Nuclear Physics B364 (1991) 703–733
North-Holland

CRITICAL PROPERTIES OF TWO-DIMENSIONAL SIMPLICIAL QUANTUM GRAVITY

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Received 26 June 1990
(Revised 24 April 1991)

Simplicial higher-derivative quantum gravity is investigated in two dimensions for a manifold of toroidal topology. The manifold is dynamically triangulated using Regge's formulation of gravity, with continuously varying edge lengths and fixed coordination number. Critical exponents are estimated by computer simulation on lattices with up to 786432 edges, and compared to the continuum conformal field theory results for central charge zero (pure gravity), one half (Ising model coupled to gravity), one and two (massless scalar field coupled to gravity). The dependence of critical properties on the coefficient of the curvature squared term and the gravitational functional measure is investigated, suggesting universal critical behavior at least within a certain class of measures. In the case of pure gravity, we have computed the string susceptibility exponent for both the torus and the sphere, and our estimates agree with the exact result of KPZ. The fluctuations in the area density are consistent with the behavior expected for a massless scalar field, the Liouville mode. In the case of gravity coupled to a massless scalar field, we have computed what corresponds to the fractal dimension of the surface, and found it to be infinite. The critical exponents associated with the Ising model coupled to gravity on a torus are found to be the same as for the Ising model in flat space.

1. Introduction

Understanding properties of two-dimensional lattice quantum gravity is likely to be an essential step on the way to formulating a lattice theory of four-dimensional quantum gravity. In addition, two-dimensional quantum gravity is closely related to the theory of random surfaces embedded in D-dimensional flat space, as well as to the theory of bosonic strings.

We will concentrate here on the simplicial formulation of quantum gravity developed by Regge [1–12], but it is known that there are other possible approaches such as the random triangulation approach for two-dimensional surfaces [13–17], as well as the hypercubic lattice formulation [18, 19]. It would not be too surprising if these three approaches are someday shown to be equivalent, since after all they correspond to different discretizations of the same original theory. On the other hand, their quantum continuum limit could in principle exhibit rather different properties, and in particular there is certainly no general proof yet of the recovery of general coordinate invariance.

One of the advantages of the simplicial Regge calculus approach lies in the fact that it can be formulated in any space-time dimension (including the physically relevant case of four dimensions), and in the fact that it can be shown to be classically equivalent to general relativity, as a consequence of several more or less rigorous convergence proofs for (arbitrarily) triangulated smooth manifolds [2–5, 7, 9, 12]. In addition, the correspondence between the lattice and continuum quantities is clear, and therefore the interpretation of the terms in the action as well as the identification and separation of, for example, the measure contribution is unambiguous [6]. For a review of Regge gravity, and a more complete list of early references, we refer the reader to ref. [6].

Approximate general coordinate (or re-parametrization) invariance corresponds to variations in the edge lengths that leave the geometry of the underlying manifold unchanged. For the limiting case of flat space there are clearly infinitely many edge length assignments which reproduce equally well the underlying manifold. This situation can be contrasted to the case of a triangulation with fixed edge lengths and a varying incidence matrix, for which there is no reparametrization invariance (except for the trivial one, corresponding to a relabeling of the lattice vertices). As one moves away from flat space though, different edge lengths assignments will in general correspond to physically inequivalent manifolds.

In continuum gravity the fundamental degrees of freedom are represented by the metric field $g_{\mu\nu}(x)$. In a two-dimensional piecewise linear space the elementary building blocks are triangles, and the relative position of points on the lattice is therefore completely specified by the incidence matrix and the edge lengths, which in turn induces a metric structure on the piecewise linear space. In order to obtain non-degenerate simplicial complexes, the edge lengths have to obey triangle inequalities, which ensure that the triangles areas are positive. General coordinate transformations correspond (at least approximately) to variations of the edge lengths, as well as appropriate modifications of the incidence matrix. But since in general different complexes will correspond to physically distinct manifolds, one expects classically general coordinate invariance to be recovered only in the continuum limit, where a continuous smooth manifold can be covered by many different almost geometrically equivalent triangulations. In the quantum theory the hope is to find a non-trivial fixed point where general coordinate invariance is recovered.

A detailed description of the construction of the action for simplicial lattice gravity without and with matter fields can be found in the literature [5, 6, 9, 11], and therefore only a brief summary will be presented here. Ref. [9] addressed the specific case of pure gravity in two dimensions, and discusses some analytic and numerical results related to the phase diagram of pure gravity. We recall the geometric correspondence between continuum and lattice quantities in two dimensions [5, 9],

$$\int d^2x \sqrt{g(x)} \to \sum_i A_i ,$$

$$\int d^2x \sqrt{g(x)}\, R(x) \to 2\sum_i \delta_i ,$$

$$\int d^2x \sqrt{g(x)}\, R^2(x) \to 4\sum_i \delta_i^2/A_i , \tag{1.1}$$

where δ_i is the deficit angle at the vertex i,

$$\delta_i = 2\pi - \sum_{\substack{\text{triangles} \\ \text{meeting on } i}} \theta_d \tag{1.2}$$

and θ_d is the dihedral angle associated with the triangle at that vertex. A_i is the area associated with the site i [5, 9], which is not unique since the lattice can be subdivided in more than one way, for example using a dual lattice or a baricentric subdivision. Given reasonable geometric and positivity properties, universality is expected to lead to the same results in the continuum. Note that in the simplicial lattice formulation, as in the continuum, the local curvature $R(x) \sim 2\delta_i/A_i$ is a continuous function of the relevant edge lengths and can take on any real value, positive or negative. In two dimensions the discrete analogue of the Gauss–Bonnet theorem holds

$$\sum_i \delta_i = 2\pi\chi , \tag{1.3}$$

where χ is the Euler characteristic of the surface (two minus twice the number of handles). In this paper we will consider only simplicial complexes topologically equivalent to the torus and the sphere.

The guiding principle in constructing physical quantities in simplicial gravity is that they should have geometric significance. It will distinguish objects which are lattice structure independent for a given physical manifold (at least for sufficiently smooth manifolds in some continuum limit) from other functions of the edge lengths which have no particular geometric meaning, and whose limiting values will therefore depend on the specific way in which the triangulation is refined. The Euler characteristic in two dimensions, expressed as a function of the edge lengths, is a clear and illustrative example of what is meant by this statement. Another clear example is the total area of the simplicial complex: if it is defined as the sum of the triangle areas (where these are very specific functions of the edge lengths), then as the triangulation is refined its limit is well defined, and agrees with the continuum definition of what is meant by the total physical area of the manifold.

Having made some general considerations regarding the action for gravity, let us now turn to the issue of the measure. The form of the measure for the $g_{\mu\nu}$ fields in continuum gravity appears not to be unique, and the topic has been discussed recurrently in the literature [20–23]. The reason for the ambiguities appears to be a lack of a clear definition of what is meant by $\prod_x$ in the functional measure. Thus it is expected that the ambiguities will persist in *any* lattice formulation of quantum gravity, unless an exact lattice invariance is found, which then uniquely selects one privileged measure. However, as will be discussed below, different measures differ by the power of $\sqrt{g}$ in the prefactor, which corresponds to some product of volume factors on the lattice.

One popular (pure) gravitational measure in the continuum is the Misner scale-invariant measure [20], which in d space-time dimensions takes the form

$$\mathrm{d}\mu[g] = \prod_x g^{-(d+1)/2} \prod_{\mu \geqslant \nu} \mathrm{d}g_{\mu\nu} . \tag{1.4}$$

The above measure is unique if the product in eq. (1.4) is interpreted over "physical" points, and invariance is imposed at one and the same "physical" point. On the other hand, if one introduces a super-metric over metric deformations, then another measure arises naturally for pure gravity [21]. Considering the simplest local form for the norm-squared of the metric deformation

$$\|\delta g\|^2 = \tfrac{1}{2}\int \mathrm{d}^d x \sqrt{g} \left[g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} + \lambda g^{\mu\nu} g^{\alpha\beta} \right] \delta g_{\mu\nu} \, \delta g_{\alpha\beta} \qquad (\lambda \neq -2/d)$$

$$\equiv \int \mathrm{d}^d x \sqrt{g}\, G^{\mu\nu,\alpha\beta} \, \delta g_{\mu\nu} \, \delta g_{\alpha\beta} , \tag{1.5}$$

then, according to DeWitt, the functional measure is given by

$$\mathrm{d}\mu[g] = \prod_x \left[\det G^{\mu\nu,\alpha\beta}\right]^{1/2} \prod_{\mu \geqslant \nu} \mathrm{d}g_{\mu\nu} , \tag{1.6}$$

where the determinant of the super-metric G is given by

$$\left[\det G^{\mu\nu,\alpha\beta}\right]^{1/2} = (-1)^{d-1} \left(1 - \tfrac{1}{2} d\lambda\right) g^{(d-4)(d+1)/8} . \tag{1.7}$$

If matter fields are present, then the gravitational measure has to be further modified. Other forms of the measure for the gravitational field have also been suggested, inspired by the canonical quantization approach to gravity [22].

On the simplicial lattice the edge lengths are the elementary degrees of freedom which uniquely specify the geometry for a given incidence matrix, and over which one should perform the functional integral [5, 6, 8, 9]. This is supported by the fact that the induced metric on a simplex i with edges $l_{i,i+\mu}$ is given by $g_{\mu\nu}(i) = \frac{1}{2}[l^2_{i,i+\mu} + l^2_{i,i+\nu} - l^2_{i+\mu,i+\nu}]$. In addition, one might want to sum over complexes with varying local coordination number. But it would seem that any smooth curved manifold can be arbitrarily well approximated by a simplicial complex with fixed coordination number by adjusting the edge lengths and refining the grid, thus presumably saturating the functional measure. This would suggest that the last step is then perhaps redundant, at least within our formalism.

One can argue that the edge lengths, being invariant quantities, are not referred to any specific coordinate systems. On the other hand, they provide for an explicit coordinatization of the manifold, once the incidence matrix is specified as well. It is clear from looking at the example of flat space that there can be an infinite number of edge length assignments that correspond to the same physical manifold. Therefore in the continuum limit the edge lengths cannot really be considered as invariants under some (approximate) lattice diffeomorphism group.

Previously [5, 6, 8, 9] we have employed the measure

$$\int d\mu_\epsilon[l] = \prod_{\text{edges } ij} \int_0^\infty \frac{dl^2_{ij}}{l^2_{ij}} F_\epsilon[l], \tag{1.8}$$

where $F_\epsilon[l]$ is a function of the edge lengths with the property that it is equal to one whenever the triangle inequalities are satisfied, and zero otherwise. A parameter ϵ can be introduced as an ultraviolet cutoff at small edge lengths: the function $F_\epsilon[l]$ is then chosen to be zero if any of the edges are equal or less than ϵ; in the following we will take $\epsilon = 0$. The above measure is correct in the weak-field limit [2, 9], where all continuum measures agree as well.

It is of interest though to explore the sensitivity of the results to the type of gravitational measure employed. Another class of pure gravity measures is obtained by considering the "volume associated with an edge" V_{ij}, and writing in two dimensions

$$\int d\mu_\epsilon[l] = \prod_{\text{edges } ij} \int_0^\infty V_{ij}^{2\sigma}\, dl^2_{ij}\, F_\epsilon[l] \tag{1.9}$$

with $\sigma = -\frac{1}{2}$ for the lattice analog of the Misner measure, and $\sigma = -\frac{1}{4}$ for a lattice analogue of the DeWitt measure. Note that the "Misner" and dl/l measure share the property of being scale invariant. Most of our results will be based on the measure dl/l. But we will explore the properties of these two other lattice measures as well, and shall return to the issue of the measure when we discuss the coupling of gravity to a scalar field, and how the results depend on the form of the measure and on σ. Our results suggest that different measures, within a certain universality class, will give the same results for infrared sensitive quantities, like correlation functions at large distances and critical exponents. We believe though that the lattice path integral might not be meaningful for certain values of σ. We have found in particular that if σ is too negative, then the measure factors tend to favor configurations of triangles which are long and thin, with a small area and a large perimeter. In this case the lattice tends to degenerate into an almost one-dimensional manifold, a situation far from the desired continuum limit, and which one would like to avoid.

In two dimensions a measure for gravity has been given by Polyakov [24–27], following the DeWitt approach. In pure two-dimensional gravity, and for vanishing renormalized cosmological constant $\lambda \to \lambda_c$, one can write for the path integral

$$\int d\mu[g] \exp(-I_G) = \int d\mu[\tilde{g}] \Delta_{FP}[\tilde{g}] \int [d\varphi] \exp(-I_G - 26 I_L) \qquad (1.10)$$

with the Liouville action contribution I_L arising from the conformal anomaly

$$I_L = \frac{1}{96\pi} \int d^2x \sqrt{\tilde{g}} \left(\tilde{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + 2\tilde{R}\varphi \right). \qquad (1.11)$$

Here $g_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x) e^{\varphi(x)}$, and $\tilde{g}_{\mu\nu}(x)$ is some reference metric. Thus for vanishing renormalized cosmological constant one expects the relevant gravitational degrees of freedom to be represented by a free scalar field. On the lattice the conformal factors correspond to local area fluctuations, $e^{\varphi(x)} \approx V(x)/V_0$. In the following sections we will discuss some of the properties of the density fluctuation field φ in simplicial gravity, which are in agreement with the continuum predictions.

Let us now turn to presentation of our results. In sect. 2 we will discuss properties of pure gravity, including the critical exponent γ_χ for the torus and the sphere, and the Liouville field susceptibility. Then we will present results for a model of gravity coupled to a D-component massless scalar field; and make contact with models of random surfaces (here without extrinsic curvature terms). We will consider the cases $D=1$ and 2, as well as the case $D=0$, which corresponds an absence of feedback of the matter fields on the geometry. Finally we will compute the critical exponents for an Ising model coupled to gravity on a torus. Sect. 5 presents our conclusions.

2. Pure gravity

We will begin by considering a higher-derivative lattice action for pure gravity in the form

$$I=\sum_i\left[\lambda A_i-k\delta_i+a\frac{\delta_i^2}{A_i}\right]. \tag{2.1}$$

For a detailed description of the construction of curvature terms on the simplicial lattice we refer to ref. [5]. In the following we will adopt a baricentric subdivision for assigning area elements to the lattice sites and edges. One could use a dual lattice subdivision as well, but it would lack some desirable positivity properties for the dual areas in some cases [5]. In the limit of small fluctuations around a smooth background, the above lattice action can be shown to correspond to the continuum action

$$I=\int d^2x\sqrt{g}\left[\lambda-\frac{k}{2}R+\frac{a}{4}R^2\right]. \tag{2.2}$$

For a manifold of fixed topology the term proportional to k can be dropped. The higher-derivative term proportional to a can be useful in controlling the fluctuations in the intrinsic local curvature, although it is expected to be irrelevant as far as critical properties are concerned. While it prevents the appearance of conical singularities where the gaussian curvature might become very large, it does not prevent "folding" singularities, corresponding to singular structures in the manifold which appear in embedding space only (to control the latter an extrinsic curvature term seems to be required). For $a\to\infty$ the manifold approaches a flat limit, whereas for $a\to 0$ local fluctuations in the curvature become more pronounced. In ref. [9] it was shown that no sensible ground state exists for $a<0$ (unless there are additional terms in the action), and in the following we will therefore only discuss the case $a\geqslant 0$.

Classically the continuum equations of motion lead to a constant curvature solution $R_0(x)=\pm\sqrt{4\lambda/a}$, (there being no real solution for $\lambda<0$). On the torus on the other hand, the only consistent classical solution is $R_0(x)\equiv 0$, an identity which remains true for the average when quantum fluctuations are included, as a consequence of the Gauss–Bonnet theorem.

In ref. [9] the canonical ensemble (with fixed bare cosmological constant λ) was considered. On the other hand, in order to compare with the exact results of KPZ [26], it is useful to consider an ensemble where the total area A is kept fixed instead. In such an ensemble the limit $A\to\infty$ corresponds to $\lambda\to\lambda_c$ in the canonical ensemble. One approaches the limit $A\to\infty$ by letting the number of sites $N\to\infty$ for fixed elementary triangle areas (to approach the continuum limit with a fixed total physical area, the elementary triangle areas would have to be scaled to zero as N is increased, which is not what we shall do here). We will therefore consider the lattice analogue of

$$Z[A]=\int \mathrm{d}\mu[g]\delta\left(\int\sqrt{g}-A\right)\exp(-I[g]), \tag{2.3}$$

which for large area is expected to behave asymptotically as

$$Z[A] \underset{A\to\infty}{\sim} A^{-3+\gamma_\chi}\exp(-(\lambda-\lambda_0)A), \tag{2.4}$$

where λ_0 affects the renormalization of the cosmological constant. The exponent $\gamma_\chi=\frac{1}{2}\chi(\gamma-2)+2$, with $\gamma=\frac{1}{12}(D-1-\sqrt{(D-1)(D-25)})$, is the "string susceptibility" exponent [26–28]. Note that the exponent γ_χ is not truly universal, since it depends on the boundary conditions through χ. Pure gravity without matter fields then corresponds to the case $D=0$, and in particular on the torus one has the prediction $\gamma_\chi=2$, independent of D.

It is easy to see that the exponent γ_χ can be related to a finite-size correction. By doing an infinitesimal scale transformation on $Z[A]$, with the action given by eq. (2.2), one obtains the identity

$$\frac{\partial \ln Z[A]}{\partial A}=-\frac{1}{A}+\frac{a}{4}\frac{\langle\int\sqrt{g}R^2\rangle_A}{A}+\lambda_0'-\lambda, \tag{2.5}$$

where λ_0' depends, among other things, on the specific form of the measure. Using eq. (2.4) for $Z[A]$ one then has

$$\frac{a}{4}\frac{\left\langle\int\sqrt{g}R^2\right\rangle_A}{A} \underset{A\to\infty}{\sim} \mathrm{const}-\frac{2-\gamma_\chi}{A}+\dots. \tag{2.6}$$

Thus the critical exponent γ_χ can be obtained by investigating the area dependence of the expectation value of R^2. In particular on the torus one has the prediction that the correction proportional to $1/A$ must have a vanishing coefficient. On the lattice the appropriate quantity to measure is

$$\frac{(1/N)\left\langle\Sigma_i\delta_i^2/A_i\right\rangle}{(1/N)\langle\Sigma_i A_i\rangle}\sim\frac{1}{4}\frac{\left\langle\int\sqrt{g}R^2\right\rangle_A}{A}, \tag{2.7}$$

where N is the total number of sites. Without loss of generality one can choose the total area such that $N=A$.

We have computed the coefficient of the finite-size correction to R^2 by investigating both the torus and the sphere, using as a background space a network of

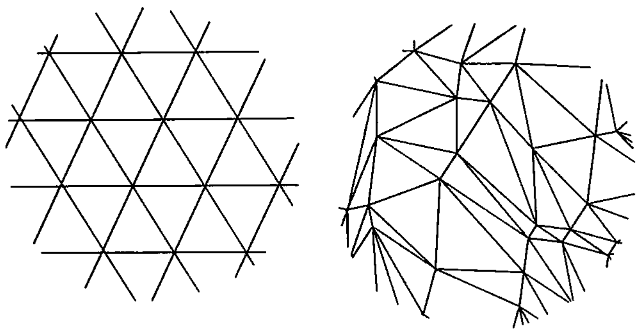

Fig. 1. Two equivalent triangulation of a smooth manifold, using a lattice of coordination number six.

unit squares divided into triangles by drawing in parallel sets of diagonals, as shown in fig. 1. (For the sphere the two poles have to be treated separately in the program due to the presence of a coordinate or incidence matrix singularity.) Ideally one would like to use a random lattice [30], but this presents additional computational problems, so we have opted for the moment for the simpler approach of using a regular lattice. The random lattice might appear more satisfactory from a conceptual point of view, since it incorporates, for smooth manifolds, the invariance under "large" lattice diffeomorphisms, whereas in the regular lattice only "small" lattice diffeomorphisms are allowed. Thus the two different lattices induce quite a different cutoff structure in orbit space. Eventually we hope to redo all our calculations for such a random lattice. On the basis of universality one would expect the results to be independent of the specific lattice structure chosen. Our results indicate that the expected result is clearly obtained for the torus, and with somewhat larger errors for the sphere as well, with a choice of fixed coordination number and varying edge lengths.

In both cases the lattices considered contained from 48 to 12288 edges (corresponding to lattices with 4^2, 5^2, 6^2, 8^2, 10^2, 12^2, 16^2, 32^2 and 64^2 sites). The lengths of the runs varied between 4700k sweeps on the small lattice and 20k sweeps on the largest lattice. The high accuracy was needed in order to reliably extract the finite-size correction. The coupling a was set equal to 1, and we used the measure $\mathrm{d}l/l$ of eq. (1.8). The results for R^2 versus $1/A$ are shown in table 1, and in figs. 2 and 3. In the case of the torus the results for the coefficient of the $1/A$ term are quite accurate, and consistent with zero to within a few percent: we obtain from a straight line fit approximately $2-\gamma_\chi = 0.025(7)$ from lattices $8-64$, and $2-\gamma_\chi = 0.047(13)$ for lattices $10-32$, leading to a combined estimate $2-\gamma_\chi = 0.025(22)$. We therefore conclude that for the torus $2-\gamma_\chi$ is very close to, and given our statistical errors almost consistent with, zero. Note that if we were describing a free massless scalar field for example, the nature of the finite-size corrections would be quite different.

TABLE 1
Average of the curvature squared as a function of the total area for the torus and the sphere

A	$\langle \Sigma_i \delta_i^2 / A_i \rangle / A$ torus	$\langle \Sigma_i \delta_i^2 / A_i \rangle / A$ sphere
16	0.40977(19)	1.10286(48)
25	0.41155(17)	0.74376(65)
36	0.41088(19)	0.60415(35)
49	–	0.53100(34)
64	0.40968(08)	0.48931(38)
81	–	0.46446(17)
100	0.40963(08)	0.45000(10)
121	–	0.44145(22)
144	0.40982(09)	0.43585(13)
196	–	0.42982(08)
256	0.40983(09)	0.42753(29)
576	–	0.42469(11)
1024	0.41010(09)	0.42290(26)
2304	–	0.42054(16)
4096	0.40995(10)	0.41843(22)

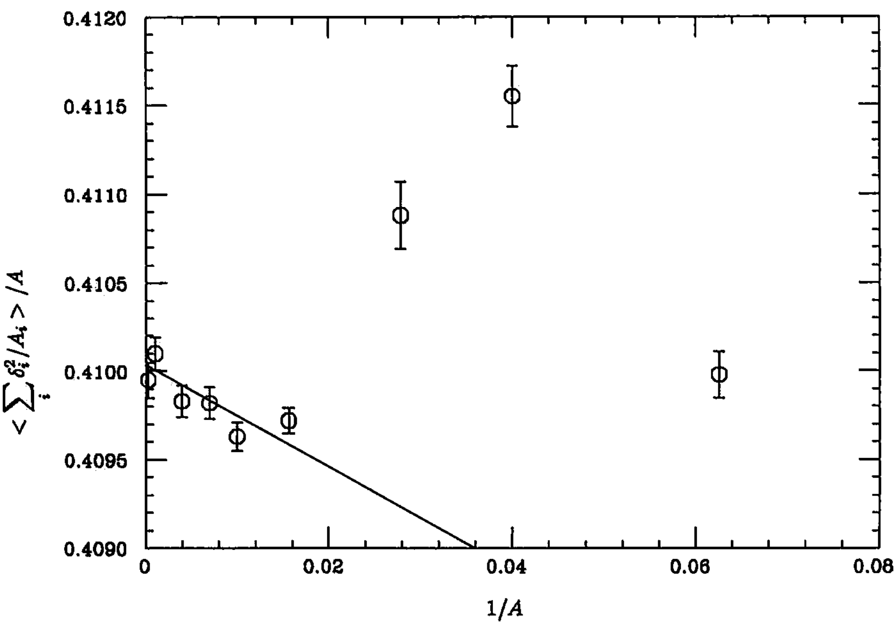

Fig. 2. Average curvature squared for the torus as a function of the lattice area for $a = 1.0$.

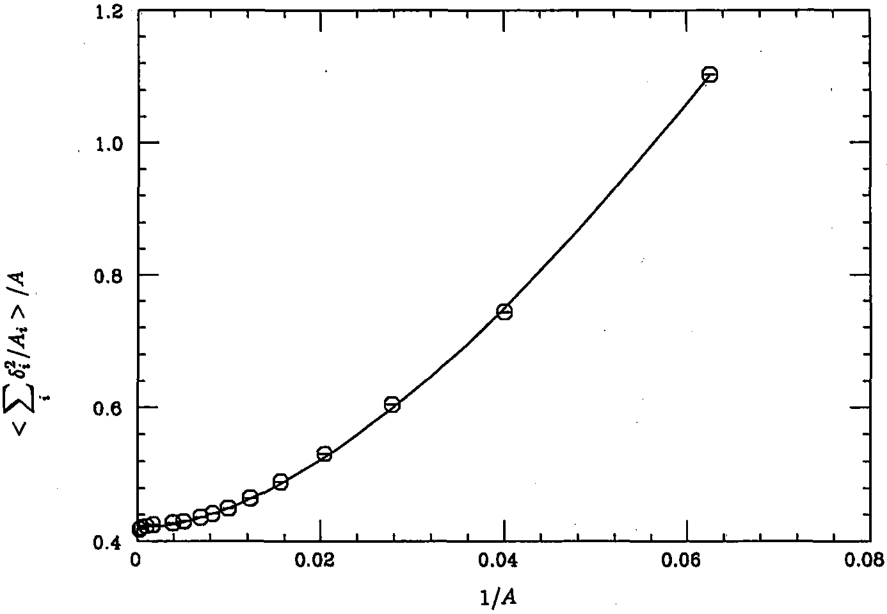

Fig. 3. Average curvature squared for the sphere as a function of the lattice are for $a = 1.0$.

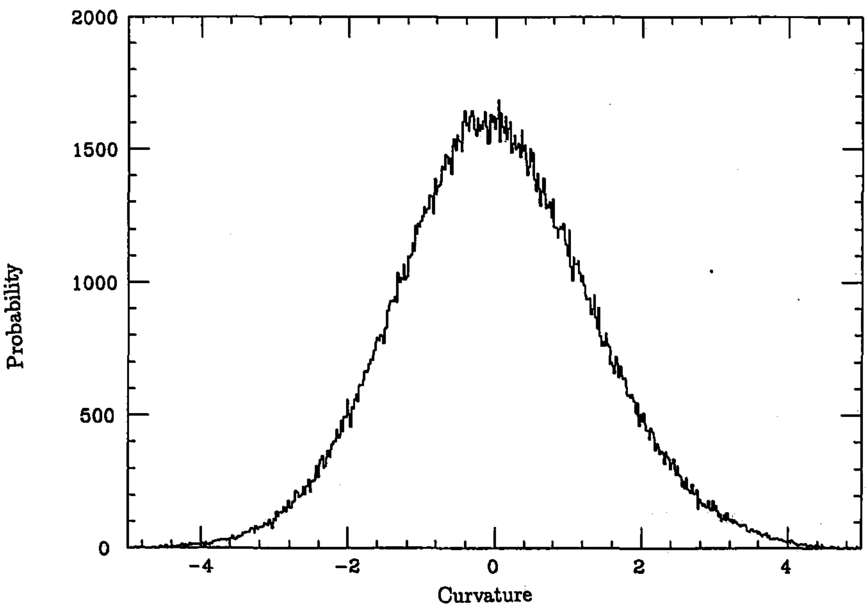

Fig. 4. Distribution of curvatures $\delta_i \sim \sqrt{g}\,R$ on a lattice with $512^2 = 262\,144$ sites.

In fig. 4 we show as an example the distribution of curvatures δ_i (corresponding to the continuum $R\sqrt{g}$) on a lattice with 262144 sites, with $a = 0.001$. Note that the distribution is a rather smooth function, as is the distribution of edge lengths and triangle areas.

Let us now turn to a discussion of the case of the sphere. Our choice of coordinates is such that we triangulate everywhere the sphere by a regular mesh of coordination number six (just like in the case of the torus), except at the two poles where we allow $(\sqrt{N})$ edges to meet at one point. We continue to use the same expression for the curvature and action at those points, since the lattice formulae are entirely geometric, and should therefore be valid irrespective of the local coordination number (in the measurements we of course include the pole contribution as well). On this (almost) regularly triangulated sphere, the results for γ_χ are not quite as accurate as for the torus, since the coordinate singularities introduced at the poles, where many $(\sqrt{N})$ edges meet at one point, introduce a finite-size correction proportional to $A^{-3/2}$, which is absent in the case of the torus. This could be avoided by using a different triangulation for the sphere which does not exhibit this feature. Still, by fitting the results for R^2 to $c_0 + c_1/A + c_2/A^{3/2}$ we have obtained $c_0 = 0.423(2)$, $c_1 = -2.55(22)$ and $c_2 = 53.6(1.2)$, and from the coefficient of the $1/A$ term the estimate $\gamma = -0.55 \pm 0.22$, to be compared to the expected KPZ answer of $\gamma = -\frac{1}{2}$.

Let us say a few words about how these estimates and errors are obtained. The χ^2 in the fits in general varies between 10 and 200. The error estimates are obtained by comparing fits that include all or only part of the data points, with proper weighting in the χ^2 function to take into account the statistical uncertainty in the Monte Carlo data. Thus if we include all the data points ($L = 4 - 64$) we get for example $c_1 = -2.57$, while if we remove the points at $L = 4$ and $L = 64$ we get $c_1 = -2.75$. If other points are removed we get estimates in the same range. If a lower statistics point (10k iterations) at $L = 128$ is also included in the fit, we obtain $c_1 = -2.55$, showing presumably a trend in the right direction for even larger lattices. We have checked that our results are stable when we add an additional term, $+c_4/A^2$, for whose coefficient we find $c_4 = 1.2(1.3)$, indeed a relatively small correction in the range of A considered. In this case a fit including all points from $L = 4$ to $L = 64$ gives $c_1 = -2.53$, consistent with previous values. By combining these values we then obtain the above estimates and uncertainties. It is fair to say that in the case of the sphere the results can only be shown to be roughly consistent with the KPZ exact result, although they clearly seem to exclude at this point for example the semiclassical result, $\gamma = 0$. In order to significantly sharpen our results much larger statistics is needed on the larger lattices, which is beyond the scope of this paper. The preceding results would suggest a restoration of general coordinate invariance at large distances or low momenta in the lattice theory. In the case of the sphere the results appear to be consistent with the KPZ result, but the errors are quite large. Further tests can be performed by embedding

TABLE 2
Liouville field or area density susceptibility as a function of lattice size. $D=0$ refers to pure gravity, and $D=\frac{1}{2}$ to gravity coupled to an Ising model

L	$\chi_\varphi(L)$		
	$D=0$	$D=\frac{1}{2}(J_c)$	$D=\frac{1}{2}$ (all J)
8	1.58(18)	1.8(2)	1.57(18)
16	10.18(3.2)	7.2(5)	9.9(3.2)
32	49.3(14.0)	22.5(4.0)	30.0(16.2)
64	113.87(33.0)	211.0(50.0)	129.0(60.0)
128	409.19(220.0)	60.0(70.0)	83.0(51.0)

the surface and measuring its extent in embedding space, as will be discussed in sect. 3.

We have also investigated the critical properties of the area fluctuation or Liouville field $\varphi(x)$, again in the case of pure gravity. We define the discrete analogue of the continuum Liouville field $\varphi(x)=\ln\sqrt{g(x)}$ as $\varphi_i=\ln A_i$, and compute the Liouville-field susceptibility on a finite lattice

$$\chi_\varphi(L)=A\left[\langle\varphi^2\rangle-\langle\varphi\rangle^2\right] \tag{2.8}$$

with

$$\varphi=\frac{1}{A}\sum_i \ln A_i . \tag{2.9}$$

For the measure $\mathrm{d}l/l$ at $a=0$ we find, using finite-size scaling on tori of sizes $L=\sqrt{A}=8-128$ (see table 2 and fig. 5),

$$\ln\chi_\varphi(L)\underset{L\to\infty}{\sim} c+(2-\eta_\varphi)\ln L \tag{2.10}$$

with an exponent $2-\eta_\varphi=2.08(12)$. This result is consistent with the expected free field behavior of the massless Liouville mode ($\eta_\varphi=0$), and again suggests a restoration of general coordinate invariance in the quantum theory. Note that even for the scale invariant measure we have breaking of scale invariance by the fact that the total area is fixed and as a consequence the average edge length takes some finite value, which then provides an ultraviolet cutoff. (When we include an R^2 term, we also break scale invariance due to its dimensionful coupling.) The natural expectation would be that if reparametrization invariance is not recovered, the Liouville mode acquires a mass of the order of the ultraviolet cutoff, which is of the order of the inverse average edge length. Thus we believe that the result that the Liouville mode is massless in our model is non-trivial. We do expect the

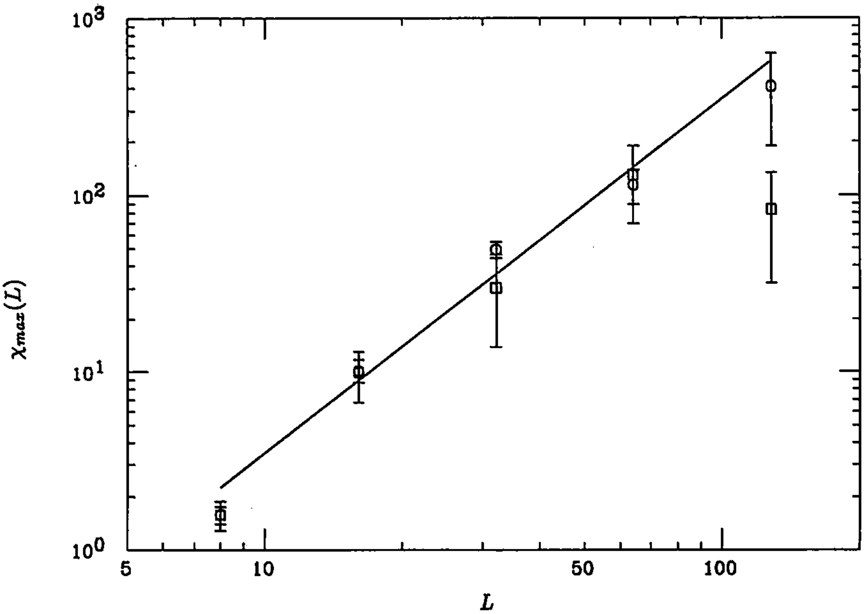

Fig. 5. Area fluctuation or Liouville susceptibility, as a function of lattice size. The circles correspond to pure gravity and $a=0$ ($D=0$), and the squares to gravity with $a=0$ coupled to an Ising field ($D=\frac{1}{2}$). The straight line on the logarithmic scale indicates a growth proportional to L^2.

same result for some class of not too singular lattice measures, and this is indeed supported by preliminary results with the $\mathrm{d}l^2$ measure.

3. Gravity coupled to a massless scalar field

Matter fields are introduced in a straightforward way. Consider a D-component scalar field ϕ_i^a, $a=1,\dots,D$. Define the scalar fields at the vertices of the triangles, and introduce finite lattice differences defined in the usual way [6, 11]

$$(\Delta_\mu \phi^a)_i = \frac{\phi_{i+\mu}^a - \phi_i^a}{l_{i,i+\mu}}. \tag{3.1}$$

The index μ labels the possible directions in which one can move from a point in a given triangle, and $l_{i,i+\mu}$ is the length of the edge connecting the two points. As an action we choose

$$I[\phi] = \frac{1}{2} \sum_{\text{edges } ij} V_{ij} \left(\frac{\phi_i^a - \phi_j^a}{l_{ij}} \right)^2, \tag{3.2}$$

where V_{ij} is the volume associated with the edge ij, via a baricentric subdivision,

$$V_{ij} = \sum_{\text{triangles } t \supset ij} \tfrac{1}{3} V_t . \tag{3.3}$$

The above lattice action then corresponds to the continuum expression

$$\tfrac{1}{2} \int \mathrm{d}^2 x \sqrt{g}\, g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^a . \tag{3.4}$$

Other forms for the scalar field action have been suggested [11], and we expect them to be equivalent in some continuum limit. The discretized partition function is then given by

$$Z = \int \mathrm{d}\mu[l]\, \mathrm{d}\mu[\phi] \exp(-I[l] - I[\phi]) . \tag{3.5}$$

Let us now consider what the appropriate lattice measure should be. For a D-component scalar field the continuum scalar field measure [21]

$$\mathrm{d}\mu[\phi] = \prod_x g^{D/4}(x) \prod_a \mathrm{d}\phi^a(x) \tag{3.6}$$

leads to a combined continuum measure for the gravitational and scalar degrees of freedom

$$\mathrm{d}\mu[g] = \prod_x \prod_{\mu \geqslant \nu} g^\sigma(x)\, \mathrm{d}g_{\mu\nu}(x) \prod_x \prod_a \mathrm{d}\phi^a(x) \tag{3.7}$$

with $\sigma = (D-1)/4$ for the DeWitt measure, and $\sigma = (D-2)/4$ for the Misner measure. Their discretized form then reads

$$\int \mathrm{d}\mu_\epsilon[l] \int \mathrm{d}\mu[\phi] = \prod_{\text{edges } ij} \int_0^\infty V_{ij}^{2\sigma}\, \mathrm{d}l_{ij}^2\, F_\epsilon[l] \prod_{\text{sites } i} \prod_a \int_{-\infty}^{\infty} \mathrm{d}\phi_i^a . \tag{3.8}$$

Again we have also considered the simpler $\mathrm{d}l/l$-type measure [6, 8, 9]

$$\int \mathrm{d}\mu_\epsilon[l] \int \mathrm{d}\mu[\phi] = \prod_{\text{edges } ij} \int_0^\infty \frac{\mathrm{d}l_{ij}^2}{l_{ij}^2} F_\epsilon[l] \prod_{\text{sites } i} \prod_a \int_{-\infty}^{\infty} \mathrm{d}\phi_i^a . \tag{3.9}$$

All the above forms of the lattice measure can be recast in the form

$$\prod_{\text{edges } ij} \int_0^\infty \frac{\mathrm{d}l_{ij}^2}{l_{ij}^2} F_\epsilon[l] \prod_{\text{sites } i} \prod_a \int_{-\infty}^{\infty} \mathrm{d}\phi_i^a \exp\Big(\sum_{ij} \ln\big[l_{ij}^{2\alpha} V_{ij}^{2\sigma} \big] \Big) \tag{3.10}$$

with $\alpha = \sigma = 0$ for the dl/l measure, and $\alpha = 1$ and σ equal to the values mentioned above for the lattice analogues of the DeWitt and Misner measures. Most of our simulation results to be presented later refer to the dl/l measure. But we will show results and give arguments which suggest that all three measures lead, in the cases we have considered, to the same critical properties.

In order to study the properties of the scalar field coupled to gravity, and attempt to compare with related work, we have measured the discrete analog of the coordinate invariant quantity

$$\langle \phi^2 \rangle = \frac{1}{D} \frac{\left\langle \int \sqrt{g} \left(\phi^a - \bar{\phi}^a \right)^2 \right\rangle}{\langle \int \sqrt{g} \rangle} \tag{3.11}$$

with

$$\bar{\phi}^a = \frac{\int \sqrt{g}\, \phi^a}{\int \sqrt{g}} . \tag{3.12}$$

Since we are working in an ensemble in which the total area is fixed, and equal to the number of sites, we have $\int \sqrt{g} = \sum_i A_i = N = A$. On the lattice we measure therefore

$$\langle \phi^2 \rangle = \frac{1}{DN} \left\langle \sum_i A_i \left(\phi_i^a - \bar{\phi}^a \right)^2 \right\rangle \tag{3.13}$$

with

$$\bar{\phi}^a = \frac{1}{N} \sum_i A_i \phi_i^a . \tag{3.14}$$

We have considered the cases $D = 0$ (no feedback of the scalar field on the geometry), $D = 1$ and $D = 2$. For the coefficient of the R^2 term in eq. (2.1) we have taken $a = 0.1$ and $a = 0.001$, motivated by our intention to explore the sensitivity of the results to what is expected to be an irrelevant term. We have considered lattices ranging in size from $8^2 = 64$ to $512^2 = 262\,144$ sites. The scalar field updates have been performed both by Metropolis Monte Carlo as well as by a heatbath, with compatible results. The number of sweeps for the scalar as well as the gravitational fields varied between at least 500k for the 64-site lattice to at least 3.5k on the 262 144-site lattice. It is only on the largest lattice that we have observed appreciable signs of critical slowing down, leading to somewhat larger errors. The results are shown in table 3 and in figs. 6 ($D = 0$), 7 ($D = 1$) and 8 ($D = 2$).

We have fitted the numerical results for $\langle \phi^2 \rangle$ to several functional forms, using a standard error-weighted least square algorithm. We estimate the errors in the fits by both computing the intrinsic uncertainty in the fitting parameters, as well as by

TABLE 3
Expectation values $\langle\phi^2\rangle$ as a function of lattice size, for different values of a and D. The column labeled with a * corresponds to the case of the DeWitt (dl^2) measure

$A=L^2$	$\langle\phi^2\rangle$ $D=0$, $a=0.1$	$D=0$, $a=0.001$	$D=1$, $a=0.1$	$D=1$, $a=0.001$	$D=1$, $a=0^*$	$D=2$, $a=0.1$
64	0.902(8)	0.915(6)	0.919(1)	0.949(1)	0.916(4)	0.938(1)
256	1.164(6)	1.196(6)	1.186(1)	1.214(2)	1.162(4)	1.210(1)
1024	1.420(4)	1.455(6)	1.443(3)	1.487(4)	1.402(4)	1.478(3)
4096	1.698(21)	1.702(12)	1.702(6)	1.743(9)	1.646(9)	1.734(6)
16384	1.928(47)	1.984(35)	1.919(17)	1.923(22)	1.906(30)	2.012(17)
65536	2.321(161)	2.117(39)	2.340(65)	2.165(47)	2.212(70)	2.191(65)
262144	2.44(23)	2.50(13)	2.327(27)	2.424(32)	2.36(8)	-

comparing the variations in the parameters as the number of points used in the fit is changed (we have typically six to seven values of $L=\sqrt{A}$, ranging from 8 to 512, at our disposal). The results clearly suggest a linear behavior of $\langle\phi^2\rangle$ in $\ln A$ for all D and a

$$\langle\phi^2\rangle \underset{A\to\infty}{\sim} c_0 + c_1 \ln A\,, \tag{3.15}$$

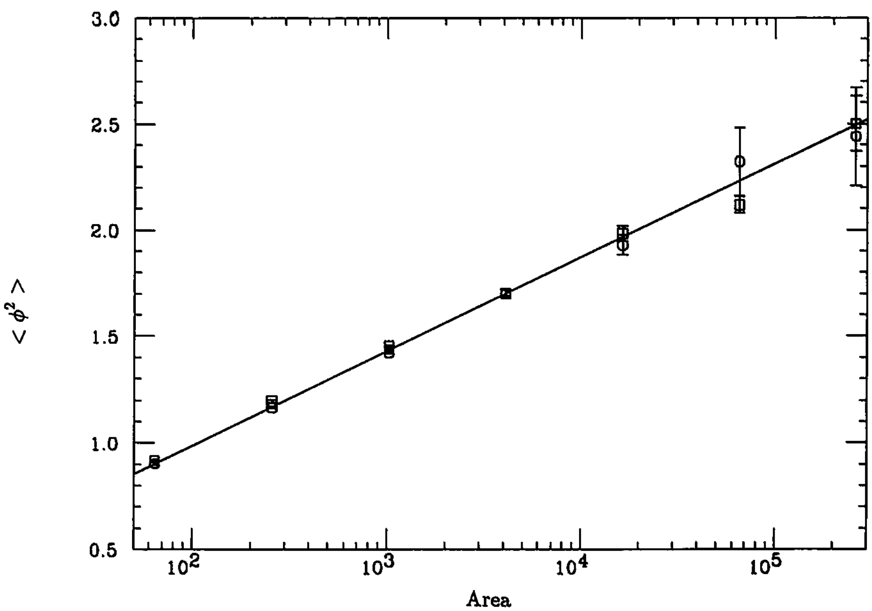

Fig. 6. Scalar field average $\langle\phi^2\rangle$ for pure gravity ($D=0$), with $a=0.1$ (circles) and $a=0.001$ (squares). The straight line corresponds to a logarithmic divergence, or $d_H=\infty$.

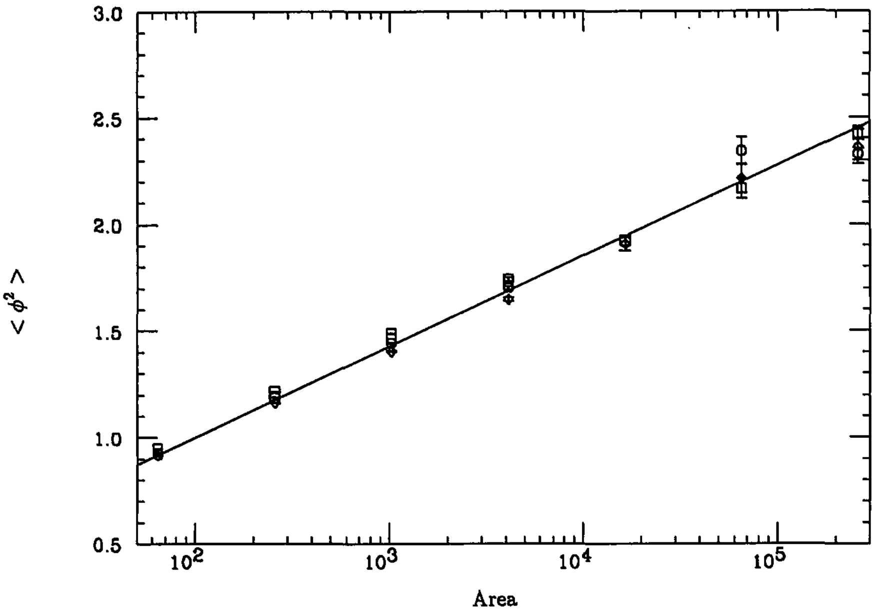

Fig. 7. Scalar field average $\langle\phi^2\rangle$ for gravity coupled to a massless scalar field ($D=1$), with $a=0.1$ (circles) and $a=0.001$ (squares). The diamonds indicate results for the DeWitt measure (dl^2), with $a=0$. The straight line corresponds to a logarithmic divergence, or $d_H=\infty$.

but we have tried other fits as well. Assuming the above functional form, we find the results given in table 4. In the fits typically the χ^2's per degree of freedom are of order one. Note that the coefficients c_0 and c_1 are *not* universal (in particular the coefficient of the ln A term can be shown to be proportional to the area of an elementary triangle on the lattice, which is clearly not a universal quantity). For the pure ln A fit there seems to be very little dependence of the fitted coefficients on D.

The above results are not surprising, since without an extrinsic curvature term in the action

$$I_{ec}=\kappa\int d^2x\sqrt{g}\,[\Delta\phi^a]^2, \tag{3.16}$$

where Δ is the covariant laplacian

$$\Delta=\frac{1}{\sqrt{g}}\partial_\mu g^{\mu\nu}\sqrt{g}\,\partial_\nu, \tag{3.17}$$

the surface is expected to completely fold onto itself in embedding space, leading to an infinite fractal dimension, even when the gaussian curvature is zero every-

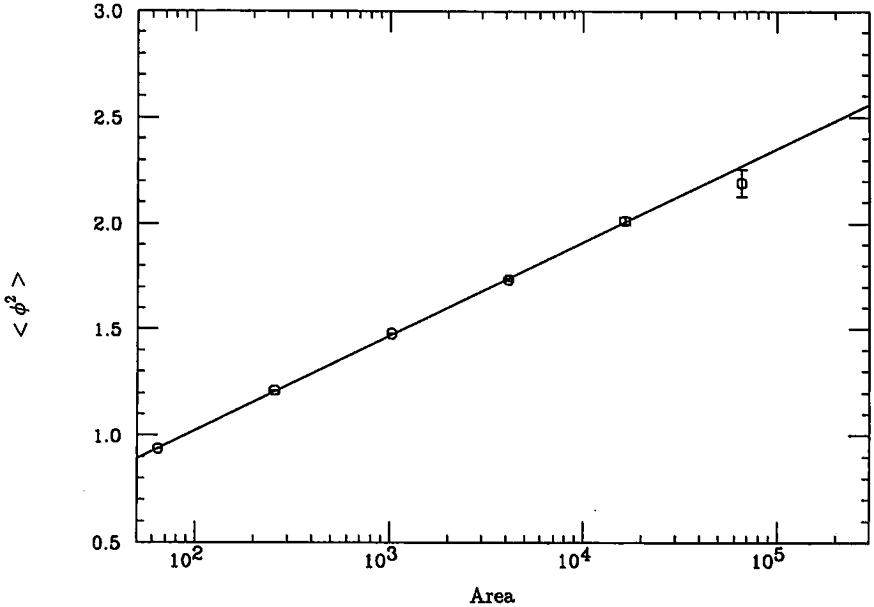

Fig. 8. Scalar field average $\langle \phi^2 \rangle$ for gravity coupled to a two-component massless scalar field ($D = 2$), with $a = 0.1$. The straight line corresponds to a logarithmic divergence, or $d_H = \infty$.

where ($a \to \infty$) [29,31,32]. If such an embedding in flat euclidean space is considered, then the field ϕ^a is known to play the role of the coordinate X in embedding space, as discussed by Polyakov [24]. As far as gravity is concerned, this situation of infinite fractal dimension is of limited consequence, since one does not require the existence of a continuum limit in embedding space, but only for *intrinsic* properties of the manifold. From an intrinsic point of view, the manifold is quite smooth, since $\langle \phi^2 \rangle$ shows a free-field type behavior, and the intrinsic or gaussian curvature (as measured by R^2) is bounded. On the other hand, critical properties of two-dimensional quantum gravity will not necessarily be independent of the

TABLE 4
Summary of least square fits for the coefficients c_0 and c_1 in eq. (3.15) for $\langle \phi^2 \rangle$, for different values of D and a

D	a	c_0	c_1
0	0.1	0.13(3)	0.187(3)
0	0.001	0.17(3)	0.185(3)
1	0.1	0.15(3)	0.187(3)
1	0.001	0.16(3)	0.190(3)
2	0.1	0.14(3)	0.193(3)

existence of an extrinsic curvature term, since correlation functions of ϕ are affected by it [31,32].

If we attempt to fit $\langle\phi^2\rangle$ to a power of L instead (as suggested for example by the authors of ref. [28]), namely

$$\ln\langle\phi^2\rangle \underset{A\to\infty}{\sim} c_0 + c_1 \ln A\,, \tag{3.18}$$

we find values for $c_1 = 2/d_{\mathrm{H}}$, where d_{H} is the fractal dimension of the surface, in the range 0.15(2), or d_{H} about 13. On the other hand, the χ^2 parameter for the fits is always at least an order of magnitude larger than in the previously discussed case, and furthermore d_{H} shows a clear trend towards an increase with lattice size. From this we conclude that the fractal dimension is always infinite in our model. This should be contrasted to the numerical results of the third paper in ref. [14], which gave a finite value for the fractal dimension (but on significantly smaller lattices than the ones we have treated here, and therefore with understandable uncertainties). Also it is clear that in order to reconcile our results, as well as the semi-classical limit $D \to -\infty$, the power-law term discussed in the second paper of ref. [28] must have a vanishing coefficient.

We have also investigated the possibility that on the torus and for $D = 1$ there might be a $(\ln A)^2$ term as well, as suggested again by the authors of the second paper of ref. [28],

$$\langle\phi^2\rangle \underset{A\to\infty}{\sim} c_0 + c_1 \ln A + c_2 (\ln A)^2. \tag{3.19}$$

Both for $a = 0.1$ and $a = 0.001$ we obtain the bound

$$c_2 \leqslant 0.002(2)\,, \tag{3.20}$$

suggesting that if such a term is present, its coefficient must be rather small. On the other hand, there is no convincing numerical evidence for such a term in any other model for two-dimensional surfaces. Our results seem to suggest that for the torus the semi-classical ($D \to -\infty$) result is exact, at least for the values of D which we have explored.

It is of interest to explore how some of the above results depend on the gravitational measure. For $D = 1$ and $a = 0$ we have repeated the simulation using the lattice analogue of the DeWitt measure (eq. (3.8)). In the case $D = 1$ this particular measure becomes quite simple, $\prod \mathrm{d}l^2$, since all the volume factors cancel out. The computed values for $\langle\phi^2\rangle$ are well described by eq. (3.15) (see also fig. 7), and we find

$$c_0 = 0.19(3)\,, \qquad c_1 = 0.175(8)\,, \tag{3.21}$$

which confirms the fact that a change in the measure changes the non-universal coefficients c_0 and c_1, but leaves the functional dependence on A unchanged, and in particular the result $d_H \equiv \infty$.

It is of interest to investigate further the dependence of physical results, like the fractal dimension d_H, on both the gravitational measure (i.e. the parameter σ in eqs. (3.8) and (3.10)) and the number of components of the scalar field D, perhaps in some more extreme limits, like large D and more singular measures. To this end we have run a number of long simulations for $D = 4$, $D = 8$ and $D = 12$ on lattices varying in size between 64 and 4096 sites with $a = 0.001$, using the "flat" measure $\mathrm{d}l/l$ (for which therefore no dependence on D is included). We have found that as D increases, the coefficient of the $\ln A$ term in $\langle \phi^2 \rangle^{10}$ increases (from 0.24 to 0.33 to 3.36) until for $D = 12$ the behavior is more consistent with a power-law in the area. In this last case a power-law fit gives $d_H \approx 4.4$, which is close the fractal dimension for branched polymers (trees), $d_H = 4$. The large error bars in the data stem from the fact that for $D = 12$ the model has entered into a new phase, in which relaxation times are extremely long. Indeed the step size in the simulation has to be decreased by four orders of magnitude to keep the acceptances of order one, which suggests that the model we are considering is probably not even appropriate for this phase. Another indication that this is the case comes from the fact that a number of edges start to become quite long, while others get quite short; regions develop where the curvature is very large in magnitude, and it becomes increasingly difficult to get rid of these "defects", especially on the larger lattices.

There are a number of ways by which one can try to locate more accurately the transition. Indeed on a finite lattice a sharp transition between a phase in which $d_H = \infty$ and $d_H = 4$ will be somewhat broadened. One way then is to try, say by a linear fit to the inverse of the coefficient of the $\ln A$ term, to determine approximately where it diverges

$$\langle \phi^2 \rangle \underset{A \to \infty}{\sim} c_0 + c_1 \ln A \,, \qquad c_1 \sim \frac{c}{D - D_c} \tag{3.22}$$

which gives from our data $D_c = 13.1 \pm 0.6$. On the other hand, by fitting the data to a finite power of A instead, one can estimate D_c from where the effective power start to become very small: one finds $D_c = 14.1 \pm 1.4$, consistent with the previous estimate. Thus for the measure $\mathrm{d}l/l$ there is a transition somewhere close to $D = 13$. One can also try to extract a value for the susceptibility exponent γ_χ by using eq. (2.6). One finds for $D = 4, 8, 12$ $\gamma_\chi \sim 1.8, 1.7, -3.8$, which is on the one hand consistent for small D with the previous results for $D = 0, 1, 2$ ($\gamma_\chi = 2$, torus), and perhaps for larger D with the fact that γ might eventually become negative. On the other hand, we do not believe that our results are accurate for $D > D_c$, for the reasons mentioned above.

One would like then to understand how the previous results depend on the gravitational measure, the R^2 term in the action, and D. Since performing a full exploration of the phase diagram using a set of different lattice sizes is quite time consuming, we have restricted ourselves to a lattice of 64 sites, and have defined an effective fractal dimension $d_H(A)$ via

$$d_H(A) = 2\left[\frac{\ln\langle\phi^2\rangle_A}{\ln A}\right]^{-1} \underset{A\to\infty}{\sim} d_H . \tag{3.23}$$

We have explored the measure dl/l for $a = 0.001, 0.1, 1.0$, the measure $dl/l \times A_l^{D/2}$ for $a = 0.001$, and the lattice analogue of the DeWitt measure, $dl^2 \times A_l^{(D-1)/2}$, also with $a = 0.001$. We varied D between 0 and 32 in intervals of 2, and on each lattice we performed 400 + 600 iterations, starting always from flat space (thus all data points are statistically uncorrelated). We have found that the transition at finite D seems to disappear when the correct DeWitt weighting factor for the scalar field measure $\sqrt{g}^{D/2}$ is taken into account for large D (for small D its effect appears to be negligible). On the other hand, if the coefficient of the R^2 term a is varied for the dl/l measure, then it seems that D_c can be shifted by one or two units. In other words, the location of the transition in D seems to be non-universal and dependent on a.

A similar situation is encountered for $D = 0$ when the parameter σ of the measure (see eq. (3.10), $\alpha = 0$) is varied. In this case we write the measure as $dl/l \times A_l^{2\sigma}$, and vary 2σ between 0 and -12, setting $a = 0.001$ and $a = 1.0$. As the measure becomes increasingly singular (large negative σ), we again encounter a transition to the branched polymer (tree) phase, with d_H approaching four. Due to the smallness of the lattice we observe some substantial rounding, which presumably will sharpen as one goes to larger lattices. For $a = 0.001$ the location of the transition can be estimated at $2\sigma_c \approx -12$, by comparing to the analogous behavior for the transition in D discussed above. If a is larger, then it seems that the transition moves to even more negative values for σ, as expected from the effect of the R^2 term which tends to suppress singularities in the curvature.

Our results for the phase diagram of two-dimensional gravity coupled to a D-component scalar field, with the measure of eq. (3.10) and a higher-derivative (regulator) term, can then be summarized as follows. We expect a whole line of phase transitions for all D's considered here ($0 \geqslant D \geqslant 32$), which crosses the $\sigma = 0$ axis only for larger (~ 12) values of D. For the DeWitt measure we find no transition in the region considered, but we cannot exclude one for even larger values of D ($D > 32$), even though we are more inclined to believe that such a transition never takes place for the DeWitt measure, which after all is perhaps the more credible gravitational measure in the presence of scalar fields.

4. Ising spins coupled to gravity

In order to study a model for gravity coupled to Ising spins, we consider the pure gravity action of eq. (2.1), with the additional term

$$\frac{J}{2}\sum_{\text{edges } ij} V_{ij}\left(\frac{S_i - S_j}{l_{ij}}\right)^2, \tag{4.1}$$

which is analogous in form to the scalar field action (3.2), except for the replacement of the scalar field multiplet ϕ_i^a by the Ising spin variables $S_i = \pm 1$.

For the case of the sphere, the Ising model on a dynamically triangulated lattice has been solved exactly by exploiting the equivalence to a large-N matrix model [17]. The critical exponents $\alpha = -1$, $\beta = \frac{1}{2}$, $\gamma = 2$, $\eta = \frac{2}{3}$ and $\nu = \frac{3}{2}$ agree with the conformal field theory estimates of ref. [26]. Here we will investigate again the case of a lattice with the topology of a torus (periodic boundary conditions for all fields), and show that for the torus the semiclassical ($D = -\infty$) results $\alpha = \alpha_0, \ldots,$ where $\alpha_0 = 0, \ldots$ are the pure Ising critical exponents, appear to hold.

In our simulation we have measured the discrete analogues of the invariant spontaneous magnetization per spin

$$M = \frac{\langle \int \sqrt{g}\, S \rangle}{\langle \int \sqrt{g}\, \rangle} \tag{4.2}$$

and of the zero field susceptibility

$$\chi = \frac{\langle \int \sqrt{g}\, S \int \sqrt{g}\, S \rangle}{\langle \int \sqrt{g}\, \rangle} - \frac{\langle \int \sqrt{g}\, S \rangle^2}{\langle \int \sqrt{g}\, \rangle}. \tag{4.3}$$

In a fixed-area ensemble with $A = N = L^2$, these formulae simplify to

$$M = \langle m \rangle, \qquad m = \frac{1}{A}\sum_i A_i S_i, \tag{4.4}$$

$$\chi = A\left[\langle m^2 \rangle - \langle m \rangle^2\right], \tag{4.5}$$

respectively. Since on a finite lattice the spontaneous magnetization will vanish identically even at low temperatures, we have found it convenient to also define the quantities M' and χ', which differ from the above expressions by the replacement of m with $|m|$. In addition we have computed the fluctuation in the Ising energy

$$C_{\text{Is}} = \frac{1}{A}\left[\left\langle (E_{\text{Is}})^2 \right\rangle - \langle E_{\text{Is}} \rangle^2\right], \tag{4.6}$$

with

$$E_{\text{Is}} = \tfrac{1}{2} \sum_{\text{edges } ij} V_{ij} \left(\frac{S_i - S_j}{l_{ij}} \right)^2 . \tag{4.7}$$

If all the edges are taken to be of equal length, then the system reduces to a pure Ising model on a triangular lattice, for which $J_c = \frac{1}{2}\sqrt{3}\,\ln 3 = 0.9514\ldots$. This has provided us with the possibility of a useful comparison of our results with and without gravity on finite lattices. In our simulations we have at first used again the gravitational measure dl/l of eq. (1.8), and have set the higher-derivative coupling to $a = 0.001$. We have investigated lattice sizes varying from $8^2 = 64$ sites to $128^2 = 16384$ sites. The length of our runs varies in the critical region ($J_c \approx 1.03$) between 400k sweeps on the smallest lattice and 50k sweeps on the largest lattice.

Let us begin by discussing the magnetization results. Since close to J_c we expect $M \sim (J - J_c)^{\beta}$, we have that $M^{1/\beta}$ should appear close to linear. In fig. 9 we show the magnetization M' squared (suggested by the assumption $\beta = \frac{1}{2}$), and raised to

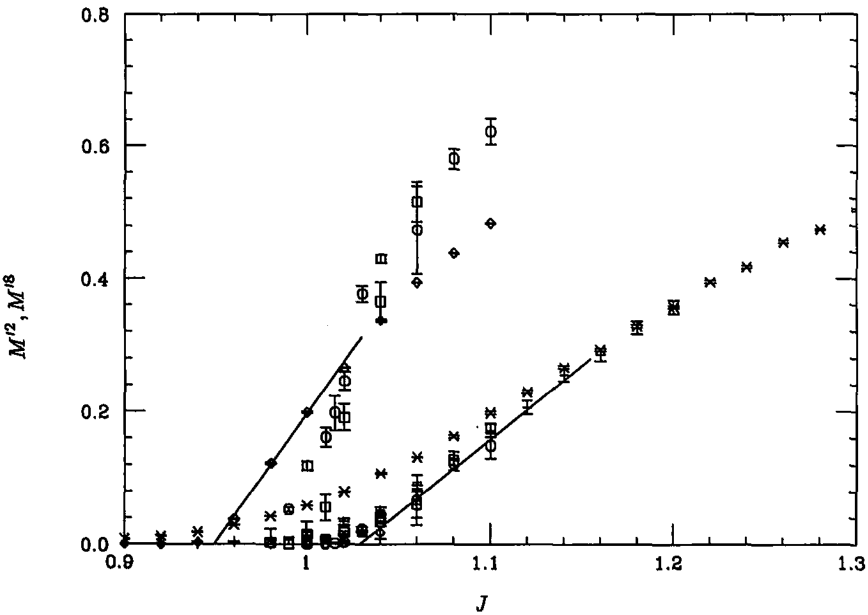

Fig. 9. Ising magnetization M' squared and raised to the eight power, for different lattice sizes ($L = 8$ (diagonal crosses), 16 (crosses), 32 (squares), 64 (circles), 128 (diamonds)). The pure Ising magnetization on an $L = 64$ lattice, raised to the eight power, (diamonds) is shown for comparison.

TABLE 5
Peak values for the Ising susceptibilities χ and χ', and for the specific heat C, as a function of lattice size

L	$\chi_{max}(L)$	$\chi'_{max}(L)$	$C_{max}(L)$
8	24.6(10)	4.5(2)	4.16(5)
16	97.3(50)	14.5(5)	4.89(5)
32	236.0(60.0)	51.0(4.0)	5.74(10)
64	710.0(120.0)	179.0(50.0)	6.62(60)
128	359.0(150.0)	342.0(70.0)	6.01(1.2)

the eight power (suggested by the assumption $\beta = \frac{1}{8}$), for different size lattices. Our results clearly favor the pure Ising exponent, in spite of the uncertainty in J_c. In the case of the susceptibility we expect from finite-size scaling a scaling form

$$\chi(L,J) = L^{2-\eta}\bar{\chi}\left(L^{1/\nu}(J-J_c)\right). \tag{4.8}$$

Thus in particular the peak in χ (or χ') should scale like $L^{2-\eta}$ for large L. In table 5 and in fig. 10 we show the computed peaks in χ and χ' as a function of L, on a

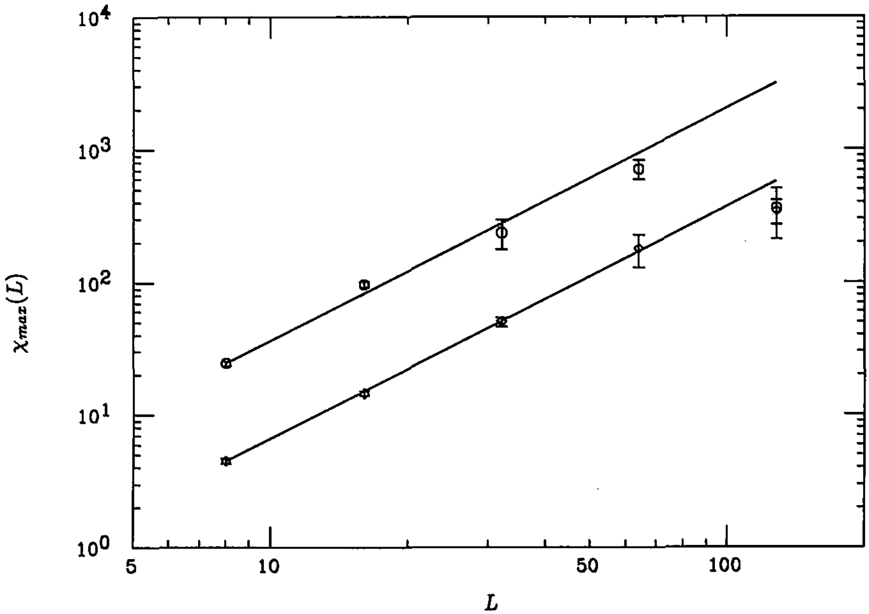

Fig. 10. Peak values for the Ising susceptibilities χ (circles) and χ^* (diamonds) for different size lattices. The straight lines correspond to an exponent $2-\eta = \frac{7}{4}$.

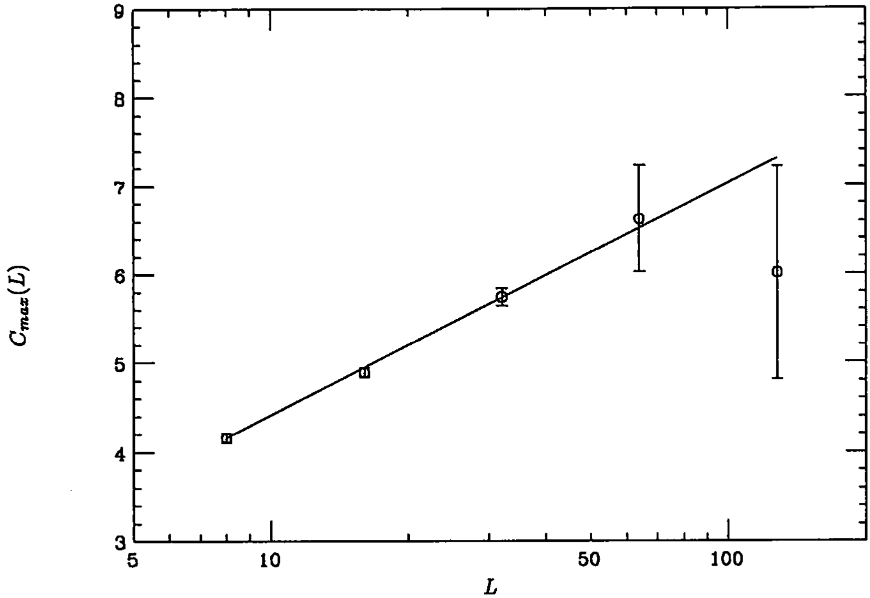

Fig. 11. Peak values for the Ising specific heat C. The straight line corresponds to a logarithmic divergence ($\alpha = 0$).

logarithmic scale in the figure. Using a least-square fit we estimate

$$2-\eta = 1.78(6) \quad (\text{from } \chi),$$

$$2-\eta = 1.68(3) \quad (\text{from } \chi'), \tag{4.9}$$

which is completely consistent with the pure Ising results $2-\eta = \gamma/\nu = \frac{7}{4}$.

The results for the peak in the Ising specific heat C_{Is} as a function of lattice size L are shown again in table 5 and in fig. 11. They resemble closely what is expected from a logarithmic divergence $C_{\text{Is}} \sim \ln L$ ($\alpha/\nu = 0$). From the combined magnetization, susceptibility and specific heat results, we estimate the critical Ising coupling in the infinite-volume limit at

$$J_c = 1.030(5). \tag{4.10}$$

One can improve on the estimate of J_c by considering the fourth-order cumulant [33]

$$U_L(J) = 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2} \tag{4.11}$$

with scaling form

$$U_L(J) = \bar{U}\big(L^{1/\nu}(J - J_c)\big). \tag{4.12}$$

The curves $U_L(J)$, for different and sufficiently large values of L, should intersect at a common point J_c, where the theory exhibits scale invariance, and where U takes on the fixed-point value U^*. We have found that indeed the curves meet very close to a common point, and from the intersection of the curves for $L = 8$ to 128 we estimate

$$J_c = 1.025(3), \tag{4.13}$$

which is consistent with the previous estimate of the critical point. We also determine $U^* = 0.59(2)$, which is again consistent with the pure Ising model estimate for the invariant charge $U^* \approx 0.58$ [33]. We conclude that for the Ising model coupled to gravity on a simplicial lattice with the topology of a torus, the exponents are, within our errors, the same as in the pure Ising model.

In order to check again the dependence of our results on the specific measure ($\mathrm{d}l/l$) used, we have also performed a simulation with the "DeWitt" measure appropriate for one field ($\mathrm{d}l^2$), and with the "Misner" measure $\mathrm{d}l^2/\sqrt{V_l}$. We have done the two simulations on a lattice of $32^2 = 1024$ sites, and have chosen $a = 0$. Our conclusions therefore rely mostly on a comparison with similar results with the $\mathrm{d}l/l$ measure, and on the same size lattice. While we find that the critical point moves to a larger value ($J_c \approx 1.04$), we find that the peak values in the susceptibility and specific heat are comparable to the values for the $\mathrm{d}l/l$ measure on the same size lattice. In table 6 we compare the "DeWitt" ($\mathrm{d}l^2$) measure for $a = 0$ and $L = 32$ to the "Misner" ($\mathrm{d}l^2/\sqrt{V_l}$) measure for $a = 0$, and to the $\mathrm{d}l/l$ measure results for $a = 0$ and $a = 0.001$. Our results show again little dependence on the measure, at least within our errors and for the lattice sizes considered. A more detailed comparison would involve several lattice sizes, which is outside the scope of the present work. (We did check though that for the "DeWitt" measure the results on a larger lattice, 64^2, combined with the results on the smaller lattice, are again consistent with pure Ising exponents: for example we estimate $2 - \eta \approx$

TABLE 6
Results for the peak values of the Ising susceptibilities χ and χ', and of the specific heat C, on a 32^2 lattice and for several forms of the gravitational measure

Measure	$\chi_{\max}$	$\chi'_{\max}$	$C_{\max}$
DW, $a = 0$	279(40)	49(5)	6.4(4)
M, $a = 0$	332(40)	51(6)	6.8(6)
$\mathrm{d}l/l$, $a = 0$	353(51)	37(6)	5.5(6)
$\mathrm{d}l/l$, $a = 0.001$	331(55)	40(6)	5.1(5)

1.75(14).) We conclude that the choice of measure does not seem to affect our conclusion that the critical exponents are pure Ising-like.

We have also investigated the critical properties of the Liouville field in the case of gravity coupled to an Ising model. We have computed the Liouville-field susceptibility $\chi_\varphi(J, L)$ discussed previously (in the context of pure gravity) and have found the critical exponent η_φ to be still consistent with 0. In particular for the measure dl/l with $a = 0$ and at the Ising transition, we find using finite-size scaling on sizes 8–64 (see fig. 5) for the exponent $2 - \eta_\varphi = 2.19(19)$, a result which is consistent with the expected free-field behavior of a massless Liouville mode. As we move away from the Ising critical point, we still find a similar result, namely that the Liouville susceptibility grows close to linearly in L^2 as L is increased, indicating that the Liouville field behaves like a massless free field for all J.

5. Conclusions

In the preceding sections we have discussed results relevant for a model of simplicial quantum gravity, and mostly on a manifold with the topology of a torus. It is characteristic of our model that the variations in the geometry of space are described by fluctuating edge lengths on a lattice with fixed coordination number. We have considered the case of pure gravity ($D = 0$), gravity coupled to a scalar field ($D = 1$, 2), and gravity coupled to an Ising model ($D = \frac{1}{2}$). We have investigated in detail how the results for critical properties depend on what are expected to be irrelevant (R^2-type) terms, as well as on the form of the gravitational measure.

In the case of pure gravity we have computed the exponent γ_χ for both the torus and the sphere, and found good agreement with the expected exact answers from conformal field theory. For the torus we have computed critical properties of the Liouville field, which corresponds to the area density fluctuations on the lattice, and found them to be in agreement with the expectation that the Liouville field behaves like a free massless field for $\lambda \to \lambda_c$, or $A \to \infty$.

By adding a D-component scalar field to the model, we have been able to compute the average $\langle \phi^2 \rangle$, and therefore make contact with the results on models of random surfaces. These predict, among other things, an infinite fractal dimension for the surface, at least in the absence of extrinsic curvature terms. We found that for the torus the fractal dimension is indeed infinite for $D = 0$, 1 and 2. For $D = 1$ we have looked for an $(\ln A)^2$ term, but have found its amplitude to be quite small and consistent with zero. Our results seem to be insensitive, within the range of parameters explored, to the presence of an R^2-type term in the action or to the detailed form of the measure. We have argued that for sufficiently singular measures though, the triangulation will tend to collapse into a degenerate configuration of edges.

Finally we have studied the case of gravity coupled to an Ising field. By computing the Liouville field susceptibility in the presence of the Ising spins as well (both at the Ising transition and away from it), we have established that the Liouville field stays massless. On the other hand, a computation of the magnetization, susceptibility and specific heat suggests that for the torus the exponents are the same as for the pure Ising model (which is basically the semi-classical $D \to -\infty$ result). There are two possible explanations for this last finding. The first is that there is some flaw in our model (in the action or in the measure), and that in some other model for lattice gravity coupled to Ising spins the correct gravitational exponents will be obtained. But it appears difficult to reconcile this conclusion with the fact that we seem to obtain the correct value for the pure gravity string susceptibility exponent γ on the sphere, and the fact that the lattice analogue of the Liouville field appears massless (a lack of reparametrization invariance is expected in general to lead to massive excitations only, with masses of the order of the ultraviolet cutoff). An alternative possible explanation lies perhaps with the peculiar properties of the toroidal topology.

In conclusion it is clear that it would be of interest to investigate in greater detail a topology different from the torus, like a sphere or surfaces with boundaries. Even more challenging are of course possible applications to four-dimensional gravity.

The authors have benefited from conversations with J. Ambjørn, F. David, V. Kazakov, J. Klauder, P. Menotti, A. Migdal and P. Rakow. A major part of the numerical computations were performed at the San Diego Supercomputer Center. The research of H.W.H. was supported in part by the National Science Foundation under grant NSF-PHY-8906641. The research of M.G. was supported in part by a California State University SCA award.

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