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SIMPLICIAL QUANTUM GRAVITY FROM TWO TO FOUR DIMENSIONS*

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Recent work on Regge's lattice formulation of quantum gravity is reviewed. The problem of the lattice transcription of the action and the measure is discussed, and a comparison is made to the expected results in the continuum. The recovery of general coordinate invariance in the continuum is illustrated in the two-dimensional case, where critical exponents can be compared to the exact continuum conformal field theory results of KPZ. In four dimensions the lattice results strongly suggest that the pure Einstein theory is not defined even at the non-perturbative level. The addition of higher derivative terms in the pure gravity theory appears to cure the unboundedness problem, but the nature of the ground state and the fixed point structure remains an open question.

INTRODUCTION

While classical general relativity is considered as a rather solid theory, the same is not true for the quantum theory, for which there is no clear prescription as to how one should proceed from the classical theory. It has been known for some time that if one attempts to quantize the Einstein theory of gravity one encounters two major difficulties. The field equations for the metric are derived from an action that is unbounded from below, and the path integral is therefore mathematically ill-defined[1]. Furthermore the coupling constant in Einstein gravity (Newton's constant) has dimension of inverse mass squared (in the usual units $h = c = 1$), and this leads to a non-renormalizable quantum theory, as can be verified by doing explicit Feynman diagram perturbation theory[2,3].

One might hope that some of these problems will be resolved in the framework of some grand unified theory which includes gravity. Even if this were to be the case, a non-perturbative formulation of quantum gravity could present some conceptual and computational advantages, just like lattice gauge theories provide a rigorous mathematical basis for the continuum theory, and at the same time allow one to use non-perturbative methods like mean field, strong coupling expansions and numerical simulations. As far as a comparison of quantum gravity predictions with the real

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world is concerned, it is clear that such a comparison will be rather difficult except possibly for some cosmological implication, especially those related to the (almost) vanishing of the renormalized cosmological constants and the appearance of close to flat space-time on macroscopic scales.

Here we will concentrate on the simplicial formulation of quantum gravity[4−16], but one should mention that there are other possible approaches such as the hypercubic lattice formulation[17,18] à la Wilson, and the random triangulation approach for two-dimensional surfaces[19]. In addition to problems common to all formulations of lattice gravity (like the problem of transcribing the continuum action, establishing the invariance group of the lattice theory and the invariant lattice functional measure), the hypercubic lattice formulation suffers from the problem of graviton doubling, which might or might not disappear in the full quantum theory, depending on the nature of the phase diagram and possible non-trivial fixed points. The random triangulation method on the other hand is up to now limited to two dimensions, and it is at first not clear how, even in principle, one could obtain continuous deformations of the metric corresponding to graviton-like excitations, given that there are no continuous deformations of a randomly triangulated lattice even in flat space (in particular the curvature at a vertex can only take discrete values, determined by the local coordination number). In two dimensions this fact is not of great concern since pure gravity corresponds to a theory with minus one degree of freedom (there are more constraints than degrees of freedom to start with) and there is really no physical graviton excitation. In four dimensions the situation is of course quite different, where the quantized fluctuation in the metric are expected to give rise to a physical massless spin two particle. On the other hand the results obtained in the discrete two-dimensional models of random surfaces are encouraging since they seem to indicate a restoration of general coordinate invariance at the quantum level, since some critical exponents agree with the results of conformal field theory.

THE LATTICE ACTION

A rigorous mathematical basis for the Minkowski path integral is usually provided by the euclidean approach, and it seems sensible to proceed along the same lines in the case of quantized gravity. Consider the euclidean Einstein action without a cosmological constant term

$$I_E = -\frac{1}{16\pi G} \int d^4x \sqrt{g}\, R. \tag{1.1}$$

where G is Newton's constant, $\sqrt{g}$ is the determinant of the metric $g_{\mu\nu}$, and R is the scalar curvature. For most of the following we will not consider boundary terms and couplings to matter fields, although their inclusion is straightforward, except possibly for fermions fields. If one attempts to write down a path integral of the form

$$Z = \int_\Gamma d\mu[g]\; e^{-I_E[g]} \tag{1.2}$$

(which will in general depend on a specified boundary three-geometry, here denoted by Γ) one soon realizes that it appears ill defined due to the fact that the scalar curvature can become arbitrarily positive (or negative). This in turn is a consequence of the fact that while gravitational radiation has positive energy, gravitational potential energy is negative because gravity is attractive. Thus the gravitational action is unbounded from below and the functional integral strongly depends on how the unboundedness is cut off. This is clearly seen by considering a conformal transformation on the metric $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ where Ω is a positive function[1]. Then the pure Einstein action transforms into

$$I_E(\tilde{g}) = -\frac{1}{16\pi G} \int d^4x \sqrt{g}\, (\Omega^2 R + 6\, g^{\mu\nu} \partial_\mu \Omega \partial_\nu \Omega). \tag{1.3}$$

which can be made arbitrarily negative by choosing a rapidly varying conformal factor Ω: the kinetic term for the conformal mode has the wrong sign. Unless other operators are added, it is usually quite difficult to make sense of contributions of this type, at least in ordinary euclidean field theory. A possible solution to the unboundedness problem has been described by Hawking[1], who suggests performing the integration over all metrics by first integrating over complex conformal factors, followed by an integration over conformal equivalence classes of metrics. A second possibility, to be further discussed below, is to add to the Einstein action extra terms, including higher derivative ones like R^2, in a carefully chosen combination which makes the total euclidean action bounded from below[20,21].

A second serious problem of the pure Einstein action is connected to the fact that the coupling constant G^{-1} has dimension of mass to the power $(d-2)$ and suggests that the theory is not perturbatively renormalizable above two dimensions (even though it can perhaps be defined perturbatively in $2+\epsilon$ dimensions[21]). In order to renormalize the theory close to four dimensions one needs at one loop to introduce higher derivative counterterms, which are needed to cancel the divergences proportional (in dimensional regularization) to[2]

$$\Delta I = \frac{1}{8\pi^2 (d-4)} \int d^4x \sqrt{g}\, \left[\frac{1}{240} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \frac{1}{80} R^2 \right] \tag{1.4}$$

It has been argued that the theory could still make sense non-perturbatively (as in the case of other non-renormalizable theories like the Gross-Neveu model above two dimensions), but this would require a more sophisticated calculational scheme that provides for some kind of resummation of the perturbative series. As far as the lattice approach is concerned, there is no indication yet that this is the case, and we shall return to this point later.

On the other hand it can be shown, at least in perturbation theory, that only up to fourth derivative terms need to be considered in order to cure the renormalizabilty problem[22], and then the unboundedness problem is resolved as well. Thus one is led to consider the extended higher derivative gravitational action

$$I = \int d^4x \, \sqrt{g} \, \left[\lambda - k\, R + b\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \frac{1}{2}(a - 4b)\, C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right] \tag{1.5}$$

with a cosmological constant term (proportional to λ), the Einstein term ($k = 1/16\pi G$, where G is the bare Newton constant), and two higher derivative terms with additional dimensionless coupling constants a^{-1} and b^{-1}. (Even though in four dimensions there are four possible higher derivative terms which do not give rise to topological invariants, only two are independent for a manifold of fixed topology, if one uses the identities relating the Riemann tensor to the integral expression for the Euler characteristic[23]). Remarkably the resulting theory is also asymptotically free[24].

The attractive features of higher derivative gravity are the solution of the unboundedness problem and renormalizability, but a less attractive feature is the lack of perturbative unitarity. In the weak field approximation it is known that the graviton propagator contains ghosts. In momentum space and for $\lambda = 0$ it has the form

$$\frac{1}{k} < h_{\mu\nu}(q) h_{\rho\sigma}(-q) > = \frac{2P^{(2)}_{\mu\nu\rho\sigma}}{q^2 + \frac{a}{k} q^4} + \frac{P^{(0)}_{\mu\nu\rho\sigma}}{-q^2 + \frac{2b}{k} q^4} + gauge\ terms \tag{1.6}$$

and clearly the higher derivative terms improve the ultraviolet behavior of the theory since the propagator now falls of as $1/q^4$ for large q^2. Understanding what might happen to the ghosts beyond perturbation theory is a rather difficult question, but it has been suggested that the ghosts are 'confined' in the sense that they might not contribute to physical amplitudes if the latter evaluated in the full theory. Alternatively the higher derivative terms can be regarded as a specific form of a non-perturbative regulator.

In the continuum the fundamental degrees of freedom are represented by the metric $g_{\mu\nu}$. On the simplicial lattice the corresponding quantities are the lengths of all the edges, as well as the incidence matrix specifying the overall triangulation of the manifold. In piecewise linear spaces [25] the elementary building blocks for d-dimensional space-time are simplices of dimension d, where a d-simplex is a d-dimensional object with $d+1$ vertices and $d(d+1)/2$ edges connecting them. It has the important property that the lengths of its edges specify the shape (and therefore the relative angles) uniquely. A simplicial complex can be viewed as a set of simplices glued together to each other, in such a way that either two simplices are disjoint or they touch at a common face. The relative position of points on the lattice is thus completely specified by the incidence matrix (it tells which point is next to which) and the edge lengths, and this in turn induces a metric structure on the piecewise linear space. The polyhedron constituting the union of all the simplices of dimension d forms then the geometrical complex or skeleton. In order to obtain non-degenerate simplicial complexes, the edge lengths have to obey triangle inequalities and their higher dimensional analogues, which ensure that for example the triangle areas are positive.

General coordinate transformations correspond then (at least approximately) to variations of the edge lengths, as well as appropriate modifications of the incidence matrix. But since in general different complexes will correspond to physically distinct manifolds, one expects general coordinate invariance to be recovered only in the continuum limit, where a continuous smooth manifold can be covered by many different almost geometrically equivalent triangulations. (In the special case of flat space it is clear that there is an infinite number of triangulations, even for a fixed incidence matrix, which correspond to the same continuous manifold.)

Since a detailed description of the construction of the action for higher derivative lattice gravity can be found in the original papers[10,12–13], only a summary will be given here. The simplicial lattice transcription of the Einstein action was given some time ago by Regge[4].

$$I_R = \sum_{hinges\ h} A_h^{(d-2)} \, \delta_h \tag{1.7}$$

where $A_h^{(d-2)}$ is the volume of the 'hinge' and δ_h is the deficit angle at the same 'hinge'. The 'hinges' are points in two, edges in three and triangles in four dimensions, respectively. Regge's action corresponds to the simplicial decomposition of

$$I_E = \frac{1}{2} \int d^d x \sqrt{g} \, R \tag{1.8}$$

and indeed it has been shown that I_R tends to the continuum expression as the mesh size tends to zero[5,8,9]. In two dimension the discrete analogue of the Gauss-Bonnet theorem holds

$$I_R = \sum_h \delta_h = 2\pi\chi \tag{1.9}$$

where χ is the Euler characteristic (two minus twice the number of handles of the surface). This remarkable identity ensures that two-dimensional lattice R gravity is as 'trivial' as the continuum theory, in the sense that the action is a constant for a manifold of fixed topology.

The guiding principle in constructing physical quantities in simplicial gravity is that they should have geometric significance. This will distinguish objects which are lattice structure independent for a given physical manifold (at least for sufficiently smooth manifolds in some continuum limit) from other functions of the edge lengths which have no particular geometric meaning, and whose limiting values will therefore depend on the specific way in which the triangulation is refined. The Euler characteristic in two dimensions, expressed as a function of the edge lengths, is a clear and illustrative example of what is meant by this statement. Another clear example is the total area of the simplicial complex: if it is defined as the sum of the triangle areas (where these are very specific functions of the edge lengths), then as the triangulation is refined its limit is well defined, and agrees with the continuum definition of what is meant by the total area.

The higher derivative terms for pure gravity can be written down once one recognizes that the deficit angle δ_h is related to the components of the Riemann tensor through[12].

$$R^{(h)}_{\mu\nu\rho\sigma} = \frac{\delta_h}{A_{\Gamma_h}} \; U^{(h)}_{\mu\nu} U^{(h)}_{\rho\sigma} \tag{1.10}$$

where A_{Γ_h} can be taken to be the dual area associated with the hinge, and $U^{(h)}_{\mu\nu}$ is a bivector orthogonal to the hinge h, defined in four dimensions by

$$U^{(h)}_{\mu\nu} = \frac{1}{2A_h} \; \epsilon_{\mu\nu\rho\sigma} \; l^{\rho}_{(a)} l^{\sigma}_{(b)} \tag{1.11}$$

and $l^{\rho}_{(a)}$ and $l^{\rho}_{(b)}$ are the vectors forming two sides of the hinge h. Then the discrete analog of the higher derivative action was written[12] as

$$I = \sum_{hinges\ h} \left[\lambda \, V_h - 2k \; \delta_h A_h + 4b \; \frac{A_h^2 \delta_h^2}{V_h} \right]$$

$$+\frac{1}{3}(a-4b) \sum_{sites\ p} V_p \sum_{hinges\ h,h' \supset p} \epsilon_{h,h'} \; \left[\omega_h \frac{A_h \delta_h}{V_h} - \omega_{h'} \frac{A_{h'} \delta_{h'}}{V_{h'}} \right]^2 \tag{1.12}$$

The numerical factor $\epsilon_{h,h'}$ is equal to 1 if the two hinges h, h' have one edge in common and -2 if they do not. The last term introduces a short range coupling between deficit angles and has the remarkable property that it requires neighboring deficit angles to have similar values, but it does require them to be small. The convergence of the higher derivative terms to the continuum values was considered for the regular tessellations of the two-, three- and four-sphere[10]. In addition there are some results for the weak field limit in two dimensions[12], and for arbitrarily fine tessellations of the two-sphere[16].

In four dimensions the (classical) continuum limit is taken by requiring that the local curvature be small on the scale of the local lattice spacing, which is equivalent to imposing

$$\left| \frac{A_h \delta_h}{V_h} \right| \ll \frac{1}{\sqrt{V_h}} \tag{1.13}$$

and implies

$$\frac{A_h^2 \delta_h^2}{V_h} \ll 1 \tag{1.14}$$

This condition can be met by having the coefficient of the curvature squared terms large. Otherwise the results are expected to depend strongly on the detailed structure of the ultraviolet cutoff (i.e. choice of lattice structure and lattice transcription of the continuum action).

Matter fields can also be introduced in a straightforward way. We will mention here one of the possible lattice expressions for the free scalar field action[12,13]. Other forms have also been suggested[15,34]. Define the scalar fields at the vertices of a simplex, and consider the following expression

$$\sum_{simplices\ s} V_s \ [\ g^{ij}(s) (V_s)^{2/d}\ ]\ \Delta_i \phi_s\ \Delta_j \phi_s \tag{1.15}$$

with the finite lattice differences defined as usual by

$$\Delta_i \phi_s = \frac{\phi_{s+i} - \phi_s}{l_{s,s+i}} \tag{1.16}$$

The index i labels the possible directions in which one can move from a point in a given simplex, and $l_{s,s+i}$ is the length of the edge connecting the two points. The induced metric $g_{ij}(s)$ within a simplex s can be taken to be

$$g_{ij}(s) = \frac{1}{2}\ [\ l^2_{s,s+i} + l^2_{s,s+j} - l^2_{s+i,s+j}\ ] \tag{1.17}$$

Then the above lattice action corresponds, up to a constant of proportionality, to the continuum expression

$$I_{sf} = \frac{1}{2} \int d^d x \sqrt{g}\ g^{\mu\nu}\ \partial_\mu \phi\ \partial_\nu \phi \tag{1.18}$$

Alternatively one could consider the expression suggested by the random lattice approach[34], and which should be equivalent to the previous lattice action in the continuum limit,

$$\frac{1}{2} \sum_{edges\ ij} V_{ij} \left(\frac{\phi_i - \phi_j}{l_{ij}} \right)^2 \tag{1.19}$$

where l_{ij} is the length of the edge connecting site i to site j, and V_{ij} is the volume associated with it, via a dual, baricentric or other reasonable geometric subdivision. Using the baricentric subdivision, one has

$$V_{ij} = \sum_{simplices\ s \supset ij} \frac{2}{d(d+1)} V_s \tag{1.20}$$

The situation for fermions is somewhat more involved, since one has to introduce frames and vierbeins in each simplex in order to define the spin connection.

THE LATTICE MEASURE

The form of the measure for the $g_{\mu\nu}$ fields in continuum gravity is not well understood. One popular measure is the so-called Misner measure[26].

$$d\mu[g] = \prod_x g^{-(d+1)/2} \prod_{\mu \geq \nu} dg_{\mu\nu} \tag{2.1}$$

which has the important property of being scale invariant. The prefactors proportional to the determinant of the metric to some power may be quite important, since it has been argued that the measure can play a delicate role in canceling spurious divergences in loop diagrams, which arise when a continuous local symmetry (here general coordinate invariance) is explicitly broken. The above measure is unique if the product in (2.1) is interpreted over 'physical' points, and invariance is imposed at one and the same 'physical' point. On the other hand if one introduces a super-metric over metric deformations, then another measure arises naturally for pure gravity[27]. Consider the simplest (local) form for the norm squared of the metric deformation

$$\begin{aligned} \|\delta g\|^2 &= \frac{1}{2} \int d^d x \sqrt{g} \left[g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} + \lambda g^{\mu\nu} g^{\alpha\beta} \right] \delta g_{\mu\nu} \delta g_{\alpha\beta} \quad (\lambda \neq -\frac{2}{d}) \\ &\equiv \int d^d x \sqrt{g}\; G^{\mu\nu,\alpha\beta}\; \delta g_{\mu\nu} \delta g_{\alpha\beta} \end{aligned} \tag{2.2}$$

Then according to De Dewitt the functional measure is given by

$$d\mu[g] = \prod_x \left[\det G^{\mu\nu,\alpha\beta} \right]^{1/2} \prod_{\mu \geq \nu} dg_{\mu\nu} \tag{2.3}$$

with the determinant of the super-metric G given by

$$\left[\, \det G^{\mu\nu,\alpha\beta} \, \right]^{1/2} = (-1)^{d-1} (1 + \frac{d\lambda}{2}) \; g^{(d-4)(d+1)/8} \tag{2.4}$$

Note that this measure is clearly quite different from the measure of eq. (2.1). Specializing to the case of four dimensions one then obtains the particularly simple result

$$d\mu[g] = \prod_x \prod_{\mu \geq \nu} dg_{\mu\nu} \tag{2.5}$$

If matter fields are present, then the gravitational measure has to be further modified. Other forms for the measure for the pure gravitational field have also been suggested[28,29].

On the simplicial lattice one can argue that the edge lengths, being invariant quantities, are not referred to any specific coordinate systems. On the other hand they provide for an explicit coordinatization of the manifold, once the incidence matrix is specified as well. It is clear from looking at the example of flat space that there can be an infinite number of edge length assignments that correspond to the same physical manifold. Therefore in the continuum limit the edge length cannot really be considered as invariants under some (approximate) lattice diffeomorphism group. This situation is illustrated in Fig. 1.

Since the continuum theory provides limited guidance as to the form of the lattice gravitational measure, and since the lattice theory has no obvious exact invariance (except for flat space and a few other special cases), one has to rely more on

Figure 1

concepts of analogy and simplicity. On the simplicial lattice the edge lengths are the elementary degrees of freedom which uniquely specify the geometry, and over which it would seem that one should integrate over. From the relationship between edge lengths and metric in a simplex (eq. (1.17)) one notices that each edge is shared between several contiguous simplices, and that an integration over the edges is not simply related to an integration over the metric (even though there are $d(d+1)/2$ edges for each simplex just as there are $d(d+1)/2$ independent components for the metric tensor in d dimensions). Thus in ref. [12] the measure

$$\int d\mu_\epsilon[l] = \prod_{edges\ ij} \int_0^\infty \frac{dl_{ij}^2}{l_{ij}^2} \, F_\epsilon[l] \tag{2.6}$$

was suggested, where $F_\epsilon[l]$ is a function of the edge lengths with the property that it is equal to one whenever the triangle inequalities and their higher dimensional analogues for the simplicial complex are satisfied, and zero otherwise (the inequalities ensure that the edge lengths, triangle areas, tetrahedron and four-simplex volumes are all positive). The positive real parameter ϵ can be introduced as an ultraviolet cutoff at small edge lengths: the function $F_\epsilon[l]$ is zero if any of the edges is equal or less than ϵ; in the following we will take $\epsilon = 0$. We notice that this measure is clearly correct in the weak field limit, where all continuum measures also agree. The same measure was also used in the work of ref. [14].

Of course the measure suggested above is not unique, but is certainly a rather attractive one, since it is local and scale invariant as the continuum measure of eq. (2.1), and integrates directly over the elementary lattice degrees of freedom, the edge scale factors $\phi_{ij} = \ln(l_{ij}/l_0)$. Other measures one might consider would involve an integration over edge lengths divided by some volume to the appropriate power, such that the total measure is perhaps again scale invariant. However there are several volumes that are touching a given edge, and the measure then becomes rather complicated, involving some odd powers of volumes in the denominator[12,13]. One simple alternative possibility is to consider the 'volume associated with an edge'

V_{ij}, and write

$$\int d\mu_\epsilon[l] = \prod_{edges\ ij} \int_0^\infty \; V_{ij}^\sigma \, dl_{ij}^2 \; F_\epsilon[l] \tag{2.7}$$

with $\sigma = -2/d$ for the lattice analog of the Misner measure, and $\sigma = (d-4)/2d$ for the De Dewitt measure.

Ambiguities associated with the measure in the continuum can be traced back to a lack of a rigorous definition of the path integral for quantum gravity, and in particular to the difficulties associated with defining what is meant by singular objects such as derivatives of delta functions and similar distributions[27]. An interesting point of view on the relevance of the specific form of the measure in the quantum theory is presented in ref. [3]. There it is argued that ambiguities in the definition of the measure reflect the lack of a unique definition for the metric tensor at short distances. Eventually the hope is that different measures, within a certain universality class, will give the same results for infrared sensitive quantities, like correlation functions at large distances and critical exponents. These concepts can presumably be tested, at least in the framework of two-dimensional gravity, where some exact results are known in the continuum from conformal field theory.

In two dimensions a measure for gravity has been given by Polyakov[30,31], following the De Dewitt approach. In pure two-dimensional gravity one can write

$$\int d\mu[g] \; e^{-I_G} = \int d\mu[\tilde g] \; \Delta_{FP}[\tilde g] \int [d\phi] \; e^{-I_G - 26 I_L} \tag{2.8}$$

with the Liouville action contribution I_L arising from the conformal anomaly

$$I_L = \frac{1}{96\pi} \int d^2x \sqrt{\tilde g} \; (\tilde g^{\mu\nu} \partial_\mu \phi \; \partial_\nu \phi + 2 \tilde R \; \phi) \tag{2.9}$$

with $g_{\mu\nu}(x) = \tilde g_{\mu\nu}(x) e^{\phi(x)}$. On the lattice the conformal factors correspond to local volume (=area) fluctuations $e^{\phi(x)} \approx V(x)/V_0$, but an identity of the type

$$\int \prod_{ij} \frac{dl_{ij}^2}{l_{ij}^2} \; \underset{long\ wavelengths}{\sim}$$

$$\int \prod_s \frac{dV_s}{V_s} e^{-\frac{26}{96\pi} (\sum_{s,s'} V_{s,s'} [\, \ln (V_s / V_{s'}]^2 + \sum_s 2 R_s \ln V_s)} \prod [d \; moduli] \prod [d \; diff.] \tag{2.10}$$

appears not too easy to prove analytically, because of the rather complicated dependence of the triangle areas on the various edge lengths. An equivalent way to prove the identity is by computing a critical exponent, which will be sensitive to the restoration of general coordinate invariance at large distances.

GRAVITY IN TWO DIMENSIONS

Two-dimensional quantum gravity presents an ideal laboratory for testing the approach described in the previous sections, since some exact results are known[31,32]. Here we will consider the higher derivative lattice action

$$I = \sum_h \left[\lambda \; A_h - 2k \; \delta_h + 4a \; \frac{\delta_h^2}{A_h} \right] \tag{3.1}$$

(for the specific formulation of higher derivative terms baricentric lattice volumes[10] will be assumed in the following). In the limit of small fluctuations around a smooth background, this lattice action can be shown to correspond to the continuum action

$$I = \int d^2 x \sqrt{g} \left[\lambda - k R + a R^2 \right] \tag{3.2}$$

In two space-time dimensions the Einstein action is a topological invariant, both in the continuum (because of the Gauss-Bonnet theorem) and on the lattice, since $\sum_h \delta_h = 2\pi\chi$, where χ is the Euler characteristic. Therefore for a manifold of fixed topology the term proportional to k can be dropped. The higher derivative term is necessary to control the fluctuations in the local curvature, and its presence is implicit in the approach of ref. [19] as well. Furthermore, in two dimensions there is only one independent higher derivative term, so the R^2 term which we have written down is the only possible term of dimension four.

Consider now the path integral

$$Z\ [\lambda, a, b, \epsilon] = \int d\mu[\, l\,]\ e^{-I[\, l\,]} \tag{3.3}$$

Because of the scale invariance of the measure, all the edge lengths can be rescaled $l_i \to (a/\lambda)^{1/4}\ l_i$, and one obtains

$$Z\ [\lambda, a] = Z\ [\sqrt{a\lambda}, \sqrt{a\lambda}] \tag{3.4}$$

The following expectation values are of interest, determining the average area and average curvature squared, respectively

$$\begin{aligned} < A > &= \frac{1}{N_h} < \sum_h A_h > \\ < R^2 > &= \frac{4}{N_h} < \sum_h \frac{\delta_h^2}{A_h} > \end{aligned} \tag{3.5}$$

where $N_h = N^2$ is the number of hinges. From the scale invariance of the measure one then obtains the exact identity

$$\frac{1}{4} \frac{< R^2 >}{< A >} = \frac{\lambda}{4a} \tag{3.6}$$

In order to compare with the exact results of KPZ[32], it is on the other hand useful to consider an ensemble where the total area A is kept fixed. KPZ consider the partition function for fixed area

$$Z\ [A] = \int d\mu[g]\ \delta(\int \sqrt{g} - A)\ e^{-I[g]} \tag{3.7}$$

which for large area behaves like

$$Z\ [A] \underset{A\to\infty}{\sim} A^{-1+\chi(\gamma-2)/2}\ e^{-(\lambda-\lambda_0)A} \tag{3.8}$$

where λ_0 affects the renormalization of the cosmological constant, and $\gamma = \frac{1}{12}(D - 1 - \sqrt{(D-1)(D-25)})$ is the string susceptibility exponent. In our case, since we are dealing with pure gravity for the moment, one expects $D = 0$ and $\gamma = -\frac{1}{2}$.

By doing again an infinitesimal scale transformation on $Z[A]$, with the action given by eq. (3.2), one obtains

$$\frac{\partial \ln Z[A]}{\partial A} = -\frac{1}{A} + a \, \frac{< \int \sqrt{g} \, R^2 >_A}{A} + \lambda_0 - \lambda \tag{3.9}$$

and therefore

$$a \, \frac{< \int \sqrt{g} \, R^2 >_A}{A} \underset{A \to \infty}{\sim} \lambda_0 + \frac{\chi(\gamma - 2)}{2A} + \cdots \tag{3.10}$$

Thus the critical exponent $\chi(\gamma - 2)/2$ can be obtained by investigating the area dependence of the expectation value of R^2. We have carried out such a determination by investigating both the torus and the sphere[33], using as a background space a network of unit squares divided into triangles by drawing in parallel sets of diagonals, as shown in Fig. 1. Ideally one would like to use a random lattice[34], but this would present further computational problems, so we have opted for the moment for the simpler approach of using a regular lattice. In both cases the lattices considered contained from 48 to 49152 edges. In the case of the torus the results for $\chi(\gamma - 2)/2$ are quite accurate and consistent with zero to within a few percent. They already would suggest a restoration of general coordinate invariance at large distances (or low momenta). In the case of the sphere the results appear to be also consistent with the KPZ result, but the errors are larger. Further tests can be performed by embedding the surface and measuring the extent of the surface and the associated Hausdorff dimension, which could then be compared to the results of refs. [32].

GRAVITY IN FOUR DIMENSIONS

The four-dimensional case is substantially more complex that the two-dimensional one, since there are more terms in the pure gravity action, there are no exact results in the full theory to compare with, the lattice structure is more complex and in addition the lattices that have been studied up to now are quite small. In addition there is the conceptual issue of what physical quantities should be measured, and given which boundary conditions. Only a small set of these questions have been addressed up to now, mostly pertaining to an exploration of the phase diagram and the location of possible renormalization group fixed points. We will therefore limit the discussion here to some general qualitative features that have been observed[12], and possible future directions.

Let us give first some details about how the numerical computations are performed. (Some analytical results for simple geometries can be found in refs. [12]). Up to now we have considered the action of eq. (1.12) with $a = 4b$ only (no Weyl term). For simplicity, and as in the two-dimensional case, the lattice was chosen to be regular and built out of rigid hypercubes. Again this choice is not unique, and is dictated mostly by a criterion of simplicity, but it has the advantage that such a lattice can be used to study rather large systems with little modification. Using scaling arguments one can show that whenever the functional integral exists, all the edge lengths can be rescaled $l_i \to (k/\lambda)^{1/2} \, l_i$, and using the scale invariance of the measure one gets

$$Z \, [\lambda, k, a, b, \epsilon] = Z \, [\frac{k^2}{\lambda}, \frac{k^2}{\lambda}, a, b, (\frac{\lambda}{k})^{1/2} \epsilon] \tag{4.1}$$

If ϵ can be sent to zero, then Z can depend only on the dimensionless couplings k^2/λ, a and b, once all lengths are expressed in units of the length scale $l_0 \equiv (k/\lambda)^{1/2}$.

If the functional integral exists for $\epsilon = 0$, then the scale invariance of the measure implies the identity

$$k < \sum_h \delta_h A_h > = \lambda < \sum_h V_h > \tag{4.2}$$

In the numerical simulations that were done the lattice was chosen of size $N \times N \times N \times N$ with $15N^4$ edges, and only the cases $N = 2$ (240 edges) and $N = 4$ (3840 edges) were considered, which corresponded to rather small lattices. Periodic boundary conditions were used, and the topology was therefore restricted to a hypertorus; other topologies can be studied by changing the boundary conditions.

In the case in which all the couplings are zero ($a = b = k = \lambda = 0$) the total action is zero, and variations in the edge lengths are only constrained by the (non-trivial) measure of eq. (1.17). The edges then perform a constrained random walk, and the situation corresponds to what might be called strong coupling and disordered spacetime. Quantities of interest are the average curvature $\mathcal{R}$

$$\mathcal{R} \ = < l^2 > \frac{< 2 \sum_h \delta_h A_h >}{< \sum_h V_h >} \tag{4.3}$$

and the average curvature squared $\mathcal{R}^2$

$$\mathcal{R}^2 \ = < l^2 >^2 \frac{< 4 \sum_h \delta_h^2 A_h^2 / V_h >}{< \sum_h V_h >} \tag{4.4}$$

which are both dimensionless quantities, since they have been expressed in units of the average edge length. One finds that at strong coupling the system tends to develop an average negative curvature. Also, the value of $\mathcal{R}^2$ is quite large, indicating a significant deviation from flat space behavior. A measure of the 'roughness' of space-time is given by the dimensionless ratio of curvature over square root of curvature squared

$$\frac{\mathcal{R}}{\sqrt{\mathcal{R}^2}} = \frac{< 2 \sum_h \delta_h A_h >}{< \sum_h V_h >} \Bigg/ \left(\frac{< 4 \sum_h \delta_h^2 A_h^2 / V_h >}{< \sum_h V_h >} \right)^{1/2} \tag{4.5}$$

Adding the Einstein term ($k \neq 0$) does not improve the situation, and the curvature remains large with large fluctuations, which is presumably a reflection of the unbounded fluctuations in the conformal modes found in the continuum. For $2k = 1$ and for small higher derivative coupling ($a = 4b = 0.005$) the average curvature $\mathcal{R}$ appears to depend very strongly on the value of the bare cosmological constant λ. Large values for the curvature squared $\mathcal{R}^2$, (at least for $\lambda = 1.0$ and 0.5) are found and indicate that, for this choice of coupling constants, the geometry of space-time is *not* well approximated by a smooth metric. This is in turn an indication that with the Einstein and cosmological constant term only, one is in general far from the lattice continuum limit[12–14]. Perhaps the average curvature can be made to vanish by choosing λ appropriately, but this would require fine-tuning. For larger $\lambda = 1.5$ the curvature was found to be significantly smaller, but the jump in $\mathcal{R}$ is so large, that it appears to be indicative of a *discontinuous* transition. The transition could be connected with the lowest eigenvalue of the quadratic fluctuation matrix becoming zero and then negative, as in the case of the regular tessellation $\alpha_{\mathbf{5}}$[11,12].

Thus it appears that in order to obtain a sensible path integral for pure lattice gravity other terms need to be added to the action. Similar results for the pure Einstein action with a cosmological constant term have recently been obtained also in the framework of the hypercubic lattice model, where again no non-trivial fixed point and therefore no sensible continuum limit appears to exist[18].

On the other hand, for larger higher derivative coupling ($a = 4b = k^2/\lambda$ and λ $= 0.5$, 1.0 and 1.5) it was found that the curvature $\mathcal{R}$ is quite uniformly small and negative, and appears to still decrease slightly when going from the 2^4 lattice to the 4^4 lattice. $\mathcal{R}^2$ is now substantially smaller, an indication that the field configurations are becoming smoother. More details about the results of the simulations can be found in [12].

To compute the renormalized, effective low energy, cosmological constant in units of the Planck mass one needs to determine the renormalized value of Newton's constant. Experimentally it is known that at large distances the dimensionless ratio λ_R/k_R^2 is about 10^{-120} or less[35]. The renormalized cosmological constant λ_R can be obtained from the average curvature $\mathcal{R}$. On the other hand one way of extracting the renormalized Newton's constant is via the connected edge (or curvature) two-point function at geodesic distance d

$$G_{\alpha\beta} \;=\; \frac{1}{< l^2 >^2} < l_\alpha^2(d)\; l_\beta^2(0) >_c \;\rightarrow\; T_{\alpha\beta}\, \frac{k_R^{-1}}{d^2} \quad \text{as} \quad d \rightarrow \infty \tag{4.6}$$

where α and β label the different edge types at one point on the lattice (body principal, face diagonal, etc.), and $T_{\alpha\beta}$ is the appropriate spin-two projection matrix. (If particles of other spin are contained in the correlation function (4.36), they can be isolated by diagonalizing the propagation matrix $G_{\alpha\beta}$). On the other hand in analogy to ordinary lattice gauge theories, the curvature correlations would seem to have more physical content, and probably do not require some additional form of gauge fixing. Because of the asymptotic freedom of higher derivative gravity theory, the physical dimensionless ratio λ_R/k_R^2 should be a computable number, and could turn out to be fairly insensitive to the value of the bare couplings. Given the small lattices that were employed, it appears rather difficult to reliably extract a value for k_R. Still, the results on the small lattices suggest that the curvature expressed in units of k_R (i.e. $\mathcal{R}$ over k_R) is a perhaps a rather small number. But this could just be an artifact of the lattice structure used and/or the small overall size. More detailed and careful computations are needed to better understand and settle this important issue. Also, little progress has been made yet in trying to address the issue of unitarity and the positivity of correlation functions at large (compared to the ultraviolet cutoff) distances. Eventually matter fields will have to be included as well, and they could play a role similar to the higher derivative terms in stabilizing the ground state.

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