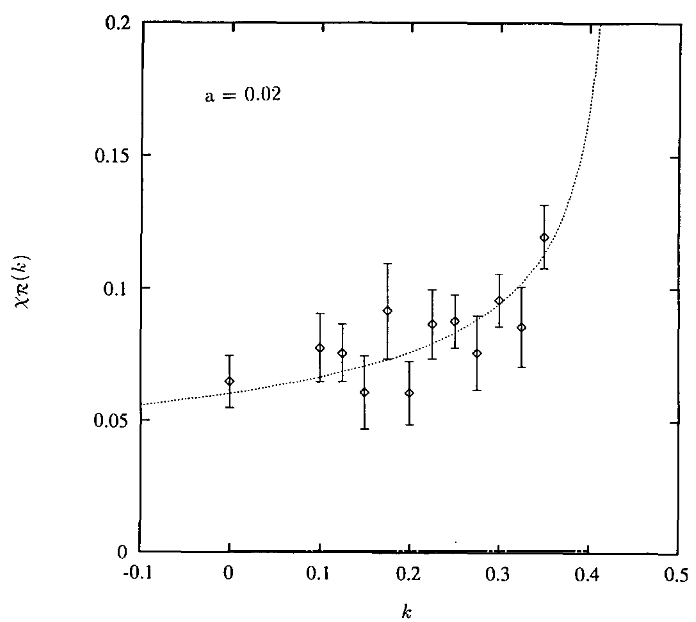

Fig. 13. Same as in fig. 12, but now for $a = 0.02$.

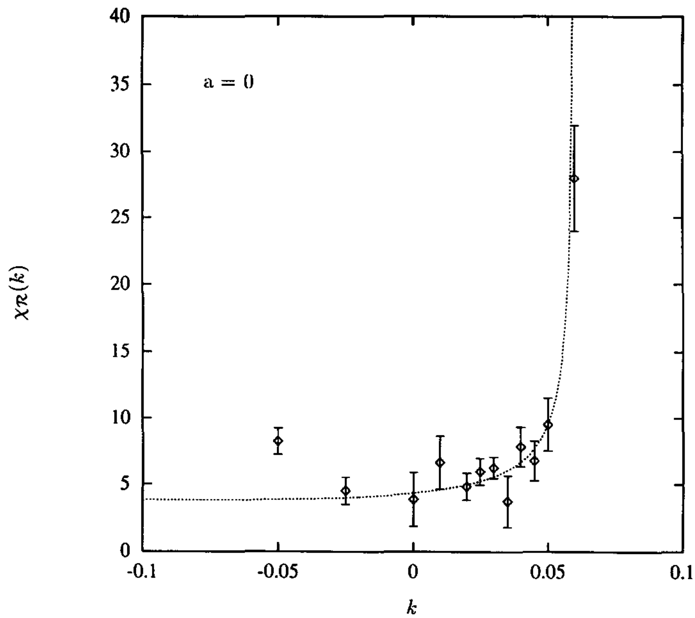

Fig. 14. Same as in fig. 12, but now for $a = 0$. The line represents a fit assuming a purely logarithmic singularity in $\mathcal{R}(k)$ (see eqs. (3.23) and (3.24) in the text).

close to being purely logarithmic; for the curvature fluctuation this would imply

$$\chi_{\mathcal{R}}(k,a) \underset{k\to k_c(a)}{\sim} A_\chi(a)\;(k_c(a)-k)^{-1}\left[-\ln(k_c(a)-k)\right]^{-2}, \tag{3.24}$$

which is certainly quite consistent with the numerical results (see fig. 14).

Close to the transition at k_c the average volume per site V_s, expressed in units of the average lattice spacing $l_0 = \sqrt{\langle l^2\rangle}$, shows no appreciable singularity when the critical point is approached from the smooth phase ($k < k_c$), as can be seen from fig. 15. On the other hand in the rough phase ($k > k_c$) the volume per site seems to approach smaller and smaller values as the lengths of the runs are extended. In fact it would seem that in the rough phase the volume per site can be made to approach zero, at least for some simplices. One could therefore alternatively refer to this phase as the "collapsed" phase. Furthermore the relaxation times become very long, with the system getting stuck in some configurations without being able to get out of it again. Also, it is difficult to see how the collapse of the simplices could be averted by choosing a different lattice structure (for example a random lattice), since its properties seem to be unaffected by changes in the measure or the action, at least to the extent we have investigated them [2,9,17]. Indeed the collapsed phase appears even in the simplest models based on a regular tessellation of the four-sphere [2,4]. Of course the existence of such a diseased phase is not completely unexpected, and seems

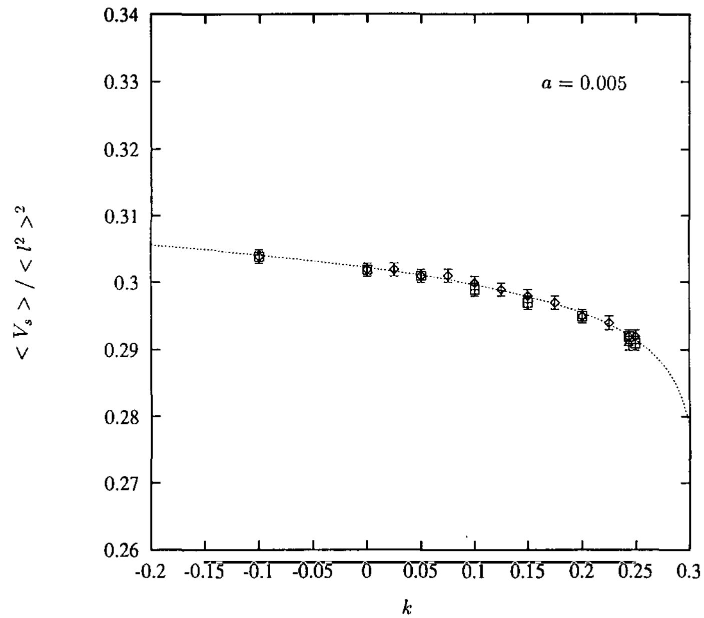

Fig. 15. Average volume per site, in units of the average edge length $\sqrt{\langle l^2 \rangle}$, as a function of k and for $a = 0.005$. The line is a polynomial fit to the data. There is no sign of a singularity for $k \leqslant k_c \approx 0.244$.

to be a reflection of the unbounded fluctuations in the conformal mode expected for sufficiently large k. Indeed unbounded fluctuations in the conformal mode in the continuum correspond to rapid fluctuations in the simplicial volumes, and this is what is observed on the lattice for $k > k_c$, namely a rapid variation of simplicial volumes when going from one simplex to a neighboring one (this does not happen in the smooth phase). This phase is somewhat reminiscent of the collapsed phase found in two-dimensional gravity in the Regge model (as well as in the DTRS model) for sufficiently large $D > 1$ [9], and which corresponds physically to branched polymers or trees. The analogy will become clearer below when we discuss the fractal dimension in this phase (which was found to be less than four [1]).

If one computes the volume susceptibility χ_V (see eq. (3.5) and fig. 16), one finds quite clearly that it approaches a finite value at k_c, suggesting the absence of critical volume fluctuations, for all values of a investigated. This situation should be contrasted with the two-dimensional case, where volume fluctuations (corresponding to the Liouville mode) are found to be massless, as expected from continuum arguments, and is more similar to the three-dimensional case, discussed in ref. [17]. Such a result is not unexpected in the case of gravity, since fluctuations in the volume correspond in the continuum to fluctuations in

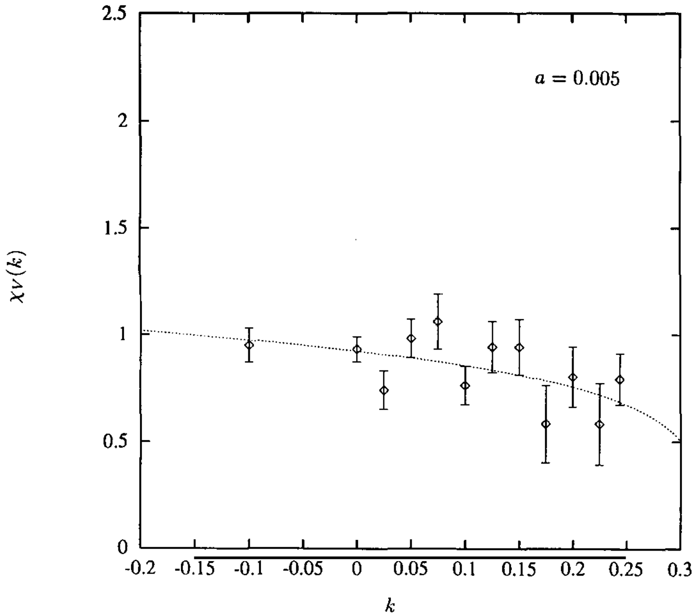

Fig. 16. Volume fluctuation χ_V measured directly, as a function of k and for $a = 0.005$. The line is a fit to the data assuming an algebraic singularity. There is no sign of divergence for $k < k_c \approx 0.244$; the volume fluctuations approach a constant at the critical point.

the square root of the determinant of the metric tensor, which couples to scalar modes only (and therefore not to the graviton). Up to now we have seen no sign of excitations that could be associated with a scalar particle. Critical fluctuations in the curvature at the critical point, accompanied by the lack of any sign of critical fluctuations in the volume, seem consistent with the appearance of a massless graviton.

3.3. VOLUME DEPENDENCE

Since the critical exponents play such a central role in determining the existence and nature of the continuum limit, it appears desirable to have an independent way of estimating them, which either does not depend on any fitting procedure, or at least analyzes a completely different set of data. By studying the dependence of averages on the physical size of the system, one can independently estimate the critical exponents. Finite size scaling tells us that at the critical point $k = k_c$ the coherence length saturates at a value comparable to the linear system size, $\xi = m^{-1} \sim L$. Thus if we set $m \sim (k_c - k)^{\nu}$, we should obtain $\mathcal{R} \sim (k_c - k)^{\delta} \sim L^{-\delta/\nu}$. Similar arguments can then be repeated for other observables. As discussed previously, the dependence of the results for the average

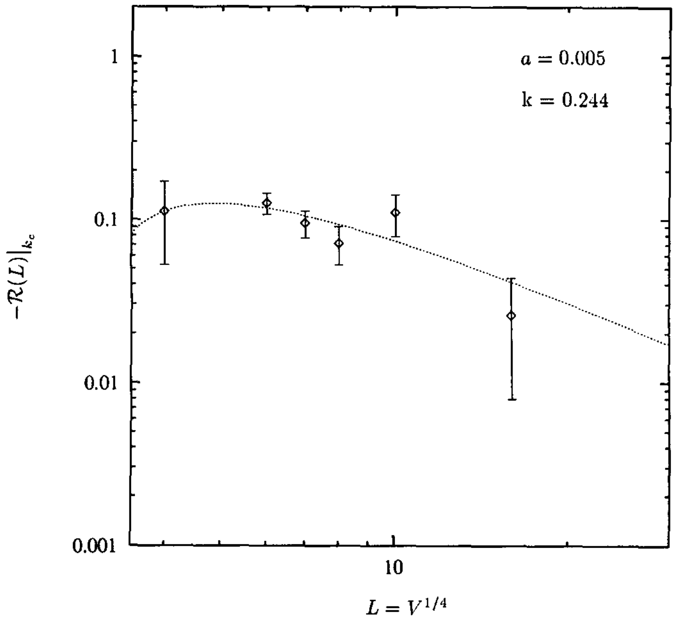

Fig. 17. Volume dependence of the average curvature at the critical point, $\mathcal{R}(k_c)$, for $a = 0.005$ and $k_c = 0.244$. For large L the slope of the line corresponds to $\delta = 0.626$, as obtained from the curvature (figs. 5 and 7).

curvature on the size of the system is quite small except close to the critical point, where the correlation length becomes comparable to the linear system size. A more careful analysis in the vicinity of the critical point for $a = 0.005$ shows that the average curvature at $k_c \approx 0.244$ decreases as the system size is increased (see fig. 17). In the figure the data points correspond to $L = 4, 6, 7, 8, 10, 16$; we prefer not to use results for smaller lattices since the higher-derivative term (proportional to a) contains next-nearest neighbor couplings. The lengths of the runs varied between about $60k$ iterations on the $L = 4$ lattice to $0.6k$ on the $L = 16$ lattice; the largest lattices were obtained originally by duplication from the smaller previously thermalized lattices. If we describe the decrease of the average curvature $\mathcal{R}$ at k_c as a function of $L = V^{1/4}$ (here $V = N_0$ is simply the number of sites in the system) by a critical exponent σ,

$$\mathcal{R}(L)\big|_{k_c} \underset{L\to\infty}{\sim} -(c_1 + c_2/L)\, L^{-\sigma}, \tag{3.25}$$

then we find $c_1 \approx 3.3$, $c_2 \approx -9.4$, and $\sigma \approx 1.55(18)$. Ideally one would have hoped for a straight line in fig. 17, but this cannot be expected for such small systems, and the regular pre-factor has to be included to account for the transients. Standard scaling arguments at a second order phase transition would suggest that σ is related to δ by $\sigma = \delta d/(1+\delta)$, which gives $\delta = 0.63(11)$, indeed

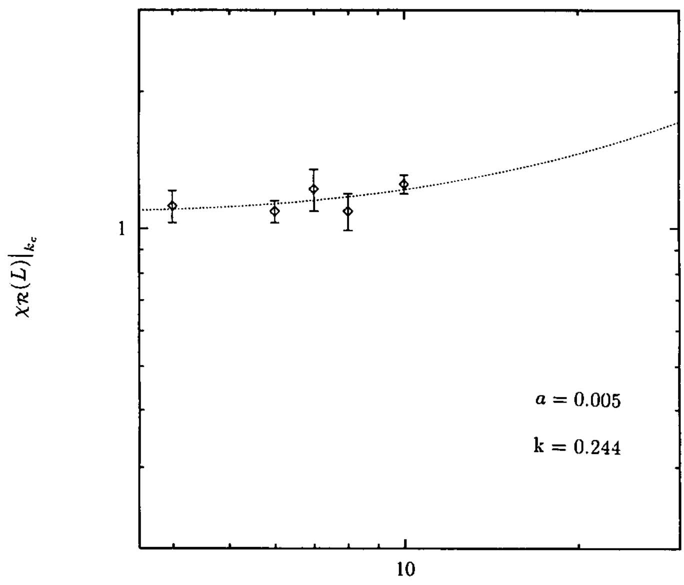

Fig. 18. Volume dependence of the curvature fluctuation at the critical point, $\chi_{\mathcal{R}}(k_c)$, again for $a = 0.005$ and $k_c = 0.244$. The lines indicate the expected slope corresponding to $1 - \delta = 0.374$, as obtained from the curvature (figs. 5 and 7).

completely consistent with the previous determination of δ.

A similar analysis can be performed for the curvature fluctuation. For a finite system one does not expect any real divergence, but rather a peak that grows sharper as the lattice volume is increased. As the infinite-volume limit is taken, the height of the peak should grow like some power of the lattice volume. Thus one expects again

$$\chi_{\mathcal{R}}(L)\big|_{k_c} \underset{L\to\infty}{\sim} (c_1' + c_2'/L)\, L^{\sigma'} . \tag{3.26}$$

The numerical results are shown in fig. 18. Again there is some curvature in the data, and we see indications of some transients.The same type of transients were also found in three dimensions, where the estimates for δ from fits to $\mathcal{R}(k)$ had much smaller errors than the finite size scaling estimates [17]. Here one finds $c_1' \approx 0.16$, $c_2' \approx 1.11$, and $\sigma' = 0.64(11)$. Again standard scaling arguments at a continuous phase transition suggest that σ' is also connected to δ by the relation $\sigma' = \alpha/\nu = d(1-\delta)/(1+\delta)$, which would give $\delta = 0.72(4)$, which is quite consistent with the estimate $\delta \approx 0.63$ obtained previously, given the uncertainties of the method. Furthermore one would expect $\sigma' + 2\sigma = d = 4$, and we obtain for the sum 3.74(36) which is adequate. It is interesting to note that if the transitions had been first order, with a finite correlation length at the

TABLE 2
Summary of the results for the critical exponents of pure four-dimensional simplicial quantum gravity

Method	Observable	a	Measure	δ	ν	α/ν
Fit	$\mathcal{R}(k)$	0	$(\mathrm{d}l^2)$	0.01(2)	0.25(3)	4.0(2)
	$\mathcal{R}(k)$	0.005	$(\mathrm{d}l^2)$	0.62(2)	0.41(1)	0.93(6)
	$\mathcal{R}(k)$	0.02	$(\mathrm{d}l^2)$	0.63(3)	0.41(1)	0.91(7)
	$\mathcal{R}(k)$	0.1	$(\mathrm{d}l^2)$	0.76(12)	0.44(3)	0.55(33)
Finite size	$\mathcal{R}_L$	0.005	$(\mathrm{d}l^2)$	0.63(11)	0.41(3)	0.91(31)
	χ_L	0.005	$(\mathrm{d}l^2)$	0.72(4)	0.43(2)	0.65(10)
Average $(a > 0)$				0.67(8)	0.42(3)	0.79(21)
1st order				0	1/4	4
1-c scalar				1	1/2	0

critical point, one would have expected $\sigma' = d = 4$ [20], which is ruled out by many standard deviations for $a = 0.005$. For a scalar field in four dimensions, $\sigma' = 0$, which is also ruled out by our results (in particular $\sigma' = 0$ would correspond to $\delta = 1$, which is inconsistent with all the previous results).

An alternative way of determining the size dependence is via the use of a scaling function. One writes $\mathcal{R}(k,l)L^{\delta/\nu} = f((k_c - k)L^{1/\nu})$ close to k_c and for large L, where the critical point and the exponents are free parameters. Using a scaling function of the form $a + bx^c$, one finds for $a = 0.005$ $\delta = 0.64(3)$ in good agreement with the previous determinations. Finally it is expected that the value of k_c itself should depend on the size of the system. Indeed such a dependence is found when comparing k_c (as obtained from the algebraic singularity fits discussed previously) on different lattice sizes. One writes

$$k_c(L) \underset{L\to\infty}{\sim} k_c(\infty) + c\, L^{-1/\nu}. \tag{3.27}$$

For $a = 0.005$ and using lattice sizes in the range $L = 2 - 16$ one finds $\nu = 0.49(22)$. Again the estimate is not very accurate, but is consistent with the scaling expected at a second order critical point (from $\delta = \nu d - 1$, one gets $\delta = 0.96(88)$).

In conclusion the estimates for δ obtained from finite size scaling at $a = 0.005$ are in agreement with the results quoted previously from the fits to $\mathcal{R}(k)$ at $a = 0.005$ and $a = 0.02$. On the other hand the finite-size scaling estimates fail to be more accurate than the fit results, since there is some curvature in the data of fig. 17 and 18. The results from the fits to $\mathcal{R}(k)$ for $a = 0.005$ and $a = 0.02$ appear for now to give by far the best estimates for δ, with the smallest statistical errors and with the least systematic uncertainties. Table 2 summarizes the results for the critical exponents obtained so far.

It would seem that in the smooth phase the system develops a mass gap, since one does not observe any critical fluctuations for $k < k_c$, and the correlation length seems to be finite there. One can determine the scaling exponent for this dynamically generated mass from δ by a simple scaling argument (if the transition is continuous, as it seems to be for sufficiently large a), and one can attempt to calculate this mass scale directly by computing an invariant correlation function. If one calls the dynamically generated mass m, then one has $m \sim (k_c - k)^\nu \sim (-\mathcal{R}(k))^{\nu/\delta}$, with $\nu = (1 + \delta)/d$, and with a calculable constant of proportionality close to k_c. After restoring the correct dimensions for $\mathcal{R}$, which has dimensions of an inverse length squared, one obtains

$$\mathcal{R} \sim -c\, \Lambda^{2-\delta/\nu}\, m^{\delta/\nu}, \tag{3.28}$$

where c is a dimensionless constant dependent on the higher-derivative coupling a, and Λ the ultraviolet cutoff. The dynamically generated mass can be calculated in principle from the edge or curvature correlation functions at fixed geodesic distance (to be discussed below). Alternatively it can be extracted from the physical size dependence of averages of local operators. For example on the torus one expects in the presence of a mass gap

$$\mathcal{R}_L(k) - \mathcal{R}_\infty(k) \underset{L \gg 1/m(k)}{\sim} m(k)^{1/2}\, L^{-3/2}\, \mathrm{e}^{-m(k)L}, \tag{3.29}$$

where here $L = V^{1/4}$ is the physical "linear size" of the system, and $m(k)$ is a physical mass. If the boundary conditions are not periodic, then the finite-size corrections are not exponential in the system size, and the above formula is no longer valid. In fig. 19 we show results for the mass parameter extracted from the finite-size corrections to $\mathcal{R}(k)$ at $a = 0.005$. Not unexpectedly the errors are large, but the results are roughly consistent with the expected decrease as k_c is approached. The exponent though is presumably known, since $\nu = (1 + \delta)/4 \approx 0.41$. For k close to, but less than, k_c one can write

$$m(k,a) \underset{k \to k_c(a)}{\sim} A_m(a)\, (k_c(a) - k)^\nu, \tag{3.30}$$

and from fig. 19 one estimates $A_m \approx 1.9(3)$ (in the graph, the mass is in units of lattice spacings $(=1)$, which have to be multiplied by the average edge length $l_0 = \sqrt{\langle l^2 \rangle} \approx 2.25$ in order to obtain "physical" distances).

The dimensionless ratio of mass squared to curvature is then given simply in terms of the average curvature and the ultraviolet cutoff, with an exponent related to δ,

$$\frac{m^2}{\mathcal{R}} \sim -\left(-\frac{\mathcal{R}}{\Lambda^2}\right)^{(1-\delta)/2\delta}. \tag{3.31}$$

Close to the fixed point, the ultraviolet cutoff can be traded for a renormalized, effective Newton's constant $G_{\text{eff}} \sim \Lambda^{2-d}$ (which can be extracted in a number of different ways, for example from the renormalized propagators at fixed geodesic

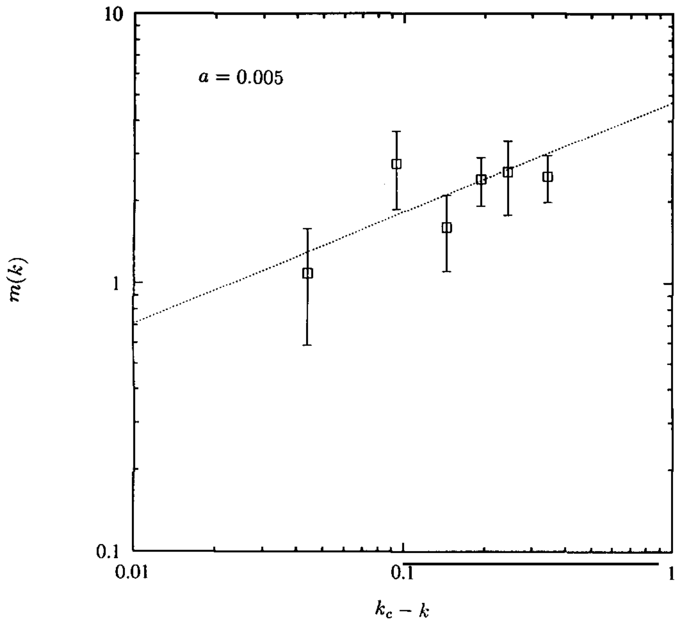

Fig. 19. Mass parameter (defined in eq. (3.29)), as determined from the volume corrections to the average curvature, as a function of $k_c - k$ for $a = 0.005$. The line indicates the expected slope corresponding to an exponent $\nu = 0.407$, as inferred from the curvature (figs. 5 and 7).

distance [1,2], or from correlations of Wilson lines). So the above relation between the mass scale, the curvature and Newton's constant becomes

$$\frac{m^2}{\mathcal{R}} \sim -C \left(-\mathcal{R}\, G_{\text{eff}}\right)^{(1-\delta)/2\delta} . \tag{3.32}$$

In ref. [1] G_{eff} was estimated to be a number of order one in lattice units ($G_{\text{eff}} \sim 0.13$), which would imply that the constant of proportionality in eq. (3.32) is also of order one, $C \sim 12$). For our calculated value of δ, the exponent on the r.h.s. is about 3/10. If the average curvature and the physical G are small, then this mass scale is exceedingly small. Of course at the fixed point, where the continuum limit is recovered, both m and $\mathcal{R}$ are zero, while G_{eff} appears to approach a finite limit. We shall leave further investigations of this issue for future work.

3.4. FRACTAL DIMENSIONS AND PHASE DIAGRAM

Some further insight into the nature of the two phases of quantum gravity can be obtained by exploring correlations which are of a purely geometric nature. Previously we found indications that some geometric properties of the discrete simplicial manifold appear to be close to euclidean in the smooth phase [1,2],

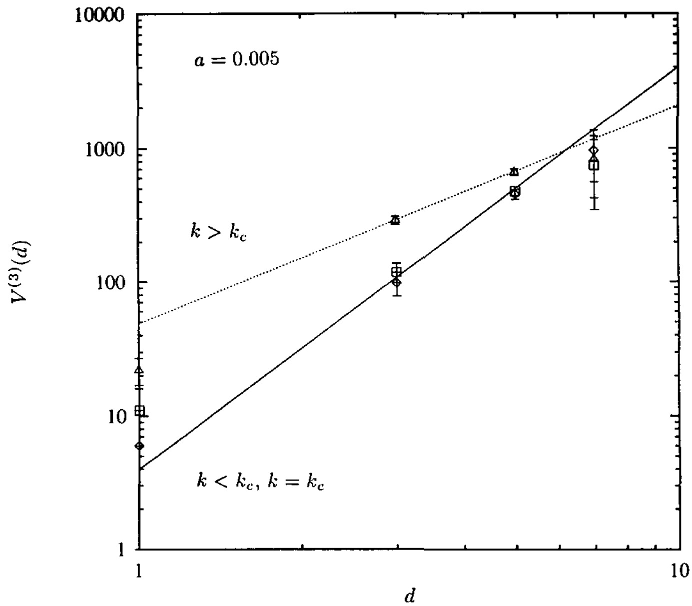

Fig. 20. Growth of the surface volume of a three-sphere as a function of geodesic distance from the center, in the smooth phase ($k < k_c$) (squares), at the critical point ($k = k_c$) (diamonds), and in the rough (collapsed) phase ($k > k_c$) (triangles). The continuous line corresponds to fractal dimension $d_H = 4.0$, the dotted line to $d_H = 2.6$.

in spite of the fact that the curvature fluctuations diverge at the critical point. Continuing the investigation of ref. [1], we have considered how the number of points within geodesic distance d and $d + \Delta d$ (where Δd is of the order of the average edge length) scales with the geodesic distance itself. This quantity is equivalent, up to a constant dependent on the average lattice spacing $\sqrt{\langle l^2 \rangle}$, to the physical extent of the "surface" within geodesics distance d and $d + \Delta d$. Finding the shortest path between two points on the curved lattice is quite time consuming, but for relatively small distances one finds

$$N(d) \underset{d \to \infty}{\sim} d^{\,d_v}, \tag{3.33}$$

with $d_v = 3.1(1)$ for $k < k_c$ (smooth phase), $d_v = 3.2(2)$ at $k = k_c$, and $d_v \approx 1.6(2)$ for $k > k_c$ (rough phase). The actual data is displayed in fig. 20. As discussed in refs. [2,4], in the rough phase the lattice tends to collapse into a degenerate configuration with many thin elongated simplices of small volume. This situation is reflected in the fact that d_v is much smaller than 3, the euclidean value. We conclude that this phase does not lead to a physically acceptable continuum limit. In the smooth phase on the other hand, as well as at the fixed point, the fractal dimension of space time is consistent with the flat space value of four ($d_v = d - 1 = 3$). It appears therefore that in this model lies a suggestion for the mechanism by which a nearly flat space-time can emerge from the strong short-distance fluctuations of the metric in quantum gravity.

Turning to the phase diagram, we observe that for different values of a the curvature vanishes along some line in the (k, a) plane, and for some small negative $a = a_0 \approx -0.0011$ a stable ground state ceases to exist (in other words, one crosses over into the rough phase). This not unexpected, since for sufficiently negative a the higher derivative term can completely cancel some of the higher-order lattice corrections present in the Regge action (which in turn is only an approximation to the pure Einstein action for small curvatures). This phenomenon is already seen in the weak-field expansion, which gives the correct order of magnitude for a_0. The leading higher-order corrections $O(k^4)$ have small coefficients in the Regge–Einstein action, while the corresponding terms have a relatively large coefficient $(O(400))$ in the higher-derivative action contribution [18]. On the other hand the higher-order lattice and radiative corrections to the pure Regge–Einstein action $(a = 0)$ seem to stabilize the theory, at least for the dl^2 measure.

Quantitatively a_0 can be estimated in the following way. For different values of a the average curvature vanishes along a line in the (k, a) plane which resembles quite closely a parabola, at least for small a,

$$a(k_c) = a_0 + a_1 k_c^2 + \cdots \tag{3.34}$$

Assuming this form, one finds $a_0 = -0.0011(5)$ and $a_1 = 0.119(6)$. Alternatively one can try to determine a_0 by assuming that the average curvature amplitude (close to k_c) diverges for small a like

$$A_{\mathcal{R}}(a) \sim A_0 (a - a_0)^{-\sigma} \tag{3.35}$$

From the results in table 1 one finds $a_0 = -0.0011(6)$ and $\sigma = 1.47(9)$, in good agreement with the previous estimate for a_0.

It is also of interest to consider what happens for large a. The results of ref. [15] would suggest that, for sufficiently large a, the scaling laws should be determined by asymptotic freedom in a. Thus for a typical mass scale $\mu(a)$ one would expect the asymptotic freedom scaling predictions

$$\mu(a) \underset{a\to\infty}{\sim} \exp(-\beta_0^{-1} a + O(\ln a)) \tag{3.36}$$

where $\beta_0 = 133/(160\pi^2)$ is the one-loop beta function coefficient for a, computed in [15]. As a consequence, one would perhaps expect that the curvature amplitude should follow the same law,

$$A_{\mathcal{R}}(a) \sim \mu(a)^{\delta/\nu} \underset{a\to\infty}{\sim} A_0 \left[\exp(-\beta_0 a + O(\ln a))\right]^{\delta/\nu} \tag{3.37}$$

The results for $A_{\mathcal{R}}(a)$ are shown in fig. 21. For now it appears that there is no sign of any exponential scaling for large a, even though the rapid decrease of the amplitude for large a is not inconsistent with such a scaling, with $A_0 \approx 0.48$. It

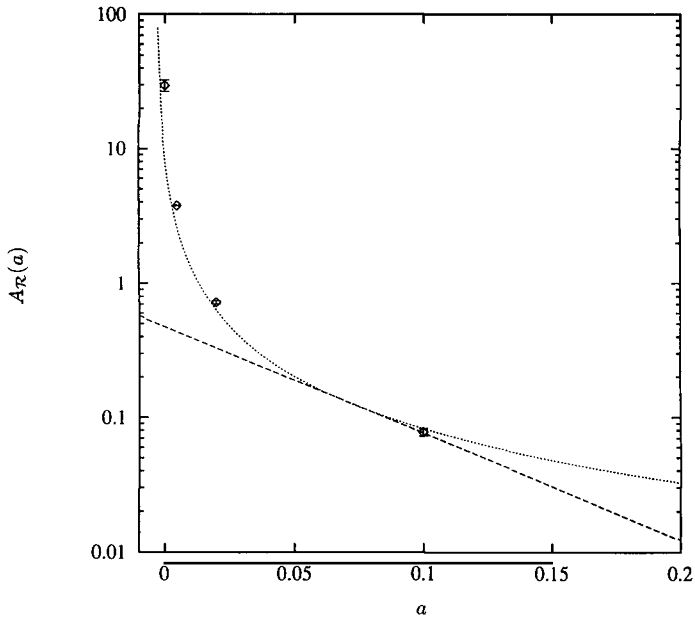

Fig. 21. Curvature amplitude $A_{\mathcal{R}}(a)$ (defined in eq. (3.15)), as determined as a function of a. The dotted line corresponds to the power law of eq. (3.35), while the straight line corresponds to the exponential asymptotic freedom prediction of eq. (3.37). There is no sign (yet) of exponential scaling.

remains to be seen whether even larger values of a (which can only be studied on larger lattices) will show an onset of exponential scaling behavior.

Returning to the discussion of the phase diagram, we can illustrate our conclusions so far by referring to fig. 22. We have discussed previously that for $a = 0.005$ and $a = 0.02$ the estimates for the critical exponents rule out a first-order phase transition by several standard deviations. There are a number of ways of characterizing a first-order transition. The criterion of ref. [20] suggests that the leading thermal exponent should be proportional to the dimensionality of space time (or $\nu = 1/d$) along the fluctuation-induced first order transition line. So one would expect $\delta = 0$, $\nu = 1/4$ and $\alpha/\nu = 4$, which seems to be completely excluded for $a = 0.005$ and $a = 0.02$ (and presumably also for $a = 0.1$).

On the other hand for $a = 0$ the situation is quite different, and the results discussed previously suggest indeed that the transition is close to first order, since $\delta \sim 0$. In general it is difficult to entirely exclude the presence of a weak first-order phase transition, if it has a very small latent heat. Indeed for $k \sim 0$ and $a \sim -a_0$ a sharp discontinuity in the average curvature develops in our model (it jumps from zero to infinity). But the results we have found in the pure Regge

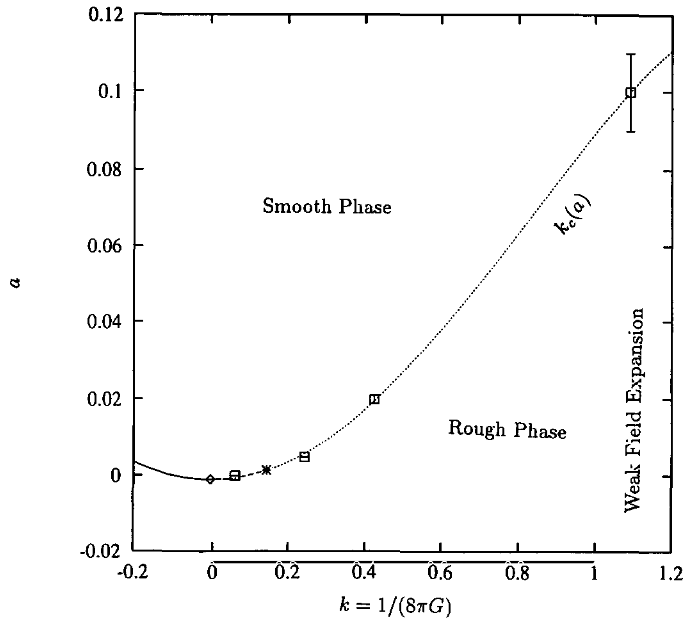

Fig. 22. Phase diagram for pure four-dimensional simplicial gravity with a higher-derivative coupling a (for fixed $\lambda = 1$). The squares correspond to computed values of $k_c(a)$. The curve therefore represents an estimate for the phase transition line $a(k_c)$, along which the curvature vanishes and the curvature fluctuations diverge, and which separates the "smooth" from the "rough" (or collapsed) phase of gravity. Along the continuous line the curvature is infinite. Along the dotted line the transition is continuous, with a well defined lattice continuum limit; along the dashed line the transition appears to be first order, with no continuum limit. The point marked by a star represents the approximate location of a multicritical point.

gravity case ($a = 0$) suggest the presence of some sort of multicritical behavior, with a line of first-order transitions (leading to no lattice continuum limit, since the correlation length is finite at the critical point) separated from a line of second order transitions by a tricritical point located somewhere between $a = 0$ and $a = 0.005$. We should stress that while the critical exponents in general are universal quantities, the location of critical points is not universal. Therefore the location of critical lines, multicritical points etc. refers to our specific action and the dl^2 measure. In particular our results do not preclude that with a different measure or a modified action the first-order transitions can be turned into second order ones. But within the context of our model, with its action and measure, a non-trivial continuum limit is only obtained if a small higher-derivative term is added to the pure Regge action. On the other hand this is no loss, since we have argued that the lattice gravitational action is not unique.

3.5. CORRELATIONS

The previous discussion has dealt almost exclusively with averages of invariant local operators such as the volume, the curvature and their fluctuations. We have shown that a great deal of information about the theory can be obtained by considering just these local quantities. But in general the information obtained is restricted only to the leading long-distance properties, and higher-order corrections as well as additional information can only be obtained by considering correlations between operators separated by some geodesic distance. In gravity complications arise since the distance between two points is a fluctuating quantity, and the Lorentz group used to classify spin states is meaningful only as a local concept. In addition the simplicial formulation is completely coordinate independent, and the introduction of the local Lorentz group requires the definition of a vierbein within each simplex, and the notion of a spin connection to describe the parallel transport between flat simplices.

For a set of local operators $\tilde{O}_\alpha$ we consider the connected correlations at fixed geodesic distance d,

$$G_{\alpha\beta}(d) = \langle \tilde{O}_\alpha(x)\, \tilde{O}_\beta(y)\, \delta(|x-y|-d)\rangle_c . \tag{3.38}$$

A suitable set of local operators in the continuum is represented for example by the (fourteen) algebraically independent coordinate scalars which can be constructed from the components of the Riemann tensor

$$R(x),\ \ R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}(x),\ \ R_{\mu\nu}R^{\mu\nu}(x),\ \ldots \tag{3.39}$$

On the lattice one can construct discrete approximations to these operators [2]. Since the deficit angles are proportional to the gaussian curvatures associated with the hinges, in general more than one hinge has to be considered and the corresponding lattice operators are not completely local, in the sense that they can involve a number of neighboring hinges as well as the angles describing their relative orientation

$$\begin{aligned} \sqrt{g}R(x) &\sim \sum_{h\supset x} \delta_h A_h , \\ \sqrt{g}R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}(x) &\sim \sum_{h\supset x} (\delta_h A_h)^2/V_h \ \ldots \end{aligned} \tag{3.40}$$

But one can argue that the lattice theory is already formulated exclusively in terms of coordinate invariant quantities, the edge lengths, and all the possible angles which are uniquely determined from them (the underlying lattice "structure" is reflected only in the local coordination number). It is therefore legitimate to consider for example connected correlations between edges or curvatures of the type

$$G^{(l)}_{\alpha\beta}(d) \equiv \langle l^2_\alpha(x)\, l^2_\beta(y)\, \delta(|x-y|-d)\rangle_c$$

$$G^{(R)}_{\alpha\beta}(d) \equiv \langle \delta_\alpha A_\alpha(x)\, \delta_\beta A_\beta(y)\, \delta(|x-y|-d)\rangle_c \tag{3.41}$$

Here the indices α,β label the edges and hinges (triangles) within a hypercube respectively, but more complicated extended operators can also be considered. With the specific simplicial lattice subdivision we are considering, $G^{(l)}_{\alpha\beta}(d)$ is a 15×15 matrix, while $G^{(R)}_{\alpha\beta}(d)$ is a 50×50 matrix if we restrict ourselves to variables within one hypercube. The correspondence between the length of the edge connecting point a to point b, and the continuum metric $g_{\mu\nu}(x)$ is

$$l_{ab} = \int_a^b \mathrm{d}s = \int_a^b \mathrm{d}\tau \big(g_{\mu\nu}(x) \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\big)^{1/2} \tag{3.42}$$

so in the first case one is considering correlations in the metric of the type

$$\langle \big[\int \mathrm{d}\tau \big(g_{\mu\nu}(\xi)\frac{\mathrm{d}\xi^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}\xi^\nu}{\mathrm{d}\tau}\big)^{1/2}(x)\big]^2 \big[\int \mathrm{d}\sigma \big(g_{\mu\nu}(\eta)\frac{\mathrm{d}\eta^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}\eta^\nu}{\mathrm{d}\sigma}\big)^{1/2}(y)\big]^2 \delta(|x-y|-d)\rangle_c \tag{3.43}$$

and the integrations are restricted to small regions surrounding x and y. In the second case we use the fact that for a small closed loop

$$\delta_h U^{(h)}_{\mu\nu} \sim g_{\nu\sigma} \oint \Gamma^\sigma_{\mu\lambda}\mathrm{d}x^\lambda \tag{3.44}$$

where $U^{(h)}_{\mu\nu}$ is a bivector perpendicular to the hinge h (we follow here the notation of ref. [2]). Therefore the second correlation function in equation eq. (3.41) is equivalent to

$$\langle A_C \oint_C (\Gamma_{\cdot\lambda}\mathrm{d}\xi^\lambda)_\perp(x)\; A_{C'} \oint_{C'} (\Gamma_{\cdot\sigma}\mathrm{d}\eta^\sigma)_\perp(y)\; \delta(|x-y|-d)\rangle_c \tag{3.45}$$

where C and C' are two small contours of area A_C and $A_{C'}$, respectively, and the symbol $\perp$ indicates here that the two corresponding parallel transport matrices are to be projected along the bivector perpendicular to the hinge (in other words, in the direction of U). Of course rather similar correlations are obtained if $\exp(\delta) - 1$ is considered instead of δ itself. (Also, for very small loops we can rewrite the integral over the affine connection in terms of a projection of the Riemann tensor in the plane of the loop). If the deficit angles are averaged over a number of contiguous hinges which share a common vertex, we obtain

$$\langle \sum_{h\supset x} \delta_h A_h \sum_{h'\supset y} \delta_{h'} A_{h'}\, \delta(|x-y|-d)\rangle_c \tag{3.46}$$

which corresponds to correlations in the scalar curvature,

$$\sim \langle \sqrt{g}\, R(x)\, \sqrt{g}\, R(y)\, \delta(|x-y|-d)\rangle_c \tag{3.47}$$

In general the above correlations will contain particles of different spin (0,2,..), but at large distances the lightest (massless) state with spin two, the graviton, should dominate, if the theory reproduces classical general relativity at large distances. (Spin one is excluded, since the elementary lattice parallel transporters are represented by real matrices). For purely massive excitations one would expect

$$G_{\alpha\beta}(d) \underset{d \gg 1/m}{\sim} T_{\alpha\beta}(d)\, \exp(-md) + T'_{\alpha\beta}(d)\, \exp(-m'd) + \cdots \tag{3.48}$$

where m and m' are the masses associated with the lowest excitations ($m < m'$), and in general the matrices T will have some power-law dependence on d like $T \sim d^{-3/2}$. The simplest way to extract the lowest mass is perhaps from the moments of the correlation function

$$M_n = \int_0^\infty \mathrm{d}x\, x^n\, G(x), \tag{3.49}$$

and then for example one has $m \sim [(15/4)M_2/M_4]^{1/2}$. At the critical point $k = k_c$ (where the graviton mass presumably vanishes) the leading behavior should become a power law

$$G_{\alpha\beta}(d) \underset{d \to \infty}{\sim} T_{\alpha\beta}\, \frac{C}{d^\sigma} \tag{3.50}$$

where T is some numeric matrix with entries of order one, C is a constant that sets the overall scale, and σ an integer power which depends on the scaling dimensions of the chosen operators. There are some indications that at the critical point the correlations in the edge lengths, as well as in the curvature, behave according to a power law [1,24]. Furthermore one expects that correlations in the volume should behave very differently from correlations in the curvature (this is what was found for the zero momentum components, the fluctuations discussed previously). Indeed by expanding around flat space, and in the gauge $\partial_\mu h^{\mu\nu} = 0$, one obtains for the graviton propagator in momentum space [15]

$$kD_{\mu\nu\lambda\sigma}(p) = \frac{4P^{(2)}_{\mu\nu\lambda\sigma}}{p^2 + (2\overline{a}/k)p^4} + \frac{2P^{(0)}_{\mu\nu\lambda\sigma}}{-p^2 + (\overline{a}/k)p^4} \tag{3.51}$$

and therefore schematically on has, in the weak-field limit,

$$\begin{aligned} \langle \sqrt{g}(x)\sqrt{g}(y) \rangle &\sim \int \mathrm{d}p\; \mathrm{e}^{-ip(x-y)} \frac{P^{(0)}}{-p^2 + (a/k)p^4} \\ \langle \sqrt{g}R(x)\sqrt{g}R(y) \rangle &\sim \int \mathrm{d}p\; \mathrm{e}^{-ip(x-y)} \int \mathrm{d}q\; \tilde{R}(p,q)\, \frac{4P^{(2)}}{p^2 + (2a/k)p^4} \\ &\quad \times \tilde{R}(p,q)\, \frac{4P^{(2)}}{(p-q)^2 + (2a/k)(p-q)^4} \end{aligned} \tag{3.52}$$

where $\tilde{R}$ represent some momentum-dependent vertices. This result would seem to imply no propagation of the volume density fluctuations, due to the wrong sign of p^2 ("anti-ferromagnetic coupling") in the scalar component of the propagator. Some of the preceding analysis might be simplified if we take into account the fact that close to the critical point the physical geometry of space-time appears to be close to euclidean. While there are strong fluctuations in the curvature, the average separation between lattice points has a finite variance at the critical point, and the almost vanishing average curvature should allow us to introduce a global Minkowski metric at large distances.

From the preceding arguments one would expect that, for example, the largest eigenvalue $\lambda_{\max}^{(l)}(d)$ of the edge–edge correlation matrix $G_{\alpha\beta}^{(l)}(d)$ should decay like $1/d^2$ for large geodesic distances d. The quantity $C_l = 4\pi^2 d^2 \lambda_{\max}^{(l)}(d)$ should approach a constant, which can be taken as a possible definition of the effective Newton's constant in units of the ultraviolet cutoff, $1/k_{\mathrm{eff}} \equiv 8\pi G_{\mathrm{eff}} = C_l$. Alternatively, one can compute G_{eff} directly from the invariant curvature correlation function (eq. (3.46) and (3.47)). It would seem from our results that this quantity tends to a finite value as k tends to k_c. From the curvature-curvature correlations for $a = 0.005$ and close to $k = k_c = 0.244$, one finds $1/k_{\mathrm{eff}} \approx 8.3$ in lattice units, which is comparable to the bare critical value $1/k \approx 4.10$. More generally, our results seem to indicate that while the average curvature in units of the ultraviolet cutoff tends to zero as one approaches the fixed point, the effective Newton's constant approaches some finite value, which is of the same order as the cutoff.

Alternatively one could determine $1/k_{\mathrm{eff}}$ by computing the analogs of correlation between Wilson lines. Integrating the parallel transport equation for an arbitrary vector S_λ,

$$\frac{\mathrm{d}S_\mu}{\mathrm{d}\tau} = \Gamma_{\mu\nu}^{\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} S_\lambda \tag{3.53}$$

along some path C connecting x and y, we get

$$S_\mu(y) = [P \exp \int_x^y \Gamma_{\cdot\lambda}^{\cdot}(z) \mathrm{d}z^\lambda]_\mu^{\ \nu} S_\nu(x) \tag{3.54}$$

In general we can compare two vectors at different locations only if one of them is first parallel transported; the same procedure has to be followed in order to define correlation functions involving two coordinate vectors U and V

$$G_{UV}(d) = \langle U^\mu(x) \, [P \exp \int_x^y \Gamma_{\cdot\lambda}^{\cdot}(z) \mathrm{d}z^\lambda]_\mu^{\ \nu} \, V_\nu(y) \, \delta(|x-y|-d) \rangle_c \tag{3.55}$$

The correlation between the world lines of two point particles separated by an invariant distance d corresponds to the choice $U^\mu = \mathrm{d}x^\mu$ and $V^\mu = \mathrm{d}y^\mu$, and is given by the expression

$$\langle \int_{W_1} \mathrm{d}x^\mu \int_{W_2} \mathrm{d}y_\nu \, [P \exp \int_x^y \Gamma_{\cdot\lambda}^{\cdot}(z) \mathrm{d}z^\lambda]_\mu^{\ \nu} \rangle_c \tag{3.56}$$

For a single closed path C we obtain the Wilson loop

$$W[C] \equiv \langle \mathrm{Tr}[P \exp \oint_C \Gamma_{\cdot\nu}^{\cdot} dx^\nu - 1] \rangle \tag{3.57}$$

Like the deficit angle δ_h, this quantity is of course coordinate independent. In gravity the Wilson loop does not have the same interpretation as in gauge theories, since it is not associated with the newtonian potential energy of two static bodies. In ordinary gauge theories at strong coupling the Wilson loop decays like the area of the loop, due to the strong independent fluctuations of the gauge fields at different points in space-time and ensuing cancellations. In gravity the situation is quite different since the connections cannot be considered as independent variables, and the fluctuations in the deficit angles at different points in space-time are strongly correlated. Some of the above correlation functions have also been discussed recently in the context of continuum weak-field perturbation theory [21].

Since the interior regions of the simplices are flat, the elementary lattice parallel transport matrices can have non-vanishing support only at the interface between two simplices. The effect of parallel transport around a closed loop C is then described by

$$\Big[\prod_j L(j, j+1)\Big]_{\mu\nu} = \Big[\exp\left\{\delta_h U_{\cdot\cdot}^{(h)}\right\}\Big]_{\mu\nu} \tag{3.58}$$

where $L(j, j+1)$ is the elementary parallel transport matrix between contiguous simplices j and $j+1$, and $U_{\mu\nu}^{(h)}$ is a bivector perpendicular to the hinge h,

$$U_{\mu\nu}^{(h)} = \frac{1}{2A_h} \epsilon_{\mu\nu\rho\sigma} l_{(a)}^\rho l_{(b)}^\sigma \tag{3.59}$$

with $l_{(a)}^\rho$ and $l_{(b)}^\rho$ the two vectors forming two sides of the hinge h. Comparison of equations (3.54) and (3.58) means that for small loops we may make the identification

$$R_{\mu\nu\rho\sigma}\, \Sigma^{\rho\sigma} \sim \delta_h U_{\mu\nu}^{(h)} \tag{3.60}$$

where $\Sigma^{\rho\sigma}$ is the area bivector of the loop. This relation emphasizes the fact that the deficit angle gives only information about the projection of the Riemann tensor in the plane of the (small) loop C orthogonal to the hinge.

The parallel transporters around closed elementary loops satisfy the lattice analogs of the Bianchi identities, which are easily derived by considering closed paths which can be shrunk to a point without entangling any hinge [5]. Then the product of rotation matrices associated with the path has to reduce to the identity matrix. Thus, for example, the ordered product of rotation matrices associated with the triangles meeting on a given edge has to give one, since a path can be constructed which sequentially encircles all the triangles and is topologically trivial

$$\prod_{\substack{\text{hinges } h \\ \text{meeting on edge } p}} \left[e^{\delta_h U^{(h)}_{..}} \right]_{\mu\nu} = 1 \tag{3.61}$$

These relations are simply a reflection of the fact that simplicial gravity (irrespective of the specific form of the lattice action) is entirely formulated in terms of invariants.

It is convenient to consider planar loops, which are spanned by the geodesics tangent to a plane at some point in the center of the loop. On the simplicial lattice the Wilson loop is computed by evaluating the deficit angle, (and its moments), associated with a large planar loop,

$$\langle (\delta_C)^n \rangle = \langle (\sum_{s \subset C} \theta_s - 2\pi)^n \rangle \tag{3.62}$$

where s labels the simplices traversed by the loop, and θ_s is the appropriate internal dihedral angle. Using the definition of the bivector $U_{\mu\nu}$ and eq. (3.58), one finds

$$W[C] \equiv \langle \mathrm{Tr}[P \exp \int_x^{x'} \Gamma^{\cdot}_{\cdot\nu} \mathrm{d}x^\nu - 1] \rangle \sim \langle (\delta_C)^2 \rangle \tag{3.63}$$

While it is easy to compute the average deficit angle for one triangle, for a large loop the average deficit angle will fluctuate around a rather small value, due to the smallness of the average curvature. Indeed for a small loop, with area bivector $\Sigma^{\lambda\sigma}$, the change in a vector S is proportional to the local average value of the Riemann tensor

$$\Delta S_\nu = R^\mu{}_{\nu\lambda\sigma} \, \Sigma^{\lambda\sigma} \, S_\mu \tag{3.64}$$

which will be small ($\sim \mathcal{R}$) close to the critical point. The Wilson loop measurement requires therefore great accuracy in the asymptotic region ($T, L \gg 1$), and in particular close to the critical point ($k \sim k_c$) where, as discussed previously, the fluctuations in the curvature diverge in the infinite-volume limit.

Finally an entirely new set of questions can be addressed if a scalar field $\phi(x)$ is introduced, in order to mimic the effects of dynamical matter fields (as opposed to static, infinitely heavy ones, which appear in some of the correlations discussed previously). One adds to the gravitational action the contribution

$$I[\phi] = \frac{1}{2} \sum_{\langle ij \rangle} V_{ij} \left(\frac{\phi_i - \phi_j}{l_{ij}}\right)^2 + \sum_i V_i \, U(\phi_i) \tag{3.65}$$

which introduces matter vacuum polarization contributions. Here ϕ_i is defined on the sites, V_{ij} is the volume associated with the edge l_{ij}, while V_i is associated with the site i. Again there is some freedom in how these volumes are defined on the lattice (baricentric or dual subdivision [2]), but one expects that universality will lead to the same long-distance properties in the continuum limit. It is expected that the critical exponents will be affected by the presence of matter fields; if the magnitude of the matter contribution to the beta function in $2+\epsilon$ dimensions (see eq. (3.11)) can be taken as an indication for the size of the dynamical matter corrections in four dimensions, then such corrections should be rather small (we should point out though that not all calculations of the one loop perturbative beta function agree on its magnitude [8]). As an extreme case one could consider a situation in which the matter action by itself is the only action contribution, without any kinetic term for the gravitational field, but still with a non-trivial gravitational measure; integration over the scalar field would then give rise to a non-local gravitational action. The potential $U(\phi)$ could contain quartic contributions, whose effects are of interest in the context of cosmological models where spontaneously broken symmetries play an important role.

4. Conclusion

We have seen in the previous sections that pure simplicial quantum gravity, without dynamical matter fields, leads already to a rich phase diagram and a number of interesting features. In some regions of the bare coupling constant space one does not recover a sensible four-dimensional theory ("rough" or "collapsed" phase), while in other regions ("smooth" phase) one finds a well defined ground state. In the smooth phase the lattice continuum limit is characterized by a set of universal critical exponents, describing the nature of curvature fluctuations close to the fixed point. At the fixed point the curvature vanishes, suggesting the emergence of flat Minkowski space-time at large distances, in spite of strong short distance fluctuations in the metric and curvatures. The appearance of the two phases of gravity is not unexpected, if one recalls the arguments about the unboundedness of the conformal mode fluctuations in the continuum, at least for sufficiently small G. The results are very different from what is found in two dimensions in all lattice models of gravity, including the present model (where fluctuations in the volume diverge instead [9]), and resemble somewhat the three-dimensional case discussed in ref. [17]. There a phase transition is found for some $k_c(a)$, again between a smooth and a rough phase of gravity. Similar phase transitions have also been found in the context of the dynamical triangulation model in three [11], and more recently in four dimensions [12,13].

The interplay between measure and cosmological constant contribution determine the distributions of edge lengths and simplex volumes on the lattice, and the latter are shown to be well behaved up to the transition point at k_c. The shape of the curvature distribution close the critical point suggests that there are significant deviations from gaussian fluctuations. For sufficiently large higher-derivative coupling the transition is continuous, with a curvature critical exponent $\delta \approx 0.626$. These results indicate therefore that the lattice continuum limit for simplicial quantum gravity exists, and is non-trivial (in the sense that the exponents do not fall into a known universality class). The nature of the divergent fluctuations at the critical point is precisely what is expected from a massless graviton (divergent curvature fluctuations, no divergence in the volume correlations), even though we should caution that only a very limited set of observables have been computed. For the pure Regge–Einstein action the transition seems to be first order, which implies no lattice continuum limit unless a small lattice higher-derivative term is added (in other words the continuum limit cannot be taken along the axis $a = 0$).

A finite-size analysis confirms (with large errors) part of the previous conclusions. Since there is a very large number of hinges (triangles) per site (50), the lattices cannot made large enough to improve on the accuracy of the exponents obtained from the fits to the average curvature (which has in comparison rather small errors), but the exponents are certainly consistent.

In the smooth phase there is evidence that the lowest lying excitation which is responsible for critical behavior has a mass; this would presumably correspond to a dynamically generated graviton mass in the presence of a non-vanishing average curvature. At the fixed point the results are consistent with a vanishing mass. The fractal dimension of space-time is consistent with four in the smooth phase up to and including the critical point. In the rough or collapsed phase the fractal dimension drops to about two, indicating that in the lattice there appears to be a proliferation of long, elongated simplices with small volumes. Finally we have argued that while the average curvature approaches zero at the critical point, the effective Newton's constant is most likely finite there, when expressed in units of the ultraviolet cutoff.

There is a close qualitative, and in some cases even quantitative, similarity between our results and results obtained recently with the dynamical triangulation models in four dimensions [12,13]. (The hypercubic lattice model of refs. [22,23] also seems the indicate the presence of some sort of a phase transition, whose nature should be further investigated). These models represents a somewhat simplified version of Regge's original formulation, in the sense that all edge lengths are taken to be equal, and the only dynamics left is in the incidence matrices. Even classically the relationship with gravity is not completely clear, since a weak-field expansion such as the one discussed in refs. [3] and [17] is still lacking, and the models lack the classical continuous diffeomorphism invariance even in the flat limit. On the other hand in two dimensions reparametrization invariance seems to be recovered in all models at the quantum level, which represents a rather remarkable result. The loss of continuous diffeomorphism invariance in the smooth limit in these models is partly compensated by the fact that at least the calculation of curvatures and volumes becomes simply a bookkeeping problem, and leads for example to a significant simplification of the programs. The question there remains whether the physical continuous diffeomorphism, which is explicitly broken on the lattice, is recovered in some continuum limit which is non-trivial (the identification of the continuum diffeomorphism group with simply the permutations of vertices on the lattice, as advocated in ref. [13], is difficult to justify). But it appears to be a common feature of all lattice models of gravity that some local invariance is lost by going on the lattice (and some is retained exactly), and that at least some models, including the Regge discretization, suggest the possible full restoration of the diffeomorphism group in the vicinity of the non-trivial fixed point, as signaled by long-range fluctuations in the physical curvatures.

In the dynamical triangulation model a phase transition separating the "smooth" from the "rough" phase of space-time was found recently, very similar in nature to the one discussed here, as well as in refs. [1,2]. This would represent a very encouraging result, since it would suggest that the two discrete lattice models belong, as perhaps expected, to the same universality class, and therefore have the same lattice continuum limit. It appears that there is even some quantitative agreement, as far as the critical exponents are concerned, between the results of ref. [12] and the results presented here and in ref. [1]. The importance of having two entirely different regularization schemes which give rise to the same lattice continuum limit need not be emphasized here. On the other hand the largest lattices used in ref. [13] to estimate the critical exponent δ are comparable in size to the smallest one employed in this work, and a detailed quantitative comparison might therefore be premature at this point (while the systematics of the two models might be quite different, we find that our results would be inconclusive on such small lattices). In addition the curvatures are continuous in the Regge model, while they are discrete in the dynamical triangulation model, and one would therefore expect the approach to the continuum limit to be slower in the latter case. Also, in the present case periodic boundary conditions are used, which are known to minimize boundary effects (see discussion in the previous sections); for the topology of the sphere boundary effects could be larger. One should also be reminded that no solution to the classical Einstein equations with a cosmological term exist on the torus, and no solutions without one exist on the sphere. It should not come as a surprise therefore that different models employing different boundary conditions lead to different values for the average curvature at the critical point; it is not obvious after all that the average curvature at the critical point should be a universal quantity.

Many questions have remained open. We have not investigated yet in detail how the results depend on the choice of invariant measure. In two and three dimension the detailed features of the invariant measure over the l^2's seem to play no role as far as the continuum limit is concerned [9,17], and one might expect a similar result to remain true also in four dimensions. Indeed on the basis of universality of the lattice continuum limit one would expect that the results for exponents and other infrared sensitive quantities should not be affected, as long as the measure does not become singular (or non-local). In addition we mentioned in the previous section why it would be of interest to investigate some of the issues discussed in the present paper in the presence of dynamical matter fields. Finally, the results we have presented for correlation functions at fixed geodesic distance have been very limited and rather preliminary, due to the technical difficulty of computing these correlations accurately. We hope to return to these questions in a future publication [24].

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