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Inconsistencies from a Running Cosmological Constant

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ABSTRACT

We examine the general issue of whether a scale dependent cosmological constant can be consistent with general covariance, a problem that arises naturally in the treatment of quantum gravitation where coupling constants generally run as a consequence of renormalization group effects. The issue is approached from several points of view, which include the manifestly covariant functional integral formulation, covariant continuum perturbation theory about two dimensions, the lattice formulation of gravity, and the non-local effective action and effective field equation methods. In all cases we find that the cosmological constant cannot run with scale, unless general covariance is explicitly broken by the regularization procedure. Our results are expected to have some bearing on current quantum gravity calculations, but more generally should apply to phenomenological approaches to the cosmological vacuum energy problem.

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1 Introduction

It is a common feature of the renormalization group approach to quantum field theory that coupling constant are generally scale dependent: they run with momentum scale in a way that is determined by the Callan-Symanzik beta function and its nonperturbative extensions. The physical reasons for the scale dependence of couplings is generally well understood. It arises because of the effects of virtual quanta that either screen (as in QED) or anti-screen (as in QCD) the fundamental bare charge, with the (anti)screening scale generally determined by a relevant infrared cutoff. Gravity itself is not immune from such effects, which arise from graviton and matter vacuum polarization contributions, but the issues has been clouded for some time as a consequence of the well-known perturbative non-renormalizability problem. Nevertheless, a number of physically relevant results have been obtained by either applying Wilson's $2+\epsilon$ dimensional expansion method, or via the 4d lattice formulation of gravity developed by Regge and Wheeler. In either case, definite predictions arise for the scale dependence of Newton's constant G, which are generally consistent between the two approaches. In particular, the lattice theory predicts a slow rise in the gravitational coupling with scale, similar to the well-known anti-screening effect of non-Abelian gauge theories. In either case the cosmological constant cannot be made to run with scale, and it emerges instead naturally as a nonperturbative renormalization group invariant scale, formally an integration constant of the renormalization group equations. A key ingredient in both theories is the preservation of local diffeomorphism invariance, which would otherwise spoil this last result.

In this paper we address the issue of how general the result is that the cosmological constant cannot be momentum-scale dependent, if manifest general covariance is strictly maintained. To do so, we first examine the question of the dependence of the renormalization group equations on the bare cosmological constant. Within the manifestly covariant functional integral approach to gravity, it is then easy to show (both in the Euclidean and in the Lorentzian formulation) that the bare cosmological can be entirely scaled out, so that physical invariant correlations cannot depend on it. The same is found to be true in the lattice formulation of gravity, where again the bare cosmological can be scaled out, and thus set equal to one in units of the ultraviolet cutoff, without any loss in generality. One concludes therefore that a running of lambda is meaningless in either formulation.

The above conclusions are reinforced by a study of perturbative gravity in $2+\epsilon$ dimensions. Here radiative corrections are computed using the background field method and dimensional regularization, by suitably performing a formal double expansion in $d-2$ and G. In this work we will point out that these results clearly show that the renormalization of the cosmological constant is gauge dependent. Furthermore, this spurious renormalization entirely disappears once a suitable rescaling of the metric is performed in order to remove the unwanted gauge dependence.

An alternative approach to the problem of the running cosmological constant is via a set of manifestly covariant effective field equations, constructed so as to incorporate the running of Newton's constant G, and in a way that is consistent with the results from the manifestly covariant functional integral method described earlier. Again we find that within such a framework it is nearly impossible to accommodate a scale dependence of the cosmological constant, for the simple reason that covariant derivatives of the metric tensor vanish identically. A similar result is later obtained from a slightly different approach, centered on an effective action of quantum gravity.

An outline of the paper is as follows. Sec. 2 discusses scaling properties of the continuum functional integral for gravity, and in particular how it transforms under a uniform rescaling of the metric. Sec. 3 recalls how the problem of the renormalization of the cosmological constant is resolved in the perturbative treatment of gravity. Sec. 4 shows that the bulk of the conclusions of Sec. 2 for the continuum are still valid on the lattice, and in particular the fact that the functional integral does not depend on the specific value for the bare cosmological constant, as long as it is positive, and irrespective of the choice of functional measure. The role of the new fundamental nonperturbative scale ξ that arises both in the lattice and in the continuum treatment is emphasized. Sec. 5 points out similarities between gravity and non-Abelian gauge theories, and in particular the important role played, in the nonperturbative treatment of QCD, by the dynamically generated length scale, seen to be related in a simple way to the color vacuum condensate. Sec. 6 summarizes the key points that lead to the identification, in the framework of nonperturbative gravity, of the renormalization-group invariant scale ξ with a gravitational condensate, and thus with the observed cosmological constant. Sec. 7 discusses the running of the cosmological constant from the perspective of a set of manifestly covariant, but nonlocal, effective field equations. Here it is shown that the cosmological constant, in this framework, cannot run. Sec. 8 analyzes the same problem by considering the implications of an effective action formulation, thereby reaching the same conclusions as in the previous section. Sec. 9 contains a summary of our results.

2 Gravitational Functional Integral

In this section we recall some basic properties of the functional integral for gravity, which will lead to the conclusion that the bare cosmological constant can largely be scaled out of the problem. Formally, the Euclidean Feynman path integral for pure Einstein gravity with a cosmological constant term can be written as [3]

$$Z \;=\; \int [d\, g_{\mu\nu}] \; \exp \left\{ - \lambda_0 \int dx \, \sqrt{g} \,+\, \frac{1}{16 \pi \, G} \int dx \, \sqrt{g} \, R \right\} \;\; . \tag{1}$$

The above state sum involves a functional integration over all metrics, with measure given by a suitably regularized form of

$$\int [d\, g_{\mu\nu}] \;\equiv\; \int \prod_x \; [g(x)]^{\sigma/2} \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \;, \tag{2}$$

as given in Eqs. (12) and (13) below, and with σ some real parameter. The value of σ will play no significant role in the following, as long as the relevant integrals are known to exist. For geometries with boundaries, further terms will need to be added to the action, representing the effects of those boundaries. Here we will consider the above expression in the absence of such boundaries.

Let us first focus on some basic scaling properties of the gravitational action. One first notices that in pure Einstein gravity, with Lagrangian density

$$\mathcal{L} \;=\; - \frac{1}{16 \pi \, G} \, \sqrt{g} \, R \;\;, \tag{3}$$

the bare coupling G can be completely reabsorbed by a suitable field redefinition

$$g_{\mu\nu} = \omega \, g'_{\mu\nu} \tag{4}$$

with ω a constant. It follows that in a quantum formulation the renormalization properties of G have no physical meaning for this theory, at least until some other terms are added, to be discussed below. The reason of course is that the term $\sqrt{g}R$ is homogeneous in $g_{\mu\nu}$, which is quite different from the Yang-Mills case. The situation changes though when one introduces a second dimensionful quantity to compare with. In the gravity case, this contribution is naturally supplied by matter, or by a cosmological constant term proportional to λ_0, [4]

$$\mathcal{L} \;=\; - \frac{1}{16 \pi \, G} \, \sqrt{g} \, R \,+\, \lambda_0 \, \sqrt{g} \; . \tag{5}$$

[3] Most aspects of the following discussion would remain unchanged if we were to consider instead the Lorentzian formulation. For concreteness, we will focus here almost exclusively on the Euclidean theory.

[4] In the following we will denote by λ_0 the (un-scaled) cosmological constant, and by λ the scaled one, so that $\lambda_0 \equiv \lambda / 8 \pi G$. In this work the symbol Λ will be reserved for the ultraviolet cutoff.

Under a rescaling of the metric, as in Eq. (4), one obtains

$$\mathcal{L} = - \frac{1}{16\pi \, G} \, \omega^{d/2-1} \, \sqrt{g'} \, R' \; + \; \lambda_0 \, \omega^{d/2} \, \sqrt{g'} \; , \tag{6}$$

which is seen as being equivalent to a rescaling of the two bare couplings

$$G \to \omega^{-d/2+1} G \; , \quad \lambda_0 \to \lambda_0 \, \omega^{d/2} \tag{7}$$

while at the same time leaving the dimensionless combination $G^d \lambda_0^{d-2}$ unchanged. Therefore only the latter quantity has physical meaning in pure gravity, and it would seem physically meaningless here to discuss separately the renormalization properties of G and λ_0. In particular, one can always choose the scale $\omega = \lambda_0^{-2/d}$, so as to adjust the volume term to have a unit coefficient. Then one obtains

$$\mathcal{L} = - \frac{1}{16\pi \, G \, \lambda_0^{1-2/d}} \, \sqrt{g'} \, R' \; + \; \sqrt{g'} \; . \tag{8}$$

One concludes that the only coupling that matters for pure gravity in four dimensions is $G\sqrt{\lambda_0}$, so that, without any loss of generality, it would seem one can take $\lambda_0 = 1$ in units of some UV cutoff.

Nevertheless, a discussion of the field rescaling properties of the theory is incomplete unless one also takes into account the effect of the functional measure. Following DeWitt [1], one defines an invariant norm for metric deformations as

$$\| \delta g \|^2 \; = \; \int d^d x \; \delta g_{\mu\nu}(x) \; G^{\mu\nu,\alpha\beta} \big[g(x) \big] \; \delta g_{\alpha\beta}(x) \;\; , \tag{9}$$

with the supermetric G given by the ultra-local (since it is defined at a single point x) expression

$$G^{\mu\nu,\alpha\beta} \big[g(x) \big] \; \equiv \; \tfrac{1}{2} \, \sqrt{g(x)} \; \Big[g^{\mu\alpha}(x) \, g^{\nu\beta}(x) + g^{\mu\beta}(x) \, g^{\nu\alpha}(x) + \lambda \, g^{\mu\nu}(x) \, g^{\alpha\beta}(x) \Big] \tag{10}$$

and λ a real parameter such that $\lambda \neq -2/d$. The above supermetric then defines a suitable volume element $\sqrt{\det G}$ in function space, and the functional measure over the $g_{\mu\nu}$'s takes on the form

$$\int [d \, g_{\mu\nu}] \; \equiv \; \int \prod_x \Big[\det G[g(x)] \Big]^{1/2} \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \;\; . \tag{11}$$

The assumed locality of the supermetric $G^{\mu\nu,\alpha\beta}[g(x)]$ implies that its determinant is also a local function of x only. Up to an inessential multiplicative constant one finds

$$\int [d \, g_{\mu\nu}] \; = \; \int \prod_x \, [g(x)]^{(d-4)(d+1)/8} \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \;\; \underset{d \to 4}{\to} \; \int \prod_x \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \; . \tag{12}$$

However it is not obvious that the above construction of the measure is unique; an alternative derivation starts from a slightly different supermetric, and leads to the scale-invariant functional measure

$$\int [d\, g_{\mu\nu}] \; = \; \int \prod_x \left[g(x) \right]^{-d(d+1)/8} \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \; \underset{d\to 4}{\to} \; \int \prod_x \left[g(x) \right]^{-5/2} \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \; , \tag{13}$$

which was originally suggested in [2]. For a more complete discussion of the many delicate issues associated with the formulation of the covariant Feynman path integral approach to quantum gravity the reader is referred to [3]. One can further show that if one introduces an n-component scalar field $\phi(x)$ in the functional integral, it leads to more changes in the gravitational measure. For the functional measure over ϕ one writes

$$\int [d\phi] \; = \; \int \prod_x \left[\sqrt{g(x)} \right]^{n/2} \prod_x d\phi(x) \quad , \tag{14}$$

so that the first factor gives an additional contribution to the gravitational measure. These arguments lead one to conclude that the volume factor in the measure should be written more generally as $g^{\sigma/2}$, and be included in a slightly more general form for the gravitational functional measure, as given before in Eq. (2). In principle there is no clear a priori way of deciding between the various choices for σ, and it may very well turn out to be an irrelevant parameter. The only constraint seems that the regularized gravitational path integral should be well defined, which would seem to rule out singular measures. It is noteworthy though that the $g^{\sigma/2}$ volume term in the measure is completely local and contains no derivatives, which therefore cannot affect the propagation properties of gravitons. But more importantly, for our purpose here it will be sufficient to note that under a rescaling of the metric the functional measure in Eq. (2) only picks up an irrelevant multiplicative constant. The fact that the latter depends on the specific form of the functional measure (i.e. on σ) is completely irrelevant; such a constant drops out automatically when computing averages. One reaches therefore the conclusion that the previous arguments remain largely unchanged: the functional integral for pure gravity only depends on one dimensionless combination of G and λ_0; λ_0 can be set equal to unity in units of th UV cutoff without any loss of generality [4, 5].

From a physical perspective, it might nevertheless seem more appropriate to keep the dimensions of various parameters appearing in the action unchanged. This can be achieved by explicitly introducing an ultraviolet cutoff Λ, so that the Euclidean Einstein-Hilbert action with a cosmological term is written in four dimensions as

$$I \; = \; \lambda_0 \, \Lambda^4 \int d^4 x \, \sqrt{g} \; - \; \frac{\Lambda^2}{16 \pi \, G} \int d^4 x \, \sqrt{g} \, R \; . \tag{15}$$

In this expression λ_0 is the bare cosmological constant and G the bare Newton's constant, both now written in units of the explicit ultraviolet cutoff Λ. Consequently, both of the above G and λ_0

are now dimensionless [a natural expectation is for the bare microscopic, dimensionless couplings not to be fine-tuned, and have magnitudes of order one, $\lambda_0 \sim G \sim O(1)$]. Now, again one can rescale the metric

$$g'_{\mu\nu} \; = \; \sqrt{\lambda_0} \, g_{\mu\nu} \qquad g'^{\mu\nu} \; = \; \frac{1}{\sqrt{\lambda_0}} \, g^{\mu\nu} \tag{16}$$

and thus obtain for the action

$$I \; = \; \Lambda^4 \int d^4 x \, \sqrt{g'} \; - \; \frac{\Lambda^2}{16\pi \, G \, \sqrt{\lambda_0}} \int d^4 x \, \sqrt{g'} \, R' \; . \tag{17}$$

The latter, for a given cutoff Λ, only depends on λ_0 and G through the dimensionless combination $G\sqrt{\lambda_0}$. Next consider again the Euclidean Feynman path integral of Eq. (1), here in four dimensions

$$Z \; = \; \int [d \, g_{\mu\nu}] \; \exp \left\{ - \int d^4 x \, \sqrt{g} \, \left(\lambda_0 \, \Lambda^4 \; - \; \frac{\Lambda^2}{16\pi \, G} \, R \right) \right\} \; . \tag{18}$$

Because of the scaling properties of the functional measure over metrics, and for a given cutoff Λ, Z itself also depends, up to an irrelevant overall multiplicative constant, on λ_0 and G only through the dimensionless combination $G\sqrt{\lambda_0}$; only the latter can play a role in the subsequent physics. One can then view a rescaling of the metric as simply a (largely inessential) redefinition of the ultraviolet cutoff Λ, $\Lambda \to \lambda_0^{-1/4} \Lambda$. Furthermore, the existence of a non-trivial ultraviolet fixed point for quantum gravity in four dimensions is entirely controlled by this dimensionless parameter only, both on the lattice [4, 5] and in the continuum [6].

It is clear from the structure of the path integral that the cosmological term controls the overall scale in the problem, while the curvature term provides the necessary derivative, or true coupling, term. But by a specific choice of overall scale one can set, without any loss of generality $\lambda_0 = 1$ in Eq. (18), and measure from now on all quantities in units of the UV cutoff Λ, which is the only remnant of this overall scale. [5] In addition, one can choose for ease of notation a unit cutoff $\Lambda = 1$, and later restore, if one so desires, the correct dimensionality of couplings and operators by suitably re-introducing appropriate powers of the UV cutoff Λ. Indeed this is a common, if not universal, procedure in lattice field theory and lattice gauge theory, where all quantities are measured in terms of a unit lattice spacing a. Since the total volume of space-time can hardly be considered a physical observable, quantum averages are in fact computed by dividing out by the total volume. Thus, for example, for the quantum expectation value of the Ricci scalar one writes

$$\mathcal{R} \; \equiv \; \frac{\langle \int d^4 x \, \sqrt{g(x)} \; R(x) \, \rangle}{\langle \int d^4 x \, \sqrt{g(x)} \, \rangle} \; . \tag{19}$$

[5] We recall here that our considerations here are not dissimilar from the case of a self-interacting scalar field, where one might want to introduce three couplings for the kinetic term, the mass term and the quartic coupling term, respectively. A simple rescaling of the field then reveals immediately that only two coupling ratios are in fact physically relevant.

The discussion so far has focused on pure gravity without matter fields. The addition of matter prompts one to do some further rescalings. Let us consider here for simplicity, and as an illustration, a single component scalar field, with action given by

$$I_S \;=\; \tfrac{1}{2} \int d^4x \sqrt{g} \; \{ g^{\mu\nu} \, \partial_\mu \, \phi \, \partial_\nu \, \phi \, + \, m_0^2 \, \phi^2 \, + \, R \, \phi^2 \} \tag{20}$$

and functional measure for ϕ

$$\int d\mu[\phi] \;=\; \int \prod_x \; [g(x)]^{1/4} \; d\phi(x) \; . \tag{21}$$

Then, if the the metric rescaling of Eq. (16) is followed by a field rescaling

$$\phi'(x) \;=\; \frac{1}{\lambda_0^{1/4}} \, \phi(x) \; , \tag{22}$$

one sees that the only surviving change is a rescaling of the bare mass, $m_0 \to m_0 \, \lambda_0^{1/4}$. In addition, the scalar field functional measure acquires an irrelevant multiplicative factor, which as stated before cannot affect quantum averages.

Let us add here one further comment. Pure gravity corresponds to a massless graviton, a property that is presumably preserved to all orders in perturbation theory, if diffeomorphism invariance is maintained. Nevertheless it is easy to see that formally the cosmological term, at least in the weak field limit, acts as a mass-like term. In the weak field expansion around flat space one has for the cosmological contribution in Eq. (5)

$$\sqrt{g} = 1 + \tfrac{1}{2} \, h_\mu^{\ \mu} + \tfrac{1}{8} \, h_\mu^{\ \mu} h_\nu^{\ \nu} - \tfrac{1}{4} \, h_{\mu\nu} h^{\mu\nu} + O(h^3) \; , \tag{23}$$

after setting $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $\eta_{\mu\nu}$ the flat metric. The first contribution $O(h)$ shifts the vacuum solution to de Sitter space, while the next terms $O(h^2)$ provide a mass-like quadratic contributions. It is tempting therefore to still regard, in some ways, this last term as analogous to some sort of mass term, with nevertheless the rather important property that it does not lead to an explicit breaking of general covariance. One would then expect that such a mass-like term could provide naturally, in a general renormalization framework, a suitable candidate for a renormalization group invariant quantity, in analogy to the dynamically generated mass parameter in non-Abelian gauge theories, which also generates a dynamical mass, without violating local gauge invariance. How this might come about will be expanded on further below.

3 Gauge Dependence in the Renormalization of the Cosmological Constant

Perturbation theory generally serves a very useful purpose, since it allows one to systematically track the gauge dependence of various renormalization effects, and determine what is physical and what is a spurious gauge artifact. Unfortunately Einstein gravity is not perturbatively renormalizable in four dimensions, so that easy route is not available. Nevertheless, if one goes down in dimensions it is possible to rescue in part the perturbative treatment, and therefore address some of the key issues raised earlier. One does not, of course, expect the answers to be quantitatively correct; nevertheless it will become clear below that the issue of gauge invariance does comes up, and is eventually successfully resolved. Let us emphasize here that one key aspect of the perturbative treatment via the background field method is that diffeomorphism invariance is preserved throughout the calculation, as in the case of non-Abelian gauge theories.

In two dimensions the gravitational coupling is dimensionless, $G \sim \Lambda^{2-d}$ and the theory appears perturbatively renormalizable. In spite of the fact that the gravitational action reduces to a topological invariant in two dimensions, it is meaningful to try to construct, in analogy to Wilson's original suggestion for scalar field theories, the theory perturbatively as a double series in $\epsilon = d - 2$ and G [7, 8]. The $2 + \epsilon$ expansion for pure gravity then proceeds as follows [6]. First the gravitational part of the action

$$\mathcal{L} = - \frac{\mu^{\epsilon}}{16\pi \, G} \sqrt{g} \, R \quad , \tag{24}$$

with G dimensionless and μ an arbitrary momentum scale, is expanded in the fields by setting

$$g_{\mu\nu} \; \rightarrow \; \bar{g}_{\mu\nu} = g_{\mu\nu} \, + \, h_{\mu\nu} \tag{25}$$

where $g_{\mu\nu}$ is the classical background field and $h_{\mu\nu}$ the small quantum fluctuation. The quantity $\mathcal{L}$ in Eq. (24) is naturally identified with the bare Lagrangian, and the scale μ with a microscopic ultraviolet cutoff Λ; the latter would be identified with the inverse lattice spacing in a lattice formulation. To make perturbation theory convergent requires a gauge fixing term, chosen by the authors of [6] in the form of a generalized background harmonic gauge condition,

$$\mathcal{L}_{gf} \; = \; \tfrac{1}{2} \, \alpha \, \sqrt{g} \; g_{\nu\rho} \left(\nabla_{\mu} h^{\mu\nu} - \tfrac{1}{2} \, \beta \, g^{\mu\nu} \nabla_{\mu} h \right) \left(\nabla_{\lambda} h^{\lambda\rho} - \tfrac{1}{2} \, \beta \, g^{\lambda\rho} \nabla_{\lambda} h \right) \tag{26}$$

with $h^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} h_{\alpha\beta}$, $h = g^{\mu\nu} h_{\mu\nu}$ and ∇_{μ} the covariant derivative with respect to the background metric $g_{\mu\nu}$. The gauge fixing term then requires the introduction of a Faddeev-Popov ghost contribution $\mathcal{L}_{ghost}$ containing the ghost field ψ_{μ}, so that the total Lagrangian becomes a sum of three terms, $\mathcal{L}+\mathcal{L}_{gf}+\mathcal{L}_{ghost}$. In a flat background, $g^{\mu\nu}=\delta^{\mu\nu}$, one obtains, from the quadratic part of the Lagrangian of Eqs. (24) and (26), a rather complicated expression for the graviton propagator [6]

$$\begin{aligned}\langle\, h_{\mu\nu}(k) h_{\alpha\beta}(-k)\,\rangle \;=\; & \frac{1}{k^2}\left(\delta_{\mu\alpha}\delta_{\nu\beta}+\delta_{\mu\beta}\delta_{\nu\alpha}\right)-\frac{2}{d-2}\,\frac{1}{k^2}\,\delta_{\mu\nu}\delta_{\alpha\beta} \\ & -\left(1-\frac{1}{\alpha}\right)\frac{1}{k^4}\left(\delta_{\mu\alpha}k_\nu k_\beta+\delta_{\nu\alpha}k_\mu k_\beta+\delta_{\mu\beta}k_\nu k_\alpha+\delta_{\nu\beta}k_\mu k_\alpha\right) \\ & +\frac{1}{d-2}\,\frac{4(\beta-1)}{\beta-2}\,\frac{1}{k^4}\left(\delta_{\mu\nu}k_\alpha k_\beta+\delta_{\alpha\beta}k_\mu k_\nu\right) \\ & +\frac{4\,(1-\beta)}{(\beta-2)^2}\left[2-\frac{3-\beta}{\alpha}-\frac{2\,(1-\beta)}{d-2}\right]\frac{1}{k^6}\,k_\nu k_\nu k_\alpha k_\beta \;\; . \end{aligned} \tag{27}$$

Normally it would be convenient to choose a gauge $\alpha=\beta=1$, in which case only the first two terms for the graviton propagator survive [8]. But it is in fact rather advantageous to leave the two gauge parameters unspecified, so that later the detailed gauge dependence of the result can be checked. In particular, the gauge parameter β is related to the gauge freedom associated with a rescaling the metric $g_{\mu\nu}$, as described in the previous section. For the one loop divergences associated with the $\sqrt{g}$ term they obtain

$$\lambda_0 \to \lambda_0\left[1-\left(\frac{a_1}{\epsilon}+\frac{a_2}{\epsilon^2}\right)G\right] \; , \tag{28}$$

with coefficients

$$\begin{aligned} a_1 &= -\frac{8}{\alpha}+8\,\frac{(\beta-1)^2}{(\beta-2)^2}+4\,\frac{(\beta-1)(\beta-3)}{\alpha\,(\beta-2)^2} \\ a_2 &= 8\,\frac{(\beta-1)^2}{(\beta-2)^2} \; . \end{aligned} \tag{29}$$

On the other hand, for the one-loop divergences associated with the $\sqrt{g}R$ term one finds

$$\frac{\mu^\epsilon}{16\pi\,G} \to \frac{\mu^\epsilon}{16\pi\,G}\left(1-\frac{b}{\epsilon}\,G\right) \tag{30}$$

with coefficient b given by [8]

$$b=\frac{2}{3}\cdot 19+\frac{4(\beta-1)^2}{(\beta-2)^2} \; . \tag{31}$$

Thus the one-loop radiative corrections modify the total Lagrangian to

$$\mathcal{L} \to -\frac{\mu^\epsilon}{16\pi\,G}\left(1-\frac{b}{\epsilon}G\right)\sqrt{g}\,R+\lambda_0\left[1-\left(\frac{a_1}{\epsilon}+\frac{a_2}{\epsilon^2}\right)G\right]\sqrt{g}\; . \tag{32}$$

Next one can make use of the freedom to rescale the metric, by setting

$$\left[1-\left(\frac{a_1}{\epsilon}+\frac{a_2}{\epsilon^2}\right)G\right]\sqrt{g} \;=\; \sqrt{g'}\; , \tag{33}$$

which restores the original unit coefficient for the cosmological constant term. The rescaling is achieved by the field redefinition

$$g_{\mu\nu} \;=\; \left[1-\left(\frac{a_1}{\epsilon}+\frac{a_2}{\epsilon^2}\right)G\right]^{-2/d} \, g'_{\mu\nu} \; . \tag{34}$$

By this procedure the cosmological term is brought back into its standard form $\lambda_0\sqrt{g'}$, and one obtains for the complete Lagrangian to first order in G

$$\mathcal{L} \rightarrow -\frac{\mu^{\epsilon}}{16\pi\, G}\left[1-\frac{1}{\epsilon}(b-\tfrac{1}{2}a_2)G\right]\sqrt{g'}\,R' + \lambda_0\sqrt{g'} \; , \tag{35}$$

where only terms singular in ϵ have been retained. From this last result one can finally read off the renormalization of Newton's constant

$$\frac{1}{G} \rightarrow \frac{1}{G}\left[1-\frac{1}{\epsilon}(b-\tfrac{1}{2}a_2)\,G\right] \; . \tag{36}$$

From Eqs. (29) and (31) one notices that the a_2 contribution cancels out the gauge-dependent part of b, giving for the remaining contribution $b-\frac{1}{2}a_2=\frac{2}{3}\cdot 19$. Therefore the gauge dependence has, as one would have hoped for on physical grounds, entirely disappeared from the final answer. The reason for this miraculous cancellation is of course general covariance. But the main point we wish to make here is that the results of covariant perturbation theory are, as expected, entirely consistent with the scaling argument given in the previous section: only the renormalization of G has physical meaning. Let us dwell further on this aspect.

In the presence of an explicit renormalization scale parameter μ the Callan-Symanzik β-function for pure gravity is obtained by requiring the independence of the effective coupling G from the original renormalization scale μ. One obtains to one loop order

$$\mu\frac{\partial}{\partial\mu}\,G(\mu) \;\equiv\; \beta(G) = (d-2)\,G \;-\; \beta_0\,G^2 \;+\; O(G^3,(d-2)G^2) \tag{37}$$

with here $\beta_0=\frac{2}{3}\cdot 19$ in the absence of matter. From the procedure outlined above it is clear that G is the only coupling that is scale-dependent in pure gravity. Depending on whether one is on the right ($G>G_c$) or on the left ($G<G_c$) of the non-trivial ultraviolet fixed point at

$$G_c = \frac{d-2}{\beta_0} + O((d-2)^2) \tag{38}$$

the coupling will either flow to increasingly larger values of G, or flow towards the Gaussian fixed point at $G=0$, respectively. In the following we will refer to the two phases as the strong and weak coupling phase, respectively. Perturbatively one only has control on the small G regime.

The running of G as a function of a sliding momentum scale $\mu = k$ in pure gravity is obtained from integrating Eq. (37), giving

$$G(k) \simeq G_c \left[1 \pm c_0 \left(\frac{m^2}{k^2} \right)^{(d-2)/2} + \dots \right] \tag{39}$$

with c_0 a positive constant, and $m = \xi^{-1}$ a mass scale that arises as an integration constant of the renormalization group equations. The k^2-dependent contribution on the r.h.s of Eq. (39) is the quantum correction, which at least within a perturbative framework is assumed to be small. The choice of $+$ or $-$ sign is determined from whether one is to the left (-), or to right (+) of G_c, in which case the effective $G(k)$ decreases or, respectively, increases as one flows away from the ultraviolet fixed point towards lower momenta or larger distances. Physically the two solutions represent a screening ($G < G_c$) and an anti-screening ($G > G_c$) situation. While in the above continuum perturbative calculation both phases, and therefore both signs, seem acceptable, the Euclidean and Lorentzian lattice results on the other hand rule out the weak coupling phase as pathological, in the sense that there the lattice collapses into a two-dimensional degenerate object [4, 3].

The k^2-dependent quantum correction in Eq. (39) involves a new physical, renormalization group invariant scale $\xi = 1/m$ which cannot be fixed perturbatively, and whose size then determines the distance scale relevant for quantum effects. In terms of the bare coupling $G(\Lambda)$, it is given by

$$\xi^{-1}(G) \equiv m = \Lambda \cdot A_m \exp \left(- \int^{G(\Lambda)} \frac{dG'}{\beta(G')} , \right) \tag{40}$$

with A_m a constant. Note the rather remarkable fact that one scale has disappeared (λ_0), and a new one has appeared dynamically (ξ). The above expression is obtained by integrating the RG equation $\mu \frac{\partial}{\partial \mu} G = \beta(G)$, and then choosing the arbitrary momentum scale $\mu \to \Lambda$. Conversely, $\xi^{-1} = m$ is an RG invariant and one has

$$\Lambda \frac{d}{d\Lambda} m(\Lambda, G(\Lambda)) = \mu \frac{d}{d\mu} m(\mu, G(\mu)) = 0 . \tag{41}$$

The running of $G(\mu)$ in accordance with the renormalization group equation of Eq. (37) ensures that the l.h.s. is indeed a renormalization group invariant. It is known that the constant A_m on the r.h.s. of Eq. (40) cannot be determined perturbatively, it needs to be computed by nonperturbative (lattice) methods, for example by evaluating invariant correlations at fixed geodesic distances. It is related to the constant c_0 in Eq. (39) by $c_0 = 1/(A_m^{1/\nu} G_c)$. In the vicinity of the ultraviolet fixed point at G_c, for which $\beta(G_c) = 0$, one can write

$$\beta(G) \equiv \mu \frac{\partial}{\partial \mu} G(\mu) \underset{G \to G_c}{\sim} \beta'(G_c)\,(G - G_c) + \dots \quad , \tag{42}$$

which by integration gives

$$\xi^{-1}(G) \;\propto\; \Lambda \, | \, (G - G_c)/G_c \, |^{\nu} \;, \tag{43}$$

with correlation length exponent $\nu = -1/\beta'(G_c)$; to lowest order perturbation theory $\nu = 1/(d - 2) + \ldots$. Note that the magnitude of ξ is not determined by the magnitude of G. Instead, it is determined by the distance of the bare G from the UV fixed point value G_c, and as such it can be *arbitrarily* large.

More recently the one-loop perturbative calculation described above were laboriously extended to two loops [9]. One important result that stays true to two loops is the fact that the only meaningful, gauge-independent renormalization is the one of G, which is not surprising in view of the general arguments given previously. One can then compute the roots $\beta(G_c) \; = \; 0$ and obtain the location of the ultraviolet fixed point, and from it the universal exponent $\nu = -1/\beta'(G_c)$. One finds for the scaling exponent ν in the presence of c scalar matter fields

$$\nu^{-1} \;=\; (d-2) \,+\, \frac{15}{25-c} \, (d-2)^2 \,+\, \ldots \;. \tag{44}$$

In four dimensions this gives for pure gravity without matter ($c = 0$) to lowest order $\nu = 1/2$, and $\nu = 5/22 \approx 0.23$ at the next order. Numerical simulations for the lattice theory of gravity in four dimensions give on the other hand $\nu = 1/3$ and $c_0 \approx 3.0$ [10].

In closing we mention that, so far, the discussion of quantum gravity has focused mainly on the perturbative scenario, where the gravitational coupling G is assumed to be weak, so that the weak field expansion can be pursued with some degree of reliability. Then at every order in the loop expansion the problem reduces to the evaluation of an increasingly complicated sequence of Gaussian integrals over some small quantum fluctuation in the fields. Nevertheless a bit of thought reveals that to all orders in the weak field expansion there is really no difference of substance between the Lorentzian (or pseudo-Riemannian) and the Euclidean (or Riemannian) formulation. The structure of the divergences would have been identical, and the renormalization group properties of the coupling the same (up to the trivial replacement of say the Minkowski momentum q^2 by its Euclidean expression $q^2 = q_0^2 + \mathbf{q}^2$ etc.). Thus, up to this point, no significant difference has appeared between the Euclidean and the Lorentzian treatment. We should also add that most of the above conclusions remain largely unchanged when higher derivative terms are included [11, 12, 13].

To summarize the results so far, we have shown that the path integral for pure quantum gravity depends only on one dimensionless combination of couplings, $G\sqrt{\lambda_0}$ in $d = 4$, and that the bare λ_0 can be entirely scaled out of the path integral, and out of the physics. It is also clear that the only renormalization (and beta function) that is gauge-independent and physically meaningful is the one for Newton's constant G. Finally, we have emphasized the fact that the very same, manifestly covariant, renormalization group treatment clearly shows the appearance of a new dynamically generated scale ξ [Eq. (40)].

4 Lattice Functional Integral and Role of the Volume Term

One might view some of the discussion of the previous sections as rather formal. The Feynman path integral for quantum gravitation, Eq. (1), is formally defined in the continuum and involves rather delicate expressions such as $\prod_x$ in the measure. Perturbation theory in the continuum is then done by performing Gaussian integrals for small metric perturbations, using dimensional regularization to manipulate the resulting divergent integrals. It seems useful therefore to revisit here the same kind of issues, as they arise in the context of the lattice theory. In the Regge-Wheeler formulation of lattice gravity [14, 15] the infinite number of degrees of freedom of the continuum gravitational field is restricted by considering piecewise-linear Riemannian spaces described by a finite number of variables, the geodesic distances between neighboring points. It is known to be the only lattice formulation of gravity containing transverse-traceless modes [16] and a local continuous lattice diffeomorphism invariance [16, 17]. An Euclidean path integral formulation can then be built based on a curvature action, supplemented by a cosmological term for convergence, and possibly higher derivative contributions [4, 5]. Following Regge, one writes for the Euclidean lattice action for pure gravity

$$I_R \;=\; -\,k \sum_{\text{hinges h}} \delta_h(l^2)\, A_h(l^2) \quad , \tag{45}$$

with bare coupling constant $k^{-1} = 8\pi\, G$. In four dimensions the sum over hinges h is equivalent to a sum over all lattice triangles; geometrically, the above action contains a sum of elementary loop contributions, since it contains as its primary ingredient the deficit angle δ_h associated with an elementary parallel transport loop around the hinge h. The deficit angle δ_h is related to the local scalar curvature by

$$R(h) \;=\; 2\, \frac{\delta_h}{A_C(h)} \tag{46}$$

where $A_C(h)$ is the area associated with an elementary parallel transport loop around the hinge (triangle) h, defined by joining the vertices of an elementary polyhedron C located in the dual lattice. In view of the following discussion one should note that, as in the continuum, the local lattice curvature has dimensions of length to the power minus two. The continuum curvature density $\sqrt{g}\, R$ is then obtained by multiplication with the volume element V_h associated with a hinge h, with the lattice Riemann tensor at a hinge h given by

$$R_{\mu\nu\lambda\sigma}(h) \;=\; \frac{\delta_h}{A_C(h)}\, U_{\mu\nu}(h)\, U_{\lambda\sigma}(h) \;. \tag{47}$$

Here $U_{\mu\nu}(h)$ is unit bivector defined for a single hinge h

$$U_{\mu\nu}(h) \;=\; \frac{1}{2A_h}\; \epsilon_{\mu\nu\alpha\beta}\, l_1^\alpha\, l_2^\beta \;, \tag{48}$$

with $l_1(h)$ and $l_2(h)$ two independent edge vectors associated with the hinge (triangle) h, and A_h the area of the hinge itself (a triangle in four dimensions). Again it is customary, as in lattice gauge theories, to set the lattice ultraviolet cutoff equal to one (i.e. measure all length scales in units of a fundamental lattice cutoff a or Λ; as an example, on a hypercubic lattice in d dimensions the two are simply related by $\Lambda = \pi/a$). Next consider the cosmological constant term, which in the continuum theory takes the form $\lambda_0 \int d^d x \sqrt{g}$. On the lattice it involves the total volume of the simplicial complex

$$V \;=\; \sum_{\text{simplices s}} V_s(l^2) \;. \tag{49}$$

In four dimensions the sum here is over all lattice four-simplices, the 4d analogs of tetrahedra. Thus one may regard the local volume element in d dimensions, $\sqrt{g}\, d^d x$, as being represented by V_s, centered on the simplex s. The curvature and cosmological constant terms then lead to the combined action

$$I_{\text{latt}} \;=\; \lambda_0 \sum_{\text{simplices s}} V_s(l^2) \;-\; k \sum_{\text{hinges h}} \delta_h(l^2)\, A_h(l^2) \;. \tag{50}$$

The action then only couples edges which belong either to the same simplex or to a set of neighboring simplices, and is therefore local, as the continuum action.

A lattice regularized version of the Euclidean Feynman path integral is then given by

$$Z_{\text{latt}} \;=\; \int [d\, l^2]\; \exp\left(-I_{\text{latt}}(l^2)\right) \quad , \tag{51}$$

where $[d\, l^2]$ is an appropriate functional integration measure over squared edge lengths l_{ij}^2. For concreteness, the local functional measure will be here of the form

$$\int [d\, l^2] \;\equiv\; \int_0^\infty \prod_{ij} dl_{ij}^2\, \prod_s \left[V_s(l^2)\right]^\sigma\, \Theta[l_{ij}^2] \;\; . \tag{52}$$

The last expression represents a rather non-trivial quantity, both in view of the complexity of the formula for the volume of a simplex, and because of the generalized triangle inequality constraints implicit in $[d\, l^2]$, given that the function $\Theta[l_{ij}^2]$ here represents a theta-function type constraint on the edge lengths. The latter is inserted in order to ensure that the triangle inequalities and their higher dimensional analogs are satisfied [4, 5]. The measure in Eq. (52) should then be considered the lattice analogue of the gravitational DeWitt measure of Eq. (2). Like the continuum functional measure, it is local to the extent that each edge length appears only in the expression for the volume of those simplices which explicitly contain it. Then the lattice partition function Z_{latt} should in turn be regarded as a discretized, and properly regularized, form of the continuum Euclidean Feynman path integral given in Eq. (1).

As in the continuum, the curvature contribution to the lattice action [Eq. (45)] contains the proper kinetic (derivative) term, which then leads to a set of suitable propagating degrees of freedom, the lattice transverse-traceless modes [16]. Such a term provides the necessary coupling between neighboring lattice metrics, nevertheless the interaction still remains local. Moreover, due to the presence of the triangle area term A_h, the curvature term in the action scales like a length squared: if all the edge lengths are rescaled by a common factor ω,

$$l_i \; \rightarrow \; \omega \, l_i \quad , \tag{53}$$

then the curvature part of the action is simply rescaled by an overall factor of ω^2. The latter can then be reabsorbed into a rescaling of the coupling G, just as in the continuum [see Eq. (6)]. On the other hand, the cosmological term is just the total four-volume of space-time. As such, it does not contain any derivatives (or finite differences) of the metric and is completely local; it does not contribute to the propagation of gravitational degrees of freedom and is thus more akin to a mass term (as is already clear from the weak field expansion of $\int \sqrt{g}$ in the continuum). This volume term scales like a length to the fourth power: if all the edge lengths are rescaled by a common factor ω, $l_i \rightarrow \omega \, l_i$, then the volume term is simply rescaled by an overall factor of ω^4. Again, this effect can be entirely reabsorbed into a rescaling of the bare cosmological constant λ_0, as in the continuum [see Eq. (6)]. [6]

We note now that, as in the continuum case, the above considerations regarding the scaling properties of the lattice gravitational action are not spoiled by the functional lattice measure in Eq. (52). As one can see by inspection. when all the edge lengths are rescaled by a common factor, the contribution from the functional measure is simply multiplied by a constant factor involving ω to some power (which will depend on the overall number of lattice points and on the choice measure parameter σ); such a factor then drops out when evaluating expectation values. More importantly, the overall length scale in the problem is controlled by the parameter λ_0; changing the value of λ_0 simply, and trivially, changes this overall scale, without affecting in any way the

[6] Note that convergence of the Euclidean lattice functional integral nevertheless requires a positive bare cosmological constant, $\lambda_0 > 0$.

underlying physics: any change in λ_0 simply reflects itself in a change in the average fundamental lattice spacing (or average local volume). As such, this change is *physically irrelevant*. Indeed, and in accordance with the methods of quantum field theory and statistical field theory, one would like to discuss renormalization group properties of the theory in a box of fixed total volume and fixed UV cutoff. Allowing a change in the overall volume of the box, or changing, equivalently, the value of the UV cutoff or lattice spacing, only hopelessly (and unnecessarily) confuses the whole renormalization issue. Of course, in a traditional renormalization group approach to field theory, the overall four-volume is always kept fixed while the scale (or q^2) dependence of the action and couplings are investigated.

It seems therefore again rather meaningless to allow the coupling λ_0 to run; the overall space-time volume is intended to stay fixed within the RG procedure, and not to be rescaled as well under a renormalization group transformation. Indeed, in the spirit of Wilson, a renormalization group transformation allows a description of the original physical system in terms of a new coarse grained Hamiltonian, whose new operators are interpreted as describing averages of the original system on a finer scale - but of course still within the same very large physical volume. The only scale change in this procedure is from the coarse-scale averaging, renormalization or block-spinning itself, to use here three roughly equivalent terms. The new effective Hamiltonian is then still supposed to describe the original physical system, but does so more economically in terms of a reduced set of effective degrees of freedom. Consequently one can take the lattice coupling $\lambda_0 = 1$ without *any* loss of generality, since different values of λ_0 just correspond to a trivial rescaling of the overall four-volume. Alternatively, one could even choose an ensemble for which the probability distribution in the total four-volume V is

$$\mathcal{P}(V) \;\propto\; \delta(V - V_0) \;, \tag{54}$$

in analogy with the microcanonical ensemble of statistical mechanics. We conclude that the results from the lattice theory of gravity *completely* mirror, and underpin, the discussion done for the continuum theory in Sec. (2). The lattice theory is shown to depend, in any dimension, on one coupling only, the dimensionless combination of G and λ_0; in four dimensions this quantity is given by $G\sqrt{\lambda_0}$. We have also given evidence that, without any loss of generality, one can take in the lattice regularized theory $\lambda_0 = 1$ in units of the UV cutoff, so that the theory depends simply on *one* coupling G only.

On the lattice one finds that the running of G is very similar in structure to what is obtained in the $2 + \epsilon$ expansion, even though the procedure followed in obtaining such a result is completely different: on the lattice the scale dependence of G is extracted from the cutoff (Λ) dependence of the bare coupling $G(\Lambda)$ at fixed physical correlation length ξ [10]. One finds, in the language of Eq. (39),

$$G(k) \;\simeq\; G_c \left[1 \,+\, c_0 \, \left(\xi^2 \, k^2 \right)^{-1/2\nu} \,+\, \dots \right] \tag{55}$$

with c_0 a positive constant, ξ again the length scale that arises as an integration constant of the renormalization group equations, and exponent $\nu = 1/3$. Here the k^2-dependent correction can be compared directly to the result given previously in Eq. (39). Note that on the lattice only the +-sign is realized, which corresponds to gravitational anti-screening. This is because the Euclidean lattice results show that the weak coupling phase ($G > G_c$) is pathological, corresponding not to a four-dimensional lattice but, instead, to a two-dimensional degenerate object [4, 3].

Here too the k^2-dependent quantum correction in Eq. (55) involves a new nonperturbative renormalization group invariant scale ξ, which determines the overall scale of quantum effects. In terms of the bare coupling $G(\Lambda)$, it is given, in the vicinity of fixed point k_c, by

$$\xi^{-1}(k) \;\underset{k \to k_c}{\sim}\; A_m \, \Lambda \, | \, k_c - k \, |^{\nu} \; , \tag{56}$$

with correlation length exponent $\nu = 1/3$. A determination of the constant of proportionality A_m in Eq. (56) then fixes the coefficient c_0 in the running of G Eq. (55), since the two coefficients are related by $c_0 = 1/(k_c A_m^{1/\nu})$. By large scale numerical simulations for the lattice theory of gravity in four dimensions one finds explicitly $k_c = 0.06381(9)$, $\nu = 0.332(2)$ and $c_0 = 3.02(38)$ [10].

5 The Gauge Theory Analogy

QED and QCD provide two invaluable illustrative cases where the running of the gauge coupling with energy is not only theoretically well understood, but also verified experimentally. As in QED, in QCD (and related Yang-Mills theories) radiative corrections are known to alter significantly the behavior of the static potential at short distances. Changes in the potential are best expressed in terms of the running strong coupling constant $\alpha_S(\mu)$, whose scale dependence is determined by the celebrated beta function of $SU(3)$ QCD with n_f light fermion flavors

$$\mu \, \frac{\partial \, \alpha_S}{\partial \, \mu} \;=\; 2 \, \beta(\alpha_S) \;=\; - \frac{\beta_0}{2 \, \pi} \, \alpha_S^2 \;-\; \frac{\beta_1}{4 \, \pi^2} \, \alpha_S^3 \;-\; \frac{\beta_2}{64 \, \pi^3} \, \alpha_S^4 \;-\; \dots \; . \tag{57}$$

with coefficients $\beta_0 = 11 - \frac{2}{3} n_f$, $\beta_1 = 51 - \frac{19}{3} n_f$, and $\beta_2 = 2857 - \frac{5033}{9} n_f + \frac{325}{27} n_f^2$, and $\alpha_S \equiv g^2/4\pi$. The solution of the renormalization group equation Eq. (57) then gives for the running of $\alpha_S(\mu)$

$$\alpha_S(\mu) \;=\; \frac{4 \, \pi}{\beta_0 \ln \mu^2 / \Lambda_{\overline{MS}}^2} \left[1 \;-\; \frac{2 \beta_1}{\beta_0^2} \, \frac{\ln \left[\ln \mu^2 / \Lambda_{\overline{MS}}^2 \right]}{\ln \mu^2 / \Lambda_{\overline{MS}}^2} \;+\; \dots \right] \; . \tag{58}$$

The nonperturbative scale $\Lambda_{\overline{MS}}$ appears as an integration constant of the renormalization group equations, and is therefore - by construction - scale independent. Indeed, the physical value of $\Lambda_{\overline{MS}}$ cannot be fixed from perturbation theory alone, and needs to be determined from experiment, which gives $\Lambda_{\overline{MS}} \simeq 213 MeV$. In principle, one can solve for $\Lambda_{\overline{MS}}$ in terms of the coupling at any scale, and in particular at the cutoff scale Λ, obtaining

$$
\begin{aligned}
\Lambda_{\overline{MS}} &= \Lambda \exp \left(- \int^{\alpha_S(\Lambda)} \frac{d\alpha'_S}{2\,\beta(\alpha'_S)} \right) \\
&= \Lambda \left(\frac{\beta_0 \, \alpha_S(\Lambda)}{4\,\pi} \right)^{\beta_1/\beta_0^2} e^{-\frac{2\,\pi}{\beta_0\,\alpha_S(\Lambda)}} \left[1 + O(\alpha_S(\Lambda)) \right] .
\end{aligned} \tag{59}
$$

Not all physical properties can be computed reliably in weak coupling perturbation theory. In non-Abelian gauge theories a confining potential is found at strong coupling by examining the behavior of the Wilson loop, defined for a large closed loop C as

$$
\langle \, W(C) \, \rangle \; = \; \langle \, \mathrm{tr} \, \mathcal{P} \, \exp \Big\{ ig \oint_C A_\mu(x) dx^\mu \Big\} \, \rangle \ , \tag{60}
$$

with $A_\mu \equiv t_a A_\mu^a$ and the t_a's the group generators of $SU(N)$ in the fundamental representation. In the pure gauge theory at strong coupling, the leading contribution to the Wilson loop can be shown to follow an area law for sufficiently large loops

$$
\langle \, W(C) \, \rangle \underset{A \to \infty}{\sim} \exp(-A(C)/\xi^2) \ , \tag{61}
$$

where $A(C)$ is the minimal area spanned by the planar loop C [18]. The quantity ξ is the gauge field correlation length, and is essentially the same [up to a factor $O(1)$] as the inverse of $\Lambda_{\overline{MS}}$ in Eq. (59). The point here is that non-Abelian gauge theories are known to contain a new, fundamental, dynamically generated length scale, in clear analogy to the result of Eq. (40) for gravity. The universal quantity ξ also appears in a number of other physical observables, including the exponential decay of the Euclidean correlation function of two infinitesimal loops separated by a distance $|x|$,

$$
G_{\mathrm{loop-loop}}(x) \; = \; \langle \, \mathrm{tr} \, \mathcal{P} \exp \Big\{ ig \oint_{C_\epsilon} A_\mu(x') dx'^\mu \Big\} (x) \; \mathrm{tr} \, \mathcal{P} \exp \Big\{ ig \oint_{C_\epsilon} A_\mu(x'') dx''^\mu \Big\} (0) \, \rangle_c \ . \tag{62}
$$

Here the C_ϵ's are two infinitesimal loops centered around x ands 0 respectively, suitably defined on the lattice as elementary square loops, and for which one has at sufficiently large separations

$$
G_{\mathrm{loop-loop}}(x) \underset{|x| \to \infty}{\sim} \exp(-|x|/\xi) \ . \tag{63}
$$

It is also understood that the inverse of the correlation length ξ corresponds to the lowest gauge invariant mass excitation in the gauge theory, the scalar glueball with mass $m_0 = 1/\xi$. As in the case of gravity [see for comparison Eq. (41)], the correlation length ξ, or equivalently its inverse $m \equiv 1/\xi$, is known to be a renormalization group invariant,

$$\Lambda \frac{d}{d\Lambda} m(\Lambda, g(\Lambda)) \;=\; 0 \; . \tag{64}$$

If one writes $m \equiv \Lambda \cdot F(g(\Lambda))$, one obtains immediately

$$\beta(g) \;=\; -\frac{F(g)}{F'(g)} \; , \tag{65}$$

which then relates, in a direct and explicit way, the dependence of the correlation length on the bare coupling g [through the function $F(g)$] to the Callan-Symanzik beta function determining the running of the gauge coupling g [18]. The point here is that in gauge theories the emergence of a nonperturbative length scale can be viewed as a direct consequence of the RG equations, and of the rather subtle renormalization group behavior of the gauge coupling α_S. It is the combination of these effects that then leads to an entirely new physical quantum vacuum.

6 Gravitational Wilson Loop

Since the bare cosmological constant can be entirely scaled out of the theory, the legitimate question arises: how can a non-vanishing (and indeed small) effective large-scale cosmological constant arise out of the field-theoretic treatment of quantum gravity? The key to this answer lies in the fact that the lattice field theory itself contains an entirely new dynamically generated scale ξ, see Eqs. (40) and (43).

To see this, consider elementary parallel transports on the lattice. Between any two neighboring pair of simplices $s, s+1$ one can associate a Lorentz transformation $\mathbf{R}^{\mu}{}_{\nu}(s, s+1)$, which describes how a given vector V^{μ} transforms between the local coordinate systems in these two simplices. Such a transformation is directly related to the continuum path-ordered (P) exponential of the integral of the local affine connection $\Gamma^{\lambda}_{\mu\nu}(x)$ via

$$R^{\mu}{}_{\nu} \;=\; \Big[P \, e^{\int_{\textbf{between simplices}}^{\textbf{path}} \Gamma_{\lambda} dx^{\lambda}} \Big]^{\mu}{}_{\nu} \; , \tag{66}$$

with the connection having support on the common interface between the two simplices. Next, and in analogy to gauge theories, one can consider a closed lattice path passing through a large number of simplices s, and spanning a large near-planar closed loop C. Along this closed loop the overall rotation matrix is given by

$$R^{\mu}{}_{\nu}(C) \;=\; \Big[\prod_{s \subset C} \mathbf{R}_{s,s+1} \Big]^{\mu}{}_{\nu} \tag{67}$$

In a semi-classical picture, if the curvature of the manifold is taken to be small, the expression for the full rotation matrix $\mathbf{R}(C)$ associated with the large near-planar loop can be re-written in terms of a surface integral of the large-scale Riemann tensor, projected along the surface area element bivector $A^{\alpha\beta}(C)$ associated with the loop,

$$R^{\mu}{}_{\nu}(C) \approx \left[e^{\frac{1}{2} \int_S R^{\cdot}{}_{\cdot \alpha\beta} \, A^{\alpha\beta}(C)} \right]^{\mu}{}_{\nu} \, . \tag{68}$$

Thus, on the one hand, the gravitational Wilson loop provides a way of determining the effective curvature at large distance scales.

Next consider the case of large metric fluctuations at strong coupling (large G). The expectation value of the gravitational Wilson loop was defined in [19] as

$$\langle \, W(C) \, \rangle \; = \; \langle \, \mathrm{tr}[\, B_C \, \mathbf{R}_1 \, \mathbf{R}_2 \, \ldots \ldots \, \mathbf{R}_n \,] \, \rangle \;\; , \tag{69}$$

where the $\mathbf{R}_i$s are the rotation matrices along the path, and B_C related to a constant bivector characterizing the geometric orientation of the loop C, which again is assumed to be near-planar. One can then show, by using known properties of the Haar measure for the rotation group, that, at least for strong coupling and large area, the Wilson loop follows an area law, $\langle \, W(C) \, \rangle \sim \exp(-\, \mathrm{const.} \, A_C)$. This last result follows from tiling the interior of the given loop by a minimal surface built up of elementary transport loops, in close analogy to the gauge theory case. For strong coupling (large G) one can write more generally the result as [19]

$$\langle \, W(C) \, \rangle \; \underset{A \to \infty}{\sim} \; \exp(-\, A_C / \xi^2) \tag{70}$$

with ξ determined, by scaling and dimensional arguments, to be the nonperturbative gravitational correlation length [see Eq. (40)], and again in close analogy to the gauge theory result of Eq. (61). In the following we shall assume, in close analogy to what is known to happen in non-Abelian gauge theories due to scaling, that even though the above form for the gravitational Wilson loop was derived in the extreme strong coupling limit, it will remain valid throughout the whole strong coupling phase and all the way up to the nontrivial ultraviolet fixed point, with the correlation length $\xi \to \infty$ the only relevant, and universal, length scale in the vicinity of such a fixed point. The evidence for the existence of such an UV fixed point comes from three different sources, which have recently been reviewed, for example, in [3] and references therein. In the gravitational Wilson loop result of Eq. (70), ξ is therefore identified with the renormalization group invariant quantity obtained by integrating the β-function for the Newtonian coupling G, see Eqs. (40) and (43).

One can now compare the quantum result at strong coupling, Eq. (70), with the semiclassical result that follows from Eq. (68). The latter gives

$$W(C) \sim \mathrm{Tr} \left(B_C \, \exp \left\{ \tfrac{1}{2} \int_{S(C)} R^{\,\cdot}_{\;\cdot\,\mu\nu} \, A^{\mu\nu}_C \right\} \right) \, . \tag{71}$$

where again B_C is constant bivector characterizing the orientation of the near-planar loop C. Then for a smooth background classical manifold with constant or near-constant large-scale curvature,

$$R_{\mu\nu\lambda\sigma} = \tfrac{1}{3} \, \lambda \, (g_{\mu\nu} \, g_{\lambda\sigma} \; - \; g_{\mu\lambda} \, g_{\nu\sigma}) \tag{72}$$

one immediately obtains from the identification of the area terms in the two Wilson loop expressions the following result for the average semi-classical curvature at large scales

$$\bar{R} \; \sim \; + 1/\xi^2 \; . \tag{73}$$

Note that a key ingredient in the derivation is the fact that both in the quantum result of Eq. (70) and in the semi-classical result of Eq. (71) the exponent contains the *area* of the loop. An equivalent way of phrasing the statement of Eq. (73) uses the classical field equations in the absence of matter, $R = 4\lambda$. The latter suggests one should view $1/\xi^2$, up to a constant of proportionality of order one, as the observed scaled cosmological constant,

$$\tfrac{1}{3} \, \lambda_{obs} \; \simeq \; + \frac{1}{\xi^2} \; . \tag{74}$$

This last quantity can then be considered as a measure of the gravitational vacuum energy, in analogy to the (by now well established) non-Abelian gauge theory vacuum condensate result, $\langle F^2_{\mu\nu} \rangle \; \simeq \; 1/\xi^4$, whose gravity analog can be written, equivalently, as

$$\langle \, R \, \rangle \; \propto \; \frac{1}{\xi^2} \; . \tag{75}$$

The nonperturbative treatment of lattice quantum gravity has thus added one more ingredient to the puzzle: while the bare cosmological constant λ_0 can be completely scaled out of the problem, a new RG invariant scale ξ of Eq. (40) appears, and is here identified with the *effective* cosmological constant. The fact that the RG invariant quantity ξ presents a natural candidate for the observed cosmological constant was proposed independently in [20], one of the first papers to bring up the cosmological consequences of such an identification.

7 Effective Field Equations

An important physical consequence, implied by the identification of the RG invariant ξ in Eq. (40) with the effective, long distance $1/\sqrt{\lambda}$, is that one expects (as in gauge theories) ξ to determine the scale dependence of the effective Newton's constant G appearing in the field equations. The latter is a solution of the renormalization group equations for G, given in Eqs. (39), (40) and (43). Specifically, if one follows Eq. (39), one obtains a momentum-dependent $G(k)$. This needs to be reexpressed in a covariant way, so that effects from it can be computed consistently for general problems, involving arbitrary background geometries. The first step in analyzing the consequences of a running of G is therefore to rewrite the expression for $G(k)$ in a manifestly coordinate-independent way. This can be done either by the use of a nonlocal Vilkovisky-type effective gravitational action [21, 22, 23, 24, 25], or by the use of a set of consistent effective field equations [26]. In either case one goes from momentum to position space by applying the prescription $k^2 \to -\Box$. This then gives for the quantum-mechanical running of the gravitational coupling the replacement

$$G \;\to\; G(k) \;\to\; G(\Box) \;. \tag{76}$$

As a consequence, the running of G in the vicinity of the UV fixed point is of the form

$$G(\Box) \;=\; G_0 \left[1 \,+\, c_0 \left(\frac{1}{-\xi^2 \, \Box} \right)^{1/2\nu} \,+\, \dots \right] \;, \tag{77}$$

where $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the covariant d'Alembertian, and the dots represent higher order terms in an expansion in $1/(\xi^2\,\Box)$. Note that $G_0 \equiv G_c$ in the above expression should be identified to a first approximation with the laboratory scale value of Newton's constant, $\sqrt{G_c} \sim 1.6 \times 10^{-33} cm$, whereas $\xi \sim 1/\sqrt{\lambda/3} \sim 1.51 \times 10^{28}$cm. Current numerical evidence from Euclidean lattice gravity gives $c_0 \simeq 3.0 > 0$ (implying infrared growth) and $\nu = \frac{1}{3}$ [10].

It is worth mentioning here that one could consider an infrared regulated version of $G(\Box)$, where the infrared cutoff $\mu \sim \xi^{-1}$ is introduced, so that in Fourier space $k > \xi^{-1}$ and thus spurious infrared divergences at small k are removed. This can be achieved by the (QCD renormalon-inspired) replacement $k^2 \to k^2 + m^2$ in Eq. (39) with $m = 1/\xi$ as the infrared cutoff. In position space this then leads to the IR regulated form of Eq. (77)

$$G(\Box) \;=\; G_0 \left[1 \,+\, c_0 \left(\frac{1}{-\,\xi^2\,\Box + 1} \right)^{1/2\nu} \right] \;. \tag{78}$$

Nevertheless, in the following it will be adequate to just consider the expression in Eq. (77), although most of the discussion given below is quite general, and does not hinge on this specific choice.

One possible approach to develop an effective theory is then to write down a set of classical effective, but nonlocal, field equations of the form

$$R_{\mu\nu} \,-\, \tfrac{1}{2}\, g_{\mu\nu}\, R \,+\, \lambda\, g_{\mu\nu} \;=\; 8\pi\, G(\Box)\; T_{\mu\nu} \tag{79}$$

with $\lambda \simeq 3/\xi^2$ and $G(\Box)$ given above, and a strong nonlocality from the $G(\Box)$ term. From this the running of G can then be worked out in detail for specific coordinate choices. For example, in the static isotropic case one finds a gradual slow rise in G with distance

$$G \;\to\; G(r) \;=\; G \left(1 \,+\, \frac{c_0}{3\,\pi}\, \frac{r^3}{\xi^3}\, \ln \frac{\xi^2}{r^2} \,+\, \dots \right) \tag{80}$$

in the regime $r \gg 2\,M\,G$ with $2MG$ is the horizon radius [27].

To aid in the interpretation of the physical content of the theory, one notes that the nonlocal effective field equation of Eq. (79) can be recast in a form very similar to the classical field equations, but with an additional source term coming from the vacuum polarization contribution [29]. For this purpose it is useful to decompose the full source term in the effective field equations by first writing

$$G(\Box) = G_0 \left(1 \,+\, \frac{\delta G(\Box)}{G_0} \right) \qquad \text{with} \qquad \frac{\delta G(\Box)}{G_0} \equiv c_0 \left(\frac{1}{-\xi^2\,\Box} \right)^{1/2\nu} \, . \tag{81}$$

Then the full source term can be written as a sum of two parts,

$$\left(1 + \frac{\delta G(\Box)}{G_0} \right) T_{\mu\nu} \;=\; T_{\mu\nu} + T_{\mu\nu}^{vac} \, . \tag{82}$$

The second, vacuum part involves the nonlocal term

$$T_{\mu\nu}^{vac} \;\equiv\; \frac{\delta G(\Box)}{G_0}\, T_{\mu\nu} \, . \tag{83}$$

with the covariant d'Alembertian operator $\Box \;=\; g^{\mu\nu}\, \nabla_\mu \nabla_\nu$ acting here on the second rank tensor $T_{\mu\nu}$,

$$\begin{aligned} \nabla_\nu T_{\alpha\beta} \;&=\; \partial_\nu T_{\alpha\beta} - \Gamma^\lambda_{\alpha\nu} T_{\lambda\beta} - \Gamma^\lambda_{\beta\nu} T_{\alpha\lambda} \;\equiv\; I_{\nu\alpha\beta} \\ \nabla_\mu \left(\nabla_\nu T_{\alpha\beta} \right) &=\; \partial_\mu I_{\nu\alpha\beta} - \Gamma^\lambda_{\nu\mu} I_{\lambda\alpha\beta} - \Gamma^\lambda_{\alpha\mu} I_{\nu\lambda\beta} - \Gamma^\lambda_{\beta\mu} I_{\nu\alpha\lambda} \, , \end{aligned} \tag{84}$$

In this picture, therefore, the running of G can be viewed as contributing to a sort of vacuum fluid, introduced in order to account for the gravitational quantum vacuum-polarization contribution. Consistency of the full covariant, nonlocal field equations then requires that the sum of the two $T_{\mu\nu}$ contributions be conserved,

$$\nabla^\mu \left(T_{\mu\nu} + T_{\mu\nu}^{vac} \right) = 0 \, , \tag{85}$$

in consideration of the contracted Bianchi identity satisfied by the Ricci tensor. Due to the appearance, in $G(\Box)$ of Eq. (77), of the inverse of the covariant Laplacian raised to a fractional power, it seems wise to consider a regulated version that can be used reliably for practical calculations. One possibility is to compute the effect of $\Box^n$ for positive integer n, and then analytically continue the results to $n \to -1/2\nu$, as was done in [26]. Alternatively, $G(\Box)$ can be defined via a regulated parametric integral representation [28]. In view of the discussion to follow, it will be advantageous to write the relevant nonlocal part of $G(\Box)$ as

$$\left(\frac{1}{-\Box(g) + \mu^2} \right)^{1/2\nu} = \frac{1}{\Gamma(\frac{1}{2\nu})} \int_0^\infty d\alpha \, \alpha^{1/2\nu - 1} \, e^{-\alpha \, [-\Box(g) + \mu^2]} \, , \tag{86}$$

where $\mu \to 0$ is a suitable infrared regulator.

Next consider what happens in the case of a running cosmological constant entering the effective field equation of Eq. (79). Earlier in this work we discussed the fact that a running cosmological constant $\lambda(k)$ is both inconsistent with the overall scaling properties of the gravitational functional integral in the continuum and on the lattice [see Secs. (2) and (4)], and with gauge invariance in the perturbative treatment about two dimensions [Sec. (3)]. The expectation is therefore that serious inconsistencies will arise when a running cosmological constant is formulated within a fully covariant effective theory approach. The first step is therefore to promote again an RG running in momentum space to a manifestly covariant form, $\lambda(k) \to \lambda(\Box)$ in the effective field equation of Eq. (79). To be more specific, consider the case of a scale dependent $\lambda(k)$, which we will write here as $\lambda = \lambda_0 + \delta\lambda(k)$. We will also assume, again for concreteness, that $\delta\lambda(k) \sim c_1 (k^2)^{-\sigma}$, where c_1 and σ are some constants. Then make again the transition to coordinate space by replacing $k^2 \to -\Box$. This leads to

$$\delta\lambda(\Box) \;\sim\; (-\Box(g) + \mu^2)^{-\sigma} \, , \tag{87}$$

where we have been careful and used again the infrared regulated expression given in Eq. (86). The effective field equations in Eq. (79) then contain the following additional (running cosmological) term

$$\delta\lambda(\Box) \cdot g_{\mu\nu} \;=\; c_1 \, \frac{1}{\Gamma(\sigma)} \int_0^\infty d\alpha \, \alpha^{\sigma - 1} \, e^{-\alpha \, (-\Box(g) + \mu^2)} \cdot g_{\mu\nu} \;=\; c_1 \, (\mu^2)^{-\sigma} \cdot g_{\mu\nu} \, . \tag{88}$$

The result therefore is still a numerical constant multiplying the metric $g_{\mu\nu}$. Use has been made here of the key result that covariant derivatives of the metric tensor vanish identically,

$$\nabla_\lambda \, g_{\mu\nu} \;=\; 0 \, . \tag{89}$$

The conclusion of this exercise is therefore that λ cannot run. Note also another key aspect of the derivation: what matters is not just the form of $\lambda(\Box)$, but also the object it acts on. This last aspect is missed completely if one just focuses on $\lambda(k)$. Moreover, the above rather general argument applies also to possible additional contributions to the vacuum energy from various condensates and nonzero vacuum expectation values of matter fields, such as the QCD color field condensate, the quark condensate and the Higgs field. One is lead therefore to the conclusion that, quite generally, a running of λ in the effective field equations inevitably ends up in conflict with general covariance, in essence by virtue of Eq. (89).

8 Effective Action

The previous section discussed how the RG running of $G(\Box)$ can be incorporated in a set of manifestly covariant effective field equations. It was also shown that a running of the cosmological constant in the same equations is essentially ruled out by the requirement of general covariance. One main advantage of Eq. (79) is that it is actually tractable, and leads to a number of reasonably unambiguous predictions for homogeneous isotropic and static isotropic background metrics [27].

In this section we will approach the same problem from a slightly different perspective, namely from the point of view of an effective gravitational action. In view of the discussion presented earlier in this work, it should be clear that such an effective action will depend on the two renormalized, dimensionful parameters G and ξ. Note that we will focus here mostly on the case of pure gravity, as the addition of matter will leave most of the main conclusions unchanged (as was the case in the previous section). Within the framework of an effective action approach, the running of the coupling constants can be implemented by the use of a manifestly covariant effective gravitational action [21, 22, 23, 24, 25]. First consider the cosmological term, for which we write again

$$\lambda_0 \;\; \rightarrow \;\; \lambda_0(k) \;\; \rightarrow \;\; \lambda_0(\Box) \; . \tag{90}$$

It is then easy to see that

$$\lambda_0 \int d^4x \, \sqrt{g} \;\; \rightarrow \;\; \int d^4x \, \sqrt{g} \;\, \lambda_0(\Box) \cdot 1 \tag{91}$$

is meaningless, as $\lambda_0(\Box)$ has nothing to act on. Therefore the λ_0 term in the gravitational action cannot be made to run, no matter how hard one tries. The implication again here is that if λ_0 is somehow made to run, this can only be achieved by an explicit breaking of general covariance. [7]

One further notices that this is clearly *not* the case for the rest of the gravitational action, and in particular for the running of G, as given in Eq. (77). Indeed, consider here for concreteness the following nonlocal effective gravitational action

$$I \; = \; - \frac{1}{16\pi \, G} \int d^4x \sqrt{g} \; \sqrt{R} \; (1 \; - \; A(\Box)) \; \sqrt{R} \tag{92}$$

[7] Of course one could *force* λ_0 to run by writing for the integrand $f(R)^{-1} A(\Box) R f(R)$ where f is some arbitrary function of the scalar curvature, or of any other quantum field for that matter. But to us this procedure seems entirely artificial.

with [see Eq. (77)]

$$A(\Box) \;\equiv\; c_0 \left(-\xi^2 \Box\right)^n \tag{93}$$

and G a true constant. In last expression n is taken to be an integer, with $n \to -1/2\nu$ at the end of the calculation. The next step is to compute the variation of the above effective action. Note that another possibility would have been to have the $G(\Box)$ act on the matter term, $\frac{1}{2} \int \sqrt{g}\, G(\Box)\, g^{\mu\nu}\, T_{\mu\nu}$, but we will not pursue this possibility here. The expression inside the integral requires the evaluation of four separate variation terms,

$$\begin{aligned} &- \tfrac{1}{2} \sqrt{g}\, \delta g^{\mu\nu}\, g_{\mu\nu} \sqrt{R}\, (1 - A(\Box)) \sqrt{R} + \sqrt{g}\, \delta\sqrt{R}\, (1 - A(\Box)) \sqrt{R} \\ &- n \sqrt{g} \sqrt{R}\, \frac{A(\Box)}{\Box} (\delta \Box) \sqrt{R} + \sqrt{g} \sqrt{R}\, (1 - A(\Box))\, \delta\sqrt{R}\,. \end{aligned} \tag{94}$$

These in turn require the following elementary variations,

$$\delta\sqrt{g} \;=\; -\tfrac{1}{2} \sqrt{g}\, g_{\mu\nu}\, \delta g^{\mu\nu} \tag{95}$$

and

$$\delta R \;=\; g^{\mu\nu}\, \delta R_{\mu\nu} + R_{\mu\nu}\, \delta g^{\mu\nu} \tag{96}$$

with

$$\delta R_{\mu\nu} \;=\; \nabla_\alpha \left(\delta \Gamma^\alpha{}_{\mu\nu}\right) - \nabla_\mu \left(\delta \Gamma^\alpha{}_{\alpha\nu}\right)\;, \tag{97}$$

for which one needs

$$\delta \Gamma^\alpha{}_{\mu\nu} \;=\; \tfrac{1}{2} g^{\alpha\beta} \left[\nabla_\mu\, \delta g_{\beta\nu} + \nabla_\nu\, \delta g_{\beta\mu} - \nabla_\beta\, \delta g_{\mu\nu}\right]\,. \tag{98}$$

It then follows that in Eq. (96)

$$g^{\mu\nu}\, \delta R_{\mu\nu} \;=\; \nabla_\mu \nabla_\nu \Big(- \delta g^{\mu\nu} + g^{\mu\nu}\, g_{\alpha\beta}\, \delta g^{\alpha\beta} \Big) \;=\; g_{\alpha\beta}\, \Box\, \delta g^{\alpha\beta} - \nabla_{(\mu} \nabla_{\nu)}\, \delta g^{\mu\nu}\;, \tag{99}$$

which gives the second and last terms in Eq. (94). Use has been made here of $\delta g_{\mu\nu} = -g_{\mu\alpha}\, g_{\nu\beta}\, \delta g^{\alpha\beta}$. Note that in general $\Box\, \nabla_\mu \neq \nabla_\mu \Box$, and that $\Box\, g_{\mu\nu} = 0$ but $\Box\, \delta g_{\mu\nu} \neq 0$. For the variation of the covariant d'Alembertian

$$\delta(\Box) \;=\; \delta g^{\mu\nu}\, \nabla_\mu \nabla_\nu - g^{\mu\nu}\, \delta \Gamma^\sigma_{\mu\nu}\, \nabla_\sigma \tag{100}$$

one needs the variation of $\Gamma^\sigma_{\mu\nu}$ given in Eq. (98), which leads to

$$\delta(\Box) \;=\; \delta g^{\mu\nu}\, \nabla_\mu \nabla_\nu + (\nabla_\mu\, \delta g^{\mu\nu})\, \nabla_\nu - \tfrac{1}{2} g_{\mu\nu}\, g^{\alpha\beta}\, (\nabla_\beta\, \delta g^{\mu\nu})\, \nabla_\alpha\;. \tag{101}$$

Generally one encounters expression that need to be properly symmetrized, as in the case of

$$\delta(\Box^n) \;\to\; \sum_{k=1}^{n} \Box^{k-1}\, (\delta\, \Box)\, \Box^{n-k}\,. \tag{102}$$

Eventually this leads to rather lengthy and complicated expressions; although these can be worked out in detail, in the following we will first consider, by choice, only one such ordering, namely $\delta(\Box^n) \to n\,\Box^{n-1}\,\delta(\Box)$, which then gives simply $\delta(A) = n[A(\Box)/\Box]\,\delta(\Box)$. Several integrations by parts need to be performed next, involving both $\Box^n$ (with integer n) and $g_{\mu\nu}\,\Box - \nabla_{(\mu}\nabla_{\nu)}$, required in order to isolate the $\delta g^{\mu\nu}$ term. In general one has to be careful about the ordering of covariant derivatives, whose commutator is non-vanishing

$$[\nabla_\mu, \nabla_\nu]\, T^{\alpha_1\,\alpha_2 \dots}{}_{\beta_1\,\beta_2 \dots} \;=\; -\sum_i R_{\mu\nu\sigma}{}^{\alpha_i}\, T^{\alpha_1 \dots \sigma \dots}{}_{\beta_1 \dots} - \sum_j R_{\mu\nu\beta_j}{}^{\sigma}\, T^{\alpha_1 \dots}{}_{\beta_1 \dots \sigma \dots} \tag{103}$$

with the σ index in T in the i-th position in the first term, and in the j-th position in the second term. As a consequence, the $O(R)$ commutator terms generally give rise to higher derivative terms in the effective field equations, due to the fact that the zeroth order terms in the action are already $O(R)$. After all these manipulations, the effective field equations for zero cosmological constant take on the form

$$\begin{aligned} &\left(R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}\, R\right) \left(1 - \frac{1}{\sqrt{R}}\, A(\Box)\, \sqrt{R}\right) \\ &- \left(g_{\mu\nu}\,\Box - \nabla_{(\mu}\nabla_{\nu)}\right) \left(\frac{1}{\sqrt{R}}\, A(\Box)\, \sqrt{R}\right) + n \left(\nabla_\mu\, \frac{A(\Box)}{\Box}\, \sqrt{R}\right) \left(\nabla_\nu\, \sqrt{R}\right) \\ &- \tfrac{1}{2}\, n\, g_{\mu\nu} \left\{ \left(\nabla^\sigma\, \frac{A(\Box)}{\Box}\, \sqrt{R}\right) \left(\nabla_\sigma\, \sqrt{R}\right) + \left(\frac{A(\Box)}{\Box}\, \sqrt{R}\right) \left(\Box\, \sqrt{R}\right) \right\} = 8\pi G\, T_{\mu\nu}\ . \end{aligned} \tag{104}$$

Note that the above effective field equations are not symmetric in $\mu \leftrightarrow \nu$ due to our specific choice of operator ordering. Note also that taking the covariant divergence of the l.h.s is expected to give zero, which is required for consistency of the field equations (for some terms it is clear that they give zero by inspection). Unfortunately the above effective field equations are still rather complicated. Note though that generally any terms of $O(R^2)$ can safely be dropped, if one is interested in the long distance, small curvature limit. For completeness, we quote here the result for arbitrary operator ordering, as in Eq. (102), where the generic term has the form $\Box^{k-1}\,\delta(\Box)\,A(\Box)\,\Box^{-k}$ with $k = 1 \dots n$. In this case the last two terms on the l.h.s. of Eq. (104) become

$$\begin{aligned} &+ \left(\nabla_\mu\, \Box^{k-1}\, \sqrt{R}\right) \left(\nabla_\nu\, \frac{A(\Box)}{\Box^k}\, \sqrt{R}\right) \\ &- \tfrac{1}{2}\, g_{\mu\nu} \left\{ \left(\nabla^\sigma\, \Box^{k-1}\, \sqrt{R}\right) \left(\nabla_\sigma\, \frac{A(\Box)}{\Box^k}\, \sqrt{R}\right) + \left(\Box^{k-1}\, \sqrt{R}\right) \left(\Box\, \frac{A(\Box)}{\Box^k}\, \sqrt{R}\right) \right\}\ . \end{aligned} \tag{105}$$

The full effective field equations are then obtained by summing over k, as in Eq. (102).

One more possibility is to generalize the effective action in Eq. (92) to the form

$$I \;=\; -\frac{1}{16\pi\, G} \int d^4x \sqrt{g}\, R^{1-\alpha} \, (1 \;-\; A(\Box))\, R^{\alpha} \;, \tag{106}$$

which now depends on a parameter α taking values between zero and one; the previous case then corresponds to the symmetric choice $\alpha = 1/2$. This last action can also be symmetrized between the first and second curvature terms, but we shall not pursue that here in order to keep things simple. Following the same procedure that lead to Eq. (104), one obtains for the field equations with zero cosmological constant the following expression

$$\begin{aligned}
& R_{\mu\nu} \;-\; \tfrac{1}{2}\, g_{\mu\nu}\, R \;+\; \tfrac{1}{2}\, g_{\mu\nu}\, R \cdot \frac{1}{R^{\alpha}}\, A(\Box)\, R^{\alpha} \\
& - R_{\mu\nu} \left\{ (1-\alpha)\, \frac{1}{R^{\alpha}}\, A(\Box)\, R^{\alpha} \;+\; \alpha\, \frac{1}{R^{1-\alpha}}\, A(\Box)\, R^{1-\alpha} \right\} \\
& - \left(g_{\mu\nu}\, \Box \;-\; \nabla_{(\mu} \nabla_{\nu)} \right) \left\{ (1-\alpha)\, \frac{1}{R^{\alpha}}\, A(\Box)\, R^{\alpha} \;+\; \alpha\, \frac{1}{R^{1-\alpha}}\, A(\Box)\, R^{1-\alpha} \right\} \\
& + n \left(\nabla_{\mu}\, \frac{A(\Box)}{\Box}\, R^{1-\alpha} \right) \left(\nabla_{\nu}\, R^{\alpha} \right) \\
& - \tfrac{1}{2}\, n\, g_{\mu\nu} \left\{ \left(\nabla^{\sigma}\, \frac{A(\Box)}{\Box}\, R^{1-\alpha} \right) \left(\nabla_{\sigma}\, R^{\alpha} \right) \;+\; \left(\frac{A(\Box)}{\Box}\, R^{1-\alpha} \right) \left(\Box\, R^{\alpha} \right) \right\} = 8\pi G\, T_{\mu\nu}
\end{aligned} \tag{107}$$

which incidentally shows that the choice of either $\alpha = 1$ or $\alpha = 0$ is problematic.

Then one final technical question remains, namely what is the relationship between the above effective field equations [Eq. (104) or Eq. (107] and the clearly more economical field equations given earlier in Eq. (79). Obviously, the equations obtained here from a variational principle are significantly more complicated. They contain a number of non-trivial terms, some of which are reminiscent of the $1 \,+\, A(\Box)$ term in Eq. (79), and others with a rather different structure (such as the $g_{\mu\nu}\, \Box \;-\; \nabla_{(\mu} \nabla_{\nu)}$ term). Note that many of the new additional curvature terms that appear on the l.h.s. of the effective field equations in Eqs. (104) and (107) can be moved, equivalently, to the r.h.s.. To do so, one makes use of the fact that to zeroth order in the quantum correction proportional to $A(\Box)$

$$R \;=\; 4\, \lambda \;-\; 8\pi\, G\, T^{\lambda}_{\;\;\lambda} \;, \tag{108}$$

which then allows, again, a separation of the source term in pure matter ($T_{\mu\nu}$) and vacuum polarization ($T^{vac}_{\mu\nu}$) contributions, as was done earlier in Eq. (82). Generally, some of the issues that come up in comparing effective field equations reflect an ambiguity of where, and at what stage, the replacement $G \to G(\Box)$ is performed. Nevertheless one would hope that when asking the right physical question the answer would largely be unambiguous. It is of course possible that, when restricted to specific metrics such as the Robertson-Walker one and its perturbations, the two sets of effective field equations will ultimately give similar results, but in general this remains still a largely open question. One possibility is that both sets of field equations describe the same running of the gravitational coupling, up to curvature squared (higher derivative) terms, which then become irrelevant at very large distances. In any case, the main purpose of our exercise here was to show that in either case [via Eq. (79) or Eq. (104)] the running of $G(\Box)$ clearly leads to non-vanishing effects which are non-trivial.

9 Renormalization via Continuum Truncation Methods

A number of approximate continuum renormalization group methods have been developed, which can be used to construct RG flows and thus estimate the scaling exponents. Let us mention here one example, as an illustration for the kind of rather delicate issues that might arise. An approach closely related to the $2+\epsilon$ perturbative expansion for gravity discussed earlier is the derivation of approximate RG flow equations from the changes of the effective action with respect to an infrared cutoff μ. In some ways the method is a variation of Wilson's original momentum slicing technique, originally developed to obtain approximate renormalization group recursions for the couplings. As an example, in the case of a scalar field one starts from the partition function

$$\exp(W[J]) \;=\; \int [d\phi] \, \exp\left\{ -\tfrac{1}{2}\phi \cdot C^{-1} \cdot \phi \,-\, I_\Lambda[\phi] + J \cdot \phi \right\} \;, \tag{109}$$

where the $C(k,\mu)$ term introduces an 'infrared cutoff term'. In order for it to act as an infrared cutoff, it needs to be small for $k<\mu$, tending to zero as $k \to 0$, with $k^2 C(k,\mu)$ large when $k>\mu$. Since the method is useful in the vicinity of the fixed point, where physical relevant scales are much smaller than the ultraviolet cutoff Λ, it is argued that the detailed nature of this cutoff is not too relevant. Taking a derivative of the function $W[J]$ with respect to μ gives

$$\frac{\partial W[J]}{\partial \mu} \;=\; -\tfrac{1}{2}\left[\frac{\delta W}{\delta J} \cdot \frac{\partial C^{-1}}{\partial \mu} \cdot \frac{\delta W}{\delta J} \,+\, \mathrm{tr}\left(\frac{\partial C^{-1}}{\partial \mu} \frac{\delta^2 W}{\delta J \, \delta J} \right) \right] \;. \tag{110}$$

The latter can then be expressed in terms of the Legendre transform $\Gamma[\phi] = -W[J] - \frac{1}{2}\phi \cdot C^{-1} \cdot \phi + J \cdot \phi$ as

$$\frac{\partial \, \Gamma[\phi]}{\partial \mu} \;=\; -\tfrac{1}{2}\,\mathrm{tr}\left[\frac{1}{C}\frac{\partial C}{\partial \mu} \cdot \left(1 \,+\, C \cdot \frac{\delta^2 \Gamma}{\delta\phi\,\delta\phi} \right)^{-1} \right] \;. \tag{111}$$

Here $\phi \equiv \delta W / \delta J$ is regarded as the classical field, and the traces are later simplified by going to momentum space. One issue that needs to be settled in advance is the choice for the cutoff function $C(k, \mu)$. Then given this choice one computes the effective action $\Gamma[\phi]$ in a derivative expansion containing terms $\partial^n \phi^m$ and calculable μ-dependent coefficients. It is also customary to write the cutoff function as $C(k, \mu) = \mu^{\eta-2} C(k^2/\mu^2)$, so as to anticipate an anomalous dimensions (η) for the ϕ field, and assume for the remaining function (now of a single variable) that $C(q^2) = q^{2p}$ with p a non-negative integer [30].

In the gravitational case one proceeds more or less in a similar way. First note that the gravity analog of Eq. (111) is

$$\frac{\partial \, \Gamma[g]}{\partial \mu} \;=\; -\tfrac{1}{2} \, \mathrm{tr} \left[\frac{1}{C} \frac{\partial C}{\partial \mu} \cdot \left(1 \, + \, C \cdot \frac{\delta^2 \Gamma}{\delta g \, \delta g} \right)^{-1} \right] \tag{112}$$

where $g_{\mu\nu} \equiv \delta W / \delta J_{\mu\nu}$ is the classical metric. The effective action then contains an Einstein and a cosmological term

$$\Gamma_\mu[g] \;=\; - \frac{1}{16 \pi \, G(\mu)} \int d^d x \sqrt{g} \, [\, R(g) \; - \; 2 \lambda(\mu) \,] + \ldots \; , \tag{113}$$

as well as gauge fixing and possibly higher derivative terms [31, 33]. After the addition of a suitable background harmonic gauge fixing term with gauge parameter α, the choice of a (scalar) cutoff function is required, $C^{-1}(k, \mu) = (\mu^2 - k^2) \theta(\mu^2 - k^2)$ [for more details see for example [33]]. The latter is then inserted into the path integral

$$\int [dh] \, \exp \left\{ - \tfrac{1}{2} \, h \cdot C^{-1} \cdot h \; - \; I_\Lambda \, [g] \; + \; J \cdot h \right\} \; . \tag{114}$$

Note that the additional momentum-dependent cutoff term violates both the weak field general coordinate invariance, as well as the general rescaling invariance of Eq. (6). Subsequently the solution of the resulting renormalization group equation for the two couplings $G(k)$ and $\lambda(k)$ is truncated to the Einstein and cosmological terms, a procedure which is more or less equivalent to the derivative expansion discussed previously for the scalar case. A nontrivial fixed point in the couplings (G^*, λ^*) is then found, generally with complex eigenvalues ν^{-1}, and some dependence on the gauge parameters [32].

There seem to be two problems with the above approach (apart from the reliability and convergence of the truncation procedure, which is an entirely separate issue). The first problem is an explicit violation of the scaling properties of the gravitational functional integral, see Eqs. (6),(7) and (8) in the continuum, and of the corresponding result in the lattice theory of gravity, Eq. (53).

As a result of this conflict, it seems now possible to find spurious gauge-dependent separate renormalization group trajectories for $G(k)$ and $\lambda(k)$, in disagreement with most of the arguments presented previously in this work, including the explicit gauge-independence of the perturbative result of Eq. (36). In light of these issues, it would seem that the RG trajectory for the *dimensionless* combination $G(k)\lambda(k)$ should be regarded as more trustworthy. The second problem is that the running of $\lambda(k)$ claimed in this approach seems largely accidental, presumably due to the diffeomorphism violating cutoff. The latter allows such a running, in spite of the fact that, as we have shown earlier, it is inconsistent with general covariance. One additional and somewhat unrelated problem is the fact that the above method, at least in its present implementation, is essentially perturbative and still relies on the weak field expansion. It is therefore unclear how such a method could possibly give rise to an explicit nonperturbative correlation length ξ [see Eq. (40)], which after all is non-analytic in G.

10 Conclusions

In this paper we have examined the issue of whether the cosmological constant of quantized gravitation can run with scale. The relevance of this problem arises at a fundamental level, but has also possible implications for observational cosmology, where a scale dependence of λ in the form of a $\lambda(a(t))$ is sometimes assumed. We have examined this issue from a variety of viewpoints, which included the continuum and lattice formulations for the gravitational path integral, with various scaling properties that come with it; the perturbative treatment of gravity; and finally from insights gained through the formulation of manifestly covariant effective actions and effective field equations.

The key message seems that the cosmological constant cannot run with scale, if general covariance is preserved. Instead, evidence from the nonperturbative path integral treatment of quantum gravity points to the fact that the observed effective long-distance cosmological is a renormalization group invariant quantity, related to the fundamental RG scale ξ, and thus to a vacuum condensate of the gravitational field. In analogy to the corresponding scale for non-Abelian gauge theories, ξ cannot run, and represents instead a dynamically generated, nonperturbative mass-like parameter. That this is possible is a highly non-trivial result of the renormalization group treatment, of the Callan-Symanzik RG equations for G, and of the phase structure of four-dimensional gravity.

In closing, let us pursue here again what appears as a deep analogy between gravity on the one hand, and gauge theories and magnets on the other. First write down the three field equations for gravity, quantum electrodynamics (made massive via the Higgs mechanism) and a scalar field. They read

$$
\begin{aligned}
R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} &= 8\pi G T_{\mu\nu} \\
\partial^\mu F_{\mu\nu} + \mu^2 A_\nu &= 4\pi e j_\nu \\
\partial^\mu \partial_\mu \phi + m^2 \phi &= \frac{g}{3!} \phi^3 ,
\end{aligned}
\tag{115}
$$

and are used here to represent the field equations relevant for a boson-mediated long range force. Now, all three mass-like parameters on the left (λ, μ and m) are considered RG invariants (this is well known for the last two cases), whereas all three couplings of the r.h.s. are known to be scale-dependent. Furthermore, in all three cases the relevant renormalized mass parameter is related to the fundamental correlation length, $m = 1/\xi$. More generally, in non-Abelian gauge theories the nonperturbative mass parameter (sometimes referred to as the mass gap) is also an RG invariant; that such a mass scale can be generated dynamically is a non-trivial result of the renormalization group.

Here we want to point out that there seems to be a fundamental relationship between the nonperturbative scale ξ (or inverse renormalized mass) and a non-vanishing vacuum condensate for the three theories,

$$
\langle R \rangle \simeq \frac{1}{\xi^2} \qquad \langle F_{\mu\nu}^2 \rangle \simeq \frac{1}{\xi^4} \qquad \langle \phi \rangle \simeq \frac{1}{\xi} . \tag{116}
$$

In all three cases the vacuum condensate's dependence on the correlation length ξ is fixed by the mass dimension of the field appearing in it. In the gauge theory case, this is due to the vanishing relevant anomalous dimension, which in turn follows from current conservation. One more notable example that comes to mind is the fermion condensate in non-Abelian gauge theories, $\langle \bar\psi \psi \rangle \simeq 1/\xi^3$. The last result listed in Eq. (116), for a scalar field with a non-vanishing vacuum expectation value, is the field theory analog of what happens in a ferromagnet. There in the magnetized phase, $T < T_c$, the general result in d dimensions is $\langle \phi \rangle \simeq 1/\xi^{\beta/\nu}$ close to the critical point, where ν and β are some exponents; then already for Ising spins in four dimensions one has $\langle S \rangle \simeq 1/\xi$, given the exponents $\nu = \beta = \frac{1}{2}$ in $d = 4$. So, in the end, the relationship between the fundamental nonperturbative correlation length ξ and the vacuum condensate starts to look a lot less exotic than what might have seemed at first sight.

Note added: After this work was submitted, we were informed that similar conclusions had been reached by the authors of [34], by considering the effective action for a self-interacting scalar field in a classical gravitational background. A closely related paper also dealt with the issue of a possible running of $\lambda(k)$ [35].

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