Simplicial quantum gravity in three dimensions: Analytical and numerical results

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The weak-field expansion and the nonperturbative ground state of three-dimensional simplicial quantum gravity are discussed. The correspondence between lattice and continuum operators is shown in the context of the lattice weak-field expansion, around a simplicial network built of rigid hypercubes, and the lattice translational zero modes are exhibited. A numerical evaluation of the discrete path integral for pure lattice gravity (with and without higher-derivative terms) shows the existence of a well-behaved ground state for sufficiently strong gravity ($G > G_c$). At the critical point, separating the smooth from the rough phase of gravity, the critical exponents are estimated using a variety of methods on lattices with up to $7 \times 64^3 = 1\,835\,008$ edges. As in four dimensions, the average curvature approaches zero at the critical point. Curvature fluctuations diverge at this point, while the fluctuations in the local volumes remain bounded.

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I. INTRODUCTION

Three-dimensional quantum gravity is of interest since it shares some common features with the four-dimensional theory (lack of perturbative renormalizability, unboundedness of the Euclidean gravitational action [1–3]), while at the same time it appears less trivial than the two-dimensional case, where the pure gravity action is a topological invariant. As is the case for other field theories, the lattice approach to gravity allows one to define the path integral for quantum gravity nonperturbatively, and explore the nature of the ground state without relying on an expansion in a small parameter [4–17]. In addition, a number of interesting exact results have appeared for pure Minkowski continuum gravity with vanishing cosmological constant [18]. These results have led to renewed interest in lattice models for three-dimensional (3D) gravity [19]. In particular, the state sum model of Turaev and Viro for a triangulated three-manifold, which was anticipated by Ponzano and Regge in what was probably the earliest work on lattice quantum gravity, has been shown to be equivalent to the exactly soluble Chern-Simons continuum theory of 3D gravity.

In this paper we concentrate on the simplicial formulation of quantum gravity, also known as Regge calculus, which was the version of lattice gravity considered by Ponzano and Regge (for reviews and a more complete list of earlier references, see Refs. [5–8]). One of the advantages of the simplicial approach lies in the fact that it can be formulated in any space-time dimension (including the physically relevant case of four dimensions), and that it can be shown to be classically completely equivalent to general relativity. This correspondence is particularly transparent in the usual weak-field expansion [9], with the edge lengths on the lattice playing the role of the components of the metric field in the continuum. Furthermore, the correspondence between lattice and continuum quantities is clear, and the interpretation of the terms in the action, as well as the identification and separation of the measure contribution, is unambiguous. The weak-field expansion has not been worked out yet for dynamically triangulated models for random surfaces [20–22], which lack classical reparametrization invariance. On the other hand, reparametrization invariance seems to be recovered in all models at the quantum level, which represents a rather remarkable result. Of course, since gravity is not well defined in the continuum, a number of difficulties, related for example to the gravitational measure factor [1], the unboundedness of the Euclidean gravitational action about two dimensions [2], and the lack of perturbative renormalizability also above two dimensions [3], persist in the lattice formulation as well.

A detailed description of the construction of the action for simplicial lattice gravity without and with matter fields can be found in the literature [4–12]. In a number of cases the correspondence with the continuum answer can be established quite rigorously and in great generality [10,11]. Given reasonable geometric and positivity properties, universality is expected to lead to the same results for quantities such as physical observables, exponents, anomalous dimensions, etc. in some continuum limit. This is known to be the case in other lattice field theories, where the physical particle spectrum is expected to be independent of specific details of the ultraviolet lattice cutoff and the specific form of the lattice action, as long as it preserves the basic symmetries; it has also been verified explicitly to some extent in the case of two-dimensional gravity [14,15], where good agreement is found between the lattice gravity results and the conformal field theory predictions. In the simplicial formulation, as in the continuum, the local curvature is a continuous function of the relevant edge lengths; the straightforward geometric correspondence of lattice and continuum quantities is one attractive feature of the Regge calculus approach, and allows one, for example, to distinguish between action and measure contributions, and between various local higher-dimensional curvature terms [12].

While simplicial quantum gravity can be formulated on a random lattice with varying edge lengths [11], it appears advantageous at least initially to adopt a regular lattice, which is much easier to work with from a computational point of view. The continuous diffeomorphism invariance discussed in [4,9], as well as in [15], is of course not lost by going to such a regular lattice. (The random lattice might appear more satisfactory from a conceptual point of view, since it incorporates, for smooth manifolds, the invariance under "large" lattice diffeomorphisms, whereas in the regular lattice only "small" lattice diffeomorphisms are allowed; thus the two different lattices induce quite different cutoff structures in orbit space.) Eventually one hopes to redo all the calculations for such a random lattice, but so far most of the numerical work on lattice gravity in four dimensions has been done for regular lattices [12–17,23,5]. In the following we will discuss results for a simplicial complex topologically equivalent to a torus in three dimensions, mainly because it is easier to work with. In the end one expects short-distance renormalization effects and critical behavior (and therefore the continuum limit as well) to be independent of the boundary conditions, and therefore of the topology.

II. GRAVITATIONAL ACTION AND MEASURE ON THE LATTICE

While the physically most interesting model of gravity is that in four dimensions, it appears worthwhile to investigate the simpler, intermediate case of three dimensions. It is less trivial than pure two-dimensional gravity, since the Einstein action is no longer a topological invariant. It is also far less complex (as far as the lattice interactions are concerned) than the four-dimensional theory, but shares some of the same problems, namely, the unboundedness of the pure gravitational action, as well as the lack of perturbative renormalizability.

The pure gravity action on the lattice will be chosen in three dimensions to be

$$I=\sum_{\text{edges } h}\left[\lambda V_h-kl_h\delta_h+a\frac{l_h^2\delta_h^2}{V_h}\right], \tag{2.1}$$

as discussed in Ref [12]. Here h denotes a hinge (=edge in three dimensions), l_h is the corresponding edge length, V_h the volume associated with that hinge in a baricentric subdivision of the simplicial lattice [12], and δ_h is the corresponding deficit angle at the hinge h, a function of the edge lengths belonging to the tetrahedra meeting on h. λ, k, and a are bare lattice coupling constants. Classically the action of Eq. (2.1) is bounded from below if $4a\lambda-k^2>0$. The lattice curvature-squared term (proportional to δ_h^2) vanishes if and only if the curvature is zero and was shown to converge to the continuum answer for the regular tessellations of the three-sphere [12]. In the following we will take the "fundamental lattice spacing" to be equal to 1; the appropriate power of the lattice spacing (which has dimensions of a length) can always be restored at the end in the relevant formulas by invoking dimensional arguments. This statement should not confuse the fact that, since the edge lengths are dynamical variables, the average physical separation between points (the "effective lattice spacing") will be some numeric factor (controlled by the bare couplings) times this "fundamental lattice spacing." In other words, the ultraviolet cutoff proportional to the inverse effective lattice spacing is dynamical, and not fixed *a priori*. In the classical continuum limit the above action is then equivalent to the continuum action

$$I=\int d^3x\sqrt{g}\left[\lambda-\frac{k}{2}R+\frac{a}{4}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}+\cdots\right], \tag{2.2}$$

with a cosmological constant term (proportional to λ), an Einstein-Hilbert term ($k=1/8\pi G$), and a higher-derivative term [24,25]. Here the dots indicate higher-order curvature corrections, as well as possible noncovariant corrections. One could consider the Regge-Einstein action by itself ($a=0$), but then the Euclidean action would be unbounded from below, and problems might arise, depending on the choice of measure (this point will be discussed further below). Some preliminary results regarding the theory defined by the action of Eq. (2.1) have also been discussed by Hamber [8]. Recently there also has been some work on three-dimensional generalizations of the original dynamical triangulation model [21,22], using equilateral tetrahedra and performing the sum over triangulations using Alexander moves [26], which are exactly what is used to prove the invariance of the Turaev-Viro state sum model [19]. This development represents an alternative and complementary approach to what is being discussed here.

An important issue which needs to be addressed is the problem of the gravitational lattice measure. In the continuum the form of the measure for the $g_{\mu\nu}$ fields appears not to be unique [1,27–29]. The reason for the ambiguity appears to be a lack of a clear definition of what is meant by $\prod_x$ in the gravitational functional measure. In spite of some recurrent claims to the contrary, it would seem that such an ambiguity persists in *all* known lattice formulations of quantum gravity. However, the difference between the various measures seems to be in the power of $\sqrt{g}$ in the prefactor, which corresponds to some product of volume factors on the lattice. These volume factors do not give rise on the lattice to coupling terms (corresponding to derivatives of the metric in the continuum), and are therefore strictly local and involve the conformal degrees of freedom only.

Different measures in the continuum give rise to different measures on the lattice [5]. DeWitt [27] has argued that the gravitational measure should be constructed by first introducing a supermetric over metric deformations, which in its simplest local form then leads to a functional measure for pure gravity in d dimensions of the type

$$d\mu[g]=\prod_x g^{(d-4)(d+1)/8}\prod_{\mu\geq\nu} dg_{\mu\nu} \ . \tag{2.3}$$

Another popular (pure) gravitational measure in the continuum is the Misner scale-invariant measure [28]

$$d\mu[g]=\prod_x g^{-(d+1)/2}\prod_{\mu\geq\nu} dg_{\mu\nu} \ . \tag{2.4}$$

It is unique if the product over x is interpreted as one over "physical" points and coordinate invariance is imposed at one and the same "physical" point. Other forms of the measure for the gravitational field have also been suggested, some inspired by the canonical quantization approach to gravity [29]. If matter fields are present, then the gravitational measure has to be further modified [27] (see further discussion of this point below).

On the simplicial lattice it is clear that the edge lengths are the elementary degrees of freedom which uniquely specify the geometry for a given incidence matrix, and over which one should perform the functional integration [6,12,5]. Indeed, the induced metric at a simplex is related to the edge lengths squared within that simplex, via the expression for the invariant line element $ds^2=g_{\mu\nu}dx^\mu dx^\nu$; the correspondence between the induced metric field defined within a simplex containing the vertex s and the lengths of the edges of that simplex is given explicitly by

$$g_{ij}(s)=\tfrac{1}{2}(l_{s,s+i}^2+l_{s,s+j}^2-l_{s+i,s+j}^2) \ . \tag{2.5}$$

Here the first two edges emanate from the given vertex, and the remaining one is opposite to the same vertex. One notices that in general each edge is shared between several contiguous simplices, and that an integration over the edges is not simply related to an integration over the metric [even though there are $d(d+1)/2$ edges for each simplex, just as there are $d(d+1)/2$ independent components for the metric tensor in d dimensions]. Originally the pure gravity measure

$$\int d\mu_\epsilon[l]=\prod_{\text{edges } ij}\int_0^\infty dl_{ij}^2 l_{ij}^{2\sigma d}F_\epsilon[l] \tag{2.6}$$

was suggested, where $\sigma d=0,-1$ for example, and $F_\epsilon[l]$ is a function of the edge lengths, with the property that it is nonvanishing and equal to 1 only whenever the triangle inequalities and their higher-dimensional analogues are satisfied. The value $\sigma d=0$ then corresponds to the DeWitt measure in four dimensions, while $\sigma d=-1$ corresponds to a scale-invariant measure such as the continuum Misner measure. The parameter ϵ can be introduced as an ultraviolet cutoff at small edge lengths: the function $F_\epsilon[l]$ is zero if any of the edges are equal to or less than ϵ. The introduction of such a cutoff seems to be necessary in four dimensions [12], but not in two [14,15]. This is a consequence of the fact that the higher-derivative terms tend to suppress configurations with small edge lengths below four dimensions; such a suppression disappears in four dimensions where the higher-derivative action contribution becomes dimensionless. Indeed, a simple scaling argument, for an edge measure of the type discussed above, gives the following estimate of the average volume per edge,

$$\langle V\rangle\sim\frac{2(1+\sigma d)}{\lambda d} \ , \tag{2.7}$$

if curvature terms in the action are neglected, and shows that the volume tends to zero for a singular measure such as the scale-invariant one ($\sigma d=-1$). (In three and four dimensions the numerical simulations agree to within a few percent with the above formula; see Sec. IV below.)

The above measure is of course correct in the weak-field limit, where all continuum measures agree as well, and integrates over coordinate-independent quantities, the invariant lengths of the edges. But since it has been argued that it is not entirely clear what the continuum measure should be, it would seem of interest to explore the sensitivity of the results to the type of gravitational measure employed. Eventually one would expect on the basis of universality of critical behavior that different measures should lead to the same continuum limit. Another class of pure gravity measures which can be written down on the lattice is obtained by considering the "volume associated with an edge" V_{ij} [which corresponds to the quantity $\sqrt{g(x)}$ in the continuum], and writing

$$\int d\mu_\epsilon[l]=\prod_{\text{edges } ij}\int_0^\infty V_{ij}^{2\sigma}dl_{ij}^2F_\epsilon[l] \ , \tag{2.8}$$

with $\sigma=-1/d$ for the lattice analogue of the Misner measure and $\sigma=(d-4)/4d$ for the lattice analogue of the DeWitt measure. If a D-component scalar field is coupled to gravity, then, due to an additional factor of $\prod_x g^{D/2}$ in the continuum gravitational measure, the power σ has to be changed to

$$\begin{aligned}\sigma&=\frac{D}{2d(d+1)}-\frac{1}{d}\underset{d=3}{\longrightarrow}\frac{1}{24}(D-8) \ ,\\ \sigma&=\frac{D}{2d(d+1)}-\frac{4-d}{4d}\underset{d=3}{\longrightarrow}\frac{1}{24}(D-2)\end{aligned} \tag{2.9}$$

for the two measures, respectively.

Eventually one would like to see how the results depend on the form of the measure and on the measure parameter σ. In the case of two-dimensional gravity extensive studies were done and compared with exact results known from conformal field theory. The numerical results appear to indicate that different measures, within a certain universality class, will give the same results for infrared-sensitive quantities, such as correlation functions at large distances and critical exponents. The lattice path integral might not be meaningful, though, for certain values of σ; in particular, if the measure parameter σ is too negative in two dimensions, then the measure factor tends to favor configurations of triangles which are long and thin, with a small area and a large perimeter. The lattice tends to degenerate into a lower-dimensional manifold, a situation far from the desired continuum limit, and therefore to be avoided. In the following we will consider the measure dl^2 ($\sigma=0$) in order to allow a direct comparison with the results in four dimensions, where the same measure was used [17,23]. In three dimensions this choice, therefore, does not correspond exactly to the DeWitt measure. In order to explore the sensitivity of our results to the choice of invariant measure, we have also considered the scale-invariant measure dl/l.

In two dimensions the results obtained seem to be largely insensitive to the choice of measure over the l's [15]. We will argue later that the same seems to be true in three dimensions.

Some useful identities can be obtained by examining the scaling properties of the action and the measure. The couplings in the above gravitational action are dimensionful in three dimensions, but one can define two dimensionless coupling constants $k/\lambda^{1/3}$ and $a\lambda^{1/3}$ and rescale the edge lengths so as to eliminate the overall length scale $\sqrt{k/\lambda}$. As a consequence, the path integral Z for pure gravity obeys an equation of the type

$$Z(\lambda,k,a)=\left[\frac{k}{\lambda}\right]^{N_1} Z\left[\left[\frac{k^3}{\lambda}\right]^{1/2},\left[\frac{k^3}{\lambda}\right]^{1/2},a\left[\frac{\lambda}{k}\right]^{1/2}\right], \tag{2.10}$$

where N_1 represents the number of edges in the lattice and we have selected here the dl^2 measure ($\sigma=0$). This equation implies, in turn, a sum rule for local averages, which (for the dl^2 measure) reads

$$3\lambda\langle V\rangle - k\langle \delta l\rangle - a\langle \delta^2 l^2/V\rangle = 2N_1/N_0 \,. \tag{2.11}$$

Here N_0 represents the number of sites in the lattice, and the averages are defined per site. The coefficients on the left-hand side (LHS) of the equation reflect the scaling dimensions of the various terms, while the RHS term arises from the scaling of the measure [in d dimensions the coefficients become d, $(d-2)$, and $(d-4)$, respectively]. This last formula can be useful in checking the accuracy of numerical calculations, since each term can be estimated independently.

III. WEAK-FIELD EXPANSION FOR SIMPLICIAL 3D GRAVITY

Here we will consider the expansion of the lattice action of Eq. (2.1) around flat space, and will compare the results with the weak-field expansion in the continuum. In the absence of the cosmological term, flat space is a solution for the higher-derivative lattice action. Following Ref. [9], we therefore take as our background space a network of unit cubes divided into tetrahedra by drawing in parallel sets of face and body diagonals, as shown in Fig. 1. With this choice, there are $2^d-1=7$ edges per lattice point emanating in the positive lattice directions: three-body principals, three face diagonals, and one-body diagonal, giving a total of seven components per lattice point.

As in Ref. [9], it is convenient to use a binary notation for edges, so that the edge index corresponds to the lattice direction of the edge expressed as a binary number:

$$\begin{aligned}&(0,0,1)\to 1,\quad (0,1,1)\to 3,\quad (1,1,1)\to 7\,,\\ &(0,1,0)\to 2,\quad (1,0,1)\to 5\,,\\ &(1,0,0)\to 4,\quad (1,1,0)\to 6\,.\end{aligned} \tag{3.1}$$

The edge lengths are then allowed to fluctuate around their flat-space values $l_i=l_i^0(1+\epsilon_i)$, and the second variation of the action is expressed as a quadratic form in ϵ:

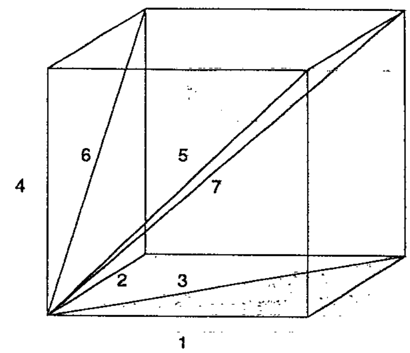

FIG. 1. One rigid cube with body principals (labeled by 1,2,4), face diagonals (labeled by 3,5,6), and one body diagonal (labeled by 7).

$$\delta^2 I=\sum_{mn}\epsilon^{(m)T}M^{(m,n)}\epsilon^{(n)}\,, \tag{3.2}$$

where n,m label the sites on the lattice and M_{mn} is some Hermitian matrix. The infinite-dimensional (but sparse) matrix $M^{(m,n)}$ is best studied by going to momentum space. Assume that the fluctuation ϵ_i at the point that is j steps from the origin in one coordinate direction, k steps in another coordinate direction, and l steps in the third coordinate direction, is related to the corresponding fluctuation ϵ_i at the origin by

$$\epsilon_i^{(j+k+l)}=\omega_1^j\omega_2^k\omega_4^l\epsilon_i^{(0)}\,, \tag{3.3}$$

with $\omega_i=e^{ik_i}$. In the smooth limit $\omega_i=1+ik_i+O(k_i^2)$, and only in this limit, are the lattice action and the continuum action expected to agree. Note also that it is convenient here to set the lattice spacing in the three principal directions equal to 1; it can always be restored at the end by using dimensional arguments. We have generated the lattice weak-field expansion coefficient both by hand (in the case of the Regge and cosmological constant term) and by computer, writing an appropriate FORTRAN code that handles the analytic weak-field expansion around an arbitrary hypercubic lattice in two, three, and four dimensions. (In four dimensions we have verified and extended to the cosmological and higher-derivative terms the formulas of Ref. [9].)

In order to indicate the comparison with the continuum results, one would like to express the lattice action in terms of variables which are closer to the continuum ones, such as $h_{\mu\nu}$ or $\bar{h}_{\mu\nu}=h_{\mu\nu}-\frac{2}{3}\eta_{\mu\nu}h_{\lambda\lambda}$. Up to terms that involve derivatives of the metric (and which reflect the ambiguity of where precisely on the lattice the continuum metric should be defined), this relationship can be obtained by considering one tetrahedron and using the expression for the invariant line elements $ds^2=g_{\mu\nu}dx^\mu dx^\nu$ and $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the diagonal flat metric.

Comparing specific values for ds^2 obtained for special choices of $dx_\mu dx_\nu$ with the physical distance as obtained from the values of the edge lengths, one finds for one tetrahedron (for the labeling see Fig. 2) the well-known simple result

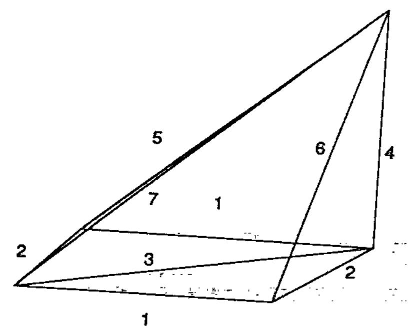

FIG. 2. Labeling of edges in two adjacent tetrahedra, as used in the construction of the metric.

$$g_{ij}=\begin{bmatrix} l_1^2 & \frac{1}{2}(l_3^2-l_1^2-l_2^2) & \frac{1}{2}(l_5^2-l_1^2-l_4^2) \\ \frac{1}{2}(l_3^2-l_1^2-l_2^2) & l_2^2 & \frac{1}{2}(l_6^2-l_2^2-l_4^2) \\ \frac{1}{2}(l_5^2-l_1^2-l_4^2) & \frac{1}{2}(l_6^2-l_2^2-l_4^2) & l_4^2 \end{bmatrix} . \tag{3.4}$$

Inserting $l_i=l_i^0(1+\epsilon_i)$, with $l_i^0=1,\sqrt{2},\sqrt{3}$ for the body principal ($i=1,2,4$), face diagonal ($i=3,5,6$), and body diagonal ($i=7$), respectively, one gets

$$\begin{aligned}
&(1+\epsilon_1)^2=1+h_{11} ,\\
&(1+\epsilon_2)^2=1+h_{22} ,\\
&(1+\epsilon_4)^2=1+h_{33} ,\\
&(1+\epsilon_3)^2=1+\tfrac{1}{2}(h_{11}+h_{22})+h_{12} ,\\
&(1+\epsilon_5)^2=1+\tfrac{1}{2}(h_{11}+h_{33})+h_{13} ,\\
&(1+\epsilon_6)^2=1+\tfrac{1}{2}(h_{22}+h_{33})+h_{23} ,\\
&(1+\epsilon_7)^2=1+\tfrac{1}{3}(h_{11}+h_{22}+h_{33})+\tfrac{2}{3}(h_{12}+h_{23}+h_{13})
\end{aligned} \tag{3.5}$$

(note that we use the binary notation for edges, but maintain the usual index notation for the field $h_{\mu\nu}$). These relationships can then be inverted to give

$$\begin{aligned}
&\epsilon_1=\tfrac{1}{2}h_{11}-\tfrac{1}{8}h_{11}^2+O(h_{11}^3) ,\\
&\epsilon_2=\tfrac{1}{2}h_{22}-\tfrac{1}{8}h_{22}^2+O(h_{22}^3) ,\\
&\epsilon_4=\tfrac{1}{2}h_{33}-\tfrac{1}{8}h_{33}^2+O(h_{33}^3) ,\\
&\epsilon_3=\tfrac{1}{4}(h_{11}+h_{22}+2h_{12})-\tfrac{1}{32}(h_{11}+h_{22}+2h_{12})^2+O(h^3) ,\\
&\epsilon_5=\tfrac{1}{4}(h_{11}+h_{33}+2h_{13})-\tfrac{1}{32}(h_{11}+h_{33}+2h_{13})^2+O(h^3) ,\\
&\epsilon_6=\tfrac{1}{4}(h_{22}+h_{33}+2h_{23})-\tfrac{1}{32}(h_{22}+h_{33}+2h_{23})^2+O(h^3) ,\\
&\epsilon_7=\tfrac{1}{6}(h_{11}+h_{22}+h_{33}+2h_{12}+2h_{13}+2h_{23})\\
&\qquad -\tfrac{1}{72}(h_{11}+h_{22}+h_{33}+2h_{12}+2h_{13}+2h_{23})^2+O(h^3) .
\end{aligned} \tag{3.6}$$

The correction terms of order $h_{\mu\nu}^2$ will be needed later for the discussion of the cosmological constant term. Note that there are seven ϵ_i variables but only six $h_{\mu\nu}$'s [in general in d dimensions we have $2^d-1\epsilon_i$ variables and $d(d+1)/2$ $h_{\mu\nu}$'s which leads to a number of redundant lattice variables for $d>2$]. [Perturbatively, the *physical* degrees of freedom in the continuum are supposed to be $\frac{1}{2}d(d+1)-1-d-(d-1)=\frac{1}{2}d(d-3)=0$ in $d=3$ for a traceless symmetric tensor, and after imposing the gauge conditions [3]; this result in general might or might not be true at the nonperturbative level.]

Thus, to lowest order in $h_{\mu\nu}$, one can perform a rotation in order to go from the ϵ_i variables to the $h_{\mu\nu}$'s (or $\bar{h}_{\mu\nu}$'s):

$$\epsilon^T M_\omega \epsilon=(\epsilon^T V^{\dagger -1})V^\dagger M_\omega V(V^{-1}\epsilon) , \tag{3.7}$$

with

$$\epsilon=V\bar{h},\quad V=U_1U_2,\quad \epsilon=U_1h,\quad h=U_2\bar{h} , \tag{3.8}$$

where V and U_1 are 7×6 matrices, while U_2 is a 6×6 matrix,

$$U_1=\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} ,$$

$$U_2=\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} , \tag{3.9}$$

$$U_1\times U_2=\begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 \\ -\frac{1}{12} & -\frac{1}{12} & -\frac{1}{3} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{12} & -\frac{1}{3} & -\frac{1}{12} & 0 & \frac{1}{2} & 0 \\ -\frac{1}{3} & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 & \frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} .$$

A. Lattice Einstein term

The first term we will consider in the action of Eq. (2.1) is the Einstein-Regge contribution

$$I_R=\sum_{\text{edges } h} l_h\delta_h . \tag{3.10}$$

In momentum space, the matrix M_ω describing the small fluctuations around flat space is given by

$$
\begin{aligned}
(M_\omega)_{1,1}&=-2\;,\\
(M_\omega)_{1,2}&=-\omega_1\omega_4-\bar\omega_2\bar\omega_4\;,\\
(M_\omega)_{1,4}&=2+2\bar\omega_2\;,\\
(M_\omega)_{1,6}&=2\omega_1+2\bar\omega_2\bar\omega_4\;,\\
(M_\omega)_{1,7}&=-3\bar\omega_2-3\bar\omega_4\;,\\
(M_\omega)_{4,4}&=-8\;,\\
(M_\omega)_{4,5}&=-4\omega_2-4\bar\omega_4\;,\\
(M_\omega)_{4,7}&=6+6\bar\omega_4\;,\\
(M_\omega)_{7,7}&=-18
\end{aligned}
\tag{3.11}
$$

(the remaining matrix elements can be obtained by permuting the appropriate indices). Because of its structure, which is of the form

$$
M_\omega=\begin{bmatrix} A_6 & b\\ b^\dagger & -18\end{bmatrix}, \tag{3.12}
$$

where A_6 is a 6×6 matrix, a rotation can be done which completely decouples the fluctuations in ϵ_7:

$$
M'_\omega=S^\dagger_\omega M_\omega S_\omega=\begin{bmatrix} A_6+\frac{1}{18}bb^\dagger & 0\\ 0 & -18\end{bmatrix}, \tag{3.13}
$$

with

$$
S_\omega=\begin{bmatrix} I_6 & 0\\ \frac{1}{18}b^\dagger & 1\end{bmatrix} \tag{3.14}
$$

$$
=\begin{bmatrix}
1&0&0&0&0&0&0\\
0&1&0&0&0&0&0\\
0&0&1&0&0&0&0\\
0&0&0&1&0&0&0\\
0&0&0&0&1&0&0\\
0&0&0&0&0&1&0\\
-\frac{1}{6}(\omega_2+\omega_4) & -\frac{1}{6}(\omega_1+\omega_4) & -\frac{1}{6}(\omega_1+\omega_2) & \frac{1}{3}(1+\omega_4) & \frac{1}{3}(1+\omega_2) & \frac{1}{3}(1+\omega_1) & 1
\end{bmatrix}. \tag{3.15}
$$

Explicitly, one has the following matrix elements for the matrix M'_ω:

$$
\begin{aligned}
&(M'_\omega)_{1,1}=-1+\frac{\omega_2}{2\omega_4}+\frac{\omega_4}{2\omega_2},\quad (M'_\omega)_{1,2}=\frac{1}{2}+\frac{\omega_1}{2\omega_2}+\frac{\omega_1}{2\omega_4}-\frac{1}{\omega_2\omega_4}-\omega_1\omega_4+\frac{\omega_4}{2\omega_2}\,,\\
&(M'_\omega)_{1,4}=1+\frac{1}{\omega_2}-\frac{1}{\omega_4}-\frac{\omega_4}{\omega_2},\quad (M'_\omega)_{1,6}=2\omega_1-\frac{1}{\omega_2}-\frac{\omega_1}{\omega_2}-\frac{1}{\omega_4}-\frac{\omega_1}{\omega_4}+\frac{2}{\omega_2\omega_4}\,,\\
&(M'_\omega)_{1,7}=0,\quad (M'_\omega)_{4,4}=-4+\frac{2}{\omega_4}+2\omega_4\,,\\
&(M'_\omega)_{4,5}=2-2\omega_2-\frac{2}{\omega_4}+\frac{2\omega_2}{\omega_4},\quad (M'_\omega)_{4,7}=0,\quad (M'_\omega)_{7,7}=-18\,.
\end{aligned}
\tag{3.16}
$$

For k going to zero it reduces to

$$
M'_\omega\underset{k\to 0}{\sim}\begin{bmatrix}0 & 0\\ 0 & -18\end{bmatrix}. \tag{3.17}
$$

It is easy to see that the matrix M'_ω has three exact zero eigenvalues, corresponding to the translational zero models for M_ω:

$$
\begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_4\\ \epsilon_3\\ \epsilon_5\\ \epsilon_6\\ \epsilon_7\end{bmatrix}
=\begin{bmatrix}
1-\omega_1 & 0 & 0\\
0 & 1-\omega_2 & 0\\
0 & 0 & 1-\omega_4\\
\frac{1}{2}(1-\omega_1\omega_2) & \frac{1}{2}(1-\omega_1\omega_2) & 0\\
\frac{1}{2}(1-\omega_1\omega_4) & 0 & \frac{1}{2}(1-\omega_1\omega_4)\\
0 & \frac{1}{2}(1-\omega_2\omega_4) & \frac{1}{2}(1-\omega_2\omega_4)\\
\frac{1}{3}(1-\omega_1\omega_2\omega_4) & \frac{1}{3}(1-\omega_1\omega_2\omega_4) & \frac{1}{3}(1-\omega_1\omega_2\omega_4)
\end{bmatrix}
\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}. \tag{3.18}
$$

In real space the above transformations become $\epsilon_1(n)=x_1(n)-x_1(n+\hat{\mu}_1)$, etc., and to this order the measure dl^2 of Eq. (2.6) is invariant under the gauge transformations. This is not expected to happen to higher order in h.

The remaining eigenvalues are -18 (once) and $O(k^2)$ (three times). Notice that one mode, corresponding to the fluctuations in the body diagonal ϵ_7, completely decouples. The eigenvalues which are of order k^3 are in general given by rather complicated expressions. For the special choice $k_1=k_2=k_3\equiv k/\sqrt{3}$ they are given by

$$\lambda_{4,5}=-\tfrac{8}{3}k^2 \quad \text{(twice)} , \qquad \lambda_6=+\tfrac{10}{3}k^2 \quad \text{(once)} . \tag{3.19}$$

The single positive eigenvalue reflects the presence of the unbounded, scalar conformal mode. One further rotation by the 6×6 matrix T_ω, defined by

$$T_\omega=\begin{bmatrix} \frac{1}{6}\omega_1 & -\frac{1}{3}\omega_1 & -\frac{1}{3}\omega_1 & 0 & 0 & 0 \\ -\frac{1}{3}\omega_2 & \frac{1}{6}\omega_2 & -\frac{1}{3}\omega_2 & 0 & 0 & 0 \\ -\frac{1}{3}\omega_4 & -\frac{1}{3}\omega_4 & \frac{1}{6}\omega_4 & 0 & 0 & 0 \\ -\frac{1}{12}\omega_1\omega_2 & -\frac{1}{12}\omega_1\omega_2 & -\frac{1}{3}\omega_1\omega_2 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{12}\omega_1\omega_4 & -\frac{1}{3}\omega_1\omega_4 & -\frac{1}{12}\omega_1\omega_4 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{3}\omega_2\omega_4 & -\frac{1}{12}\omega_2\omega_4 & -\frac{1}{12}\omega_2\omega_4 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} , \tag{3.20}$$

gives the new small fluctuation matrix

$$L_\omega=T_\omega^\dagger M'_\omega T_\omega . \tag{3.21}$$

The rotation by T_ω, of course, preserves the three translational zero modes, and the three remaining eigenvalues are still $O(k^2)$. If one then defines the matrix

$$\tilde{L}_\omega=L_\omega-\tfrac{1}{2}C_\omega^\dagger C_\omega , \tag{3.22}$$

where C_ω is introduced in order to give the lattice analogue of the harmonic gauge-fixing term, with

$$C_\omega=\frac{1}{6}\begin{bmatrix} 5(-1+\omega_1) & 1-\omega_1 & 1-\omega_1 & 6(1-1/\omega_2) & 6(1-1/\omega_4) & 0 \\ 1-\omega_2 & 5(-1+\omega_2) & 1-\omega_2 & 6(1-1/\omega_1) & 0 & 6(1-1/\omega_4) \\ 1-\omega_4 & 1-\omega_4 & 5(-1+\omega_4) & 0 & 6(1-1/\omega_1) & 6(1-1/\omega_2) \end{bmatrix} , \tag{3.23}$$

one obtains

$$\begin{aligned} (\tilde{L}_\omega)_{1,1}&=-\frac{3}{4}+\frac{1}{8\omega_1}+\frac{\omega_1}{8}+\frac{1}{8\omega_2}+\frac{\omega_2}{8}+\frac{1}{8\omega_4}+\frac{\omega_4}{8}\underset{k\to0}{\sim}\frac{-k_1^2}{8}+\frac{k_1^4}{96}-\frac{k_2^2}{8}+\frac{k_2^4}{96}-\frac{k_3^2}{8}+\frac{k_3^4}{96}+O(k^6) , \\ (\tilde{L}_\omega)_{1,2}&=\frac{3}{4}-\frac{1}{8\omega_1}-\frac{\omega_1}{8}-\frac{1}{8\omega_2}-\frac{\omega_2}{8}-\frac{1}{8\omega_4}-\frac{\omega_4}{8}\underset{k\to0}{\sim}\frac{k_1^2}{8}-\frac{k_1^4}{96}+\frac{k_2^2}{8}-\frac{k_2^4}{96}+\frac{k_3^2}{8}-\frac{k_3^4}{96}+O(k^6) , \\ (\tilde{L}_\omega)_{1,4}&=0, \quad (\tilde{L}_\omega)_{1,6}=0 , \\ (\tilde{L}_\omega)_{4,4}&=-3+\frac{1}{2\omega_1}+\frac{\omega_1}{2}+\frac{1}{2\omega_2}+\frac{\omega_2}{2}+\frac{1}{2\omega_4}+\frac{\omega_4}{2}\underset{k\to0}{\sim}\frac{-k_1^2}{2}+\frac{k_1^4}{24}-\frac{k_2^2}{2}+\frac{k_2^4}{24}-\frac{k_3^2}{2}+\frac{k_3^4}{24}+O(k^6) , \\ (\tilde{L}_\omega)_{4,5}&=0 . \end{aligned} \tag{3.24}$$

Since on the lattice derivatives are approximated by finite differences, it is not surprising to find higher-order $[O(k^4)]$ lattice corrections. Indeed, if one defines $\Sigma_\omega=\omega_1+\omega_2+\omega_4$, then

$$6-\Sigma_\omega-\bar{\Sigma}_\omega=2\sum_{\mu=1}^{3}(1-\cos k_\mu)=k_1^2+k_2^2+k_3^2-\frac{1}{12}(k_1^4+k_2^4+k_3^4)+O(k^6) . \tag{3.25}$$

Note the opposite sign of the higher-order contribution, which partially cures the unboundedness problem [thus, for example, $-\sum_\mu k_\mu^2$ is unbounded from below, but $-2\sum_\mu(1-\cos k_\mu)$ is not]. On the other hand, the higher-order lattice corrections appear, for our choice of coordinates, to break Lorentz invariance, since they are not simply functions of k^2. These result will be true in general for higher-order (in k) corrections in simplicial gravity.

For small k the Einstein-Regge action contribution can be written as

$$\tilde{L}_\omega = -\tfrac{1}{2}k^2 V \;, \qquad (3.26)$$

with $k^2 = k_1^2 + k_2^2 + k_3^2$, and the matrix V is given by

$$V = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} , \qquad (3.27)$$

and now all its eigenvalues are nonzero $(-\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1)$. Its inverse, which determines the free propagator, is given by

$$V^{-1} = \begin{bmatrix} 0 & -2 & -2 & 0 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 \\ -2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \qquad (3.28)$$

As will be shown below, the form of $\tilde{L}_\omega$ in Eq. (3.26) is precisely equivalent to the corresponding continuum expression. Here we note that the combined S_ω and T_ω rotations correspond to a rotation to lattice trace-reversed variables, as discussed previously. Indeed, one has

$$S_\omega \times T_\omega = \frac{1}{12} \begin{bmatrix} 2\omega_1 & -4\omega_1 & -4\omega_1 & 0 & 0 & 0 \\ -4\omega_2 & 2\omega_2 & -4\omega_2 & 0 & 0 & 0 \\ -4\omega_4 & -4\omega_4 & 2\omega_4 & 0 & 0 & 0 \\ -\omega_1\omega_2 & -\omega_1\omega_2 & -4\omega_1\omega_2 & 6 & 0 & 0 \\ -\omega_1\omega_4 & -4\omega_1\omega_4 & -\omega_1\omega_4 & 0 & 6 & 0 \\ -4\omega_2\omega_4 & -\omega_2\omega_4 & -\omega_2\omega_4 & 0 & 0 & 6 \\ -2\omega_1\omega_2\omega_4 & -2\omega_1\omega_2\omega_4 & -2\omega_1\omega_2\omega_4 & 2+2\omega_4 & 2+2\omega_2 & 2+2\omega_1 \end{bmatrix} , \qquad (3.29)$$

and for $k_\mu \sim 0$ one has $S \times T \sim U_1 \times U_2$ [see Eq. (3.9)].

B. Continuum weak-field expansion for Einstein, cosmological constant, and curvature-squared term

As in four dimensions, the continuum weak-field expansion for the action of Eq. (2.2) is generated by defining the small fluctuation $h_{\mu\nu}$ about flat space:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \;, \qquad (3.30)$$

where $\eta_{\mu\nu}$ is the flat diagonal metric. One finds

$$\begin{aligned} \sqrt{g}R &= \tfrac{1}{2}h_{\sigma\sigma,\kappa}h_{\kappa\lambda,\lambda} - \tfrac{1}{2}h_{\lambda\sigma,\sigma}h_{\lambda\kappa,\kappa} \\ &\quad + \tfrac{1}{4}h_{\lambda\sigma,\kappa}h_{\lambda\sigma,\kappa} - \tfrac{1}{4}h_{\sigma\sigma,\kappa}h_{\lambda\lambda,\kappa} + O(h^3) \;, \\ \sqrt{g}R^2 &= (h_{\lambda\lambda,\kappa\kappa} - h_{\lambda\kappa,\lambda\kappa})^2 + O(h^3) \;, \\ \sqrt{g}R_{\lambda\mu\nu\kappa}R^{\lambda\mu\nu\kappa} &= \tfrac{1}{4}(h_{\nu\lambda,\mu\kappa} + h_{\mu\kappa,\lambda\nu} - h_{\mu\nu,\lambda\kappa} - h_{\kappa\lambda,\mu\nu})^2 \\ &\quad + O(h^3) \;. \end{aligned} \qquad (3.31)$$

For the curvature-squared terms one has in three dimensions the identities

$$\begin{aligned} &R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} - 3R^2 = 0 \;, \\ &C_{\mu\nu\lambda\sigma}C^{\mu\nu\lambda\sigma} = 0 \;, \end{aligned} \qquad (3.32)$$

and therefore only two independent terms need to be considered. It is useful to introduce the trace-reversed variables

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \tfrac{2}{3}\eta_{\mu\nu}\bar{h} \;. \qquad (3.33)$$

Then one has

$$\begin{aligned} \sqrt{g}R &= \tfrac{1}{6}\bar{h}_{\kappa\lambda,\lambda}\bar{h}_{,\kappa} - \tfrac{1}{2}\bar{h}_{\lambda\sigma,\sigma}\bar{h}_{\lambda\kappa,\kappa} \\ &\quad + \tfrac{1}{4}\bar{h}_{\lambda\sigma,\kappa}\bar{h}_{\lambda\sigma,\kappa} - \tfrac{5}{36}\bar{h}_{,\lambda}\bar{h}_{,\lambda} + O(\bar{h}^3) \;, \\ \sqrt{g}R^2 &= (\bar{h}_{\lambda\kappa,\lambda\kappa} + \tfrac{1}{3}\bar{h}_{,\kappa\kappa})^2 + O(\bar{h}^3) \;, \\ \sqrt{g}R_{\lambda\mu\nu\kappa}R^{\lambda\mu\nu\kappa} &= \bar{h}_{\nu\lambda,\mu\kappa}\bar{h}_{\nu\lambda,\mu\kappa} + \bar{h}_{\nu\lambda,\mu\kappa}\bar{h}_{\mu\kappa,\nu\lambda} \\ &\quad - 2\bar{h}_{\nu\lambda,\mu\kappa}\bar{h}_{\mu\nu,\lambda\kappa} - \tfrac{4}{3}\bar{h}_{,\mu\kappa}\bar{h}_{\mu\kappa,\nu\nu} \\ &\quad + \tfrac{8}{3}\bar{h}_{,\mu\kappa}\bar{h}_{\mu\nu,\nu\kappa} + \tfrac{4}{9}\bar{h}_{,\nu\nu}\bar{h}_{,\kappa\kappa} \\ &\quad - \tfrac{8}{9}\bar{h}_{,\nu\kappa}\bar{h}_{,\nu\kappa} + O(\bar{h}^3) \;. \end{aligned} \qquad (3.34)$$

A gauge-fixing term can be added to the Einstein term in the form

$$+\tfrac{1}{2}C_\mu^2 , \tag{3.35}$$

with

$$C_\mu = h_{\mu\nu,\nu} - \tfrac{1}{2}h_{\nu\nu,\mu} = \bar{h}_{\mu\nu,\nu} - \tfrac{1}{6}\bar{h}_{\nu\nu,\mu} . \tag{3.36}$$

The the R term can be written in the more compact form [30]

$$\begin{aligned}\sqrt{g}R + \tfrac{1}{2}C_\mu^2 &= \tfrac{1}{4}h_{\lambda\sigma,\kappa}h_{\lambda\sigma,\kappa} - \tfrac{1}{8}h_{\sigma\sigma,\kappa}h_{\lambda\lambda,\kappa} + O(h^3) \\ &= \tfrac{1}{2}h_{\alpha\beta,\lambda}V_{\alpha\beta\mu\nu}h_{\mu\nu,\lambda} + O(h^3) ,\end{aligned} \tag{3.37}$$

with

$$V_{\alpha\beta\mu\nu} = \tfrac{1}{2}\eta_{\alpha\mu}\eta_{\beta\nu} - \tfrac{1}{4}\eta_{\alpha\beta}\eta_{\mu\nu} . \tag{3.38}$$

With the index correspondence between components of $h_{\mu\nu}$ and row and/or column indices of V given by

$$11\to 1,\ \ 22\to 2,\ \ 33\to 3\ ,\ \ 12\to 4,\ \ 13\to 5,\ \ 23\to 6\ , \tag{3.39}$$

the matrix V is the same as the one defined in Eq. (3.27). Because of the presence of the gauge-fixing term, there are of course no zero eigenvalues. Also, the matrix V is unchanged when going to trace-reversed variables. In the form of Eq. (3.37), the Einstein action contribution is identical to the lattice results, Eq. (3.26), after integrating by parts and going to momentum space.

The continuum cosmological term can also be expressed in terms of the matrix V,

$$\sqrt{g} = 1 + \tfrac{1}{2}h_{\mu\mu} - \tfrac{1}{2}h_{\alpha\beta}V_{\alpha\beta\mu\nu}h_{\mu\nu} + O(h^3) , \tag{3.40}$$

giving for the combined Einstein and cosmological constant terms

$$\lambda\left[1+\frac{1}{2}h_{\mu\mu}\right] + \left[-\frac{k}{2}\right]\frac{1}{2}h_{\alpha\beta}V_{\alpha\beta\mu\nu}\left[\partial^2+\frac{2\lambda}{k}\right]h_{\mu\nu} + O(h^3) . \tag{3.41}$$

Strictly speaking, the expansion about a flat-space background is no longer valid in the presence of a cosmological term, since flat space is no longer a solution of the classical equations of motion. Were it a reasonable procedure, one would obtain a contribution to the propagator involving in the denominator a "mass" $\mu^2 = -2\lambda/k$. On the other hand, it has been argued that if the tadpole term is treated properly, by expanding around the correct solution in the presence of the λ term, this mass term disappears classically, and its presence here should be considered an artifact of the (incorrect) expansion [30].

Finally, let us give the continuum weak-field expansion results for the two independent curvature-squared terms in three dimensions. In momentum space the matrix element contributions appearing in the quadratic form in $\bar{h}_{\mu\nu}(k)$ are given by

$$\begin{aligned}(\sqrt{g}R^2)_{1,1} &= \tfrac{1}{9}(3k_1^2+k^2)^2 , \\ (\sqrt{g}R^2)_{1,2} &= \tfrac{1}{9}(3k_1^2+k^2)(3k_2^2+k^2) , \\ (\sqrt{g}R^2)_{1,4} &= \tfrac{2}{3}k_1k_2(3k_1^2+k^2) , \\ (\sqrt{g}R^2)_{1,6} &= \tfrac{2}{3}k_2k_3(3k_1^2+k^2) , \\ (\sqrt{g}R^2)_{4,4} &= 4k_1^2k_2^2 , \\ (\sqrt{g}R^2)_{4,5} &= 4k_1^2k_2k_3 ,\end{aligned} \tag{3.42}$$

and by

$$\begin{aligned}(\sqrt{g}R_{\lambda\mu\nu\sigma}R^{\lambda\mu\nu\sigma})_{1,1} &= \tfrac{1}{9}[5(k^2)^2-6k^2k_1^2+9k_1^4] , \\ (\sqrt{g}R_{\lambda\mu\nu\sigma}R^{\lambda\mu\nu\sigma})_{1,2} &= \tfrac{1}{9}[9k_1^2k_2^2+2(k^2)^2-6k^2k_3^2] , \\ (\sqrt{g}R_{\lambda\mu\nu\sigma}R^{\lambda\mu\nu\sigma})_{1,4} &= \tfrac{2}{3}k_1k_2(3k_1^2-k^2) , \\ (\sqrt{g}R_{\lambda\mu\nu\sigma}R^{\lambda\mu\nu\sigma})_{1,6} &= \tfrac{2}{3}k_2k_3(3k_1^2+2k^2) , \\ (\sqrt{g}R_{\lambda\mu\nu\sigma}R^{\lambda\mu\nu\sigma})_{4,4} &= 2k_3^2k^2+4k_1^2k_2^2 , \\ (\sqrt{g}R_{\lambda\mu\nu\sigma}R^{\lambda\mu\nu\sigma})_{4,5} &= 2k_2k_3(2k_1^2-k^2) .\end{aligned} \tag{3.43}$$

C. Lattice cosmological constant term

Next we will consider the cosmological-constant contribution in the lattice action of Eq. (2.1):

$$I_V = \sum_{\text{edges } h} V_h , \tag{3.44}$$

where V_h is defined to be the volume associated with an edge h. It is obtained by subdividing the volume of each tetrahedron into contributions associated with each edge (here via a baricentric subdivision), and then adding up the contributions from each tetrahedron touched by the given edge [12]. In momentum space the matrix M_ω describing the small fluctuations around flat space is then given by

$$\begin{aligned}(M_\omega)_{1,1} &= -\tfrac{5}{3} , \\ (M_\omega)_{1,2} &= \tfrac{1}{6}(\omega_1\omega_4+\bar{\omega}_2\bar{\omega}_4) , \\ (M_\omega)_{1,4} &= 1+\bar{\omega}_2 , \\ (M_\omega)_{1,6} &= 0 , \\ (M_\omega)_{1,7} &= -\tfrac{1}{2}(\bar{\omega}_2+\bar{\omega}_4) , \\ (M_\omega)_{4,4} &= -\tfrac{16}{3} , \\ (M_\omega)_{4,5} &= -\tfrac{2}{3}(\omega_2+\bar{\omega}_4) , \\ (M_\omega)_{4,7} &= 2+2\bar{\omega}_4 , \\ (M_\omega)_{7,7} &= -9 .\end{aligned} \tag{3.45}$$

Next, the same set rotation is performed as in the Einstein-term case, in order to go from the lattice variables ϵ to the lattice trace-reversed variables $\bar{h}$. After the S_ω-matrix rotation [see Eq. (3.14)] one obtains for M'_ω the matrix elements

$$(M'_\omega)_{1,1}=-\frac{11}{6}-\frac{\omega_2}{12\omega_4}-\frac{\omega_4}{12\omega_2}\,,$$
$$(M'_\omega)_{1,2}=-\frac{1}{12}-\frac{\omega_1}{12\omega_2}-\frac{\omega_1}{12\omega_4}+\frac{1}{6\omega_2\omega_4}+\frac{\omega_1\omega_4}{6}-\frac{\omega_4}{12\omega_2}\,,$$
$$(M'_\omega)_{1,4}=1+\frac{1}{\omega_2}\,,$$
$$(M'_\omega)_{1,6}=0\,,$$
$$(M'_\omega)_{1,7}=\frac{1}{\omega_2}+\frac{1}{\omega_4}\,, \tag{3.46}$$
$$(M'_\omega)_{4,4}=-\frac{14}{3}+\frac{1}{3\omega_4}+\frac{\omega_4}{3}\,,$$
$$(M'_\omega)_{4,5}=\frac{1}{3}-\frac{\omega_2}{3}-\frac{1}{3\omega_4}+\frac{\omega_2}{3\omega_4}\,,$$
$$(M'_\omega)_{4,7}=-1-\frac{1}{\omega_4}\,,$$
$$(M'_\omega)_{7,7}=-9\,.$$

For $k_\mu \to 0$ the above matrix becomes

$$M'_\omega \underset{k\to 0}{\sim} \begin{bmatrix} -2 & 0 & 0 & 2 & 2 & 0 & 2 \\ 0 & -2 & 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & -2 & 0 & 2 & 2 & 2 \\ 2 & 2 & 0 & -4 & 0 & 0 & -2 \\ 2 & 0 & 2 & 0 & -4 & 0 & -2 \\ 0 & 2 & 2 & 0 & 0 & -4 & -2 \\ 2 & 2 & 2 & -2 & -2 & -2 & -9 \end{bmatrix}, \tag{3.47}$$

which has six negative eigenvalues and one positive one, reminding us of the classical instability of flat space in the presence of the cosmological term (the eigenvalues are approximately -13.029, -3.143, -0.764 and -5.236 twice, and 1.172).

After the second rotation by the matrix T_ω of Eq. (3.20), one finds, for the matrix elements of the transformed matrix L_ω,

$$(L_\omega)_{1,1}=-\frac{1}{2}+\frac{2}{27\omega_1}+\frac{2\omega_1}{27}+\frac{19}{216\omega_2}+\frac{1}{108\omega_1\omega_2}+\frac{19\omega_2}{216}+\frac{\omega_1\omega_2}{108}+\frac{19}{216\omega_4}+\frac{1}{108\omega_1\omega_4}-\frac{1}{54\omega_2\omega_4}+\frac{19\omega_4}{216}+\frac{\omega_1\omega_4}{108}-\frac{\omega_2\omega_4}{54}$$
$$\underset{k\to 0}{\sim}\frac{-5k_1^2}{54}-\frac{k_1k_2}{54}-\frac{17k_2^2}{216}-\frac{k_1k_3}{54}+\frac{k_2k_3}{27}-\frac{17k_3^2}{216}+O(k^4)\,,$$
$$(L_\omega)_{1,2}=\frac{1}{54\omega_1}+\frac{11\omega_1}{108}+\frac{11}{108\omega_2}+\frac{1}{108\omega_1\omega_2}+\frac{\omega_2}{54}+\frac{\omega_1\omega_2}{108}-\frac{1}{27\omega_4}-\frac{1}{216\omega_1\omega_4}+\frac{1}{27\omega_2\omega_4}-\frac{\omega_4}{27}+\frac{\omega_1\omega_4}{27}-\frac{\omega_2\omega_4}{216}$$
$$\underset{k\to 0}{\sim}\frac{1}{4}+\frac{i}{8}k_1-\frac{37k_1^2}{432}-\frac{i}{8}k_2-\frac{k_1k_2}{54}-\frac{37k_2^2}{432}-\frac{7k_1k_3}{216}-\frac{7k_2k_3}{216}+\frac{k_3^2}{216}+O(k^3)\,,$$
$$(L_\omega)_{1,4}=\frac{7}{72\omega_1}-\frac{1}{9\omega_2}+\frac{1}{24\omega_1\omega_2}-\frac{1}{72\omega_1\omega_4}-\frac{1}{18\omega_1\omega_4}+\frac{1}{18\omega_1\omega_2\omega_4}-\frac{\omega_4}{72\omega_1\omega_2}$$
$$\underset{k\to 0}{\sim}-\frac{i}{6}k_1-\frac{k_1^2}{12}+\frac{i}{12}k_2-\frac{k_1k_2}{12}+\frac{k_2^2}{24}-\frac{k_1k_3}{18}-\frac{k_2k_3}{72}+\frac{k_3^2}{72}+O(k^3)\,, \tag{3.48}$$
$$(L_\omega)_{1,6}=-\frac{11}{72\omega_2}-\frac{1}{72\omega_1\omega_2}-\frac{11}{72\omega_4}-\frac{1}{72\omega_1\omega_4}+\frac{5}{12\omega_2\omega_4}-\frac{1}{36\omega_1\omega_2\omega_4}-\frac{\omega_1}{18\omega_2\omega_4}$$
$$\underset{k\to 0}{\sim}\frac{k_1^2}{18}-\frac{i}{6}k_2-\frac{k_1k_2}{72}-\frac{k_2^2}{12}-\frac{i}{6}k_3-\frac{k_1k_3}{72}-\frac{k_2k_3}{3}-\frac{k_3^2}{12}+O(k^3)\,,$$
$$(L_\omega)_{4,4}=-\frac{7}{6}+\frac{1}{12\omega_4}+\frac{\omega_4}{12}\underset{k\to 0}{\sim}-1-\frac{k_3^2}{12}+O(k^4)\,,$$
$$(L_\omega)_{4,5}=\frac{1}{12}-\frac{\omega_2}{12}-\frac{1}{12\omega_4}+\frac{\omega_2}{12\omega_4}\underset{k\to 0}{\sim}\frac{k_2k_3}{12}+O(k^3)\,.$$

In this form the matrix L_ω is almost identical to the matrix $-V$ appearing in the weak-field expansion of the cosmological-constant term in the continuum [see Eq. (3.41)]:

$$L(k=0)=\begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} . \tag{3.49}$$

A discrepancy appears in the (1,1), (2,2), and (3,3) matrix elements, but it is easily seen that the cause lies in the fact that there are additional contributions from the terms linear in ϵ:

$$I_V \sim \sum_n (\epsilon_1^{(n)}+\epsilon_2^{(n)}+\epsilon_4^{(n)}) + \frac{1}{2}\sum_{mn,ij} \epsilon_i^{(m)T} M_{i,j}^{(m,n)} \epsilon_j^{(n)} \tag{3.50}$$

[the factor of $\frac{1}{2}$ in the second term on the RHS arises because a factor of $\frac{1}{2}$ was omitted from the small fluctuation matrix in (3.45)–(3.49) to simplify comparison with the continuum expression (3.41)]. Since the expansion of ϵ_i in terms of $h_{\mu\nu}$ contains also terms quadratic in $h_{\mu\nu}$ [see Eq. (3.6)], there are additional diagonal contributions to L_ω:

$$\begin{aligned} \epsilon_1+\epsilon_2+\epsilon_4 &= \tfrac{1}{2}(h_{11}+h_{22}+h_{33}) - \tfrac{1}{8}(h_{11}^2+h_{22}^2+h_{33}^2) + \cdots \\ &= -\tfrac{1}{2}(\bar{h}_{11}+\bar{h}_{22}+\bar{h}_{33}) - \tfrac{1}{8}(\bar{h}_{11}^2+\bar{h}_{22}^2+\bar{h}_{33}^2) + \cdots . \end{aligned} \tag{3.51}$$

These additional contributions make the lattice cosmological-constant term identical to the continuum one, for small momenta and to quadratic order in the weak-field expansion. As in the case of the Regge-Einstein term, there are higher-order lattice corrections to the cosmological-constant term of $O(k)$ (which are completely absent in the continuum, since no derivatives are present there).

D. Lattice curvature-squared term

Finally, it remains to consider the curvature-squared contribution to the action of Eq. (2.1):

$$I_{R^2} = \sum_{\text{edges } h} \frac{l_h^2 \delta_h^2}{V_h} . \tag{3.52}$$

In momentum space the matrix elements of M_ω, describing the small fluctuations around flat space, are given by

$$\begin{aligned}
(M_\omega)_{1,1} &= 744+72\omega_2+72\omega_4+72\omega_1\omega_2\omega_4+12\omega_1\omega_2^2\omega_4+12\omega_1\omega_2\omega_4^2 \\
&\quad +72\bar{\omega}_2+108\omega_4\bar{\omega}_2+72\bar{\omega}_4+108\omega_2\bar{\omega}_4+72\bar{\omega}_1\bar{\omega}_2\bar{\omega}_4+12\bar{\omega}_1\bar{\omega}_2^2\bar{\omega}_4+12\bar{\omega}_1\bar{\omega}_2\bar{\omega}_4^2 , \\
(M_\omega)_{1,2} &= 192+216\omega_1+192\omega_1\omega_4+12\omega_1^2\omega_2\omega_4+216\bar{\omega}_2+192\omega_1\bar{\omega}_2+108\omega_1\bar{\omega}_4+108\omega_2\bar{\omega}_4+192\bar{\omega}_2\bar{\omega}_4+12\bar{\omega}_1\bar{\omega}_2^2\bar{\omega}_4 , \\
(M_\omega)_{1,3} &= 192+216\omega_1+192\omega_1\omega_2+12\omega_1^2\omega_2\omega_4+108\omega_1\bar{\omega}_2+216\bar{\omega}_4+192\omega_1\bar{\omega}_4+108\omega_2\bar{\omega}_4+192\bar{\omega}_2\bar{\omega}_4+12\bar{\omega}_1\bar{\omega}_2\bar{\omega}_4^2 , \\
(M_\omega)_{1,4} &= -864-168\omega_4-168\omega_1\omega_4-24\omega_1\omega_2\omega_4-864\bar{\omega}_2-216\omega_4\bar{\omega}_2-216\bar{\omega}_4-168\bar{\omega}_2\bar{\omega}_4-168\bar{\omega}_1\bar{\omega}_2\bar{\omega}_4-24\bar{\omega}_1\bar{\omega}_2^2\bar{\omega}_4 , \\
(M_\omega)_{1,6} &= -672\omega_1-24\omega_1\omega_2-24\omega_1\omega_4-360\bar{\omega}_2-360\omega_1\bar{\omega}_2-360\bar{\omega}_4-360\omega_1\bar{\omega}_4-672\bar{\omega}_2\bar{\omega}_4-24\bar{\omega}_2^2\bar{\omega}_4-24\bar{\omega}_2\bar{\omega}_4^2 , \\
(M_\omega)_{1,7} &= 648+288\omega_1+36\omega_2+36\omega_4+1152\bar{\omega}_2+1152\bar{\omega}_4+648\bar{\omega}_2\bar{\omega}_4+288\bar{\omega}_1\bar{\omega}_2\bar{\omega}_4+36\bar{\omega}_2^2\bar{\omega}_4+36\bar{\omega}_2\bar{\omega}_4^2 , \\
(M_\omega)_{4,4} &= 3456+48\omega_1+48\omega_2+432\omega_4+288\omega_1\omega_4+288\omega_2\omega_4+48\omega_1\omega_2\omega_4 \\
&\quad +48\bar{\omega}_1+48\bar{\omega}_2+432\bar{\omega}_4+288\bar{\omega}_1\bar{\omega}_4+288\bar{\omega}_2\bar{\omega}_4+48\bar{\omega}_1\bar{\omega}_2\bar{\omega}_4 , \\
(M_\omega)_{4,5} &= 768+1728\omega_2+384\omega_1\omega_2++1728\bar{\omega}_4+768\omega_2\bar{\omega}_4+384\bar{\omega}_1\bar{\omega}_4 , \\
(M_\omega)_{4,7} &= -3168-504\omega_1-504\omega_2-72\bar{\omega}_1-72\bar{\omega}_2-3168\bar{\omega}_4-72\omega_2\bar{\omega}_4-72\omega_2\bar{\omega}_4-504\bar{\omega}_1\bar{\omega}_4-504\bar{\omega}_2\bar{\omega}_4 , \\
(M_\omega)_{7,7} &= 8424+648(\omega_1+\omega_2+\omega_4)+108(\omega_2\bar{\omega}_1+\omega_4\bar{\omega}_1+\omega_4\bar{\omega}_2)+648(\bar{\omega}_1+\bar{\omega}_2+\bar{\omega}_4)+108(\omega_1\bar{\omega}_2+\omega_1\bar{\omega}_4+\omega_2\bar{\omega}_4) .
\end{aligned} \tag{3.53}$$

After a rotation by the matrix S_ω of Eq. (3.14), one obtains for the matrix M'_ω the matrix elements

$$
\begin{aligned}
(M'_\omega)_{1,1}={}&450+\frac{36}{\omega_1}+36\omega_1-\frac{12}{\omega_2^2}-\frac{90}{\omega_2}-\frac{96}{\omega_1\omega_2}+\frac{9\omega_1}{\omega_2}-90\omega_2+\frac{9\omega_2}{\omega_1}-96\omega_1\omega_2-12\omega_2^2-\frac{12}{\omega_4^2}+\frac{12}{\omega_1\omega_2\omega_4^2}\\
&+\frac{18\omega_2}{\omega_4^2}+\frac{3\omega_1\omega_2}{\omega_4^2}+\frac{3\omega_2^2}{\omega_4^2}-\frac{90}{\omega_4}-\frac{96}{\omega_1\omega_4}+\frac{9\omega_1}{\omega_4}+\frac{12}{\omega_1\omega_2^2\omega_4}-\frac{24}{\omega_2\omega_4}+\frac{72}{\omega_1\omega_2\omega_4}-\frac{36\omega_2}{\omega_4}+\frac{18\omega_2}{\omega_1\omega_4}\\
&+\frac{18\omega_1\omega_2}{\omega_4}+\frac{18\omega_2^2}{\omega_4}+\frac{3\omega_2^2}{\omega_1\omega_4}-90\omega_4+\frac{9\omega_4}{\omega_1}-96\omega_1\omega_4+\frac{18\omega_4}{\omega_2^2}+\frac{3\omega_1\omega_4}{\omega_2^2}-\frac{36\omega_4}{\omega_2}+\frac{18\omega_4}{\omega_1\omega_2}+\frac{18\omega_1\omega_4}{\omega_2}\\
&-24\omega_2\omega_4+72\omega_1\omega_2\omega_4+12\omega_1\omega_2^2\omega_4-12\omega_4^2+\frac{3\omega_4^2}{\omega_2^2}+\frac{18\omega_4^2}{\omega_2}+\frac{3\omega_4^2}{\omega_1\omega_2}+12\omega_1\omega_2\omega_4^2\,,
\end{aligned}
$$

$$
\begin{aligned}
(M'_\omega)_{1,2}={}&51+\frac{18}{\omega_1}+54\omega_1-54\omega_1^2-\frac{54}{\omega_2^2}+\frac{18\omega_1}{\omega_2^2}+\frac{3\omega_1^2}{\omega_2^2}+\frac{54}{\omega_2}-54\omega_1\omega_2+\frac{51\omega_1}{\omega_2}+\frac{18\omega_1^2}{\omega_2}+18\omega_2+\frac{3\omega_2}{\omega_1}\\
&-54\omega_1\omega_2-\frac{6}{\omega_4^2}+\frac{18\omega_1}{\omega_4^2}+\frac{3\omega_1^2}{\omega_4^2}-\frac{6\omega_1}{\omega_2\omega_4^2}+\frac{3\omega_1\omega_2}{\omega_4^2}-\frac{72}{\omega_4}-\frac{6}{\omega_1\omega_4}-\frac{36\omega_1}{\omega_4}+\frac{18\omega_1^2}{\omega_4}+\frac{12}{\omega_1\omega_2^2\omega_4}-\frac{6\omega_1}{\omega_2^2\omega_4}\\
&+\frac{84}{\omega_2\omega_4}-\frac{72\omega_1}{\omega_2\omega_4}+\frac{6\omega_1^2}{\omega_2\omega_4}+\frac{6\omega_2}{\omega_4}+\frac{18\omega_1\omega_2}{\omega_4}-72\omega_4+\frac{6\omega_4}{\omega_1}+84\omega_1\omega_4+\frac{18\omega_4}{\omega_2^2}+\frac{6\omega_1\omega_4}{\omega_2^2}-\frac{36\omega_4}{\omega_2}\\
&+\frac{18\omega_4}{\omega_1\omega_2}-\frac{72\omega_1\omega_4}{\omega_2}-\frac{6\omega_1^2\omega_4}{\omega_2}-6\omega_2\omega_4+12\omega_1^2\omega_2\omega_4-6\omega_4^2+\frac{3\omega_4^2}{\omega_2^2}+\frac{18\omega_4^2}{\omega_2}+\frac{3\omega_4^2}{\omega_1\omega_2}-\frac{6\omega_1\omega_4^2}{\omega_2}\,,
\end{aligned}
$$

$$
\begin{aligned}
(M'_\omega)_{1,4}={}&-270-\frac{36}{\omega_1}+144\omega_1+\frac{60}{\omega_2^2}-\frac{6\omega_1}{\omega_2^2}-\frac{270}{\omega_2}+\frac{144}{\omega_1\omega_2}-\frac{36\omega_1}{\omega_2}+60\omega_2-\frac{6\omega_2}{\omega_1}-\frac{36}{\omega_4^2}-\frac{6\omega_1}{\omega_4^2}+\frac{12}{\omega_2\omega_4^2}\\
&-\frac{6\omega_2}{\omega_4^2}+\frac{186}{\omega_4}+\frac{48}{\omega_1\omega_4}-\frac{30\omega_1}{\omega_4}+\frac{12}{\omega_2^2\omega_4}-\frac{24}{\omega_1\omega_2^2\omega_4}+\frac{72}{\omega_2\omega_4}-\frac{72}{\omega_1\omega_2\omega_4}-\frac{12\omega_1}{\omega_2\omega_4}-\frac{30\omega_2}{\omega_4}-\frac{6\omega_2}{\omega_1\omega_4}\\
&+72\omega_4-\frac{12\omega_4}{\omega_1}-72\omega_1\omega_4-\frac{30\omega_4}{\omega_2^2}-\frac{6\omega_1\omega_4}{\omega_2^2}+\frac{186\omega_4}{\omega_2}-\frac{30\omega_4}{\omega_1\omega_2}+\frac{48\omega_1\omega_4}{\omega_2}+12\omega_2\omega_4-24\omega_1\omega_2\omega_4\\
&+12\omega_4^2-\frac{6\omega_4^2}{\omega_2^2}-\frac{36\omega_4^2}{\omega_2}-\frac{6\omega_4^2}{\omega_1\omega_2}\,,
\end{aligned}
$$

$$
\begin{aligned}
(M'_\omega)_{1,6}={}&156-\frac{12}{\omega_1}-264\omega_1+96\omega_1^2+\frac{48}{\omega_2^2}-\frac{30\omega_1}{\omega_2^2}-\frac{6\omega_1^2}{\omega_2^2}+\frac{42}{\omega_2}-\frac{36}{\omega_1\omega_2}+\frac{42\omega_1}{\omega_2}-\frac{36\omega_1^2}{\omega_2}+12\omega_2-12\omega_1\omega_2\\
&+\frac{48}{\omega_4^2}-\frac{30\omega_1}{\omega_4^2}-\frac{6\omega_1^2}{\omega_4^2}-\frac{12}{\omega_2\omega_4^2}+\frac{12\omega_1}{\omega_2\omega_4^2}-\frac{6\omega_2}{\omega_4^2}-\frac{6\omega_1\omega_2}{\omega_4^2}+\frac{42}{\omega_4}-\frac{36}{\omega_1\omega_4}+\frac{42\omega_1}{\omega_4}-\frac{36\omega_1^2}{\omega_4}-\frac{12}{\omega_2^2\omega_4}\\
&+\frac{12\omega_1}{\omega_2^2\omega_4}-\frac{264}{\omega_2\omega_4}+\frac{96}{\omega_1\omega_2\omega_4}+\frac{156\omega_1}{\omega_2\omega_4}-\frac{12\omega_1^2}{\omega_2\omega_4}-\frac{30\omega_2}{\omega_4}-\frac{6\omega_2}{\omega_1\omega_4}+\frac{48\omega_1\omega_2}{\omega_4}+12\omega_4-12\omega_1\omega_4\\
&-\frac{6\omega_4}{\omega_2^2}-\frac{6\omega_1\omega_4}{\omega_2^2}-\frac{30\omega_4}{\omega_2}-\frac{6\omega_4}{\omega_1\omega_2}+\frac{48\omega_1\omega_4}{\omega_2}\,,
\end{aligned}
$$

$$
\begin{aligned}
(M'_\omega)_{1,7}={}&432-\frac{36}{\omega_1}+288\omega_1-\frac{108}{\omega_2^2}-\frac{18\omega_1}{\omega_2^2}-\frac{270}{\omega_2}-\frac{108}{\omega_1\omega_2}-\frac{108\omega_1}{\omega_2}+36\omega_2-\frac{108}{\omega_4^2}-\frac{18\omega_1}{\omega_4^2}+\frac{36}{\omega_2\omega_4^2}\\
&-\frac{18\omega_2}{\omega_4^2}-\frac{270}{\omega_4}-\frac{108}{\omega_1\omega_4}-\frac{108\omega_1}{\omega_4}+\frac{36}{\omega_2^2\omega_4}+\frac{432}{\omega_2\omega_4}+\frac{288}{\omega_1\omega_2\omega_4}-\frac{36\omega_1}{\omega_2\omega_4}\\
&-\frac{108\omega_2}{\omega_4}-\frac{18\omega_2}{\omega_1\omega_4}+36\omega_4-\frac{18\omega_4}{\omega_2^2}-\frac{108\omega_4}{\omega_2}-\frac{18\omega_4}{\omega_1\omega_2}\,,
\end{aligned}
\tag{3.54}
$$

$$(M'_\omega)_{4,4}=1248-\frac{180}{\omega_1}-180\omega_1-\frac{180}{\omega_2}+\frac{24\omega_1}{\omega_2}-180\omega_2+\frac{24\omega_2}{\omega_1}+\frac{72}{\omega_4^2}+\frac{12\omega_1}{\omega_4^2}+\frac{12\omega_2}{\omega_4^2}-\frac{600}{\omega_4}+\frac{24}{\omega_1\omega_4}$$
$$+\frac{48\omega_1}{\omega_4}+\frac{24}{\omega_2\omega_4}+\frac{48}{\omega_1\omega_2\omega_4}+\frac{12\omega_1}{\omega_2\omega_4}+\frac{48\omega_2}{\omega_4}+\frac{12\omega_2}{\omega_1\omega_4}-600\omega_4+\frac{48\omega_4}{\omega_1}+24\omega_1\omega_4+\frac{48\omega_4}{\omega_2}+\frac{12\omega_1\omega_4}{\omega_2}+24\omega_2\omega_4$$
$$+\frac{12\omega_2\omega_4}{\omega_1}+48\omega_1\omega_2\omega_4+72\omega_4^2+\frac{12\omega_4^2}{\omega_1}+\frac{12\omega_4^2}{\omega_2},$$

$$(M'_\omega)_{4,5}=-300-\frac{108}{\omega_1}-108\omega_1+\frac{60}{\omega_2}+\frac{12\omega_1}{\omega_2}+360\omega_2+\frac{48\omega_2}{\omega_1}+120\omega_1\omega_2-96\omega_2^2+\frac{12\omega_2^2}{\omega_1}-\frac{96}{\omega_4^2}+\frac{12\omega_1}{\omega_4^2}$$
$$+\frac{60\omega_2}{\omega_4^2}+\frac{12\omega_1\omega_2}{\omega_4^2}+\frac{12\omega_2^2}{\omega_4^2}+\frac{360}{\omega_4}+\frac{120}{\omega_1\omega_4}+\frac{48\omega_1}{\omega_4}-\frac{96}{\omega_2\omega_4}+\frac{12\omega_1}{\omega_2\omega_4}-\frac{300\omega_2}{\omega_4}-\frac{108\omega_2}{\omega_1\omega_4}-\frac{108\omega_1\omega_2}{\omega_4}$$
$$+\frac{60\omega_2^2}{\omega_4}+\frac{12\omega_2^2}{\omega_1\omega_4}+60\omega_4+\frac{12\omega_4}{\omega_1}+\frac{12\omega_4}{\omega_2}-96\omega_2\omega_4+\frac{12\omega_2\omega_4}{\omega_1},$$

$$(M'_\omega)_{4,7}=-144+\frac{180}{\omega_1}-288\omega_1+\frac{180}{\omega_2}+\frac{36\omega_1}{\omega_2}-288\omega_2+\frac{36\omega_2}{\omega_1}+\frac{216}{\omega_4^2}+\frac{36\omega_1}{\omega_4^2}+\frac{36\omega_2}{\omega_4^2}-\frac{144}{\omega_4}-\frac{288}{\omega_1\omega_4}$$
$$+\frac{180\omega_1}{\omega_4}-\frac{288}{\omega_2\omega_4}+\frac{36\omega_1}{\omega_2\omega_4}+\frac{180\omega_2}{\omega_4}+\frac{36\omega_2}{\omega_1\omega_4}+216\omega_4+\frac{36\omega_4}{\omega_1}+\frac{36\omega_4}{\omega_2},$$

$$(M'_\omega)_{7,7}=8424+\frac{648}{\omega_1}+648\omega_1+\frac{648}{\omega_2}+\frac{108\omega_1}{\omega_2}+648\omega_2+\frac{108\omega_2}{\omega_1}+\frac{648}{\omega_4}+\frac{108\omega_1}{\omega_4}+\frac{108\omega_2}{\omega_4}+648\omega_4+\frac{108\omega_4}{\omega_1}+\frac{108\omega_4}{\omega_2}.$$

For $k_\mu \to 0$ all the matrix elements of M'_ω tend to zero except for the (7,7) entry, and the matrix reduces to

$$M'_\omega \underset{k\to 0}{\sim} \begin{bmatrix} 0 & 0 \\ 0 & 12\,960 \end{bmatrix}. \tag{3.55}$$

From the form of the matrix M'_ω one notices that the mode associated with fluctuations in ϵ_7 does not completely decouple, as was the case for the Einstein-Regge action. But its mass is enormous in lattice units, since $\lambda_7=12\,960+O(k^2)$, and it therefore still effectively decouples. A second rotation by the matrix T_ω, defined in Eq. (3.20), gives the matrix elements of the transformed matrix L_ω, whose matrix elements are now remarkably all of $O(k^4)$, due to cancellations:

$$(L_\omega)_{1,1}=96-\frac{88}{3\omega_1}-\frac{88\omega_1}{3}+\frac{4}{3\omega_1\omega_2^2}-\frac{73}{3\omega_2}+\frac{4}{3\omega_1^2\omega_2}+\frac{4}{3\omega_1\omega_2}+\frac{20\omega_1}{3\omega_2}-\frac{73\omega_2}{3}+\frac{20\omega_2}{3\omega_1}+\frac{4\omega_1\omega_2}{3}+\frac{4\omega_1^2\omega_2}{3}$$
$$+\frac{4\omega_1\omega_2^2}{3}+\frac{4}{3\omega_1\omega_4^2}+\frac{1}{3\omega_2\omega_4^2}-\frac{73}{3\omega_4}+\frac{4}{3\omega_1^2\omega_4}+\frac{4}{3\omega_1\omega_4}+\frac{20\omega_1}{3\omega_4}+\frac{1}{3\omega_2^2\omega_4}+\frac{10}{3\omega_2\omega_4}+\frac{14\omega_2}{3\omega_4}-\frac{73\omega_4}{3}+\frac{20\omega_4}{3\omega_1}$$
$$+\frac{4\omega_1\omega_4}{3}+\frac{4\omega_1^2\omega_4}{3}+\frac{14\omega_4}{3\omega_2}+\frac{10\omega_2\omega_4}{3}+\frac{\omega_2^2\omega_4}{3}+\frac{4\omega_1\omega_4^2}{3}+\frac{\omega_2\omega_4^2}{3}$$
$$\underset{k\to 0}{\sim}\frac{8k_1^4}{3}+\frac{8k_1^3k_2}{3}+\frac{28k_1^2k_2^2}{3}+\frac{8k_1k_2^3}{3}+\frac{5k_2^4}{3}+\frac{8k_1^3k_3}{3}$$
$$+\frac{2k_2^3k_3}{3}+\frac{28k_1^2k_3^2}{3}+\frac{16k_2^2k_3^2}{3}+\frac{8k_1k_3^3}{3}+\frac{2k_2k_3^3}{3}+\frac{5k_3^4}{3}+O(k^6),$$

$$
\begin{aligned}
(L_\omega)_{1,2}={}&6-\frac{31}{3\omega_1}-\frac{31\omega_1}{3}+\frac{4}{3\omega_1\omega_2^2}-\frac{31}{3\omega_2}+\frac{4}{3\omega_1^2\omega_2}+\frac{4}{3\omega_1\omega_2}+\frac{29\omega_1}{3\omega_2}-\frac{31\omega_2}{3}+\frac{29\omega_2}{3\omega_1}+\frac{4\omega_1\omega_2}{3}+\frac{4\omega_1^2\omega_2}{3}\\
&+\frac{4\omega_1\omega_2^2}{3}-\frac{2}{3\omega_1\omega_4^2}-\frac{2}{3\omega_2\omega_4^2}+\frac{53}{3\omega_4}-\frac{2}{3\omega_1^2\omega_4}-\frac{11}{3\omega_1\omega_4}-\frac{10\omega_1}{3\omega_4}-\frac{2}{3\omega_2^2\omega_4}-\frac{11}{3\omega_2\omega_4}+\frac{3}{\omega_1\omega_2\omega_4}-\frac{10\omega_2}{3\omega_4}+\frac{53\omega_4}{3}\\
&-\frac{10\omega_4}{3\omega_1}-\frac{11\omega_1\omega_4}{3}-\frac{2\omega_1^2\omega_4}{3}-\frac{10\omega_4}{3\omega_2}-\frac{11\omega_2\omega_4}{3}+3\omega_1\omega_2\omega_4-\frac{2\omega_2^2\omega_4}{3}-\frac{2\omega_1\omega_4^2}{3}-\frac{2\omega_2\omega_4^2}{3}\\
\underset{k\to 0}{\sim}{}&\frac{2k_1^4}{3}+\frac{8k_1^3k_2}{3}+\frac{37k_1^2k_2^2}{3}+\frac{8k_1k_2^3}{3}+\frac{2k_2^4}{3}-\frac{4k_1^3k_3}{3}+3k_1^2k_2k_3+3k_1k_2^2k_3-\frac{4k_2^3k_3}{3}-\frac{14k_1^2k_3^2}{3}+3k_1k_2k_3^2\\
&-\frac{14k_2^2k_3^2}{3}-\frac{4k_1k_3^3}{3}-\frac{4k_2k_3^3}{3}-\frac{4k_3^4}{3}+O(k^6)\,,
\end{aligned}
$$

$$
\begin{aligned}
(L_\omega)_{1,4}={}&7-\frac{2}{\omega_1^2}+\frac{13}{\omega_1}+2\omega_1+\frac{1}{\omega_2^2}+\frac{2}{\omega_1\omega_2^2}-\frac{13}{\omega_2}+\frac{2}{\omega_1^2\omega_2}+\frac{7}{\omega_1\omega_2}-\frac{2\omega_1}{\omega_2}+2\omega_2+\frac{\omega_2}{\omega_1}-\frac{2}{\omega_1\omega_4^2}+\frac{1}{\omega_2\omega_4^2}-\frac{1}{\omega_4}\\
&-\frac{2}{\omega_1^2\omega_4}+\frac{13}{\omega_1\omega_4}-\frac{2\omega_1}{\omega_4}+\frac{1}{\omega_2^2\omega_4}-\frac{2}{\omega_1\omega_2^2\omega_4}+\frac{7}{\omega_2\omega_4}+\frac{4}{\omega_1^2\omega_2\omega_4}-\frac{12}{\omega_1\omega_2\omega_4}-\frac{2\omega_2}{\omega_4}-12\omega_4+\frac{7\omega_4}{\omega_1}\\
&+4\omega_1\omega_4-\frac{2\omega_4}{\omega_1\omega_2^2}+\frac{13\omega_4}{\omega_2}-\frac{2\omega_4}{\omega_1^2\omega_2}-\frac{\omega_4}{\omega_1\omega_2}-\frac{2\omega_1\omega_4}{\omega_2}-2\omega_2\omega_4+\frac{\omega_2\omega_4}{\omega_1}+\frac{\omega_4^2}{\omega_1}-\frac{2\omega_4^2}{\omega_2}\\
\underset{k\to 0}{\sim}{}&4k_1^3k_2-2k_1k_2^3+4k_1^3k_3+8k_1^2k_2k_3-k_1k_2^2k_3+k_2^3k_3-8k_1k_2k_3^2-3k_2^2k_3^2-3k_1k_3^3+3k_2k_3^3-k_3^4+O(k^6)\,,
\end{aligned}
$$

(3.56)

$$
\begin{aligned}
(L_\omega)_{1,6}={}&-80+\frac{32}{\omega_1}+24\omega_1-\frac{2}{\omega_2^2}-\frac{2}{\omega_1\omega_2^2}+\frac{74}{\omega_2}-\frac{2}{\omega_1^2\omega_2}-\frac{23}{\omega_1\omega_2}-\frac{23\omega_1}{\omega_2}-\frac{2\omega_1^2}{\omega_2}-\omega_2+\frac{\omega_2}{\omega_1}\\
&+4\omega_1\omega_2-\frac{2}{\omega_4^2}-\frac{2}{\omega_1\omega_4^2}-\frac{1}{\omega_2\omega_4^2}+\frac{4}{\omega_1\omega_2\omega_4^2}+\frac{\omega_1}{\omega_2\omega_4^2}+\frac{74}{\omega_4}-\frac{2}{\omega_1^2\omega_4}-\frac{23}{\omega_1\omega_4}-\frac{23\omega_1}{\omega_4}-\frac{2\omega_1^2}{\omega_4}-\frac{1}{\omega_2^2\omega_4}\\
&+\frac{4}{\omega_1\omega_2^2\omega_4}+\frac{\omega_1}{\omega_2^2\omega_4}-\frac{80}{\omega_2\omega_4}+\frac{24}{\omega_1\omega_2\omega_4}+\frac{32\omega_1}{\omega_2\omega_4}-\frac{2\omega_2}{\omega_4}-\frac{2\omega_1\omega_2}{\omega_4}-\omega_4+\frac{\omega_4}{\omega_1}+4\omega_1\omega_4-\frac{2\omega_4}{\omega_2}-\frac{2\omega_1\omega_4}{\omega_2}\\
\underset{k\to 0}{\sim}{}&-4k_1^4+3k_1^2k_2^2+k_1k_2^3+40k_1^2k_2k_3+5k_1k_2^2k_3+4k_2^3k_3+3k_1^2k_3^2+5k_1k_2k_3^2+k_1k_3^3+4k_2k_3^3+O(k^6)\,,
\end{aligned}
$$

$$
\begin{aligned}
(L_\omega)_{4,4}={}&312-\frac{45}{\omega_1}-\frac{45}{\omega_2}+\frac{6\omega_1}{\omega_2}-45\omega_2+\frac{6\omega_2}{\omega_1}+\frac{18}{\omega_4^2}+\frac{3\omega_1}{\omega_4^2}+\frac{3\omega_2}{\omega_4^2}-\frac{150}{\omega_4}\\
&+\frac{6}{\omega_1\omega_4}+\frac{12\omega_1}{\omega_4}+\frac{6}{\omega_2\omega_4}+\frac{12}{\omega_1\omega_2\omega_4}+\frac{3\omega_1}{\omega_2\omega_4}+\frac{12\omega_2}{\omega_4}+\frac{3\omega_2}{\omega_1\omega_4}-150\omega_4+\frac{12\omega_4}{\omega_1}\\
&+6\omega_1\omega_4+\frac{12\omega_4}{\omega_2}+\frac{3\omega_1\omega_4}{\omega_2}+6\omega_2\omega_4+\frac{3\omega_2\omega_4}{\omega_1}+12\omega_1\omega_2\omega_4+18\omega_4^2+\frac{3\omega_4^2}{\omega_1}+\frac{3\omega_4^2}{\omega_2}\\
\underset{k\to 0}{\sim}{}&12k_1^2k_2^2+12k_1^2k_2k_3+12k_1k_2^2k_3+24k_1^2k_3^2+6k_1k_2k_3^2+24k_2^2k_3^2-6k_1k_3^3-6k_2k_3^3+24k_3^4+O(k^6)\,,
\end{aligned}
$$

$$
\begin{aligned}
(L_\omega)_{4,5}={}&-75-\frac{27}{\omega_1}-27\omega_1+\frac{15}{\omega_2}+\frac{3\omega_1}{\omega_2}+90\omega_2+\frac{12\omega_2}{\omega_1}+30\omega_1\omega_2-24\omega_2^2+\frac{3\omega_2^2}{\omega_1}\\
&-\frac{24}{\omega_4^2}+\frac{3\omega_1}{\omega_4^2}+\frac{15\omega_2}{\omega_4^2}+\frac{3\omega_1\omega_2}{\omega_4^2}+\frac{3\omega_2^2}{\omega_4^2}+\frac{90}{\omega_4}+\frac{30}{\omega_1\omega_4}+\frac{12\omega_1}{\omega_4}-\frac{24}{\omega_2\omega_4}+\frac{3\omega_1}{\omega_2\omega_4}\\
&-\frac{75\omega_2}{\omega_4}-\frac{27\omega_2}{\omega_1\omega_4}-\frac{27\omega_1\omega_2}{\omega_4}+\frac{15\omega_2^2}{\omega_4}+\frac{3\omega_2^2}{\omega_1\omega_4}+15\omega_4+\frac{3\omega_4}{\omega_1}+\frac{3\omega_4}{\omega_2}-24\omega_2\omega_4+\frac{3\omega_2\omega_4}{\omega_1}\\
\underset{k\to 0}{\sim}{}&6k_1^2k_2^2-6k_1k_2^3+24k_1^2k_2k_3-24k_2^3k_3+6k_1^2k_3^2+6k_2^2k_3^2-6k_1k_3^3-24k_2k_3^3+O(k^6)\,.
\end{aligned}
$$

After the T_ω rotation one finds that the matrix L_ω also has three exact zero modes, as suggested by gauge invariance; the remaining three eigenvalues are of $O(k^4)$. The nonvanishing eigenvalues of $O(k^4)$ are in general given by rather complicated expressions. For the special choice $k_1=k_2=k_3\equiv k/\sqrt{3}$ they are given by

$$\lambda_{4,5}=+150k^4 \quad \text{(twice)}\,,$$
$$\lambda_6=+132k^4 \quad \text{(once)}\,. \tag{3.57}$$

Now also the eigenvalue associated with the conformal mode is positive. Indeed, the continuum weak-field expansion for $R^2_{\mu\nu\rho\sigma}$ [see Eq. (3.43)] also leads to three zero modes, and to the additional three positive eigenvalues (again for the above-mentioned special case)

$$\lambda_{4,5}=+\tfrac{5}{3}k^4 \quad \text{(twice)}\,,$$
$$\lambda_6=+\tfrac{4}{3}k^4 \quad \text{(once)}\,. \tag{3.58}$$

Still, the eigenvalues of the lattice and continuum operators are different, and so are their individual matrix elements. In particular, the lattice operator still contains apparently noncovariant contributions such as $k_1k_2^3$, but this should not be surprising since already the Regge-Einstein term contains such contributions. By varying the coefficient of the higher-derivative term one is therefore also modifying the strength of the higher-order lattice corrections to the Regge-Einstein action.

IV. NUMERICAL STUDIES OF SIMPLICIAL 3D GRAVITY

In order to explore the ground state of three-dimensional gravity beyond perturbation theory one has to resort to numerical methods. General aspects of the method as applied to simplicial quantum gravity are discussed in Refs. [12,13,23] and will not be repeated here. In the numerical simulations presented below the simple cubic lattice was employed, with three face diagonals and a body diagonal introduced to make the cube rigid, as in the discussion of the lattice weak-field expansion of the previous section. One could perform the numerical studies with lattices of different topologies, but one expects that universal infrared scaling properties of the theory should be determined by short-distance renormalization effects, and should therefore in general be independent of the specific choice of boundary conditions. Lattices of size between 4^3 (with 448 edges) and 32^3 (with 229 376 edges) were considered; some shorter runs with lattices of size 64^3 (with 1 835 008 edges) were also done. The measure was chosen to be dl^2, in order to compare directly with results with the same measure in $d=2$ [15] and $d=4$ [17]. As mentioned previously, in order to explore the sensitivity of our results to the choice of measure, we have also done a comparison with the scale-invariant measure dl/l.

The edge lengths were updated by a standard Metropolis algorithm, generating eventually an ensemble of configurations distributed according to the action of Eq. (2.1), and with the inclusion of the appropriate generalized triangle inequality constraints. The lengths of the runs typically varied between $100k$ Monte Carlo iterations on the 4^3 lattice, $40k$ on the 8^3 lattice, $10k$ on the 16^3 lattice, and $1-2k$ iterations on the 32^3 lattice. One notices that in all runs the scaling relation of Eq. (2.11) is very well verified, as one would expect if the edge probability distribution is sampled correctly. Figures 3–5 show the local distribution of edge lengths, volumes, and curvatures throughout the lattice, as obtained for a system of size 64^3 (in this particular case with action parameters $\lambda=1$, $k=0.163$, and $a=0.005$). As can be seen, the distributions are rather smooth and well behaved, at least not too close to the critical point (to be discussed below). On the larger lattices duplicated copies of the smaller lattices are used as starting configurations for each k, allowing for additional equilibration sweeps after duplicating the lattice in all directions. One should emphasize that at this point the nature of the results is still rather preliminary, even though some effort has been made to control the systematic errors by computing the critical exponents for 3D gravity for different values of the (irrelevant) coupling a, and by a number of different methods which presumably have different (and hopefully small) systematic biases. Some of the results presented in this section were presented previously by Hamber [8].

Quantities of physical interest which have been computed include the total average curvature $\mathcal{R}$,

$$\mathcal{R}(\lambda,k,a)=\langle l^2\rangle\frac{\left\langle 2\sum_h \delta_h l_h\right\rangle}{\left\langle \sum_h V_h\right\rangle} \sim \frac{\left\langle \int\sqrt{g}R\right\rangle}{\left\langle\int\sqrt{g}\right\rangle}\,, \tag{4.1}$$

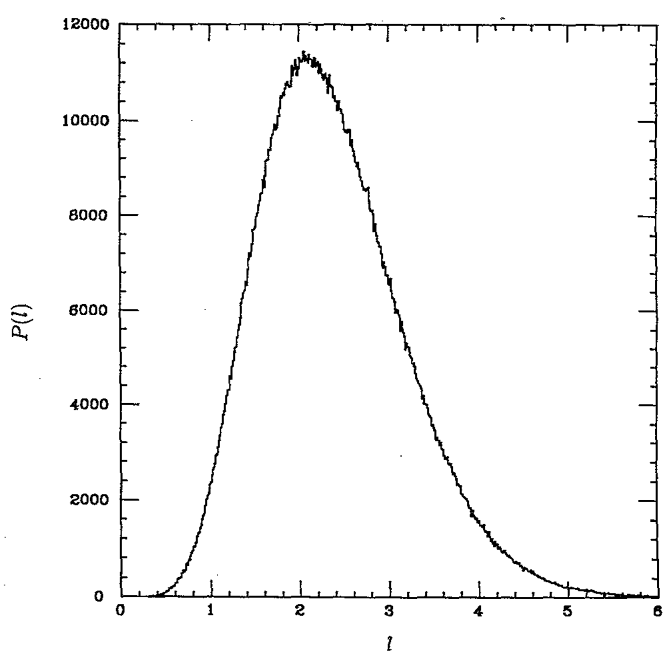

FIG. 3. Histogram of the distribution of edge lengths $P(l)$ on a 64^3 lattice for $\lambda=1, k=0.163$, and $a=0.005$ (dl^2 measure).

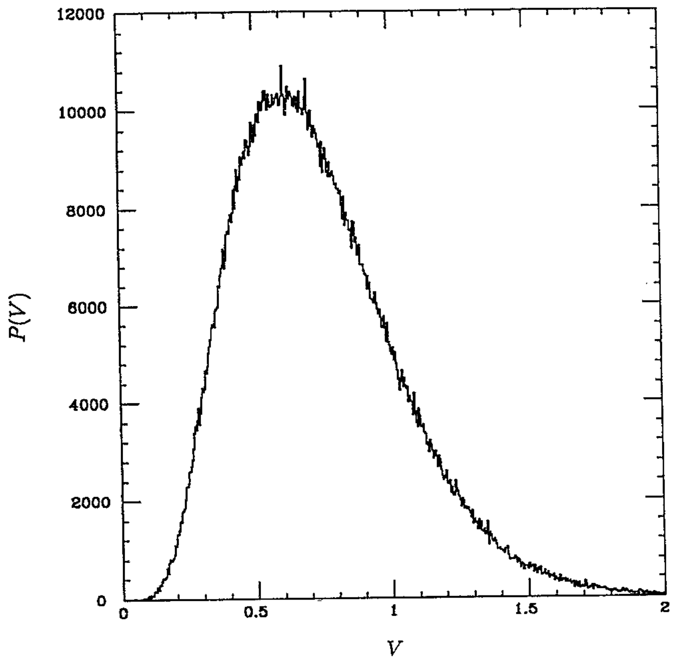

FIG. 4. Histogram of the distribution of local volumes $P(V_l)$ for the same parameters as in Fig. 3.

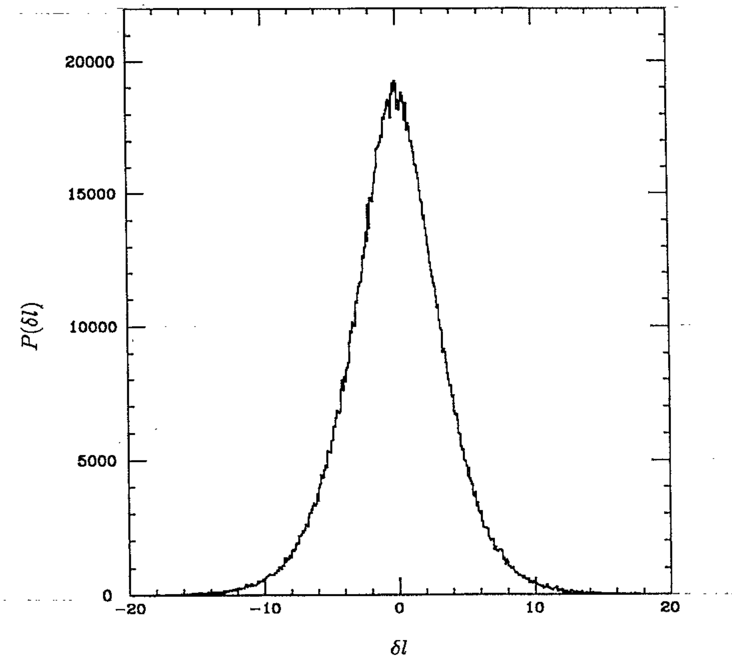

FIG. 5. Histogram of the distribution of local curvatures $P(l\delta_l)$ for the same parameters as in Fig. 3.

and the average curvature squares $\mathcal{R}^2$,

$$\mathcal{R}^2(\lambda,k,a)=\langle l^2\rangle^2\frac{\left\langle 4\sum_h \delta_h^2 l_h^2/V_h\right\rangle}{\left\langle \sum_h V_h\right\rangle}$$

$$\sim\frac{\left\langle \int\sqrt{g}\,R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right\rangle}{\left\langle\int\sqrt{g}\right\rangle}; \tag{4.2}$$

measured in units of the squared link length $\langle l^2\rangle$, and the sum over hinges h is here simply a sum over the edges in the simplicial lattice. In four dimensions it has been shown that for a sufficiently large higher-derivative coupling there is a continuous transition between a "smooth" (small negative average curvature) and a "rough" (very large positive average curvature) phase of space-time [12,13,17]. In three dimensions the method of analysis for the phase transition is rather similar. Besides $\mathcal{R}$ and $\mathcal{R}^2$, one can also estimate the lattice analogues of the fluctuations in the local curvatures,

$$\chi_{\mathcal{R}}(\lambda,k,a)=\frac{1}{\left\langle\sum_h V_h\right\rangle}\left[\left\langle\left[2\sum_h\delta_h l_h\right]^2\right\rangle\right.$$

$$\left.-\left\langle 2\sum_h \delta_h l_h\right\rangle^2\right], \tag{4.3}$$

and of the fluctuations in the local volumes,

$$\chi_V(\lambda,k,a)=\frac{1}{\left\langle\sum_h V_h\right\rangle}\left[\left\langle\left[\sum_h V_h\right]^2\right\rangle-\left\langle\sum_h V_h\right\rangle^2\right]. \tag{4.4}$$

These definitions are completely analogous to the ones previously used in four dimensions; a divergence in the fluctuation is indicative of long-range correlations (a massless particle), since the fluctuations are as usual related to the zero-momentum component of the propagator. (See also the discussion in Ref. [31].)

The derivative of the average curvature is related to the fluctuations in the curvature, since one has, from the definition of $\mathcal{R}$ in Eq. (4.1),

$$\frac{\partial\mathcal{R}}{\partial k}=\frac{2\langle\delta l\rangle}{\langle V\rangle}\frac{\partial\langle l^2\rangle}{\partial k}+\frac{2\langle l^2\rangle}{\langle V\rangle}\frac{\partial\langle\delta l\rangle}{\partial k}$$

$$-\frac{2\langle l^2\rangle\langle\delta l\rangle}{\langle V\rangle^2}\frac{\partial\langle V\rangle}{\partial k}. \tag{4.5}$$

Only the second term on the RHS is divergent as k approaches k_c from below, and this is due to a divergence in $\partial\langle\delta l\rangle/\partial k$ (the other derivatives remain quite small in comparison). Similarly, the fluctuations in the total average action

$$\langle I\rangle=\lambda\langle V\rangle-k\langle\delta l\rangle+a\langle\delta^2 l^2/V_l^2\rangle \tag{4.6}$$

are dominated by the fluctuations in the curvature as well.

It is useful at this point to recall some general features and predictions of the $2+\epsilon$ expansion of Einstein's gravity (see the penultimate reference in [24], and references therein). If one sets $k^{-1}=8\pi G$, then the physical dimensionful bare coupling is $G_0=\Lambda^{2-d}G$, where Λ is an ultraviolet cutoff (for example, of the order of the inverse average lattice spacing, $\Lambda\sim\pi/\langle l^2\rangle^{1/2}$), and G is a dimensionless bare coupling constant. Then the β function close to two dimensions has been computed to be

$$\beta(G) \equiv \frac{\partial G}{\partial \ln \Lambda} = \epsilon G - \frac{2}{3}(25-D)G^2 + O(G^3, G^2\epsilon, G\epsilon^2) , \tag{4.7}$$

where D is the number of massless scalar field (here $D=0$). The (nonuniversal) ultraviolet fixed point is, to lowest order, located at

$$G^* = \frac{3}{2}\frac{\epsilon}{25-D} + O(\epsilon^2) , \tag{4.8}$$

with the derivative of the β function given by

$$\beta'(G^*) = -\epsilon = -1/\nu . \tag{4.9}$$

Integrating close to the nontrivial fixed point one then gets

$$\mu_0 = \Lambda \exp\left[-\int^G \frac{dG'}{\beta(G')}\right] \underset{G \to G^*}{\sim} \Lambda |G-G^*|^{-1/\beta'(G^*)} \sim \Lambda |G-G^*|^{1/\epsilon} , \tag{4.10}$$

where μ_0 is an arbitrary integration constant with the dimension of a mass. Note that the coefficient of the one-loop term has disappeared from the final results (to this order), reflecting the fact that the leading term in the exponent depends only on $d=2+\epsilon$.

The lattice continuum limit then corresponds to $\Lambda \to \infty, G \to G^*$ with μ_0 held constant. It is not entirely clear what the mass scale μ_0 should correspond to in gravity; it presumably should be interpreted as a physical, renormalization-group invariant mass (or inverse length) scale. Some physical quantity such as the average curvature is then expected to be proportional to this fundamental mass scale, raised to some power determined by the mass scaling dimension (plus 2 for the average scalar curvature). The assumption of an algebraic singularity in the curvature (and its derivatives) close to the fixed point is then a natural one, at least from the point of view of the $2+\epsilon$ expansion.

The numerical results obtained for the average curvature $\mathcal{R}(k)$ are shown for the case $a=0.005$ in Fig. 6. Up to now two values of a, 0.0 and 0.005, have been studied, and $\lambda=1$ was held fixed (λ sets the overall scale in the action). The statistical errors in $\mathcal{R}(k)$ are estimated by the usual binning procedure and represent one standard deviation. As in four dimensions, one notices that as k is varied the curvature is negative for sufficiently small k, and appears to go to zero continuously at some finite value k_c. For $k \geq k_c$ the curvature is really infinite (or very large on a large lattice), and the simplices tend to collapse into degenerate configurations with very small volumes ($\langle V_h \rangle / \langle l^2 \rangle^2 \sim 0$) (this is the region of the usual weak-field expansion as $G \to 0$). For k close to, but less than, k_c (and $\lambda=1$) one can write

$$\mathcal{R}(k) \underset{k \to k_c}{\sim} A_{\mathcal{R}}(k_c-k)^\delta , \qquad \chi_{\mathcal{R}}(k) \underset{k \to k_c}{\sim} A_\chi (k_c-k)^{\delta-1} , \tag{4.11}$$

where δ is a universal exponent characteristic of the transition. After performing a simultaneous fit to $\mathcal{R}(k)$ in $A_{\mathcal{R}}$, k_c, and the exponent δ, and using the data close to k_c on the largest lattice available, one finds the results summarized in Table I (where we also include some results for the dl/l measure on a small lattice). Furthermore, one would expect that the results for $[-\mathcal{R}(k)]^{1/\delta}$ should lie close to a straight line. This appears indeed to be the case, as shown in Fig. 7. A similar analysis for the case of a higher-derivative coupling $a=0$ leads to a similar conclusions (see Figs. 8 and 9 and the results in Table I). A weighted average of all the lattice results then gives the preliminary estimate $\delta=0.80(6)$. Finally, in Fig. 10 the average curvatures are shown on a log-log plot. As can be seen from the graph, the assumption of a universal curvature critical exponent δ describing a universal type of phase transition is well supported by the results.

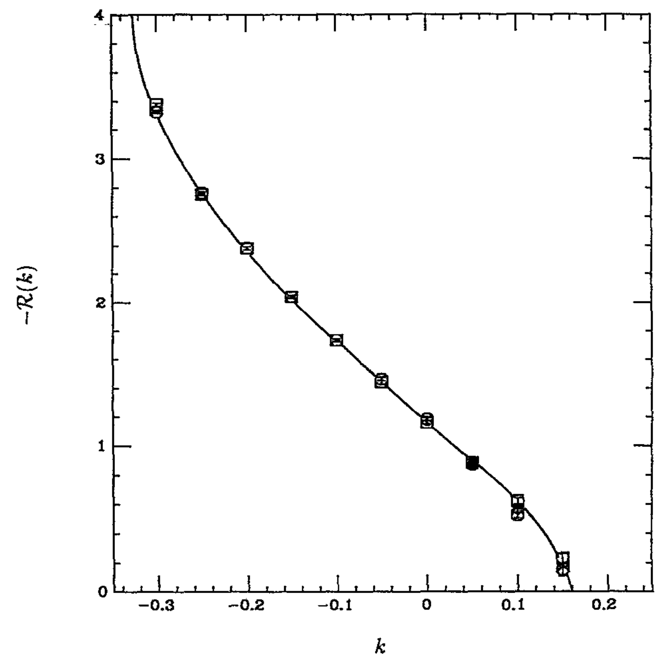

FIG. 6. Average curvature $\mathcal{R}$ as a function of k for $\lambda=1$ and $a=0.005$ (dl^2 measure). The squares refer to $L=4$, the circles to $L=8$, the diamonds to $L=16$, and the stars to $L=32$.

For different values of a the curvature vanishes along some line in the (k,a) plane, and for some small negative $a=a_0$ the ground state ceases to exist. This is not unexpected, since for sufficiently negative a the higher-derivative term can completely cancel some of the higher-order lattice corrections present in the Regge action (which in turn is only an approximation to the pure Einstein action for small curvatures). This phenomenon is already seen in the weak-field expansion discussed in the previous section, and it gives the correct sign and or-

TABLE I. Estimates, for different values of a, of the critical amplitude $A_{\mathcal{R}}$, the critical point k_c, and the critical exponent δ.

a	L	$A_{\mathcal{R}}$	k_c	δ	χ^2/N_{DF}
0.005	4–64	−5.10(16)	0.162(3)	0.80(2)	0.7
0	4–16	−14.1(12)	0.112(5)	0.78(9)	2.6
0.005 (dl/l)	4	−0.98(11)	0.512(9)	0.80(5)	4.8

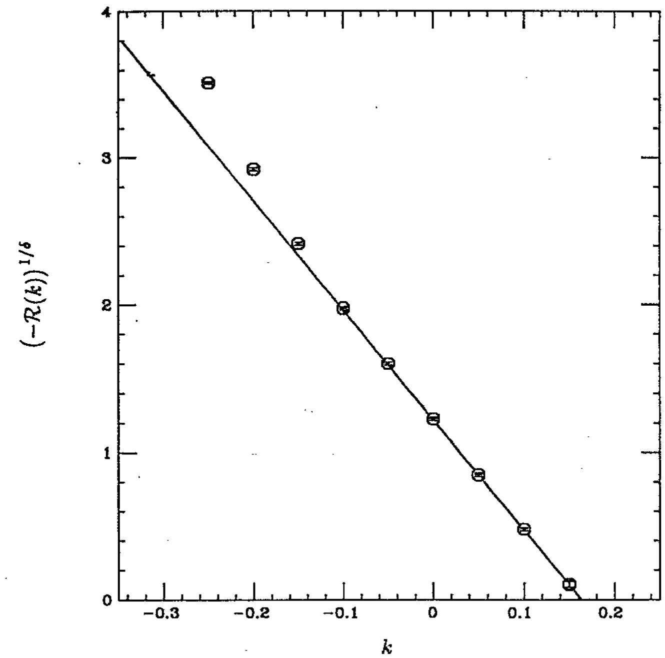

FIG. 7. Average curvature $\mathcal{R}$ raised to the power $1/\delta \sim 1/0.81$, using the data on the largest lattice available ($L = 8-32$); other parameters are the same as in Fig. 6. The computed points lie close to a straight line.

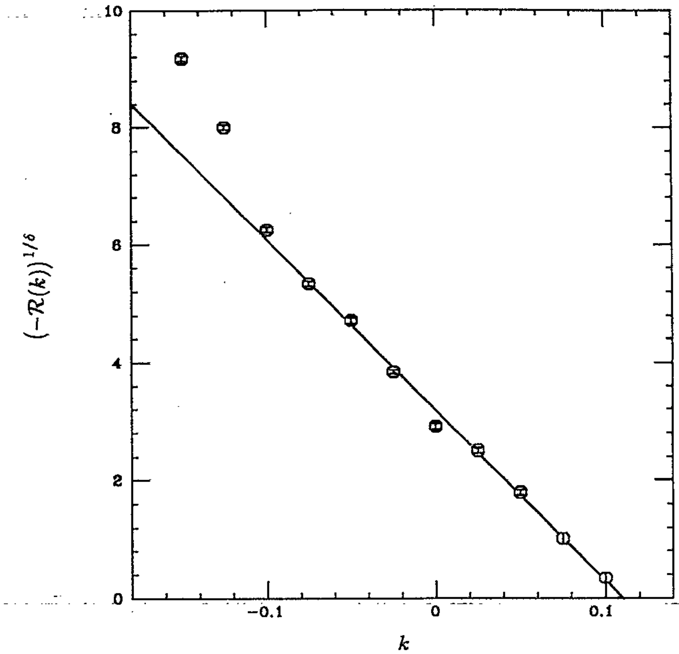

FIG. 9. Same as Fig. 7, but for $a = 0$.

der of magnitude of a_0 [the leading higher-order corrections $O(k^4)$ have small coefficients in the Regge-Einstein action, while the corresponding terms have a relatively large coefficient in the higher-derivative action, leading to a rough estimate $a_0 \sim -|k|/64$]. On the other hand, the higher-order lattice and radiative corrections to the pure Regge-Einstein action ($a = 0$) seem to stabilize the theory, at least for the dl^2 measure.

From the analysis of the curvature fluctuation $\chi_{\mathcal{R}}(k)$ one obtains similar values for δ and k_c, but with significantly larger errors. In Fig. 11 the curvature susceptibility is shown, again for $a = 0.005$. If, on the other hand, one computes the volume susceptibility χ_V (see Fig. 12), one finds that it approaches a finite value at k_c, suggesting the absence of critical volume fluctuations. This situation should be contrasted to the two-dimensional case, where the volume fluctuations (corresponding to the Liouville mode) are found to be massless, as expected from continuum arguments [15], and is some-

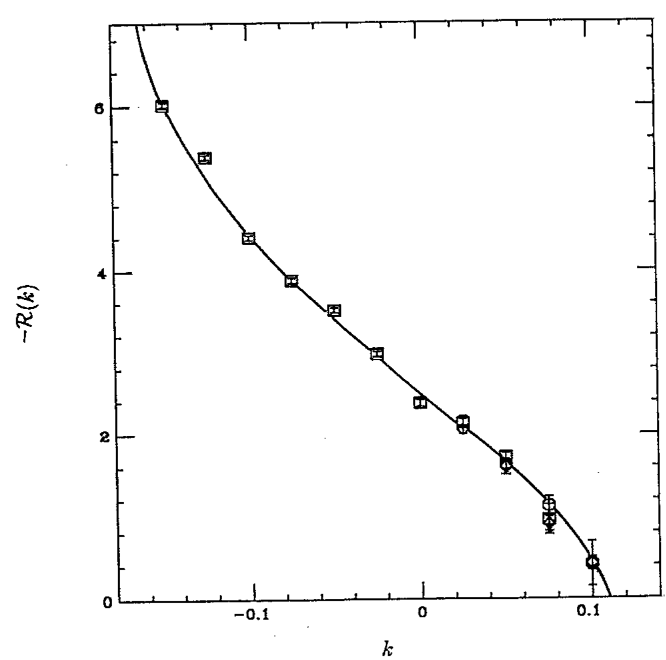

FIG. 8. Same as Fig. 6, but for $a = 0$.

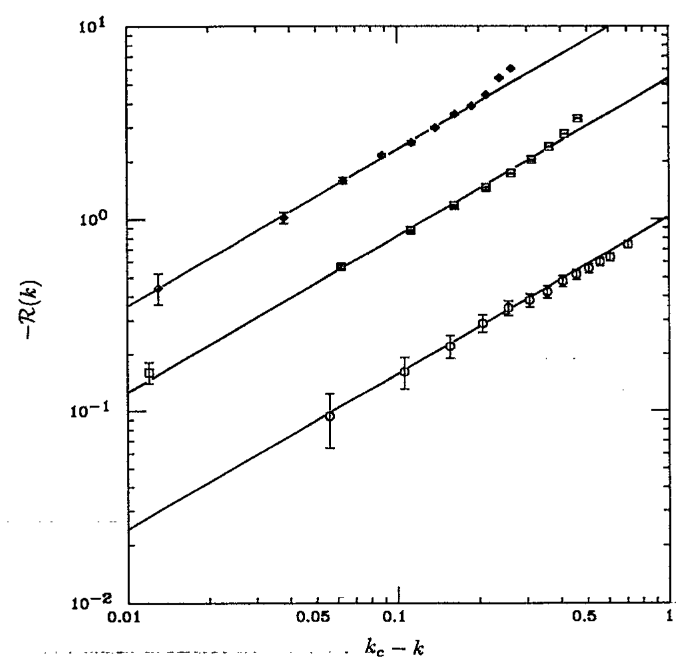

FIG. 10. Average curvature $\mathcal{R}$ as a function of $k_c - k$ on a log-log scale, for $\lambda = 1$ and $a = 0.005$ (circles) and $a = 0$ (squares) (dl^2 measure), and for $a = 0$ (dl/l measure) (diamonds).

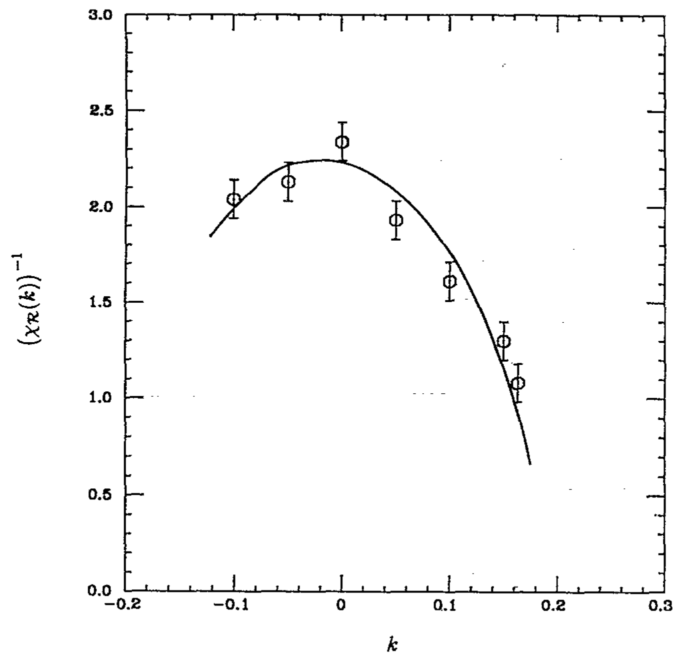

FIG. 11. Inverse of the curvature fluctuation $\chi_{\mathcal{R}}$ as a function of k for the same parameters as in Fig. 6. Note that the curvature fluctuations seem to diverge at the critical point.

what similar to the four-dimensional case, as discussed in Ref. [17]. A more careful analysis shows that the curvature fluctuation at the critical point grows with the size of the system, as expected from finite-size scaling arguments at a continuous phase transition:

$$\ln\chi_{\mathcal{R}}|_{k_c} \underset{L\to\infty}{\sim} c + \frac{\alpha}{\nu}\ln L \;, \tag{4.12}$$

with $\alpha/\nu = d(1-\delta)/(1+\delta)$. The estimate for δ obtained from finite-size scaling [0.81(5)] is in good agreement with the results quoted above from the fits to $\mathcal{R}(k)$. On the other hand, the finite-size scaling estimate fails to be more accurate than the fit results, since there is some curvature in the data of Fig. 13 (ideally one would hope to get a straight line; the curvature is presumably an indication of the presence of transients and corrections to scaling). Table II summarizes the results for the critical exponents obtained so far. Note that the estimates for the critical exponents rule out a first-order phase transition by several standard deviations. At a first-order phase transition one expects that the leading thermal exponent equals the dimensionality of space, and consequently one should have $\delta=0$, $\nu=\frac{1}{3}$, and $\alpha/\nu=3$. In Fig. 14 we show our impression for the phase diagram of 3D gravity, with a phase transition line separating the "smooth" from the "rough" phase. Figures 15 and 16 show our estimates for the critical point (for the dl^2 measure and $a=0$) and the exponent δ as a function of the dimension of space. The straight line in Fig. 16 is the results from the $2+\epsilon$ expansion (Kawai and Ninomiya [24]).

In general it is difficult to entirely exclude the presence of a first-order phase transition if it has a very small latent heat. Indeed, for $k\sim 0$ and $a\sim -a_0$ a sharp discontinuity in the average curvature develops in our model (it jumps from zero to infinity). Several scenarios are therefore possible. One possibility is that the average curvature is discontinuous only at that point; on a finite lattice one then should see some rounding off of a sharp transition, but for sufficiently large systems universal critical exponents should emerge away from the singular point.

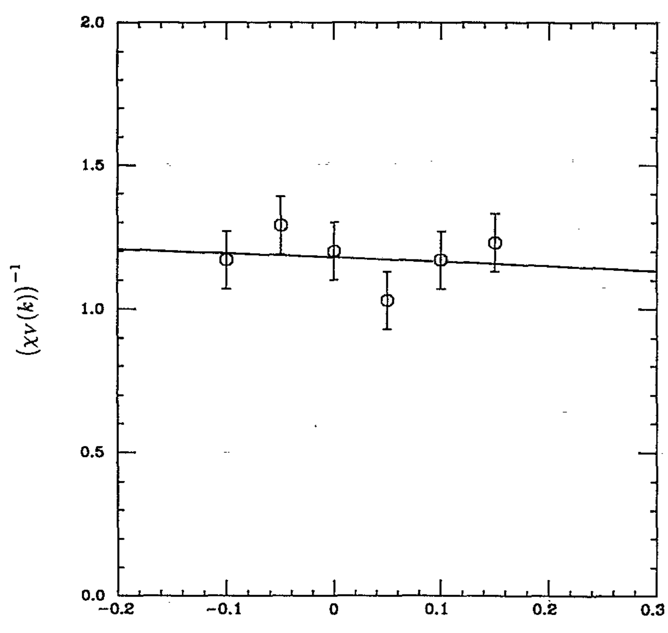

FIG. 12. Inverse of the volume fluctuation χ_V as a function of k for the same parameters as in Fig. 6. The volume fluctuations approach a constant at the critical point.

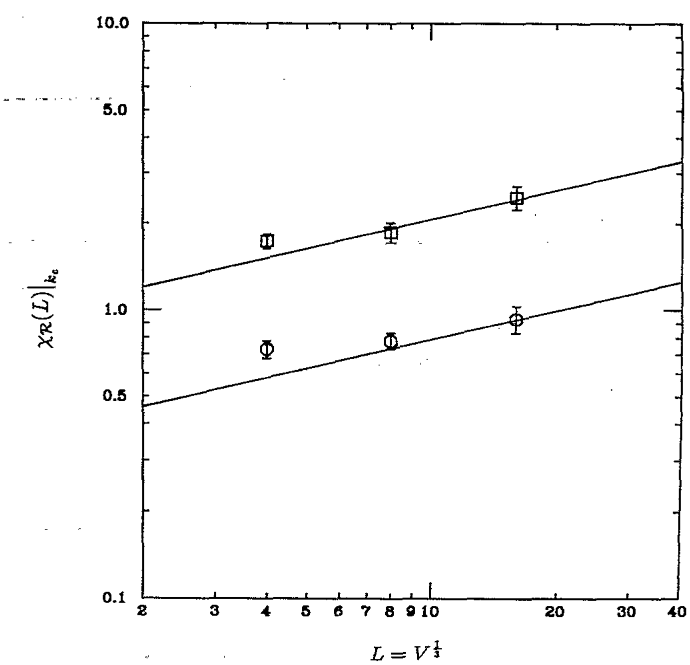

FIG. 13. Size dependence of the curvature fluctuation at the critical point $\chi_{\mathcal{R}}(k_c)$ for $a=0$ (upper curve) and $a=0.005$ (lower curve), both for the dl^2 measure. The lines indicate the expected slope corresponding to $\delta=0.81$, as obtained from the previous curvature fits (see Figs. 6 and 8).

TABLE II. Summary of the results for the critical exponents of the pure three-dimensional simplicial quantum gravity.

Method	Observable	a	Measure	δ	ν	α/ν
Fit	$\mathcal{R}(k)$	0.005	(dl^2)	0.80(2)	0.60(1)	0.33(5)
	$\mathcal{R}(k)$	0.0	(dl^2)	0.78(9)	0.59(3)	0.39(16)
	$\mathcal{R}(k)$	0.005	$(dl/l)(L=4)$	0.80(5)	0.60(2)	0.33(10)
Finite size	χ_L	0.005	(dl^2)	0.84(5)	0.61(2)	0.26(5)
	χ_L	0.0	(dl^2)	0.77(5)	0.59(2)	0.40(4)
Average				0.80(6)	0.60(2)	0.34(9)
1st order				0	$\frac{1}{3}$	3

A second possibility is that one finds some sort of multicritical behavior, with a line of first-order transitions (leading to no lattice continuum limit, since the correlation length is finite at the critical point) separated from a line of second-order transitions by a tricritical point, with universal tricritical exponents. The criterion of Ref. [32] then suggests that the leading thermal exponent should be proportional to the dimensionality of space-time (or $\nu=1/d$) along the fluctuation-induced first-order transition line. Furthermore, close to this critical line the curvature histogram should exhibit a double-peak structure (indicative of a two-phase coexistence at the critical point), and the relaxation times should grow faster than any power of the linear system size $[\tau\sim\exp(L^{d-1})]$ because of the barrier separating the two phases. Our results for now seem to indicate that the exponents are independent of a, and inconsistent with a first-order transition for the values we explored and for the chosen measure.

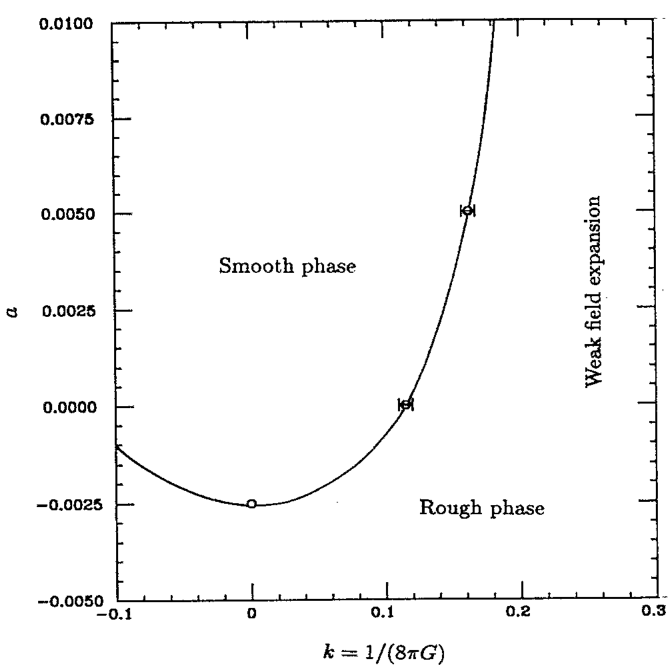

FIG. 14. Phase diagram for pure three-dimensional simplicial gravity with a higher-derivative coupling a (for fixed $\lambda=1$). The curve represents an estimate for the phase transition line $a(k_c)$, where the curvature fluctuations diverge, and which separates the "smooth" from the "rough" phase of gravity.

The results for the average curvature $\mathcal{R}$ are not inconsistent with known results within the weak-field expansion in the continuum (at least for small a). Substituting $k^{-1}=8\pi G_0$, where G_0 is the dimensionful bare Newton's constant, and setting $k_c=c\Lambda^{d-2}$, where c is a constant independent of k and Λ the ultraviolet cutoff (here of the order of the average inverse lattice spacing, $\sim\langle l^2\rangle^{-1/2}$), one obtains, from Eq. (4.11),

$$\begin{aligned}\mathcal{R}(G_0)&\sim A_{\mathcal{R}}\left[\frac{-1}{8\pi G_0}\right]^{\delta}(1-c\Lambda^{d-2}8\pi G_0)^{\delta}\\ &\sim A_{\mathcal{R}}\left[\frac{-1}{8\pi G_0}\right]^{\delta}\left[1+\delta c\Lambda^{d-2}(-8\pi G_0)+\frac{\delta(\delta-1)}{2}(c\Lambda^{d-2})^2\times(-8\pi G_0)^2+\cdots\right]. \quad (4.13)\end{aligned}$$

One can see that $\mathcal{R}(G_0)$ is possibly not analytic at $G_0=0$. Furthermore, an expansion in powers of G_0 involves (for $d>2$) increasingly higher powers of the ultraviolet cutoff Λ, as expected from a theory which is not perturbatively renormalizable in G_0 [24].

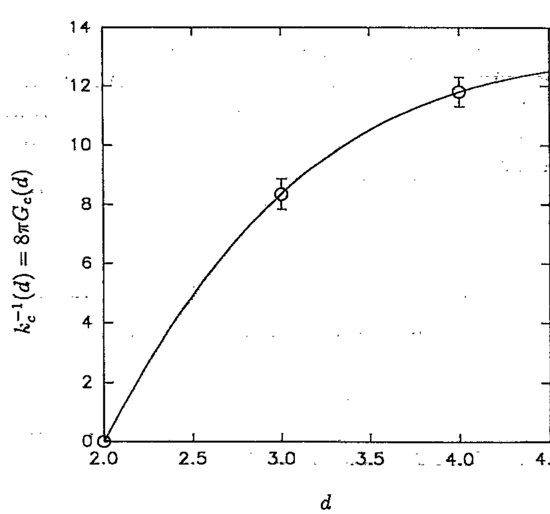

FIG. 15. Critical point $k_c=1/8\pi G_c$ as a function of the space-time dimension d for the pure Regge action ($a=0$) and for $\lambda=1$. The numerical lattice results for $d=3$ and $d=4$ are shown, together with the result $k_c^{-1}=0$ for $d=2$.

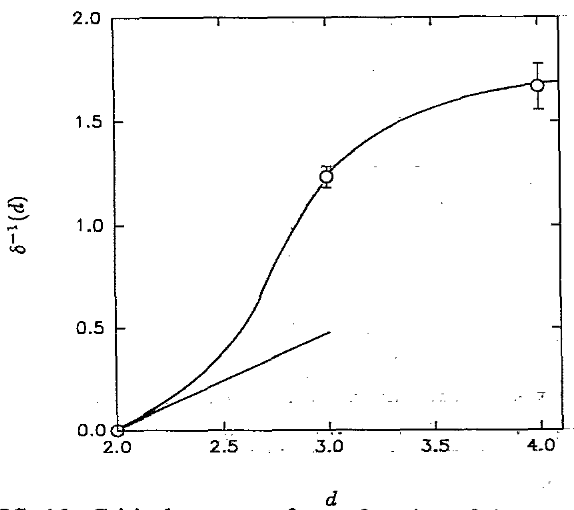

FIG. 16. Critical exponent δ as a function of the space-time dimension d. The numerical lattice results for $d=3$ and 4 are shown, together with the result in $2+\epsilon$ dimensions, $\delta \sim 2/\epsilon$.

If we call the dynamically generated mass m, then we have $m \sim (k_c - k)^{\nu} \sim \mathcal{R}^{\nu/\delta}$, with $\nu=(1+\delta)/d$, and with a calculable constant of proportionality close to k_c. Therefore, after restoring the correct dimensions for $\mathcal{R}$, which has dimensions of an inverse length squared, one obtains

$$\mathcal{R} \sim c \Lambda^{2-\delta/\nu} m^{\delta/\nu} , \tag{4.14}$$

where c is a dimensionless constant dependent on the higher-derivative coupling a. The dynamically generated mass can be calculated in principle from the edge or curvature correlation functions at fixed geodesic distance. Alternatively, it can be extracted from the physical size dependence of averages of local operators. For example, one expects in the presence of a mass gap

$$\mathcal{R}_L(k) - \mathcal{R}_\infty(k) \underset{L \gg 1/m(k)}{\sim} m(k)^{(d-3)/2} L^{(1-d)/2} e^{-m(k)L} , \tag{4.15}$$

where $L = V^{1/d}$ is the physical linear "size" of the system and $m(k)$ is a physical mass. The dimensionless ratio of mass squared to curvature is then given simply in terms of the average curvature and the ultraviolet cutoff, with an exponent related to δ and to the dimensionality of space-time:

$$\frac{m^2}{\mathcal{R}} \sim \left[\frac{\mathcal{R}}{\Lambda^2} \right]^{[2-(d-2)\delta]/d\delta} . \tag{4.16}$$

Close to the fixed point, the ultraviolet cutoff can be traded for a renormalized, effective Newton's constant $G_{\text{eff}} \sim \Lambda^{2-d}$ (which can be extracted, for example, from the renormalized propagators at fixed geodesic distance [17]).

Finally, let us mention that it is of some interest to explore correlations which are of a purely geometric nature. In our investigations we have found that some of the geometric properties of the discrete simplicial manifold appear to be close to being Euclidean in the smooth phase. As an example we have considered how the number of points within geodesic distances d and $d+\Delta d$ scales with the geodesic distance itself. This quantity is equivalent, up to a constant dependent on the average lattice spacing $\sqrt{\langle l^2 \rangle}$, to the physical extent of the "surface" within geodesic distances d and $d+\Delta d$. In practice, the geodesic distance between two arbitrary points on the simplicial lattice can be determined from a fixed configuration of edge lengths by selecting, among all the possible random walks between the two points which are less than some cutoff length ($\simeq 2L$), the one with the shortest length. A precise determination of the fractal dimension requires therefore the examination of a larger number of configurations, and an accurate unbiased determination of the asymptotic behavior of the above correlation, which is outside the scope of the present work. Still, in our model one finds, for the small distances considered in this work ($d \leq 8\sqrt{\langle l^2 \rangle} \simeq 16$), and therefore for the correspondingly few lattice spacings,

$$N(d) \underset{d \to \infty}{\sim} d^{d_v} , \tag{4.17}$$

with d_v finite and close to the flat-space value of 2 in the smooth phase ($k < k_c$), but closer to a value of 1 in the rough phase ($k > k_c$). In this latter phase the lattice tends to collapse into a degenerate configuration with thin elongated tetrahedra of small volume, just as one finds in four dimensions where a discontinuity in the local volumes at the critical point is also found [13,17].

In conclusion, the results are consistent with the picture of a vanishing curvature and a divergent curvature fluctuation at the same value of k_c, and with well-defined critical exponents. The presence of a continuous phase transition suggests the existence of a well-defined lattice continuum limit in the neighborhood of the critical point at k_c. For sufficiently large k, or for small k and sufficiently small (negative) a, the path integral ceases to exist. Since for positive or zero a the curvature approaches zero for some positive values of k, and approaches infinity for some negative value of k, and since the curvature amplitude $A_{\mathcal{R}}$ increases as a is decreased, one concludes that some rather sharp discontinuity in the curvature appears for sufficiently small negative a. Close to this point the average curvature must jump from zero (or a small finite value) to an infinite value. This suggests that one should be very careful in analyzing the results of a simulation close to this point, since fluctuations and finite-size effects will in general tend to round off sharp discontinuities.

In closing let us say a few words on the issue of unitarity in higher-derivative gravity (and in higher-derivative theories in general). Based on the (unstable) zeroth-order weak-field expansion it is often argued that the higher-derivative theories contain ghosts and/or tachyons. In our work we have made use of the higher-derivative terms as regulators, to ensure the existence of the Euclidean path integral. But the issue of unitarity in (lattice) gravity is a rather delicate one, since it involves the definition of a unitary time evolution operator as a function of physical time, again an essentially nonperturbative problem that can only be clarified through the knowledge of the nature of physical states. One can argue in general that for $a=0$ the lattice action and the local lattice measure are manifestly reflection positive, and are therefore expected to lead to a unitary continuum limit, if it exists. If the reflection positive theory and the extended higher-derivative one have a common well-defined continuum limit (characterized by critical exponents and scaling laws such as the ones we have indicated before), then this limit must be unitary. In our work we have given arguments that, at least in three dimensions, this seems to be the case.

V. CONCLUSION

In the preceding sections we have discussed results relevant for a model of simplicial quantum gravity based on Regge calculus. It is characteristic of the model that the variations in the geometry of space are described by fluctuating edge lengths on a lattice with fixed coordination number.

The weak-field expansion for the gravitational action around a regular cubic lattice divided into simplices shows the correspondence between lattice and continuum terms, and allows one to estimate the size of the lattice corrections that appear when the momenta are not too small. As in four dimensions, the computation clearly indicates the correspondence between the degrees of freedom on the lattice (the invariant edge lengths) and the continuum degrees of freedom (the metric field). In spite of the fact that the number of degrees of freedom on the lattice and in the continuum differ, the correct counting emerges for low momenta.

Contrary to two dimensions, and in analogy with four dimensions, in three dimensions the action leads to nontrivial effects due to the influence of the Regge-Einstein term. A higher-derivative term was included as a regulator, together with a cosmological-constant term. A transition between a "smooth" and a "rough" phase of pure gravity was found, as in four dimensions. The transition is a continuous one, if it is approached from the smooth phase (small k or large G) where the average curvature is negative. The critical exponents were estimated, and it appears that at the point where the average curvature vanishes, the curvature fluctuations diverge, leading to a well-defined lattice continuum limit. The results are very different from two dimensions (where fluctuations in the volume diverge instead) and resemble somewhat more the four-dimensional case.

In addition there is also a close similarity between the present results and results obtained with the dynamical triangulation models [21,22]. In these models as well, a phase transition separating a "smooth" from a "rough" phase of space-time is found, similar in nature to the one discussed in the present work. This is a very encouraging result, since it would suggest that the two discrete lattice models belong, as expected, to the same universality class, and therefore have the same lattice continuum limit. On the other hand, the phase transition is found to be perhaps of first order in the dynamical triangulation model, which could suggest that the two lattice models explore different regions of the same phase diagram, with possible multicritical behavior. The discreteness of the curvatures in the dynamical triangulation model (which are continuous instead in the Regge model) could be the reason for the appearance of a first-order freezing transition, at least in the simplest formulation of the model with no continuum limit. Of course, the presence of hysteresis effects by itself is not a reliable indicator of a first-order transition, and the critical exponents should be measured in order to find or exclude a discontinuity fixed point, as we have done here. It should be pointed out that often a continuous phase transition is found when the parameter space is extended, as we have done here by considering the effect of a higher-derivative term, and such a term was not included explicitly in the dynamical triangulation work (we have mentioned before that our work is suggestive of the presence of a complex phase diagram, possibly including a multicritical point). Thus, the fact that the transition is first order at one point in the phase diagram should not be taken yet as an indication that the model has no lattice continuum limit. The connection between the absence of gravitons in the continuum in three dimensions and the lack of a lattice continuum limit for the discrete models is not clear at this point, since the continuum considerations do not suggest that the limit does not exist, but rather that it is trivial, at least in perturbation theory.

Many questions have remained open. It would be of interest to investigate further how the results depend on the coupling a and the choice of measure, and to complete the picture for the phase diagram for pure gravity. But universality of the lattice continuum limit would suggest that the results for exponents and other infrared-sensitive quantities should not be affected. It would also be of interest to investigate these questions in the presence of matter fields, as well as for surfaces with boundaries.

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