Nuclear Physics B267 (1986) 482-496

TWO-DIMENSIONAL SIMPLICIAL QUANTUM GRAVITY

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Received 5 August 1985

Quantum gravity in two dimensions is reviewed. Its formulation on simplicial lattices is described, with an action involving cosmological constant and higher derivative terms. Results are presented for the weak field approximation and for Monte Carlo calculations.

1. Introduction

Recently it has become fashionable to study two-dimensional gravity as a toy model for the full four-dimensional theory. Two-dimensional Einstein gravity is trivial since, by the Gauss-Bonnet theorem, the Einstein action is constant provided one studies systems with fixed topology. However, if one generalizes the action to include a cosmological constant and/or a higher-derivative term, the theory may contain some features which will be of interest for gravity in four dimensions.

The success of lattice-gauge theories in providing non-perturbative methods in quantum chromodynamics, through the introduction of a natural cutoff, the lattice spacing, had led to various formulations of lattice gravity. The approach described in this paper relies on Regge's approach to gravity [1], a discrete description of general relativity in which space-time is triangulated by a simplicial lattice. The lattice thus becomes a dynamical object, with the edge lengths of the lattice describing the evolution of space-time. A formalism has been set up [2, 3] for representing higher derivative terms on such a lattice and this paper is concerned with the results of both analytic and numerical work on simplicial lattices with higher derivative actions. Before describing the present study some other work on two-dimensional gravity will be described briefly below. In sect. 2, higher derivative gravity in two dimensions will be formulated, and in sect. 3 the weak field approximation for two-dimensional latttice gravity will be discussed. This includes the expansion around flat space and around a regular tessellation of the two-sphere. In sect. 4 the results of some numerical studies for quantum gravity on a two-torus will be presented. Sect. 5 contains some concluding remarks.

Recent work on two-dimensional gravity has proceeded along both analytic and numerical lines. Results from each of these approaches will now be mentioned briefly. As we have already remarked, two-dimensional Einstein gravity is trivial because the Einstein action is constant and the Einstein tensor vanishes identically. When a cosmological constant term and a curvature-squared term are included in the action

$$I = \int d^2x \sqrt{g}[\lambda - kR + aR^2] \tag{1.1}$$

the only classical solutions have constant curvature with $R = \pm\sqrt{\lambda/a}$ (there being no real solutions for $\lambda < 0$). Although the theory with the Einstein action and a cosmological constant is metrically trivial, having neither dynamical degrees of freedom nor field equations [4], it is not topologically trivial. It has been shown [5] that quantum gravity in $1+1$ dimensions with $\lambda = 0$, exists in a disordered phase, dominated by non-trivial topological configurations. This might suggest an interpretation in terms of "space-time foam" [6] which we shall discuss again in the concluding section. Also the functional measure can lead to a non-trivial effective action [7]. However, for a system with fixed topology, the only non-classical aspects of $1+1$ dimensional gravity are fluctuations in volume [8].

Much of the analytical and numerical work relevant to two-dimensional quantum gravity has involved the study of random surfaces, with an action depending on the area (i.e., a cosmological constant type action) [9-13]. Most of these calculations have taken the random surface to be embedded in a higher-dimensional continuum space with a dimension-dependent measure. It has been argued that the Hausdorff dimension of such surfaces is infinite [9, 10], and this feature does not seem to be affected by the inclusion of a non-local curvature term in the action [10]. In the continuum limit, random planar surfaces seem to correspond to free field theories [11]. David has studied two-dimensional surfaces without reference to any embedding properties, and has shown that a universal continuum limit exists for open surfaces, which can be interpreted as a space with negative mean curvature [13].

Two-dimensional lattice gravity based on the simplicial method of Regge calculus has been used by Jevicki and Ninomiya [12] to exhibit a discrete version of the conformal trace anomaly. They discuss the relation between a mass term for the scalar fields and the discretization of an R^2 term, and also write down the lattice form for the action of the Polyakov string (see also ref. [11]).

2. Two-dimensional higher derivative gravity on a simplicial lattice

We now turn to a study of quantum gravity on a two-dimensional surface consisting of a network of flat triangles. Such a lattice may be constructed in a number of ways. Points may be distributed randomly on the surface and then joined to form triangles according to some algorithm. However, because of their computational complexity we shall not consider such random lattices here. Alternatively one can start with a regular lattice, like a regular tessellation of the two-sphere, or a lattice of squares divided into triangles by drawing in parallel sets of diagonals, and then allow the edge lengths to vary.

The Einstein action for a two-dimensional simplicial lattice is given by [1]

$$\int d^2x \sqrt{g} R \to 2 \sum_{\text{hinges h}} \delta_h \,. \tag{2.1}$$

The hinges, where the curvature is concentrated, correspond to vertices in two dimensions and δ_h is the deficit angle at a hinge, defined by

$$\delta_h = 2\pi - \sum_{\substack{\text{triangles}\\ \text{meeting at h}}} [\text{vertex angle at h}] \,. \tag{2.2}$$

According to the Gauss-Bonnet theorem, the Einstein action in two-dimensions is equal to 4π times the Euler characteristic of the surface, and so is a constant provided we consider surfaces with fixed topology.

A cosmological constant term can be included in the action in the form

$$\lambda \int d^2x \sqrt{g} \to \lambda \sum_{\text{triangles t}} A_t \,, \tag{2.3}$$

where A_t is the area of triangle t. Equivalently we may divide the triangles into areas associated with each hinge A_h and use the expression

$$\lambda \sum_{\text{hinges h}} A_h \,. \tag{2.4}$$

Methods of constructing A_h are discussed in ref. [2]: the simplest method is the barycentric one for which

$$A_h = \tfrac{1}{3} \sum_{\substack{\text{triangles t}\\ \text{meeting at h}}} A_t \,. \tag{2.5}$$

A_h can also be taken to be the area of the cell surrounding h in the dual lattice.

In constructing higher derivative terms for a simplicial lattice, we note that in two dimensions the Weyl tensor vanishes identically, and that the other curvature-squared terms are all proportional to each other

$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \tfrac{1}{2} R_{\mu\nu} R^{\mu\nu} = R^2 \,. \tag{2.6}$$

Thus we need to write down only *one* term quadratic in the curvature for the lattice action. Using the requirements that it be a sum over hinges (the only places where the curvature is non-zero), that it be quadratic in the deficit angle, and that it have the correct dimension $(\text{length})^{-2}$, we are led to postulate that

$$\int d^2x \sqrt{g} R^2 \to 4 \sum \frac{\delta_h^2}{A_h} \,. \tag{2.7}$$

It has been shown [2, 3] that this formula is *exact* for all regular tesslations of the two-sphere. One might also wish to consider terms in the lattice action which involve cross-terms from different hinges, like

$$\sum_{\text{triangles } t} \left(\sum_{h \subset t} A_h \right) \sum_{\substack{\text{hinges} \\ hh' \subset t}} \left(\omega_h \frac{\delta_h}{A_h} - \omega_{h'} \frac{\delta_{h'}}{A_{h'}} \right)^2 , \tag{2.8}$$

which includes a weighting function ω_h for the different hinges. However we do not consider such a term necessary in two dimensions.

Before discussing numerical studies of the two-dimensional lattice gravity model described here, we shall next describe some analytic calculations.

3. The weak-field limit for two-dimensional lattice gravity

One of the simplest problems which can be studied analytically in simplicial quantum gravity is the analysis of small fluctuations about some classical background solution. The second variation of the action is then related to the inverse of the free propagator. Such a calculation has been carried out in four dimensional flat background space with the Einstein action, and it was found that the Regge calculus propagator agreed exactly with the continuum result in the weak-field limit [14].

Let us now consider a two-dimensional lattice with a higher derivative action

$$I = 4a \sum_{\text{hinges } h} \frac{\delta_h^2}{A_h} . \tag{3.1}$$

Flat space is a classical solution for such an R^2 type action. We therefore take as our background space a network of unit squares divided into triangles by drawing in parallel sets of diagonals (see fig. 1).

We use the binary notation for vertices described in refs. [14]. The edge lengths are then allowed to fluctuate around their flat space values: $l_i = l_i^0(1 + \epsilon_i)$, and the

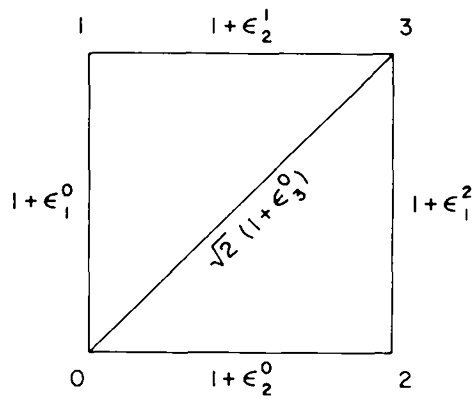

Fig. 1

second variation of the action is expressed as a quadratic form in ε

$$\delta^2 I = 4a \sum_{ij} \varepsilon_i M_{ij} \varepsilon_j , \tag{3.2}$$

where M_{ij} is the matrix which is related to the inverse of the free propagator. The fluctuation vector ε_i has three components per lattice point. The infinite-dimensional but sparse matrix M_{ij} is best studied by going to momentum space. We assume that the fluctuation ε_i at the point i, j steps in one coordinate direction and k steps in the other coordinate direction from the origin, is related to the corresponding ε_i at the origin by

$$\varepsilon_i^{(j+k)} = \omega_1^j \omega_2^k \varepsilon_1^{(0)} , \tag{3.3}$$

where $\omega_i = \mathrm{e}^{-ip_i l^{(i)}}$ and p_i is the momentum in the direction i. The matrix M reduces to a 3×3 matrix M_ω with components given by

$$\begin{aligned}
(M_\omega)_{11} &= 2 + \omega_1 - 2\omega_2 - 2\omega_1\omega_2 + \omega_1\omega_2^2 + \text{c.c.} , \\
(M_\omega)_{12} &= 2 - \omega_1 - \bar{\omega}_2 - \omega_1\omega_2 - \bar{\omega}_1\bar{\omega}_2 - \omega_1^2 - \bar{\omega}_2^2 + \omega_1^2\omega_2 + \bar{\omega}_1\bar{\omega}_2^2 + 2\omega_1\bar{\omega}_2 , \\
(M_\omega)_{13} &= 2(-1 + 2\omega_1 - \bar{\omega}_1 + \omega_2 - \bar{\omega}_2 - \omega_1\omega_2 + 2\bar{\omega}_1\bar{\omega}_2 + \bar{\omega}_2^2 - \bar{\omega}_1\bar{\omega}_2^2 - \omega_1\bar{\omega}_2) , \\
(M_\omega)_{33} &= 4(2 - 2\omega_1 - 2\omega_2 + \omega_1\omega_2 + \bar{\omega}_1\omega_2 + \text{c.c.}) .
\end{aligned} \tag{3.4}$$

(Other components may be obtained by appropriate interchanrge of ω_1 and ω_2.) In the weak-field limit, where the momentum p is small, M_ω takes the form

$$M_\omega = l^4 \begin{pmatrix} p_2^2(p_1+p_2)^2 & p_1p_2(p_1+p_2)^2 & -2p_1p_2^2(p_1+p_2) \\ p_1p_2(p_1+p_2)^2 & p_1^2(p_1+p_2)^2 & -2p_1^2p_2(p_1+p_2) \\ -2p_1p_2^2(p_1+p_2) & -2p_1^2p_2(p_1+p_2) & 4p_1^2p_2^2 \end{pmatrix} + \mathrm{O}(p^5) . \tag{3.5}$$

A simple change of variables

$$\delta_1' = \delta_1 , \qquad \delta_2' = \delta_2 , \qquad \delta_3' = \tfrac{1}{2}(\delta_1 + \delta_2) + \delta_3 \tag{3.6}$$

leads to the matrix M'_ω given by

$$M'_\omega = l^4 \begin{pmatrix} p_2^4 & p_1^2p_2^2 & -2p_1p_2^3 \\ p_1^2p_2^2 & p_1^4 & -2p_1^3p_2 \\ -2p_1p_2^3 & -2p_1^3p_2 & 4p_1^2p_2^2 \end{pmatrix} + \mathrm{O}(p^5) , \tag{3.7}$$

which is *exactly* what one obtains from the corresponding weak-field limit in the continuum theory. Define the small fluctuation $h_{\mu\nu}$ about flat space by

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu} . \tag{3.8}$$

The terms of order h^2 in $\int \mathrm{d}^2x \sqrt{g} R^2$ then come from the terms of order h in R. In two dimensions one has

$$R = h_{11,22} + h_{22,11} - 2h_{12,12} + O(h^2)\,, \tag{3.9}$$

$$\sqrt{g} = 1 + \tfrac{1}{2}(h_{11} + h_{22}) + O(h^2)\,, \tag{3.10}$$

which lead to

$$\sqrt{g}\,R^2 = (h_{11,22} + h_{22,11} - 2h_{12,12})^2 + O(h^3)\,. \tag{3.11}$$

In momentum space, each derivative ∂_ν produces a factor of p_ν, and one may write

$$\sqrt{g}\,R^2 = h_{\mu\nu} V_{\mu\nu,\rho\sigma} h_{\rho\sigma}\,, \tag{3.12}$$

where $V_{\mu\nu,\rho\sigma}$ coincides with M' above (when we have relabelled the components according to $11 \to 1$, $22 \to 2$, $12 \to 3$).

We may also consider a lattice action which includes a cosmological constant term, as well as the higher derivative term

$$I = \sum_{\text{hinges h}} \left[\lambda A_{\text{h}} + 4a \frac{\delta_{\text{h}}^2}{A_{\text{h}}} \right] \tag{3.13}$$

corresponding to a continuum action

$$I = \int d^2x \, \sqrt{g}[\lambda + aR^2]\,. \tag{3.14}$$

Expansion about a flat space background is no longer valid, strictly speaking, since flat space is not a classical solution in the presence of a cosmological constant. Were it a reasonable procedure we would obtain a contribution to the second variation of the action of

$$L_\omega = \frac{\lambda}{2} \begin{pmatrix} -1 & 0 & 1 + \bar{\omega}_2 \\ 0 & -1 & 1 + \bar{\omega}_1 \\ 1 + \omega_2 & 1 + \omega_1 & -4 \end{pmatrix} \tag{3.15}$$

from the cosmological constant term. In the weak field limit, and with the same change of variables as described for the matrix M_ω, this leads to

$$L'_\omega = \frac{\lambda}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} + O(p)\,. \tag{3.16}$$

Thus the typical behaviour of the inverse propagator in this case would be

$$-\tfrac{1}{2}\lambda + 4ap^4\,. \tag{3.17}$$

We now look briefly at the same procedure for variations about spaces with *are* classical solutions for the gravitational action *with* a cosmological constant term.

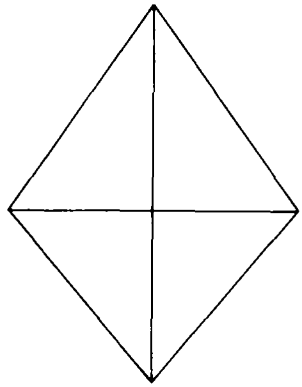

Fig. 2

Consider first the regular tessellation of the two-sphere given by a tetrahedron, octahedron and icosahedron. Suppose that the triangles in these tessellations have edge length l. One writes down the contributions from the two terms in the action and then varies the action to find the classical equations of motion. For each of the tesselations mentioned, the classical solution is of the form $l_0^4 = \alpha\pi^2(4a/\lambda)$ with $\alpha = \frac{16}{3}, \frac{4}{3}$ and $\frac{16}{75}$ respectively. Let us consider fluctuations about the classical solution for just the *tetrahedral* tessellation of S^2, which is shown in fig. 2.

We allow all the edge lengths to undergo small variations

$$l_i = l_0 \to l_i' = l_0(1+\varepsilon_i)\,. \tag{3.18}$$

The contribution to $\sum A_h$ which is quadratic in the ε_i's (the linear terms cancel later, after using the equations of motion) is

$$\sqrt{3}l_0^2\left[-\tfrac{5}{18}\sum_i \varepsilon_i^2+\tfrac{2}{9}\left(\sum_{\substack{i\neq j\\ i,j\text{ neighboring}}}\varepsilon_i\varepsilon_j-\sum_{\substack{i\neq j\\ i,j\text{ opposite}}}\varepsilon_i\varepsilon_j\right)\right] \tag{3.19}$$

and corresponding contribution from $\sum \delta_h^2/A_h$ is

$$\frac{4}{\sqrt{3}l_0^2}\left[\frac{2}{3}\left(\frac{65\pi^2}{27}-\frac{8\pi}{3\sqrt{3}}+8\right)\sum_i \varepsilon_i^2+\frac{32}{3}\left(\frac{2\pi^2}{27}+\frac{\pi}{3\sqrt{3}}-1\right)\sum_{\substack{i\neq j\\ i,j\text{ opposite}}}\varepsilon_i\varepsilon_j\right]. \tag{3.20}$$

When one adds these contributions with appropriate coefficients and uses the value of l_0 from the classical solution, one obtains from (3.13)

$$I \approx 16\pi\sqrt{a\lambda}+\frac{8\pi\sqrt{a\lambda}}{9\sqrt{3}}\left[\mu\sum_i \varepsilon_i^2+2\sum_{\substack{i\neq j\\ i,j\text{ neighboring}}}\varepsilon_i\varepsilon_j+2(2-\mu)\sum_{\substack{i\neq j\\ i,j\text{ opposite}}}\varepsilon_i\varepsilon_j\right], \tag{3.21}$$

where $\mu = 2(5\pi^2-6\sqrt{3}\pi+54)/9\pi^2 \approx 1.5919$. Therefore the "free propagator" will depend on the inverse of the matrix

$$\frac{8\pi\sqrt{a\lambda}}{9\sqrt{3}}\begin{pmatrix} \mu & 1 & 1 & 1 & 1 & 2-\mu \\ 1 & \mu & 1 & 2-\mu & 1 & 1 \\ 1 & 1 & \mu & 1 & 2-\mu & 1 \\ 1 & 2-\mu & 1 & \mu & 1 & 1 \\ 1 & 1 & 2-\mu & 1 & \mu & 1 \\ 2-\mu & 1 & 1 & 1 & 1 & \mu \end{pmatrix}. \tag{3.22}$$

Note that the λ/a dependence has disappeared. The couplings a and λ only appear in the dimensionless combination $\sqrt{a\lambda}$. The matrix is singular and the zero modes have to be extracted before it can be inverted. The eigenvalues of the matrix (forgetting about the constants in front of it) are 0 (with multiplicity 2), $2(\mu-1)$ (with multiplicity 3) and 6 (with multiplicity one). The multiplicites are in agreement with the dimensions of the irreducible representations of the symmetry group of the tetrahedron. Thus remarkably two zero modes have survived the lattice transcription of the continuum action. This is presumably a consequence of a residual symmetry ("general coordinate invariance") of the higher derivative lattice action in two dimensions.

5. Numerical studies of two-dimensional lattice gravity

We now describe the results of some numerical calculations based on the theory described in sect. 2. The lattice consists of a network of squares divided into triangles by drawing in parallel sets of diagonals. Opposite edges of the network are identified so that the lattice has the topology of a torus. The lattice action is

$$I=\sum_{\rm h}\left[\lambda A_{\rm h}-2k\delta_{\rm h}+4a\frac{\delta_{\rm h}^2}{A_{\rm h}}\right]. \tag{4.1}$$

In the limit of small fluctuations around a smooth background, this lattice action was shown above to correspond to the continuum action

$$I=\int d^2x\,\sqrt{g}[\lambda-kR+aR^2]. \tag{4.2}$$

In two space-time dimensions the Einstein action is a topological invariant, both in the continuum (because of the Gauss-Bonnet theorem) and on the lattice, since $\sum_{\rm h}\delta_{\rm h}=2\pi\chi$, where χ is the Euler characteristic. Therefore for a manifold of fixed topology the term proportional to k can be dropped. An additional possible term is given by a long-range interaction of the type [7, 10, 12, 3]

$$\tfrac{1}{2}\sum_{\rm h,h'}\delta_{\rm h}\left[\frac{1}{-\Delta+m^2}\right]_{\rm h,h'}\delta_{\rm h'}, \tag{4.3}$$

where Δ is the nearest-neighbor covariant lattice laplacian, and m^2 is an infrared mass regulator. A term of this type arises from the conformal anomaly in the continuum. It corresponds to the continuum contribution

$$\frac{1}{2}\int d^2x\, d^2y\, R\sqrt{g}(x)\langle x|\frac{1}{-\partial^2+m^2}|y\rangle R\sqrt{g}(y)\,, \tag{4.4}$$

where ∂^2 is the continuum covariant laplacian, $\partial^2 \equiv \partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu)$. This action was considered (for $m^2=0$) by Polyakov [7] in the context of the problem of random surfaces embedded in higher dimensional space. It is required, at least in its ultra-local form ($m^2\to\infty$), which is proportional to

$$b\sum_{\mathrm{h}}\delta_{\mathrm{h}}^2\,, \tag{4.5}$$

with b a dimensionless coupling, to ensure the existence of a (naive) lattice continuum limit. This limit requires that the curvature be small on the scale of the local volume [16]

$$(|\text{curvature}|)_{\mathrm{h}} = \left|\frac{\delta_{\mathrm{h}}}{A_{\mathrm{h}}}\right| \ll (\text{volume})_{\mathrm{h}}^{-1} = \frac{1}{A_{\mathrm{h}}} \qquad \text{or} \qquad |\delta_{\mathrm{h}}| \ll 1\,. \tag{4.6}$$

We are concerned with the evaluation of the path integral

$$Z = \int d\mu_\varepsilon[l]\, e^{-I[l]} \tag{4.7}$$

with the scale invariant measure

$$\int d\mu_\varepsilon[l] = \prod_i \int \frac{dl_i^2}{l_i^2} F_\varepsilon(l)\,, \tag{4.8}$$

which integrates directly over the elementary coordinate invariant lattice degrees of freedom, the edge lengths squared. The function $F_\varepsilon(l)$ vanishes if any of the edge lengths is less than ε or if any of the triangle inequalities is violated; otherwise it is unity. Thus $F_\varepsilon(l)$ introduces an ultra-violet cut-off, ε, into the calculations and also ensures that the triangle areas are always real, which means that the space has euclidean signature. Other possible scale invariant measures would be less local, in the sense that they would involve a power of some volume or volumes.

One possible method of evaluating the path intgral is to use the Monte Carlo method, in which the edge lengths are varied individually or in small groups by a small random amount and the change in the action is calculated. If the action is lowered, the new edge length is accepted; if the action is raised, the new edge length is accepted with a probability given by the exponential of the change in the action. The same procedure is then applied to another edge, and so on. After many edges have been changed, the probability distribution for the edges approaches the equilibrium one given by the exponential factor $\exp(-I[l])$. (Alternatively, one could generate the edge length distribution by using the Langevin equation, as discussed in refs. [3, 16]).

The two couplings in the action have dimensions $[\lambda] \sim \varepsilon^{-2}$ and $[a] \sim \varepsilon^2$. The cosmological constant in two dimensions has at most a quadratic divergence

$$\lambda_R = \lambda_0 + c_2 L^2 + c_0 \ln L + \cdots , \tag{4.9}$$

with $L \sim \varepsilon^{-1}$ the ultraviolet cutoff.

Consider now the path integral

$$Z[\lambda, a, b, \varepsilon] = \int \mathrm{d}\mu_\varepsilon[l]\, \mathrm{e}^{-I[l]} . \tag{4.10}$$

Because of the scale invariance of the measure, all the edge lengths can be rescaled $l_i \to (a/\lambda)^{1/4} l_i$, and one obtains

$$Z[\lambda, a, b, \varepsilon] = Z\left[\sqrt{a\lambda}, \sqrt{a\lambda}, b, \left(\frac{\lambda}{a}\right)^{1/4} \varepsilon\right] . \tag{4.11}$$

If ε can be sent to zero (in other words, if the functional integral exists with $\varepsilon = 0$), then Z depends only on $\sqrt{a\lambda}$ and b, once all lengths are expressed in units of the length scale $l_0 \equiv (a/\lambda)^{1/4}$.

Here only a lattice with the topology of the torus will be considered. The square lattice divided into triangles was chosen of size 32×32, with 3072 edge variables. The following expectation values are of interest

$$\langle A \rangle = \frac{1}{N_\mathrm{h}} \left\langle \sum_\mathrm{h} A_\mathrm{h} \right\rangle ,$$

$$\langle R^2 \rangle = \frac{4}{N_\mathrm{h}} \left\langle \sum_\mathrm{h} \frac{\delta_\mathrm{h}^2}{A_\mathrm{h}} \right\rangle , \tag{4.12}$$

where $N_\mathrm{h} = N^2$ is the number of hinges. For the formulation of higher derivative terms dual lattice volumes [2] were used in the following. A numerical evaluation of expectation values by the Monte Carlo method leads to the following results. Consider first the case $b = 0$. For $a = 0$ the results are trivial

$$\langle A \rangle = \begin{cases} 0, & \text{if } \lambda > 0 \\ \infty, & \text{if } \lambda \leq 0 \end{cases} \tag{4.13}$$

and $\langle R^2 \rangle = \infty$ for both signs of λ. These results are valid as the cutoff ε is sent to zero. For strictly positive a the path integral exists for positive λ, while for negative λ one has results similar to $a = 0$ (no nontrivial equilibrium distribution of edge lengths). In fig. 3 we show the behavior of the average edge length as a function of the Monte Carlo iterations for $4a = 1$. The top curve corresponds to $\lambda = 0$ and the bottom curve to $\lambda = 0.4$.

Other results for the case $4a = 1$ some results are displayed in table 1. (They were obtained by averaging over 1000 passes through the lattice, after discarding an initial 1000.) There $\sqrt{\langle l^2 \rangle}$ denotes the square root of the average edge length squared, and

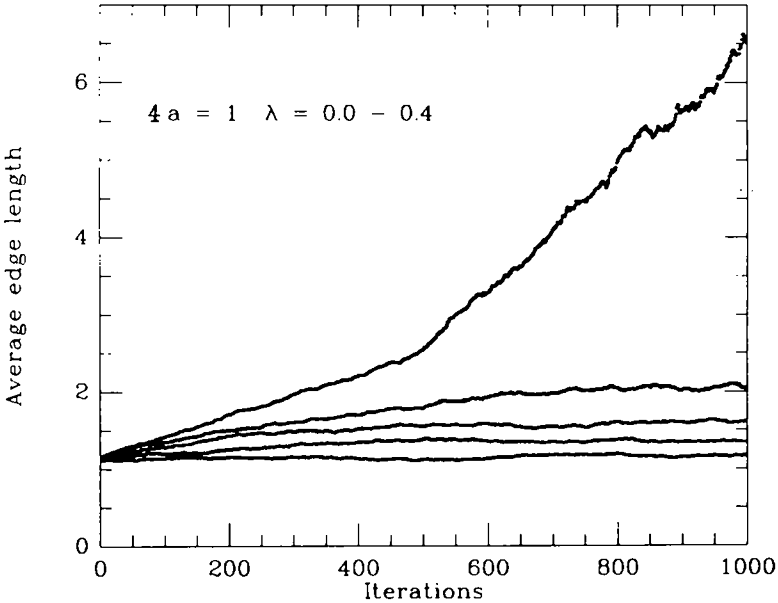

Fig. 3

TABLE 1

Results for the 2-dimensional model with $4a = 1$ and $b = 0$

λ	$\sqrt{\langle l^2 \rangle}$	$\langle A \rangle$	$\frac{1}{2}\frac{\langle A \rangle}{\langle l^2 \rangle}$	$\frac{1}{4}\langle R^2 \rangle$	$\frac{1}{4}\frac{\langle R^2 \rangle}{\langle A \rangle}$
0.4	1.28 (3)	0.96 (5)	0.292 (5)	0.408 (3)	0.42 (3)
0.3	1.52 (3)	1.34 (5)	0.289 (5)	0.396 (3)	0.30 (1)
0.2	1.78 (3)	1.82 (5)	0.287 (5)	0.371 (3)	0.20 (1)
0.1	2.34 (3)	3.12 (5)	0.285 (5)	0.324 (3)	0.10 (1)
0.0	∞	∞	0.288	0	0

$\langle A \rangle$ and $\langle R^2 \rangle$ are defined above. $\sqrt{\langle l^2 \rangle}$ here plays the role of the lattice spacing, the fundamental unit of length. A typical equilibrium distribution of edge lengths is shown in fig. 4 for $4a = 1$ and $\lambda = 0.2$. It is insensitive to a small edge length cutoff ε, unless λ is very large.

For $\varepsilon = 0$ one has from scale invariance the exact identity

$$\frac{\langle R^2 \rangle}{\langle A \rangle} = \frac{\lambda}{a}, \tag{4.14}$$

which is well satisfied, as can be seen from the table. Also, the average area of a

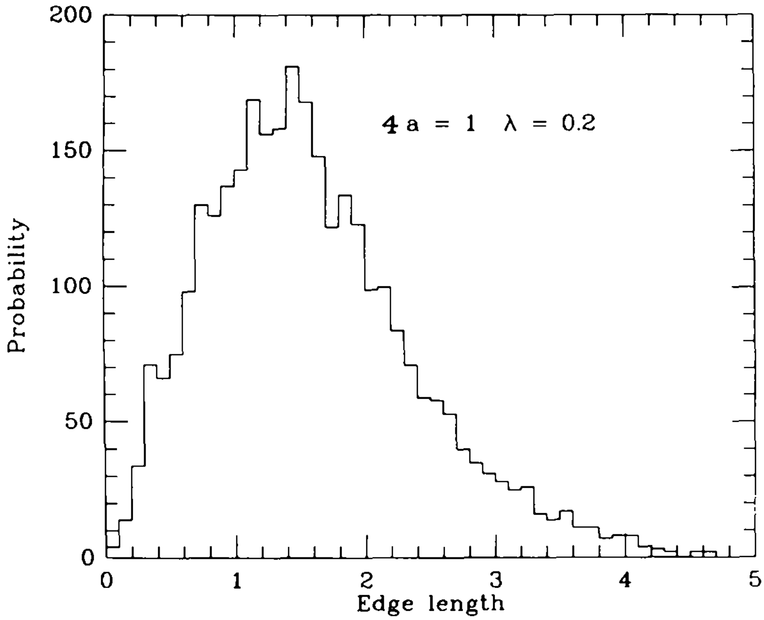

Fig. 4

triangle divided by the average edge length squared is independent of λ. One finds

$$\frac{\langle A\rangle}{2\langle l^2\rangle} = 0.288 \approx \frac{\sqrt{3}}{6}, \tag{4.15}$$

which shows that the triangles on the average are not equilateral. (For equilateral triangles the ratio is $\sqrt{3}/4 = 0.433$). In the region of λ and a considered the space-time volume and the integrated curvature squared can be reasonably well fitted (in the region $4a\lambda = 0.1 - 0.4$) by simple functions of the form

$$\lambda\langle A\rangle = \frac{2}{B + C\ln(1/4\lambda a)},$$

$$a\langle R^2\rangle = \frac{2}{B + C\ln(1/4\lambda a)}, \tag{4.16}$$

with $B = 4.0$ and $C = 1.0$, but other fits are equally possible at this point. (It is difficult to distinguish a logarithm for a small power, so another possible fit is given by $\lambda\langle A\rangle = B(4a\lambda)^C$ with $B = 0.46(3)$ and $C = 0.15(5)$.) In fig. 5 we plot $[\lambda\langle A\rangle]^{-1}$ (circles) and $[(\frac{1}{4})\langle R^2\rangle]^{-1}$ (squares) as a function of $\ln(1/4\lambda a)$ for $4a = 1$, with the values given in table 1. For comparison we also show the straight line fit of eq. (4.16) and the power law fit mentioned above. There is no clear preference in the data, which have errors of about five percent. More accurate studies could clarify

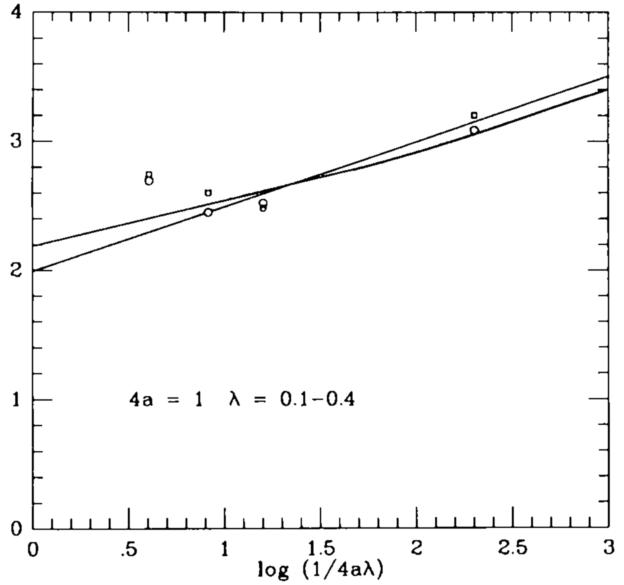

Fig. 5

this point. In any case our results suggest the presence of a non-analytic behaviour for small $\sqrt{\lambda a}$.

It should also be stressed that, because of the scale invariance of the measure (for $\varepsilon = 0$), only dimensionless quantities are of interest, like the aveage curvature squared in units of the average edge length: $\frac{1}{4}\langle l^2\rangle^2\langle R^2\rangle/\langle A\rangle = 1.14$, 1.58, 2.05, 3.11 for $\lambda = 0.4$, 0.3, 0.2, 0.1, respectively.

Physical correlation functions like the volume-volume correlation at fixed geodesic separation $D(hh')$

$$G_V(d) = \sum_{\mathrm{h,h'}} \langle V_{\mathrm{h}} V_{\mathrm{h'}} \delta(D\mathrm{h,h'}) - d)\rangle \tag{4.17}$$

can also be evaluated, but appear not to be positive definite. This is a consequence of the pure $1/p^4$ behavior of the "graviton" propagator in two-dimensional higher derivative gravity with action (3.13).

In the case of finite coupling b the sum rule (3.22) can no longer be satisfied. It is known that for sufficiently large b the edge length configurations become smooth, but there is no solution on a torus to the classical field equations of continuum higher derivative gravity with a λ term. Still, a finite result is found for the average curvature squared in units of the average edge length: $\frac{1}{4}\langle l^2\rangle^2\langle R^2\rangle/\langle A\rangle = 0.47$, 0.56, 0.64, 0.73 for $\lambda = 0.4$, 0.3, 0.2, 0.1, respectively. Here $4a = 1$ and $b = 2$, and the estimates were obtained again by averaging over 1000 passes. As can be seen, the surface has become significantly smoother in this case.

5. Conclusions

In spite of the lack of dynamical content for two-dimensional gravity, it exhibits a number of features which are instructive for future study of gravity in higher dimensions. In our numerical studies it appears that the phase diagram of a model of higher derivative gravity can be worked out with relative ease even though the detailed dependence on the bare parameters is more difficult to determine and would require further work. Scalar matter fields coupled to gravity by the lattice action discussed in ref. [3] could also be studied.

The pure gravity theory studied here does not appear to have unexpected phase transitions (which would correspond to coupling constant fixed points) or other unusual behavior except at $\lambda a = 0$ and $\lambda a = \infty$. We have argued that without the higher derivative terms two-dimensional lattice gravity leads to trivial results and does not possess a continuum limit. This is not unexpected since with the Einstein action and a cosmological term there is no mechanism that prevents the curvature for becoming *locally* arbitrarily positive or negative. The restriction to a system with fixed topology requires only that the integrated curvature be finite, which can always be achieved by having arbitrarily large positive curvatures coexist with arbitrarily negative curvatures in different regions of space-time.

In the presence of higher derivative terms contact between the lattice and continuum description of the quantum theory can be established. Unfortunately because of the lack of unitarity of the pure higher derivative theory, and the fact that by the Gauss-Bonnet theorem no R term is generated by radiative corrections, the resulting theory of two-dimensional quantum gravity is not very realistic. In this framework the question of the renormalization of the cosmologial constant cannot be addressed, since no newtonian potential, and therefore no renormalized Newton's constant, can be defined in two dimensions. The situation is of course very different in four dimensions, and will be discussed elsewhere [15, 16].

The research of H.W.H. was supported by the US Department of Energy under grant no. DE-AC02-76ER02220. R.M.W. is grateful for hospitality at the Institute for Advanced Study.

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