Quantum Gravity

on a

Lattice

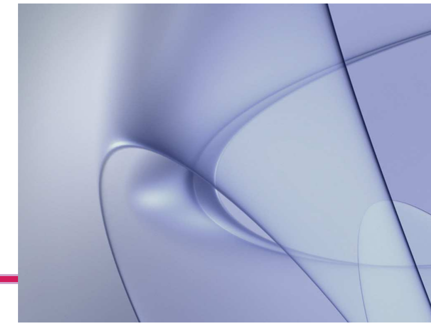
A Picture Book Description

Herbert W. Hamber

Univ. of Calif. Irvine & AEI

hep-th/0704.2895

Outline



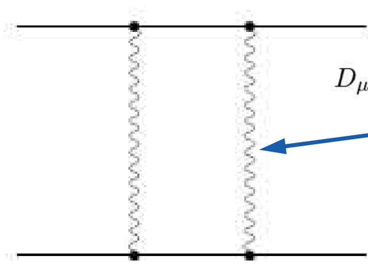
- *Motivation: Perturbative Quantum Gravity*
  - Failure of perturbative renormalization in d=4
  - Quantum Gravity in 2+ε dimensions
  - Other non-renormalizable theories: Sigma Model
- *Method: Formulation of Lattice Quantum Gravity*
  - Simplicial lattice-regularized formulation
  - Matter fields, Observables
  - Methods for determining non-trivial scaling dimensions
- *Outlook: Possible non-perturbative ground state scenarios*
  - Running of Newton’s G
  - Effective non-local covariant relativistic field equations
  - Simple (cosmological & static isotropic) solutions

Perturbative Quantum Gravity

Compute QM amplitudes by Feynman diagram perturbation theory:

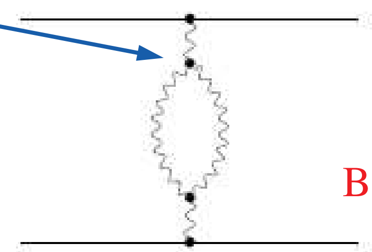
't Hooft. & Veltman, 1974

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$



$$D_{\mu\nu\rho\sigma}(p) = \frac{i}{2}\,\frac{\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}}{p^2 + i\epsilon}$$

$$\begin{aligned}
&U(q_1, q_2, q_3)_{\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3} = \\
&-i\frac{\kappa}{2}\Big[q^2_{(\alpha_1}q^3_{\beta_1)}\left(2\eta_{\alpha_2(\alpha_3}\eta_{\beta_3)\beta_2} - \tfrac{2}{d-2}\eta_{\alpha_2\beta_2}\eta_{\alpha_3\beta_3}\right) \\
&\quad +q^1_{(\alpha_2}q^3_{\beta_2)}\left(2\eta_{\alpha_1(\alpha_3}\eta_{\beta_3)\beta_1} - \tfrac{2}{d-2}\eta_{\alpha_1\beta_1}\eta_{\alpha_3\beta_3}\right) \\
&\quad +q^1_{(\alpha_3}q^2_{\beta_3)}\left(2\eta_{\alpha_1(\alpha_2}\eta_{\beta_2)\beta_1} - \tfrac{2}{d-2}\eta_{\alpha_1\beta_1}\eta_{\alpha_2\beta_2}\right) \\
&\quad +2q^3_{(\alpha_2}\eta_{\beta_2)(\alpha_1}\eta_{\beta_1)(\alpha_3}q^2_{\beta_3)} + 2q^1_{(\alpha_3}\eta_{\beta_3)(\alpha_2}\eta_{\beta_2)(\alpha_1}q^3_{\beta_1)} + 2q^2_{(\alpha_1}\eta_{\beta_1)(\alpha_3}\eta_{\beta_3)(\alpha_2}q^1_{\beta_2)} \\
&\quad +q^2\cdot q^3\left(\tfrac{2}{d-2}\eta_{\alpha_1(\alpha_2}\eta_{\beta_2)\beta_1}\eta_{\alpha_3\beta_3} + \tfrac{2}{d-2}\eta_{\alpha_1(\alpha_3}\eta_{\beta_3)\beta_1}\eta_{\alpha_2\beta_2} - 2\eta_{\alpha_1(\alpha_2}\eta_{\beta_2)(\alpha_3}\eta_{\beta_3)\beta_1}\right) \\
&\quad +q^1\cdot q^3\left(\tfrac{2}{d-2}\eta_{\alpha_2(\alpha_1}\eta_{\beta_1)\beta_2}\eta_{\alpha_3\beta_3} + \tfrac{2}{d-2}\eta_{\alpha_2(\alpha_3}\eta_{\beta_3)\beta_2}\eta_{\alpha_1\beta_1} - 2\eta_{\alpha_2(\alpha_1}\eta_{\beta_1)(\alpha_3}\eta_{\beta_3)\beta_2}\right) \\
&\quad +q^1\cdot q^2\left(\tfrac{2}{d-2}\eta_{\alpha_3(\alpha_1}\eta_{\beta_1)\beta_3}\eta_{\alpha_2\beta_2} + \tfrac{2}{d-2}\eta_{\alpha_3(\alpha_2}\eta_{\beta_2)\beta_3}\eta_{\alpha_1\beta_1} - 2\eta_{\alpha_3(\alpha_1}\eta_{\beta_1)(\alpha_2}\eta_{\beta_2)\beta_3}\right)\Big]
\end{aligned}$$



Bad high energy behavior

$$\frac{G(k^2)}{G} = 1 + \text{const.}\, G\,k^2 + \ldots$$

Non-Renormalizability in Four Dimensions

$$I \;=\; \lambda \int d^d x \sqrt{g} - \frac{1}{16\pi G} \int d^d x \sqrt{g}\, R$$

Radiative corrections generate a host of new interactions…

$$\Gamma^{(1)}_{div} = \frac{1}{4-d} \frac{\hbar}{16\pi^2} \int d^4 x \sqrt{g} \left(\frac{7}{20} R_{\mu\nu} R^{\mu\nu} + \frac{1}{120} R^2 \right)$$

$$\Gamma^{(2)}_{div} = \frac{1}{4-d} \frac{209}{2880} \frac{\hbar^2 G}{(16\pi^2)^2} \int d^4 x \sqrt{g}\, R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\kappa\lambda} R_{\kappa\lambda}{}^{\mu\nu}$$

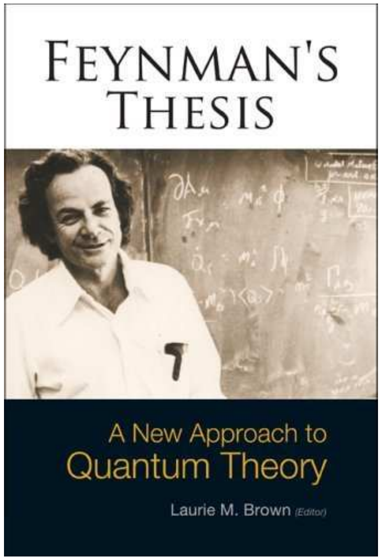
$$I \;\longrightarrow\; \lambda \int d^d x \sqrt{g} - \frac{1}{16\pi G} \int d^d x \sqrt{g}\, R + \frac{\alpha_0}{\Lambda^{4-d}} \int d^d x \sqrt{g}\, R_{\mu\nu} R^{\mu\nu} + \frac{\beta_0}{\Lambda^{4-d}} \int d^d x \sqrt{g}\, R^2 + \cdots$$

- 4-d perturbation theory in (ordinary) gravity seemingly leads to a dead end.
- Non-perturbative methods ? $\Rightarrow$ non-perturbative regularization, search for a new vacuum …

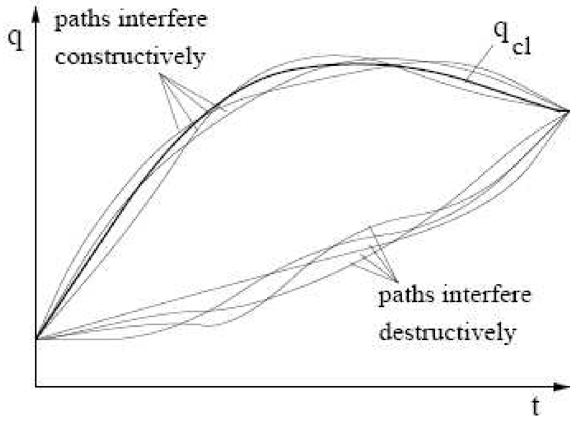
Feynman Path Integral

Reformulate QM amplitudes in terms of discrete

Sum over Paths



- non-commuting operators P,Q replaced by random Wiener paths.
- In complex time $t = -i\tau$ probabilities are <u>*real*</u> (as in statistical mechanics: $KT \rightarrow \hbar$).



$$K(q'', q'; T) = \sum_{all\ paths} A\ e^{iS(q'',q';T)/\hbar}$$

$$\Longrightarrow \quad K = \int \mathcal{D}q(t) e^{iS[q(t)]}$$

Path Integral for Quantum Gravitation

$$\|\delta g\|^2 \equiv \int d^d x \, G^{\mu\nu,\alpha\beta}[g(x)] \, \delta g_{\mu\nu}(x) \, \delta g_{\alpha\beta}(x)$$

DeWitt approach to measure: Volume element in function space obtained from *super-metric* over metric deformations.

$$G^{\mu\nu,\alpha\beta}[g(x)] = \tfrac{1}{2}\sqrt{g(x)} \left[g^{\mu\alpha}(x) g^{\nu\beta}(x) + g^{\mu\beta}(x) g^{\nu\alpha}(x) + \lambda \, g^{\mu\nu}(x) g^{\alpha\beta}(x) \right]$$

$$\int d\mu[g] = \int \prod_x \left(\det[G(g(x))] \right)^{\frac{1}{2}} \prod dg_{\mu\nu}(x)$$

$$\longrightarrow \int d\mu[g] = \int \prod_x [g(x)]^{(d-4)(d+1)/8} \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \underset{d=4}{\rightarrow} \boxed{\int \prod_x \prod_{\mu \geq \nu} dg_{\mu\nu}(x)}$$

$$\boxed{Z_{cont} = \int [d \, g_{\mu\nu}] \, e^{-\lambda \int dx \sqrt{g} + \frac{1}{16\pi G} \int dx \sqrt{g} \, R}}$$

$$I_E[h_{\mu\nu}] = \tfrac{1}{2} \int d^4 x \; h_{\mu\nu} V_{\mu\nu\lambda\sigma} h_{\lambda\sigma} \qquad V = [P^{(2)} - 2P^{(0)}] \, p^2$$

Euclidean E-H action *unbounded below* (conformal instability).

Only One Coupling

Pure gravity path integral:

$$Z = \int [d\, g_{\mu\nu}]\, e^{-I_E[g]}$$

$$I_E[g] = \lambda_0 \Lambda^d \int dx \sqrt{g} - \frac{1}{16\pi G_0} \Lambda^{d-2} \int dx \sqrt{g}\, R$$

Rescale metric (edge lengths):

$$g'_{\mu\nu} = \lambda_0^{2/d}\, g_{\mu\nu} \qquad g'^{\mu\nu} = \lambda_0^{-2/d}\, g^{\mu\nu}$$

$$I_E[g] = \Lambda^d \int dx \sqrt{g'} - \frac{1}{16\pi G_0 \lambda_0^{\frac{d-2}{d}}} \Lambda^{d-2} \int dx \sqrt{g'}\, R'$$

- In the absence of matter, only *one* dimensionless coupling:

$$\boxed{\tilde{G} \equiv G_0\, \lambda_0^{(d-2)/d}}$$

Similar to the *g* of QCD !

Functional Measure cont'd

Add volume term to functional measure (Misner 1955) ;
coordinate transformation $x^\mu + \epsilon^\mu(x)$

$$\prod_x [g(x)]^{\sigma/2} \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \rightarrow \prod_x \left(\det \frac{\partial x'^\beta}{\partial x^\alpha} \right)^\gamma [g(x)]^{\sigma/2} \prod_{\mu \geq \nu} dg_{\mu\nu}(x)$$

$$\prod_x \left(\det \frac{\partial x'^\beta}{\partial x^\alpha} \right)^\gamma = \prod_x [\det(\delta_\alpha{}^\beta + \partial_\alpha \epsilon^\beta)]^\gamma = \exp\left\{ \gamma \, \delta^d(0) \int d^d x \, \partial_\alpha \epsilon^\alpha \right\} = 1$$

[Faddeev & Popov, 1973]

Skeptics should systematically investigate (on the lattice) effects due to the addition of an ultra-local term of the type

$$\prod_x [g(x)]^{\sigma/2} = \exp\left\{ \tfrac{1}{2} \sigma \, \delta^d(0) \int d^d x \ln g(x) \right\}$$

Due to it's ultra-local nature, such a term would not be expected to affect the propagation properties of gravitons (which are det. by R-term).

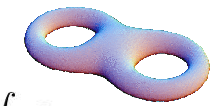
Perturbatively Non-Renorm. Interactions

Some early work :

- K.G. Wilson, *Quantum Field Theory Models in $D < 4$,* PRD 1973.
- K. Symanzik, *Renormalization of Nonrenormalizable Massless φ^4 Theory,* CMP 1975.
- G. Parisi, *Renormalizability of not Renormalizable Theories,* LNC 1973.
- G. Parisi, *Theory Of Nonrenormalizable Interactions - Large N,* NPB 1975.
- E. Brézin and J. Zinn-Justin, *Nonlinear σ Model in $2+\varepsilon$ Dimensions,* PRL 1976.
- …

Gravity in 2.000001 Dimensions

- Wilson expansion: formulate in 2+ε dimensions…



$$\chi = \frac{1}{4\pi}\int d^2x\sqrt{g}\,R$$

G becomes dimensionless in d = 2 ... “Kinematic singularities” as d → 2 make limit *very delicate*.

$$D_{\mu\nu\rho\sigma}(p) = \frac{i}{2}\,\frac{\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\frac{2}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma}}{p^2+i\epsilon}$$

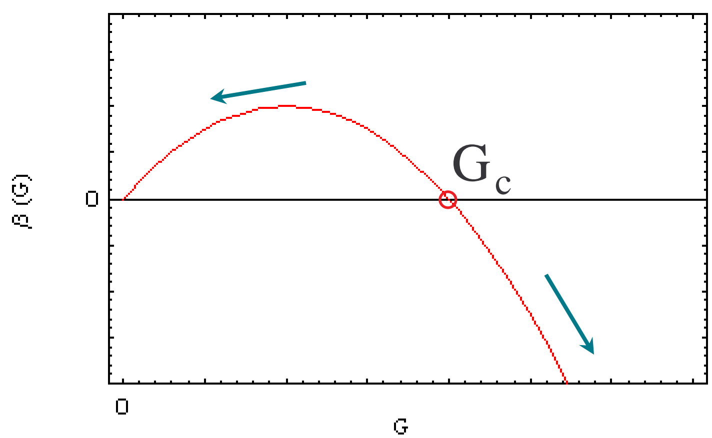
But *G* is dim-less and theory is pert. renormalizable,

GKT, CD 1974
Weinberg 1977
Kawai, Ninomiya 1995
Kitazawa, Aida 1998

$$\mu\frac{\partial}{\partial\mu}G(\mu) = \beta(G(\mu))$$

$$\beta(G) = (d-2)\,G - \frac{2}{3}(25-n_s)\,G^2 - \frac{20}{3}(25-n_s)\,G^3 + \dots$$

$$\begin{cases} G_c = \dfrac{3}{2\,(25-n_s)}(d-2) - \dfrac{45}{2(25-n_s)^2}(d-2)^2 + \dots \\ \nu^{-1} = -\beta'(G_c) = (d-2) + \dfrac{15}{25-n_s}(d-2)^2 + \dots \end{cases}$$



(two loops, manifestly covariant, gauge independent)

A phase transition…

More on 2.000001 dim's …

Graviton loops　　　　　Graviton-ghost loops

- Expansion parameter $\varepsilon = 2$ not small …
- Singularity structure in $d > 2$ unclear (Borel)…

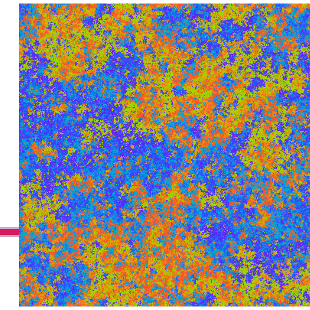
- But analytical control of UV fixed point at G_c .

$$G(k^2) \;=\; \frac{G_c}{1 \,\pm\, (m^2/k^2)^{(d-2)/2}}$$

Nontrivial scaling determined by UV FP.

$$m \sim \Lambda \exp\left(-\int^{G} \frac{dG'}{\beta(G')}\right) \underset{G \to G_c}{\sim} \Lambda\,|\,G - G_c|^{-1/\beta'(G_c)}$$

Detour : Non-linear Sigma model



- Field theory description [*O(N)* Heisenberg model] :

$$Z \;=\; \int [\,d\sigma\,] \prod_x \delta\left(\sigma^a(x)\,\sigma^a(x) \,-\, 1\right) \exp\left(-\,\frac{\Lambda^{d-2}}{g^2} \int d^d x\, \partial_\mu \sigma^a(x)\, \partial_\mu \sigma^a(x) \;+\; \int d^d x\, j^a(x) \sigma^a(x) \right)$$

Coupling *g* becomes *dimensionless* in *d = 2.*

For *d > 2* theory is not perturbatively renormalizable,

but in the 2+ ε expansion one finds:

E. Brezin J. Zinn-Justin 1975
F. Wegner, 1989
N.A. Kivel et al, 1994
E. Brezin and S. Hikami, 1996

$$\Lambda \frac{\partial\, g^2}{\partial\, \Lambda} \;\equiv\; \beta(g^2) \;=\; (d-2)\, g^2 \;-\; \frac{N-2}{2\pi}\, g^4 \;+\; O\left(g^6, (d-2) g^4\right)$$

Phase Transition = non-trivial UV fixed point; new non-perturbative mass scale.

$$\nu^{-1}(\epsilon) = \epsilon + \frac{\epsilon^2}{n-2} + \frac{\epsilon^3}{2(n-2)} - \left[30 - 14 + n^2 + (54 - 18n)\zeta(3)\right] \frac{\epsilon^4}{4(n-2)^3} + \ldots$$

$$\xi(g^2) \;\equiv\; m^{-1}(g^2) \;\simeq\; c_d \Lambda \left(\frac{1}{g_c^2} - \frac{1}{g^2} \right)^{\nu}$$

$$\langle\, \vec{S}(\mathrm{x}) \cdot \vec{S}(0)\, \rangle \;\sim\; \exp\{-|\mathrm{x}|/\xi\}$$

Renormalization Group Equations

In the framework of the *double* (*g and* $2+\varepsilon$) expansion the model looks just *like any other renormalizable theory, to every order…*

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} - \frac{n}{2}\zeta(g) + \rho(g)\, h\,\frac{\partial}{\partial h}\right]\Gamma^{(n)}(p_i, g, h, \Lambda) = 0$$

Callan-Symanzik Eq.

$$\Lambda\frac{\partial}{\partial\Lambda}|_{\text{ren.fixed}}\, g = \beta(g)$$

$$\Lambda\frac{\partial}{\partial\Lambda}|_{\text{ren.fixed}}\,(-\ln Z) = \zeta(g)$$

$$2 - d + \tfrac{1}{2}\zeta(g) + \frac{\beta(g)}{g} = \rho(g)\ .$$

$$g = (\Lambda/\mu)^{d-2} Z_g g_r \qquad \pi(x) = Z^{-1/2}\pi(x)$$

$$h = Z_h h_r \qquad Z_h = Z_g/\sqrt{Z}$$

$$\Gamma_r^{(n)}(p_i, g_r, h_r, \mu) = Z^{n/2}(\Lambda/\mu, g)\,\Gamma^{(n)}(p_i, g, h, \Lambda)$$

… but the price one pays is that now one needs $\varepsilon \longrightarrow 1$!

Similar result are obtained in large N limit [Parisi]…

But is it correct ?

Experimental test: *O(2)* non-linear sigma model describes the phase transition of *superfluid Helium*

Space Shuttle experiment (2003)

High precision measurement of specific heat of superfluid Helium He4 (zero momentum energy-energy correlation at FP)

LIPA *et al.*

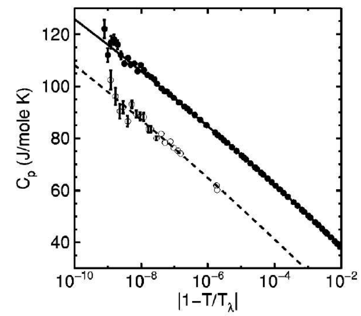


FIG. 15. Semilogarithmic plot of the specific heat vs reduced temperature over the full range measured. Below the transition the data (closed symbols) were binned with a density of 10 bins per decade, and above (open symbols) with a density of 8 bins per decade. Lines show best fits to the data.

J.A. Lipa et al, Phys Rev 2003: $\alpha = 2 - 3\nu = -0.0127(3)$

4-ε expansion to four loops, & to six loops in d=3: $\alpha = 2 - 3\nu \approx -0.0125(4)$

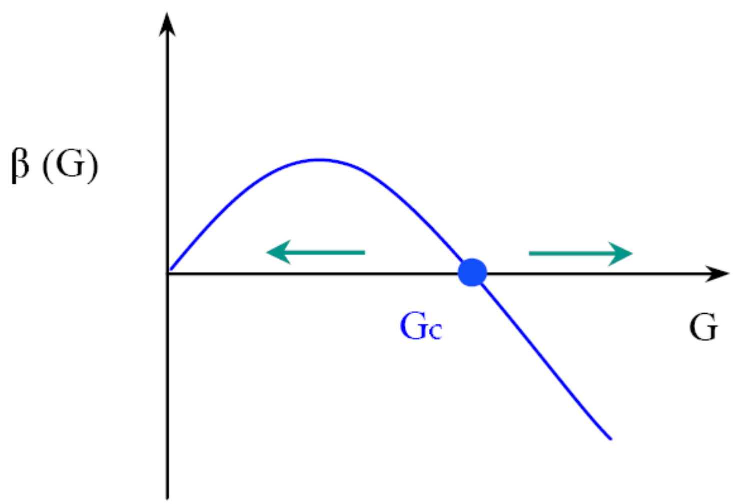
One of the most accurate predictions of QFT - Theory value reviewed in J. Zinn-Justin, 2007

The non-linear sigma model in 3d provides an explicit example of a field theory which :

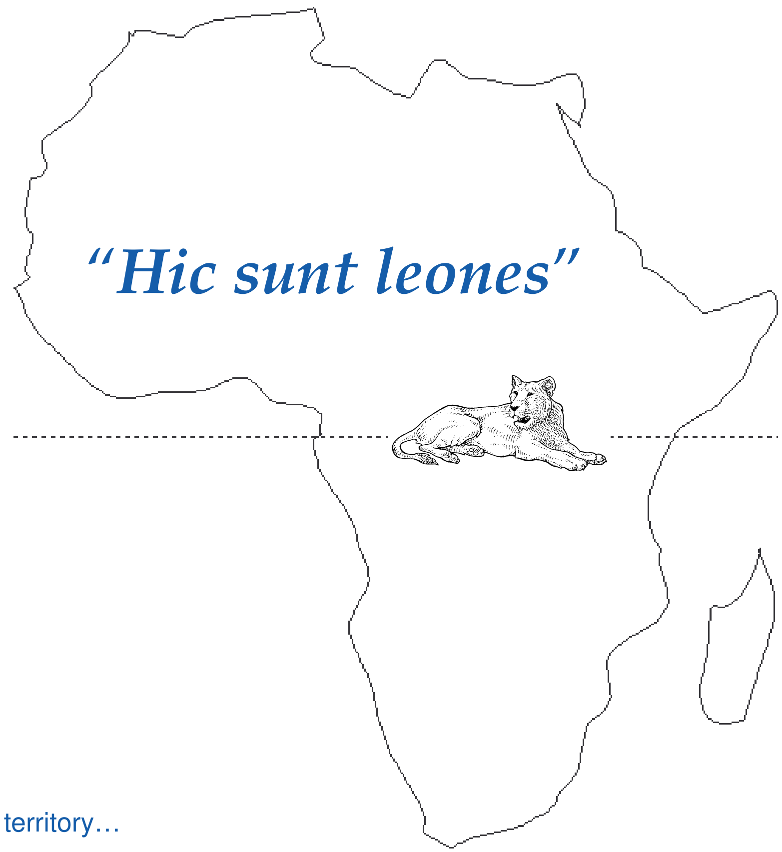
- ✓ Is *not* perturbatively renormalizable in d=3 .
- ✓ Nevertheless leads to *detailed, calculable* predictions in the scaling limit $r \gg a$ ($q^2 \ll \Lambda^2$) .
- ✓ Involves a new non-perturbative scale ξ, essential in determining the scaling behavior in the vicinity of the FP.
- ✓ Whose non-trivial, universal predictions *agree* with experiments.

Key question:

What is left of the above q. gravity scenario in 4 dimensions?



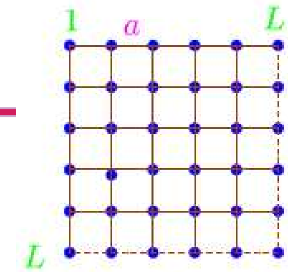
Strongly coupled gravity



The Roman's description of unknown territory…

Lattice Theory

Lattice Quantum Gravity

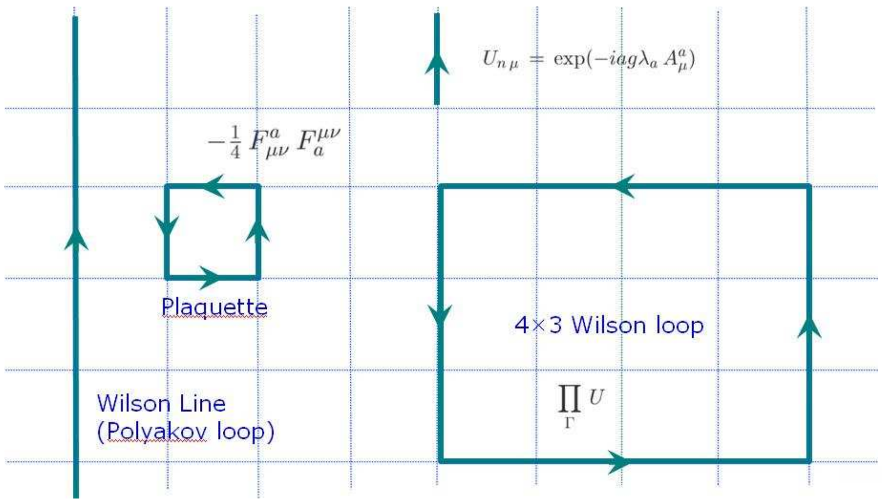


Lattice regularization provides
explicit short distance cutoff.

- Regularized theory is finite, allows non-perturbative treatment.
- Methods of statistical field theory.
- Multi-year experience with lattice QCD.
- Numerical evaluation feasible.
- Continuum limit requires UV fixed point.

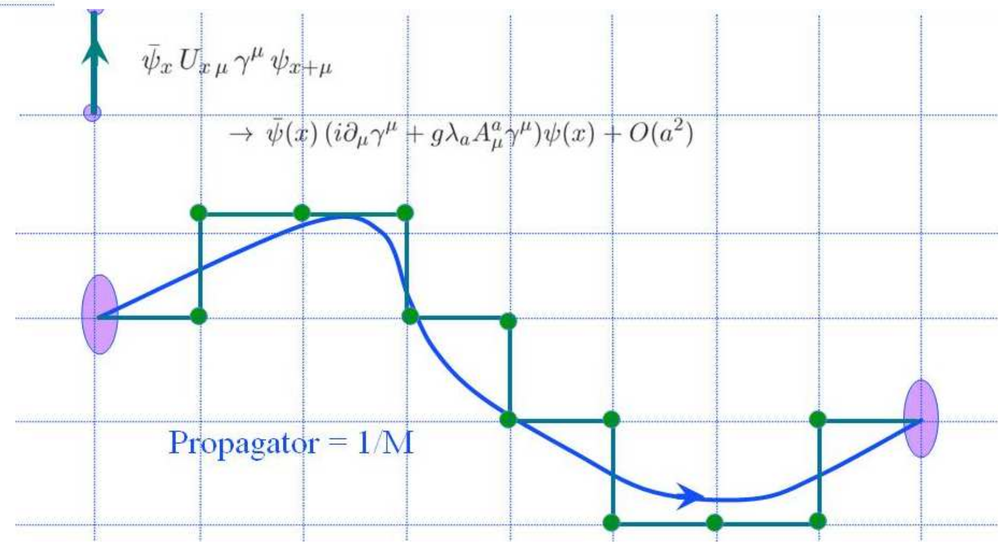
statistical mechanics	$\longleftrightarrow$	quantum field theory
ensemble	$\longleftrightarrow$	phase space
ensemble average	$\longleftrightarrow$	path integral
$exp\{-\beta_k H\}$	$\longleftrightarrow$	$exp\{-S^E\}$
$\beta_k \int d^3x \mathcal{H}$	$\longleftrightarrow$	$\int dx_4 \int d^3x \mathcal{L}$
finite β_k	$\longleftrightarrow$	finite $\int dx_4 = T$
zero temperature	$\longleftrightarrow$	infinite time extent T

Proto: Wilson' Lattice Gauge Theory



Local gauge invariance

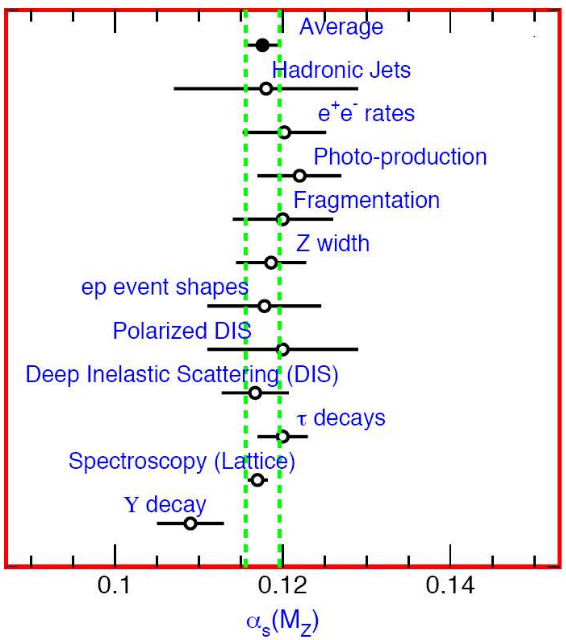
→ exact lattice Ward identities



Lattice Gauge Theory Works

$$L_{\rm QCD} = -\frac{1}{4} F^{(a)}_{\mu\nu}\, F^{(a)\mu\nu} + i \sum_q \overline{\psi}^i_q\, \gamma^\mu\, (D_\mu)_{ij}\, \psi^j_q - \sum_q m_q\, \overline{\psi}^i_q\, \psi_{qi}\ ,$$

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0\, \ln(\mu^2/\Lambda^2)} \left[1 - \frac{2\beta_1}{\beta_0^2} \frac{\ln\left[\ln(\mu^2/\Lambda^2)\right]}{\ln(\mu^2/\Lambda^2)} + \frac{4\beta_1^2}{\beta_0^4 \ln^2(\mu^2/\Lambda^2)} \times \left(\left(\ln\left[\ln(\mu^2/\Lambda^2)\right] - \frac{1}{2} \right)^2 + \frac{\beta_2\beta_0}{8\beta_1^2} - \frac{5}{4} \right) \right] .$$



Summary of the value of $\alpha_s(M_Z)$ from various processes. The values shown indicate the process and the measured value of α_s extrapolated to $\mu = M_Z$. The error shown is the *total* error including theoretical uncertainties. The average quoted in this report which comes from these measurements is also shown. See text for discussion of errors.

[Particle Data Group LBL, 2008]

Lattice gauge theory provides (so far) the only convincing evidence for *confinement* and *chiral symmetry breaking* in QCD.

Quantum Continuum Limit

- Naïve continuum limit : $a \to 0 \quad (\Lambda = \pi/a \to \infty)$
- Quantum continuum limit (based on RG) : $a \to 0 \quad g(a) \to 0$

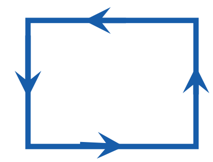
$$\xi = \frac{1}{m_{\text{phys}}} = \text{const.} \times a \exp\left\{\frac{1}{2\beta_0 g^2(a)}\right\} \quad \text{fixed}$$

or simply: $\boxed{\xi/a \to \infty}$

A *phase transition* (UV fixed point) is required for the existence of a non-trivial continuum limit [Wilson, 1974].

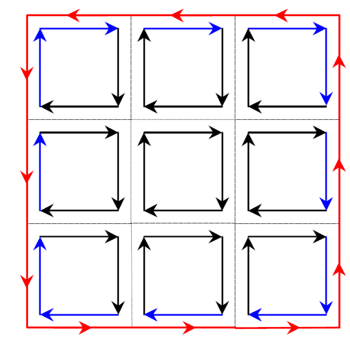
Wilson Loop in *SU(N)* Gauge Theories

- Wilson loop in Lattice Gauge Theories,



$$W(C) = \left\langle \mathrm{tr}\,\mathcal{P}\exp\left\{ig\oint_C A_\mu(x)dx^\mu\right\}\right\rangle, \quad \sim_{A\to\infty} \exp(-A(C)\boxed{/\xi^2}),$$

Gives linear confinement [textbook result, Peskin & Schroeder p. 783]



ξ = gauge correlation length

$$G_\square(x) = \left\langle \mathrm{tr}\,\mathcal{P}\exp\left\{ig\oint_{C'_\epsilon} A_\mu(x')dx'^\mu\right\}\right.$$
$$\left.\times (x)\,\mathrm{tr}\,\mathcal{P}\exp\left\{ig\oint_{C''_\epsilon} A_\mu(x'')dx''^\mu\right\}(0)\right\rangle_c . \quad \sim_{|x|\to\infty} \exp(-|x|\boxed{/\xi}).$$

… Both results are essentially geometric in nature.
They follow (almost trivially) from the use of the *SU(N)* Haar measure.

Simplicial Lattice Formulation

"General Relativity without coordinates" (T.Regge)

MTW ch 42.

- Based on a <u>dynamical lattice.</u>
- Incorporates <u>continuous local</u> invariance.
- Puts within the reach of <u>computation</u> problems which in practical terms are beyond the power of normal analytical methods.
- It affords any desired level of <u>accuracy</u> by a sufficiently fine subdivision of the space-time region under consideration.

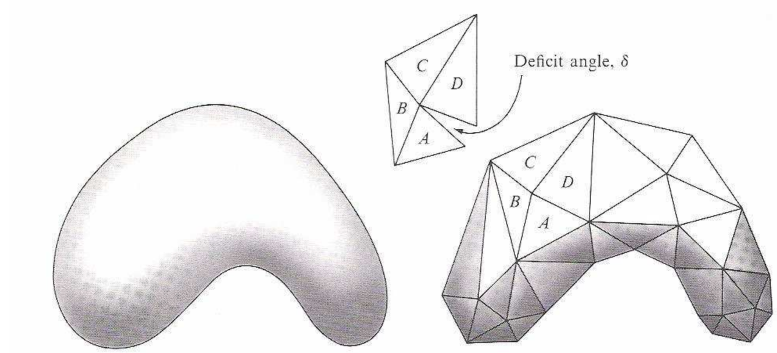
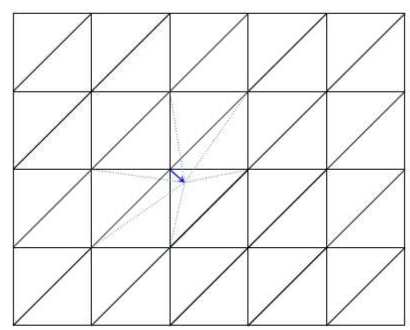
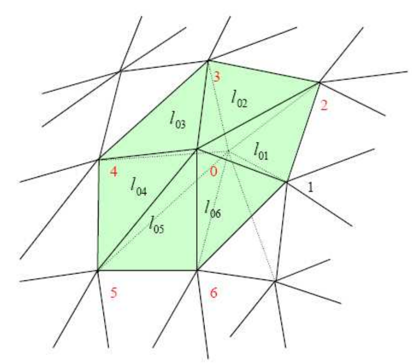
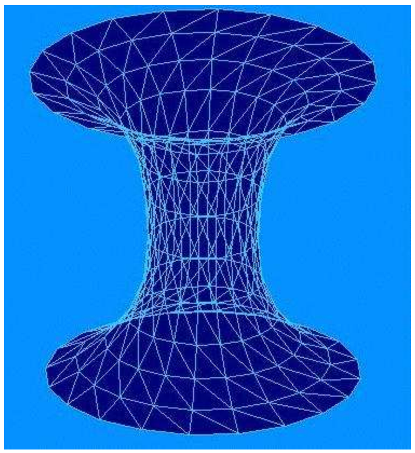


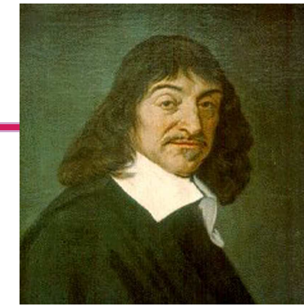
Figure 42.1.

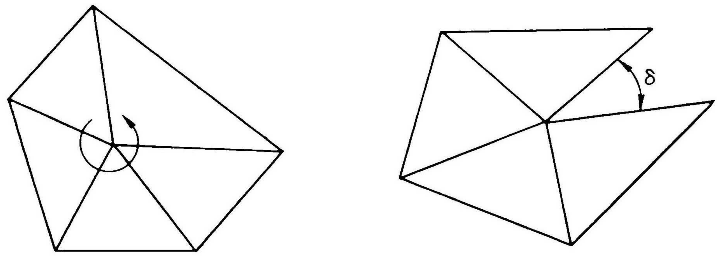




Curvature - Described by Angles







$d = 2$

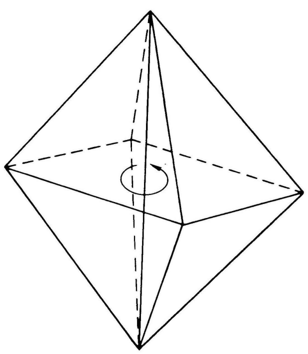
$$\chi = \frac{1}{2\pi} \sum_h \delta_h$$

$$g_{ij} = \frac{1}{2}\left(l^2_{1,i+1} + l^2_{1,j+1} - l^2_{i+1,j+1}\right)$$

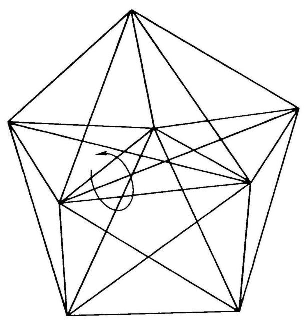
$$V_d = \frac{1}{d!}\sqrt{\det g_{ij}}$$

$$\sin\theta_d = \frac{d}{d-1}\,\frac{V_d\, V_{d-2}}{V_{d-1}\, V'_{d-1}}$$

$$\delta_h = 2\pi - \sum_{\substack{\text{d-simplices} \\ \text{meeting on h}}} \theta_d$$



$d = 3$



$d = 4$

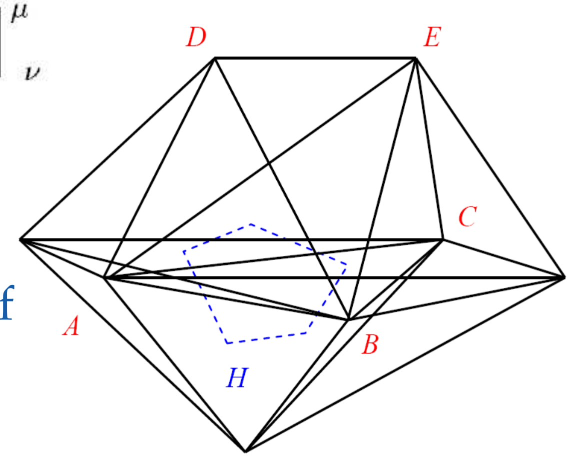
Curvature determined by edge lengths

T. Regge 1961

J.A. Wheeler 1964

Lattice Rotations

$$\phi^\mu(s_{n+1}) = R^\mu{}_\nu(P)\,\phi^\nu(s_1) \qquad R^\mu{}_\nu = \left[P\, e^{\int_{\text{between simplices}}^{\text{path}} \Gamma_\lambda dx^\lambda} \right]^\mu{}_\nu$$



Elementary polygonal path around a hinge (triangle) in four dimensions.

$$\mathbf{R}(C) = \mathbf{R}(s_1, s_n) \cdots \mathbf{R}(s_2, s_1)$$

Due to the hinge's intrinsic orientation, only components of the vector in the plane *perpendicular to the hinge* are rotated:

$$U_{\mu\nu}(h) = \mathcal{N}\,\epsilon_{\mu\nu\alpha_1\alpha_{d-2}}\, l^{\alpha_1}_{(1)} \ldots l^{\alpha_{d-2}}_{(d-2)}$$

$$R^\mu{}_\nu(C) = \left(e^{\delta U}\right)^\mu{}_\nu$$

$$R_{\mu\nu\lambda\sigma}(h) = \frac{\delta(h)}{A_C(h)}\, U_{\mu\nu}(h)\, U_{\lambda\sigma}(h)$$

$$R(h) = 2\,\frac{\delta(h)}{A_C(h)}$$

Exact lattice Bianchi identity,

$$\prod_{\substack{\text{hinges h} \\ \text{meeting on edge p}}} \left[e^{\delta(h) U(h)} \right]^\mu{}_\nu = 1$$

Lattice Action

$$\sqrt{g}\,(x) \quad \to \quad \sum_{\text{hinges}\, h \supset x} V_h$$

$$\sqrt{g}\, R(x) \quad \to \quad 2 \sum_{\text{hinges}\, h \supset x} \delta_h A_h$$

$$\sqrt{g}\, R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma}(x) \quad \to \quad 4 \sum_{\text{hinges}\, h \supset x} (\delta_h A_h)^2 / V_h$$

More than one way to *finite-difference* a continuum expression…

- Alternate actions can be a useful device for analytical estimates (i.e. large d)
- Should exhibit same continuum limit (universality)

$$I_R(l^2) \; = \; -\,k \sum_{\text{hinges h}} \delta(h)\, V^{(d-2)}(h) \qquad\qquad k = 1/(8\pi G)$$

$$I_{\rm com}(l^2) \; = \; -\,k \sum_{\text{hinges h}} \tfrac{1}{2}\, \omega_{\alpha\beta}(h)\, R^{\alpha\beta}(h) \quad \to \quad \sin\delta_p$$

hinge bivector

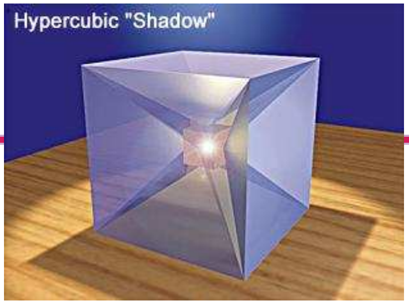
rotation matrix

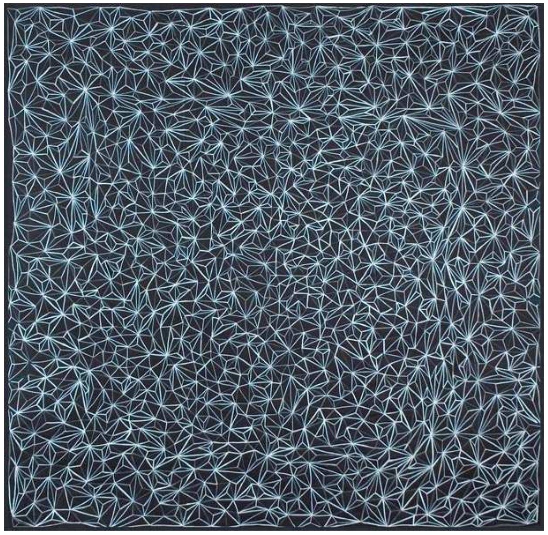
J. Fröhlich 1980

T.D. Lee 1984

Caselle, d'Adda Magnea 1989

Choice of Lattice Structure

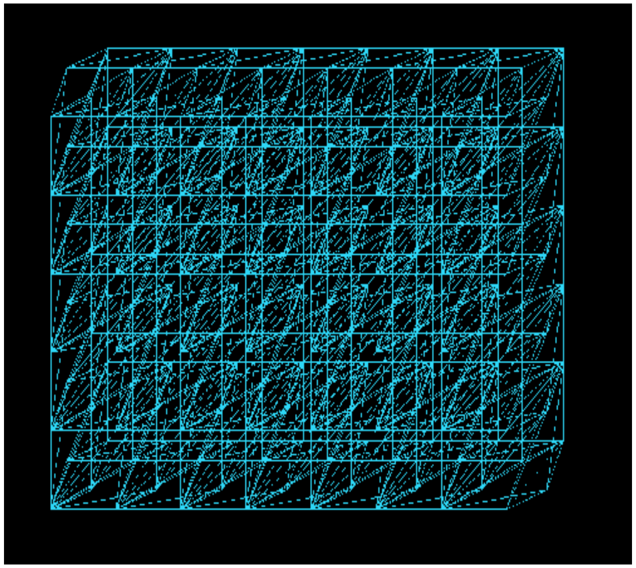




A not so regular lattice …

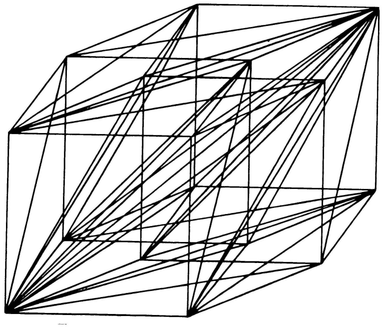
Timothy Nolan,

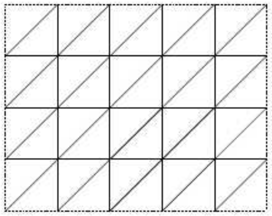
Carl Berg Gallery, Los Angeles

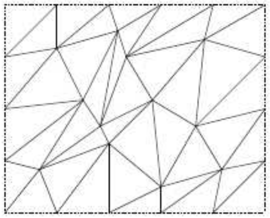


… *and a more regular one:*

Regular geometric objects (hypercubes) can be *stacked* - to form a regularly coordinated lattice of infinite extent.



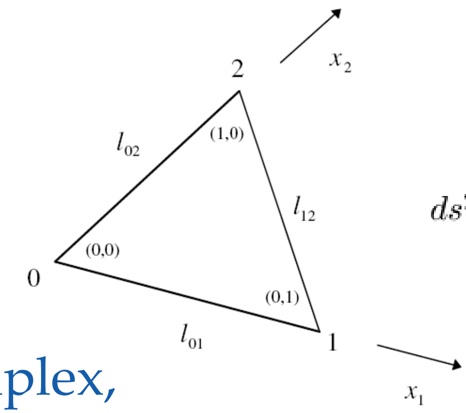




Lattice Measure

Metric deformations linearly related to *squared edge lengths*

$$\delta g_{ij}(l^2) \;=\; \tfrac{1}{2}\left(\delta l_{0i}^2 + \delta l_{0j}^2 - \delta l_{ij}^2\right)$$



$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Jacobian from g's to l's is constant within a simplex,

$$\left(\frac{1}{d!}\sqrt{\det g_{ij}(s)}\right)^\sigma \prod_{i \geq j} dg_{ij}(s) \;=\; \left(-\tfrac{1}{2}\right)^{\frac{d(d-1)}{2}} \left[V_d(l^2)\right]^\sigma \prod_{k=1}^{d(d+1)/2} dl_k^2$$

$$\Longrightarrow \quad \int [d\,l^2] \;=\; \int_0^\infty \prod_s \left(V_d(s)\right)^\sigma \prod_{ij} dl_{ij}^2 \; \Theta[l_{ij}^2]$$

CMS, 1982 ; T.D.Lee, 1982
J.Hartle, 1984 ;
H. & Williams, 1984 ;
B. Berg, 1985 .

Alternatively, can construct the discrete analog of DeWitt's (super) metric over metric deformations, and obtain same result [CMS]…

$$\|\,\delta g(s)\,\|^2 \;=\; \sum_s G^{ijkl}\left(g(s)\right)\; \delta g_{ij}(s)\, \delta g_{kl}(s)$$

Lattice Measure is Non-Trivial

There are important nontrivial constraints on the lattice gravitational measure,

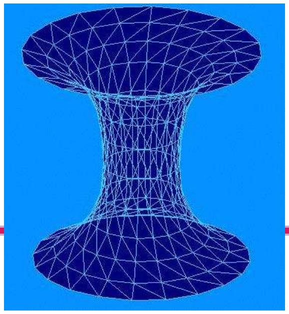
$$\int [d\,l^2] \;=\; \int_0^\infty \prod_s \,(V_d(s))^\sigma \prod_{ij} dl_{ij}^2 \;\Theta[l_{ij}^2]$$

which is generally subject to the *"triangle inequality constraints"* :

$$\begin{cases} l_{ij}^2 > 0 & \\ V_k^2 = \left(\frac{1}{k!}\right)^2 \det g_{ij}^{(k)}(s) > 0 & k = 2 \ldots d \end{cases}$$

Generally these are implied in the *continuum* functional measure as well, but are normally not spelled out in detail …

Lattice Path Integral



Lattice path integral follows from edge assignments,

$$g_{ij} = \frac{1}{2}\left(l_{1,i+1}^2 + l_{1,j+1}^2 - l_{i+1,j+1}^2\right) \qquad V_d = \frac{1}{d!}\sqrt{\det g_{ij}}$$

$$I_E[g] = \lambda_0 \Lambda^d \int dx \sqrt{g} - \frac{1}{16\pi G_0} \Lambda^{d-2} \int dx \sqrt{g}\, R \longrightarrow I_L = \lambda_0 \sum_h V_h(l^2) - 2\kappa_0 \sum_h \delta_h(l^2)\, A_h(l^2)$$

$$Z = \int [d\, g_{\mu\nu}]\; e^{-\lambda_0 \int d^d x \sqrt{g} + \frac{1}{16\pi G} \int d^d x \sqrt{g}\, R} \longrightarrow Z_L = \int [d\, l^2]\; e^{-I_L[l^2]}$$

$$\int [d\, g_{\mu\nu}] = \int \prod_x [g(x)]^{\frac{(d-4)(d+1)}{8}} \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \longrightarrow \int [d\, l^2] \equiv \int_0^\infty \prod_{ij} dl_{ij}^2 \prod_s [V_d(s)]^\sigma\; \Theta(l_{ij}^2)$$

Without loss of generality, one can set bare $\lambda_0 = 1$;

Besides the cutoff, the only relevant coupling is κ (or G).

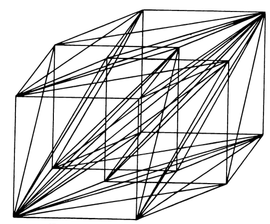
Lattice Weak Field Expansion

- Exhibits correct nature of gravitational degrees of freedoms in the *lattice* weak field limit.
- Allows *clear connection between* ***lattice*** *and* ***continuum*** *operators.*

… start from Regge lattice action

Roĉek and Williams, PLB 1981

$$- 2 \kappa_0 \sum_h \delta_h(l^2) \, A_h(l^2)$$



… call small edge fluctuations "*e*" :

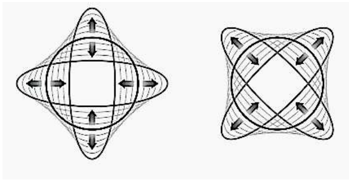
$$I_R = \frac{1}{2} \sum_{ij} e_i \, M_{ij} \, e_j$$

… then Fourier transform, and express result in terms of metric deformations :

$$\delta g_{ij}(l^2) \;=\; \tfrac{1}{2} \, (\delta l_{0i}^2 + \delta l_{0j}^2 - \delta l_{ij}^2)$$

… obtaining in the vacuum gauge precisely the familiar *TT form* in k→0 limit:

$$\frac{1}{4} \mathbf{k}^2 \bar{h}_{ij}^{TT}(\mathbf{k}) \; h_{ij}^{TT}(\mathbf{k})$$



Lattice Higher Derivative Terms

- **HDQG** is perturbatively renormalizable, asymptotically free, but contains *s=0* and *s=2* ghosts,

$$\int d^4x \sqrt{g}\, R^2$$

$$\int d^4x \sqrt{g}\, R_{\mu\nu} R^{\mu\nu}$$

$$\int d^4x \sqrt{g}\, R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma}$$

$$\int d^4x \sqrt{g}\, C_{\mu\nu\lambda\sigma} C^{\mu\nu\lambda\sigma}$$

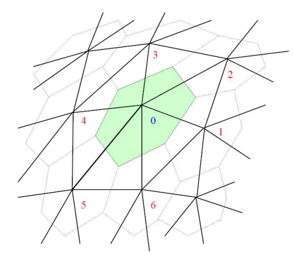
$$\int d^4x \sqrt{g}\, \epsilon^{\mu\nu\kappa\lambda} \epsilon^{\rho\sigma\omega\tau}\, R_{\mu\nu\rho\sigma} R^{\kappa\lambda\omega\tau} = 128\pi^2 \chi$$

$$\int d^4x \sqrt{g}\, \epsilon^{\rho\sigma\kappa\lambda}\, R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\kappa\lambda} = 96\pi^2 \tau$$

$$k < h_{\mu\nu}(q)\, h_{\rho\sigma}(-q) > = \; 2P^{(2)}_{\mu\nu\rho\sigma} \left[\frac{1}{q^2} - \frac{1}{q^2 + \frac{k}{a}} \right] + P^{(0)}_{\mu\nu\rho\sigma} \left[-\frac{1}{q^2} + \frac{1}{q^2 + \frac{k}{2b}} \right]$$

- *Lattice* higher derivative terms

  … involve deficit angles *squared*, as well as coupling between hinges,

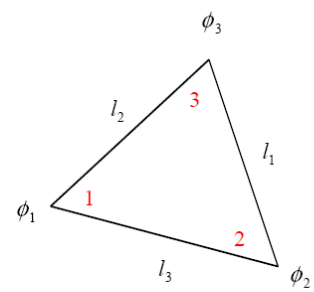


$$\frac{1}{4} \int d^d x \sqrt{g}\, R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} \;\rightarrow\; \sum_{\text{hinges h}} V_h \left(\frac{\delta_h}{A_{C_h}} \right)^2$$

$$\int d^d x \sqrt{g}\, C_{\mu\nu\lambda\sigma} C^{\mu\nu\lambda\sigma} \sim \quad \frac{2}{3} \sum_s V_s \sum_{h,h' \subset s} \epsilon_{h,h'} \left(\omega_h \left[\frac{\delta}{A_C} \right]_h - \omega_{h'} \left[\frac{\delta}{A_C} \right]_{h'} \right)^2$$

Scalar Matter

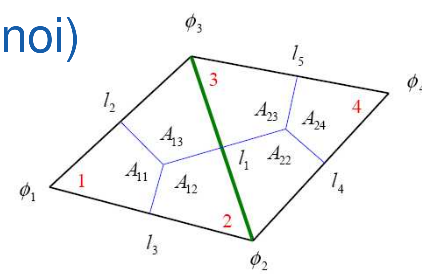
Make use of *lattice metric* to correctly define lattice field derivatives [Ninomiya 1985] …



$$\begin{aligned} g_{\mu\nu}(x) &\longrightarrow g_{ij}(\Delta) \\ \det g_{\mu\nu}(x) &\longrightarrow \det g_{ij}(\Delta) \\ g^{\mu\nu}(x) &\longrightarrow g^{ij}(\Delta) \\ \partial_\mu \phi \partial_\nu \phi &\longrightarrow \Delta_i \phi \, \Delta_j \phi \end{aligned}$$

$$g_{ij}(\Delta) = \begin{pmatrix} l_3^2 & \frac{1}{2}(-l_1^2 + l_2^2 + l_3^2) \\ \frac{1}{2}(-l_1^2 + l_2^2 + l_3^2) & l_2^2 \end{pmatrix}$$

… and obtain a simple geometric form, involving dual (Voronoi) volumes



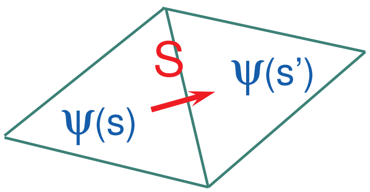
$$I(l^2, \phi) = \frac{1}{2} \sum_{<ij>} V_{ij}^{(d)} \left(\frac{\phi_i - \phi_j}{l_{ij}} \right)^2$$

$$\longrightarrow \qquad I[g, \phi] = \frac{1}{2} \int dx \sqrt{g} [g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + (m^2 + \xi R)\phi^2] + \dots$$

…which also allows correct definition of *lattice Laplacian*: $G_{ij}(l^2) = \left[\frac{1}{-\Delta(l^2) + m^2} \right]_{ij}$

Fermionic Matter

Start from continuum Dirac action



$$I = \int dx \sqrt{g}\, \bar{\psi}(x)\, \gamma^\mu\, D_\mu\, \psi(x)$$

$$\{\gamma^\mu(s), \gamma^\nu(s)\} = 2\, g^{\mu\nu}(s)$$

$$D_\mu = \partial_\mu + \tfrac{1}{2}\omega_{\mu ab}\sigma^{ab} \qquad \sigma_{ab} = \frac{1}{2i}[\gamma_a, \gamma_b]$$

Discrete action *[Drummond 1986]* involves *lattice spin connection :*

$$I = \frac{1}{2} \sum_{\text{faces } f(ss')} V(f(s,s'))\, \bar{\psi}_s\, \mathbf{S}(\mathbf{R}(s,s'))\, \gamma^\mu(s')\, n_\mu(s,s')\, \psi_{s'}$$

Potential problems with fermion doubling (as in ordinary LGT)…

Wilson Loop vs. Loop correlations

Giddings, Hartle &
Marolf PRD 2006

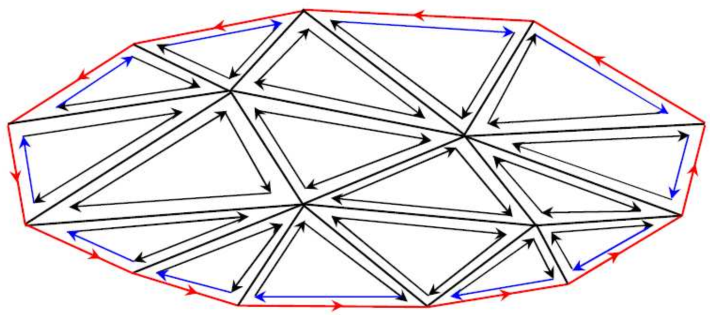


FIG. 2 (color online). Gravitational analog of the Wilson loop.

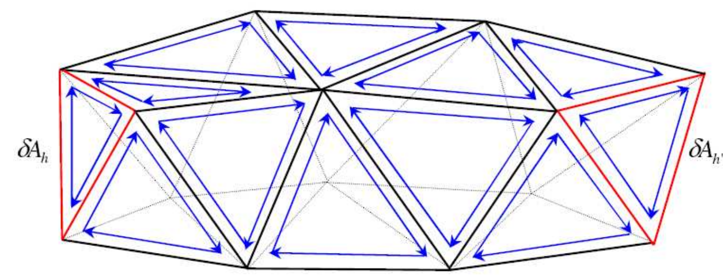
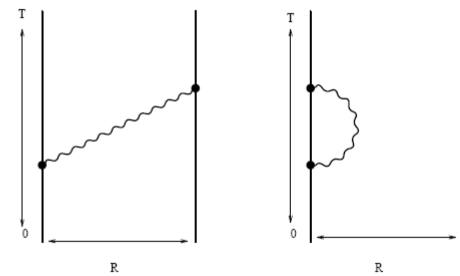


FIG. 6 (color online). Correlations between action contributions on hinge h and hinge h' arise to lowest order in the strong

$$R^{\alpha}{}_{\beta}(C) = \left[\mathcal{P}\exp\left\{\oint_{\mathbf{path\ C}} \Gamma^{\cdot}{}_{\lambda\cdot}\, dx^{\lambda}\right\}\right]^{\alpha}{}_{\beta}.$$

$$G_R(d) \sim < \sqrt{g}\, R(x)\, \sqrt{g}\, R(y)\, \delta(|x-y|-d) >_c$$

Wilson Loop does *not* give Potential



In ordinary LGT, Wilson loop gives $V(r)$

$$W(\Gamma) \; = < \exp\Big\{ ie \oint_\Gamma A_\mu(x) dx^\mu \Big\} >$$

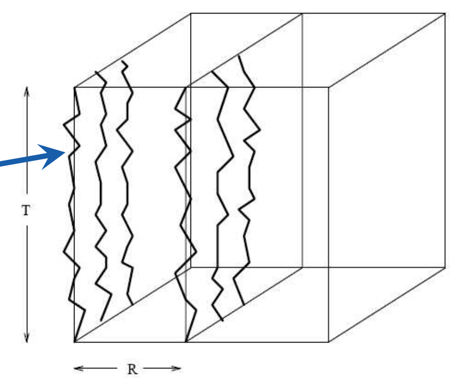
$$V(R) = - \lim_{T \to \infty} \frac{1}{T} \log < \exp\Big\{ ie \oint_\Gamma A_\mu dx^\mu \Big\} >$$

In lattice regularized gravity, potential is computed from the correlation of geodesic *line segments, associated with the particle's world line:*

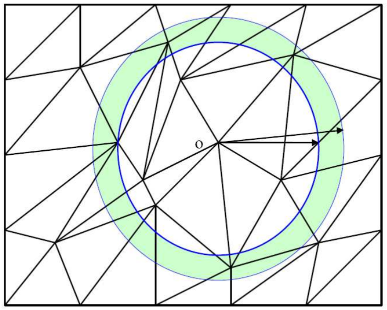
G. Modanese, PRD 1994; NPB 1995

$$\mu \int_{\tau(a)}^{\tau(b)} d\tau \sqrt{ g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} }$$

$$L(0;\; \mu_1) = \exp\Big\{ -\mu_1 \int d\tau \sqrt{ g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} } \; \Big\}$$



Correlations



… of invariant operators at fixed geodesic distance.

Distance is a function of metric, which fluctuates:

$$d(x,y\,|\,g) \;=\; \min_{\xi} \int_{\tau(x)}^{\tau(y)} d\tau \, \sqrt{g_{\mu\nu}(\xi) \frac{d\xi^{\mu}}{d\tau} \frac{d\xi^{\nu}}{d\tau}}$$

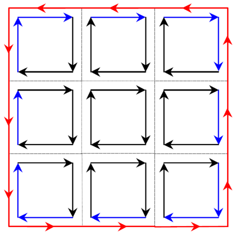
$$< \int dx \int dy \, \sqrt{g} R(x) \, \sqrt{g} R(y) \, \delta(|x-y| - d) >$$

$$\longrightarrow \quad G_R(d) \equiv < \sum_{h \supset x} \delta_h A_h \sum_{h' \supset y} \delta_{h'} A_{h'} \, \delta(|x-y| - d) >_c$$

$$G(x,y|g) = < x \,|\, \frac{1}{-\Delta(g) + m^2} \,|\, y >$$

$$\underset{d(x,y)\to\infty}{\sim} \; d^{-(d-1)/2}(x,y) \, \exp\{-m \, d(x,y)\}$$

Hypercubic Lattice Gravity



- Flat hypercubic lattice - geometric features not manifest
  e.g. Mannion &Taylor PLB 1982 ; see also Smolin 1978; Das Kaku Townsend 1982.
- Lattice discretization of the Cartan theory based on SL(2,C) ➔ SO(3,1) ➔ SO(4)

$$U_\mu(n) = \left[U_{-\mu}(n+\mu)\right]^{-1} = \exp[i B_\mu(n)] \qquad B_\mu = \tfrac{1}{2} a B_\mu^{ab}(n) J_{ba} \qquad \sigma_{ab} = \frac{1}{2i}[\gamma_a, \gamma_b]$$

Local *gauge* invariance:

$$E_\mu(n) = a\, e_\mu^{\ a}\, \gamma_a$$

$$U_\mu \to \Lambda(n)\, U_\mu(n)\, \Lambda^{-1}(n+\mu) \qquad E_\mu(n) \to \Lambda(n)\, E_\mu(n)\, \Lambda^{-1}(n)$$

$$I = \frac{i}{16\kappa^2} \sum_{n,\mu,\nu,\lambda,\sigma} \mathrm{tr}[\gamma_5\, U_\mu(n)\, U_\nu(n+\mu)\, U_{-\mu}(n+\mu+\nu)\, U_{-\nu}(n+\nu)\, E_\sigma(n)\, E_\lambda(n)]$$

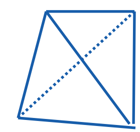
$$\longrightarrow \quad I = \frac{1}{4\kappa^2} \int d^4x\, \epsilon^{\mu\nu\lambda\sigma}\, \epsilon_{abcd}\, R_{\mu\nu}^{\ \ ab}\, e_\lambda^{\ c}\, e_\sigma^{\ d}$$

Path integral over U's (Haar) and E's:

$$Z = \int \prod_{n,\mu} dB_\mu(n) \prod_{n,\sigma} dE_\sigma(n)\, \exp\left\{-I(B,E)\right\}$$

Dynamical Triangulations

- Simplified version of Regge Gravity
- Edge lengths *fixed to unity*, vary incidence matrix [David 1984, ...]



an integer

$$V_d = \frac{1}{d!}\sqrt{\frac{d+1}{2^d}} \qquad \cos\theta_d = \frac{1}{d} \qquad \delta(h) = 2\pi - n_s(h)\,\theta_d$$

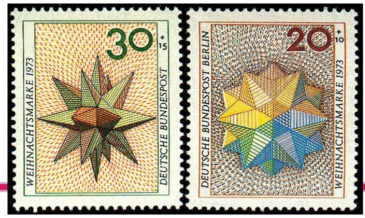
- No immediate notion of continuous metric, or continuous diffeos.
- Curvature varies in discrete steps.
- No continuous metric deformations – hence *no* w.f.e., and no gravitons (at least not in an explicit way).

Constraints on *functional measure* <u>unclear</u>, since theory has no explicit metric.

Seemingly pathological behavior of Euclidean theory [Loll et al] $\longrightarrow$

numerical Lorentzian path integral (with yet unresolved convergence issues).

Large D Limit

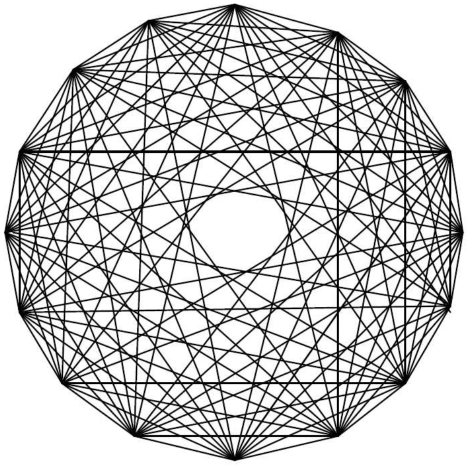


Early work in *continuum* by A. Strominger (1984, $\lambda=0$), ...

On the lattice, phase transition persists at $d=\infty$.

$$\left\{ \begin{array}{l} k_c = \dfrac{\lambda_0^{\frac{d-2}{d}}}{d^3} \left[\dfrac{2}{d} \, \dfrac{d! \, 2^{d/2}}{\sqrt{d+1}} \right]^{2/d} \\ l_0^2 = \dfrac{1}{\lambda_0^{2/d}} \left[\dfrac{2}{d} \, \dfrac{d! \, 2^{d/2}}{\sqrt{d+1}} \right]^{2/d} \end{array} \right.$$

$k_c = \sqrt{3}/(16 \cdot 5^{1/4}) = 0.0724 \text{ in } d=4$ compared to $k_c = 0.0636$



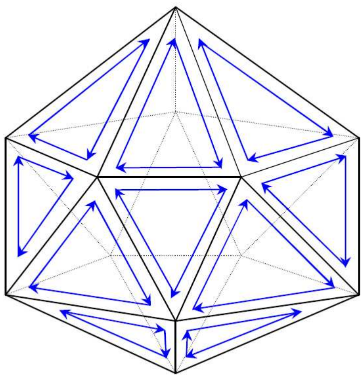
N-cross polytope, homeomorphic to a sphere

- Conformal mode instability disappears, *O(1/d).*
- At large d, partition function at large *G* dominated by *closed surfaces*, tiled with elementary parallel transport polygonal loops.

  Very large surfaces are important as $k \to k_c$.

H & Williams, PRD 2006

Large D Limit - Exponent ν



- At large d, *characteristic size* ξ of random surface diverges *logarithmically* as $G \rightarrow G_c$ (D. Gross PLB 1984).
- Suggests universal correlation length exponent $\nu = 0$.

Known results from random surface theory then imply:

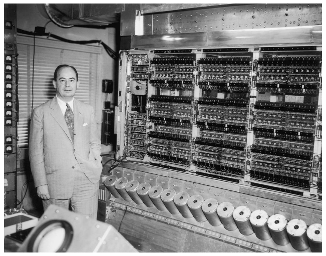
$$\xi \sim \sqrt{\log T} \underset{k \to k_c}{\sim} |\log(k_c - k)|^{1/2}$$

$\boxed{\nu = 1/(d-1)}$ ⟷ $\boxed{\nu = 1/2d}$

D. Litim PRL 2004, PLB 2007

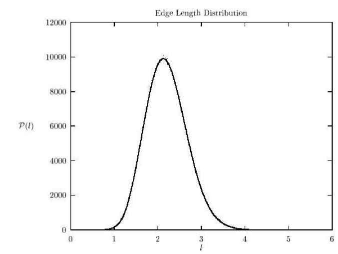
scalar field	$\nu = \frac{1}{2}$
lattice gauge field	$\nu = \frac{1}{4}$
lattice gravity	$\nu = 0$

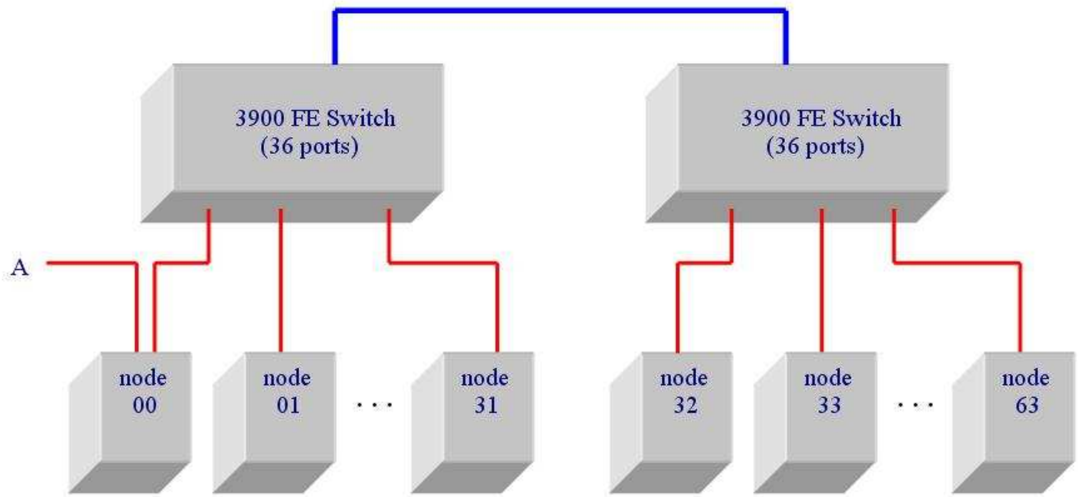
Numerical Evaluation of Z

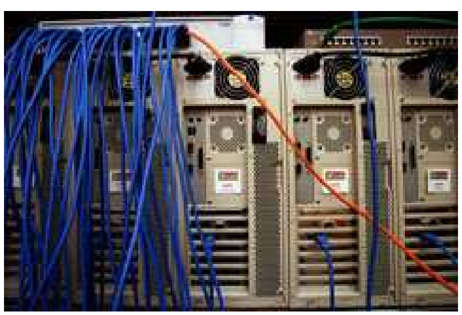


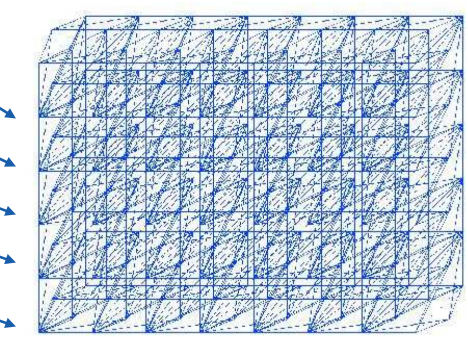
CM5 at NCSA, 512 processors

Dedicated Parallel Supercomputer

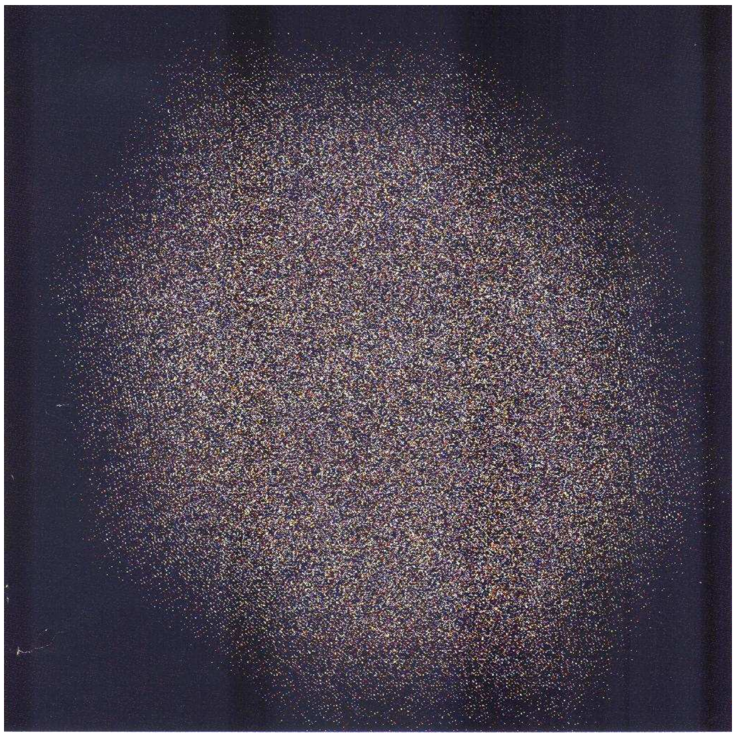




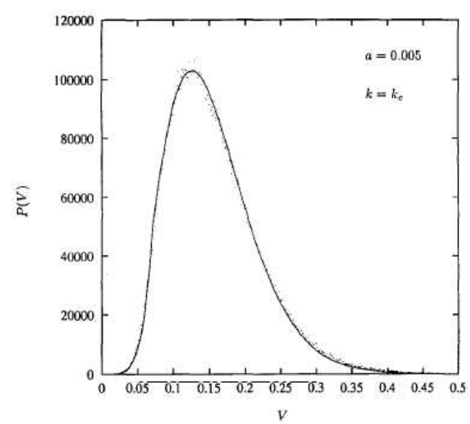


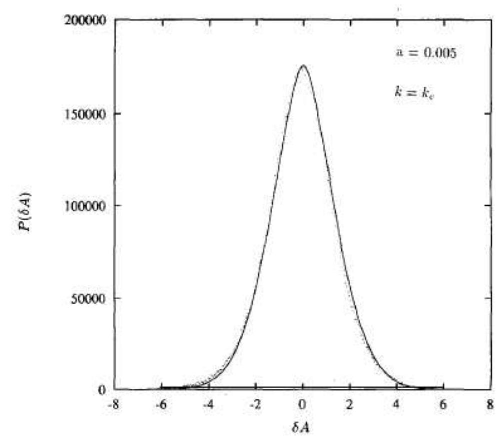


Edge length/metric distributions



- L=4 → 6,144 simplices
- L=8 → 98,304 simplices
- L=16 → 1,572,864 simplices
- L=32 → 25,165,824 simplices



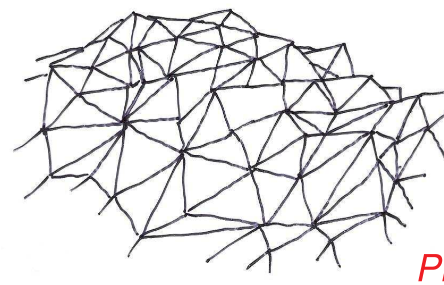


Two Phases of L. Quantum Gravity

Earliest studies of Regge lattice theories found evidence for :

$G > G_c$ Smooth phase: R ≈ 0

$$\langle g_{\mu\nu} \rangle \approx c\, \eta_{\mu\nu}$$

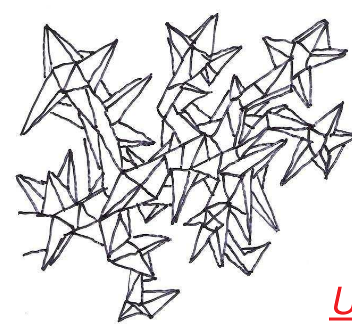


Physical

$$N(\tau) \underset{\tau \to \infty}{\sim} \tau^{\, d_v}$$

$G < G_c$ Rough phase : branched polymer, d ≈ 2

$$\langle g_{\mu\nu} \rangle = 0$$



Unphysical

Lattice manifestation of conformal instability

Similar two-phase structure also found later in some d=4 DTRS models [Migdal, …]

Invariant Averages

$$\mathcal{R}(k) \sim \frac{< \int d^4x \sqrt{g}\, R(x) >}{< \int d^4x \sqrt{g} >}$$

$$\chi_{\mathcal{R}}(k) \sim \frac{< (\int \sqrt{g}\, R)^2 > - < \int \sqrt{g}\, R >^2}{< \int \sqrt{g} >}$$

Singularities in the free energy *F* are determined from non-analiticities in invariant local averages.

$$\mathcal{R}(k) \sim \frac{1}{V} \frac{\partial}{\partial k} \ln Z_L$$

$$\chi_{\mathcal{R}}(k) \sim \frac{1}{V} \frac{\partial^2}{\partial k^2} \ln Z_L$$

- Divergent local averages provide information about non-trivial *exponents.*
- Finite Size Scaling (FSS) theory useful.

$$O(L,t) \;=\; L^{x_O/\nu} \left[\tilde{f_O}\,(L\, t^\nu) \;+\; \mathcal{O}(L^{-\omega}) \right]$$

- *Correlations* are harder to compute directly (geodesic distance).

$$\boxed{F_{sing}(G) \sim \xi^{-d}} \longleftarrow \boxed{\xi \sim t^{-\nu}}$$

"Scaling assumption" for F = ln Z

Determination of Scaling Exponents

$$\mathcal{R}(k) \underset{k \to k_c}{\sim} -A_{\mathcal{R}} \, (k_c - k)^{\delta} \qquad \nu = \frac{1+\delta}{d}$$

$$\chi_{\mathcal{R}}(k) \underset{k \to k_c}{\sim} -A_{\mathcal{R}} \, (k_c - k)^{-(1-\delta)}$$

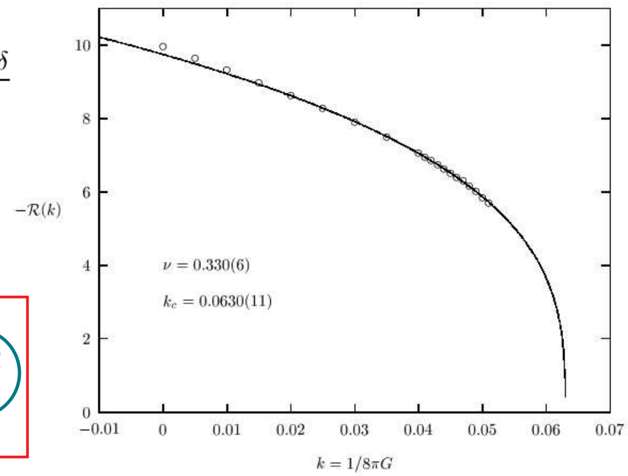
Scaling assumption: $F_{sing}(G) \sim \xi^{-d}$

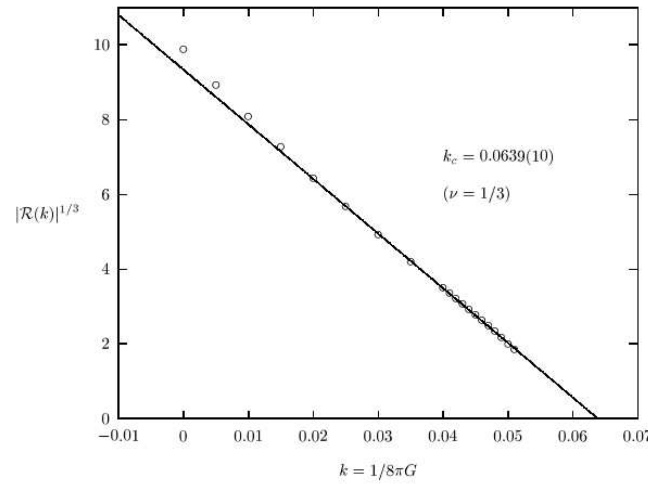
$$\xi(k) \equiv m(k)^{-1} \underset{k \to k_c}{\sim} A_{\xi} \, (k_c - k)^{-\nu}$$

Find value close to 1/3:

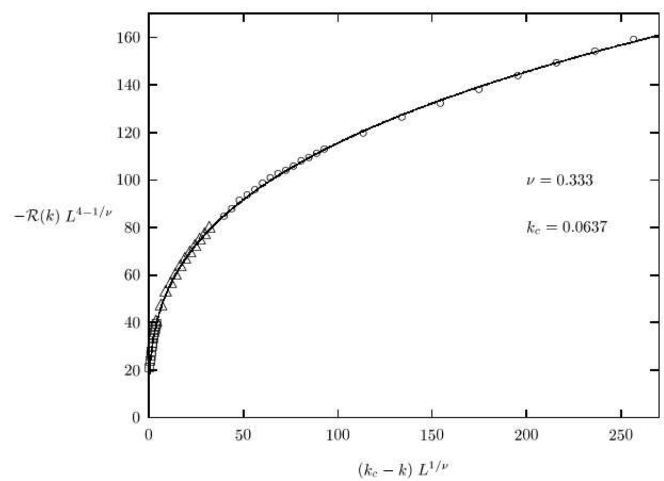
$$\boxed{k_c = 0.0636(11) \qquad \nu = 0.335(9)}$$

$$\mathcal{R}(\xi) \underset{k \to k_c}{\sim} \frac{1}{l_P \, \xi}$$



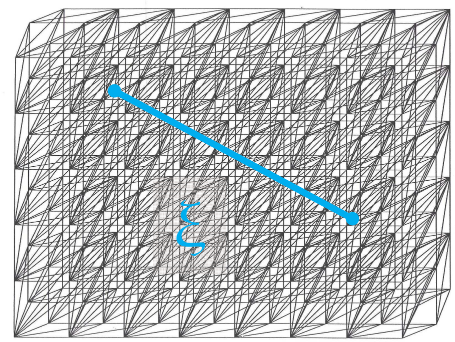


$$\boxed{\nu \approx 1/3}$$



PRD 61 2000

(Lattice) Continuum Limit $\Lambda \to \infty$



Standard (Wilson) procedure in cutoff field theory:

$$\xi = 1/m \qquad\qquad m \underset{G(\Lambda)\to G_c}{\sim} \Lambda \left[\frac{G(\Lambda) - G_c}{a_0\, G_c}\right]^{\nu}$$

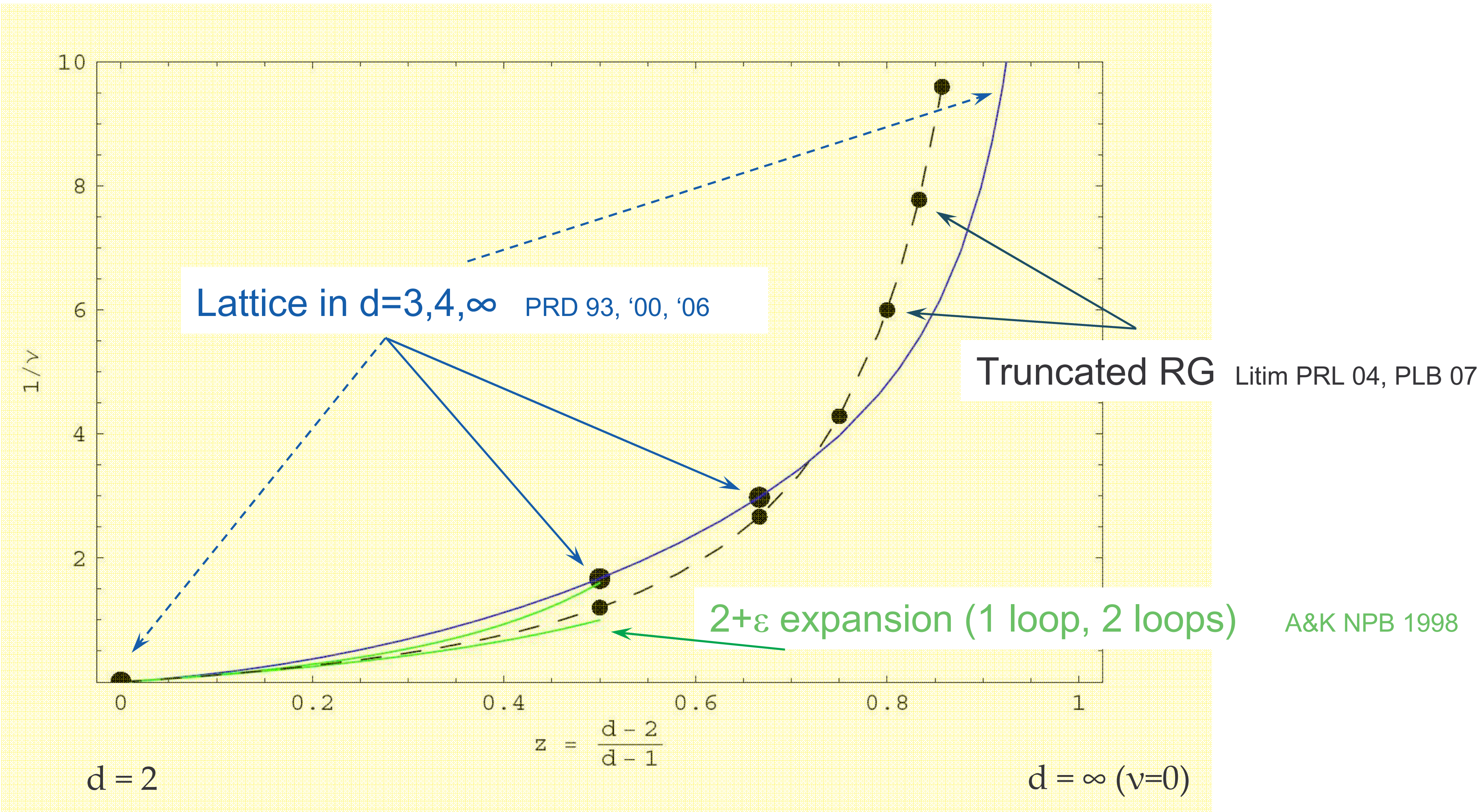
RG *invariant* correlation length ξ is kept fixed

UV cutoff $\Lambda \to \infty$ (average lattice spacing $\to 0$)

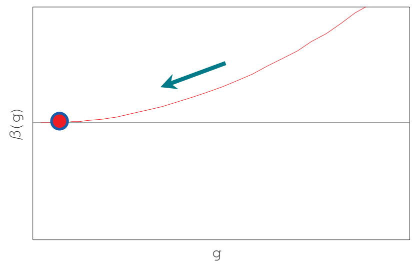
Bare G *must* approach UV fixed point at G_c

The *very same* relation gives the RG running of $G(\mu)$ close to the FP.

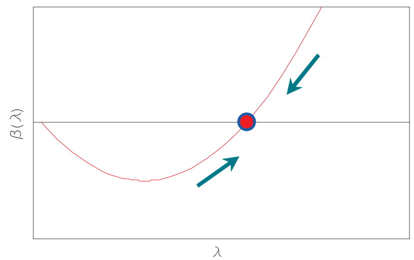
Exponent ν compared



RG Running Scenarios

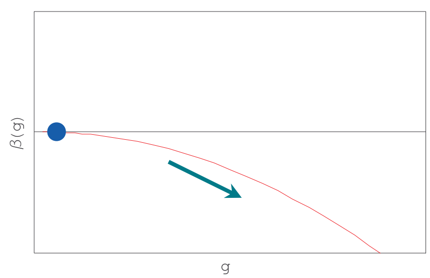


"Triviality" of lambda phi 4

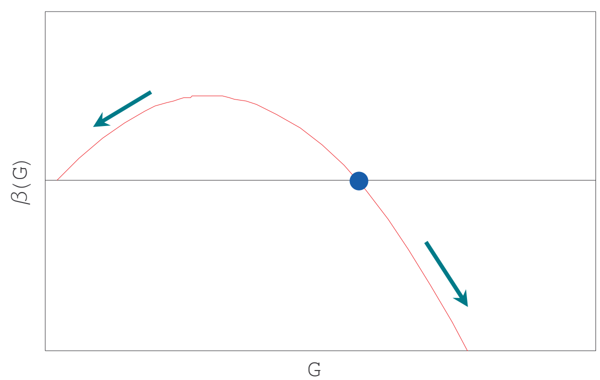


Wilson-Fisher FP in d<4

- Coupling gets *weaker at large r*
- … approaches an IR FP at large r.
- … gets *weaker at small r* : UV FP
- Both possibilities can coexist: nontrivial UV fixed point.



Asymptotic freedom of YM



Ising model, σ-model, Gravity (2+ε, lattice)

Callan-Symanzik. beta function(s):

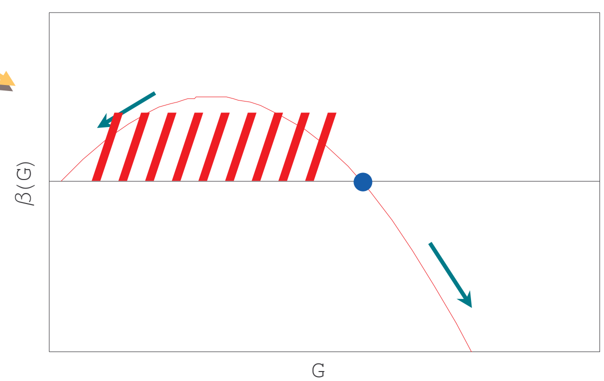
$$\boxed{\mu \frac{\partial}{\partial \mu} G(\mu) \;=\; \beta(G(\mu))}$$

Only One Phase?

Weak coupling phase is seemingly unphysical (branched polymer).

- ✓ Lattice results appear to exclude the weak coupling phase as physically relevant…
- ✓ Leads to a gravitational coupling G that *increases* with distance…

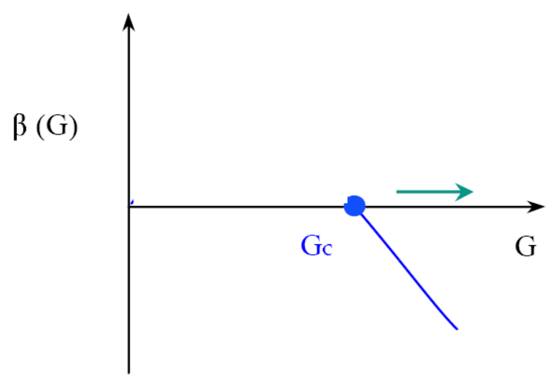
Anti-screening phase.



Gravity

New question then :

Is this new scenario physically acceptable?



Running Newton's G

- ξ is a *new invariant scale of gravity*. $m \equiv \xi^{-1} = \Lambda\, F(G)$
- Newton's constant *G must* run.
- *Cutoff* dependence determines β-function :

$$\Lambda \frac{d}{d\,\Lambda} m(\Lambda, G(\Lambda)) \; = \; 0 \quad \text{and} \quad \Lambda \frac{\partial}{\partial\,\Lambda} G(\Lambda) \; = \; \beta(G(\Lambda)) \;\longrightarrow\; \beta(G) \; = \; - \frac{F(G)}{\partial F(G)/\partial G}$$

[In fact, one can be quite specific …

$$\beta(G) \underset{G \to G_c}{\sim} -\frac{1}{\nu}(G - G_c) \quad \Longleftrightarrow \quad m \sim \Lambda \exp\left(- \int^G \frac{dG'}{\beta(G')} \right) \underset{G \to G_c}{\sim} \Lambda \, | G - G_c |^{-1/\beta'(G_c)}$$]

Running of G det. largely by ξ and ν :

$$\mu \frac{\partial}{\partial\,\mu} G(\mu) \; = \; \beta(G(\mu)) \;\Longrightarrow\; \boxed{G(k^2) \; = \; G_c \left[1 \, + \, a_0 \left(\frac{m^2}{k^2} \right)^{\frac{1}{2\nu}} \, + \, O((m^2/k^2)^{\frac{1}{\nu}}) \right]}$$

So, what value to take for ξ ?

- ξ is an RG invariant.
- $m=1/\xi$ has dimensions of a mass.

In Yang-Mills m = glueball mass

Three Theories Compared

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\partial^\mu F_{\mu\nu} + \mu^2 A_\nu = 4\pi e j_\nu$$

$$\partial^\mu \partial_\mu \phi + m^2 \phi = \frac{g}{3!} \phi^3$$

Suggests $\lambda_{phys} \simeq \frac{1}{\xi^2}$

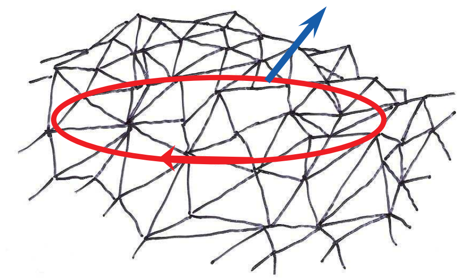
RG invariants

Running couplings

$$m = 1/\xi$$

Gravitational Wilson Loops

- In gravity, Wilson loop not related to static potential
  [G. Modanese PRD 1993; PRD 1994]
- Parallel transport of a vector done via lattice rotation matrix



$$\mathbf{R}^{\alpha}{}_{\beta} = \left[\mathcal{P} \, e^{\int_{\substack{\text{path} \\ \text{between simplices}}} \Gamma^{\lambda} dx_{\lambda}} \right]^{\alpha}_{\beta}$$

For a *large* closed circuit obtain *Wilson loop* - which can be computed at strong coupling using a first order formulation of Regge gravity [Caselle, d'Adda, Magnea PLB 1989]

$$W(\Gamma) \sim \mathrm{Tr}\, \mathcal{P} \exp \left[\int_C \Gamma^{\lambda}{}_{..} \, dx_{\lambda} \right] \sim \exp \left[\int_{S(C)} R^{\cdot}{}_{\cdot \mu\nu} \, A_C^{\mu\nu} \right] \sim \exp(-A/\xi^2)$$

- Stokes theorem -

- ξ related to curvature.
- ξ RG invariant.
- prediction of positive cosmological constant?

$$\boxed{\lambda_{phys} \simeq \frac{1}{\xi^2}}$$

"Area law" follows from loop tiling

HH&R.M.Williams, PRD 76, 2007

Vacuum Condensate Picture of QG?

- Lattice Quantum Gravity: Curvature condensate *See also J.D.Bjorken, PRD '05*

$$\mathcal{R} \simeq (10^{-30} eV)^2 \sim \xi^{-2} \qquad \lambda_{phys} \simeq \frac{1}{\xi^2}$$

- Quantum Chromodynamics: Gluon and Fermion condensate

$$\alpha_S < F_{\mu\nu} \cdot F^{\mu\nu} > \simeq (250 MeV)^4 \sim \xi^{-4}$$

$$\xi_{QCD}^{-1} \sim \Lambda_{\overline{MS}}$$

$$(\alpha_S)^{4/\beta_0} < \bar{\psi}\psi > \simeq -(230 MeV)^3 \sim \xi^{-3}$$

- Electroweak Theory: Higgs condensate

Effective Theory

Graviton Vacuum Polarization Cloud

Picture: Source mass M surrounded by *virtual graviton cloud*

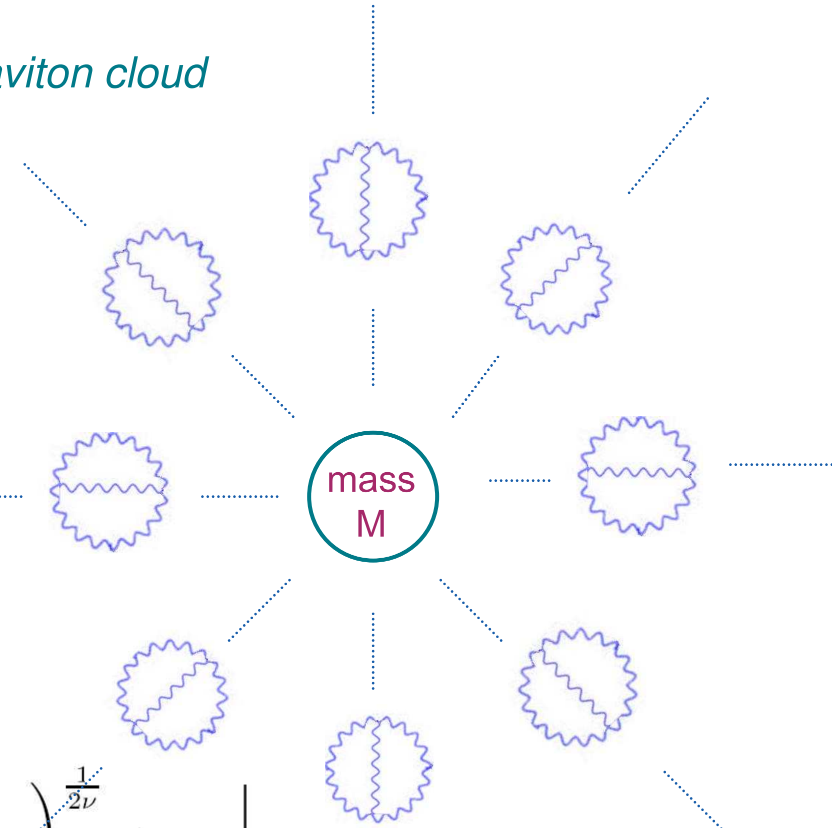
Need a *covariant* running of G.

Effective field equations:

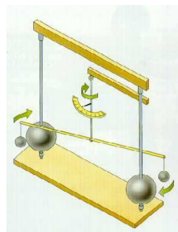
$$\boxed{R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G(\Box) T_{\mu\nu}}$$

$$\lambda \simeq 1/\xi^2 \qquad G \to G(\Box) = G \left[1 + a_0 \left(\frac{m^2}{-\Box + m^2} \right)^{\frac{1}{2\nu}} + \dots \right]$$

$$\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$$



Relative Scales in the Cutoff Theory



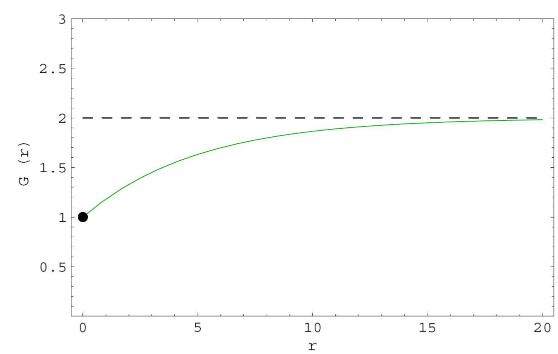
$\approx 10^{-33}\, cm$

$\approx 10^{28}\, cm$

l_P $\qquad\qquad\qquad\qquad$ ξ

$$\boxed{l_P << r << \xi}$$

$$r >> \xi$$



At the Planck length new terms appear:

higher derivative terms, string corrections, conformal anomaly contributions…

Cosmological Solutions

Explore possible effective field equations…generally covariant

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G \left(1 + A(\Box)\right) T_{\mu\nu}}$$

G. Veneziano
G.A. Vilkovisky ..

$$A(\Box) = c_{\Box} \left(\frac{1}{\xi^2 \Box} \right)^{1/2\nu}$$

$\Lambda = 0$ initially for simplicity

… for RW metric

$$ds^2 = -dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - k r^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \right\}$$

… and perfect fluid $p(t) = 0$

Consistency condition:

$$\nabla^\mu \tilde{T}_{\mu\nu} \equiv \nabla^\mu \left[\left(1 + A(\Box)\right) T_{\mu\nu} \right] = 0$$

$$\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu \qquad \Box \, T^{\alpha\beta\ldots}{}_{\gamma\delta\ldots} = g^{\mu\nu} \nabla_\mu \left(\nabla_\nu T^{\alpha\beta\ldots}{}_{\gamma\delta\ldots} \right)$$

Form of D'Alembertian depends on object it acts on …

Solution of Effective Field Equation

- Full effective field equation involves D'Alembertian on tensor

$$\boxed{R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G \left(1 + A(\Box)\right) T_{\mu\nu}}$$

$$A(\Box) = c_\Box \left(\frac{1}{\xi^2 \Box}\right)^{1/2\nu}$$

$$\Lambda = 0$$

Repeated action of D'Alembertian $\Box^n$, $n = -1/2\nu$

$$
\begin{aligned}
(\Box T_{\mu\nu})_{tt} &= 6\,[\rho(t) + p(t)] \left(\frac{\dot R(t)}{R(t)}\right)^2 - 3\,\dot\rho(t)\,\frac{\dot R(t)}{R(t)} - \ddot\rho(t) \\
(\Box T_{\mu\nu})_{rr} &= \frac{1}{1 - k r^2} \left\{ 2\,[\rho(t) + p(t)]\,\dot R(t)^2 - 3\,\dot p(t)\,R(t)\,\dot R(t) - \ddot p(t)\,R(t)^2 \right\} \\
(\Box T_{\mu\nu})_{\theta\theta} &= r^2 (1 - k r^2)\,(\Box T_{\mu\nu})_{rr} \\
(\Box T_{\mu\nu})_{\varphi\varphi} &= r^2 (1 - k r^2) \sin^2\theta\,(\Box T_{\mu\nu})_{rr}
\end{aligned}
$$

$\Longrightarrow$ existence of solution requires $\beta = -2 - 1/\nu$ as before,

and $R(t) \sim t^\alpha \sim t^{1/2}$ $\qquad \rho(t) \sim t^{-2-1/\nu} \sim (R(t))^{-2(2+1/\nu)}$

Accelerated expansion, even with c.c. Λ=0. $\qquad \rho(t) \sim t^{-2}$

Cosmological Solutions – Cont'd

- Modified FRW solution acquires a significant *radiation-like* (vac. pol.) component at large times,

$$\frac{k}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G(t)}{3}\rho(t) + \frac{1}{3\,\xi^2}$$
$$= \frac{8\pi G}{3}\left[1 + c_\xi\,(t/\xi)^{1/\nu} + \dots\right]\rho(t) + \frac{1}{3\,\xi^2}$$

t-t eq.

$$\rho_{eff}(t) = \frac{G(t)}{G}\rho(t)$$

$$\frac{k}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)} + \frac{2\ddot{a}(t)}{a(t)} = \boxed{-\frac{8\pi G}{3}\left[c_\xi\,(t/\xi)^{1/\nu} + \dots\right]\rho(t)} + \frac{1}{\xi^2}$$

r-r eq.

$$p_{eff}(t) = \frac{1}{3}\left(\frac{G(t)}{G} - 1\right)\rho(t)$$

Effective pressure term

Similarities to:

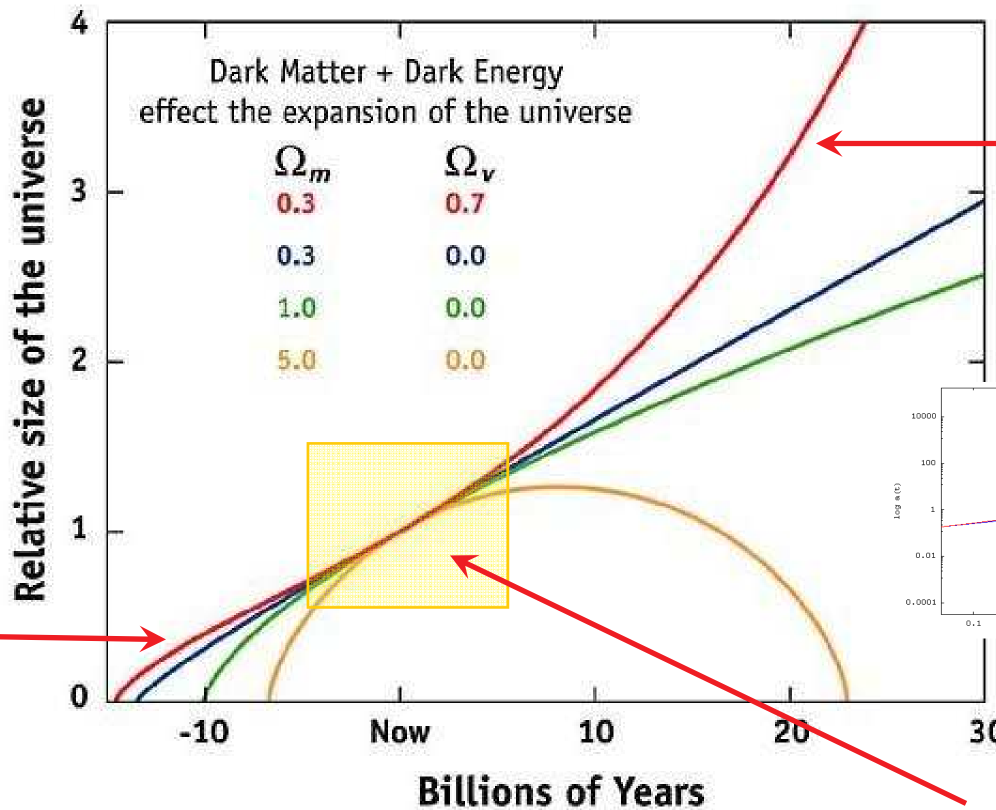
$$p(t) = \omega\,\rho(t)$$
$$\omega = 1/3$$

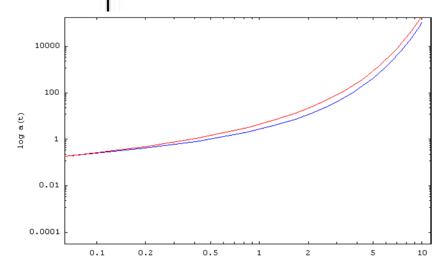
At (very) large times, G is further modified to:

$$G(t) = G\left[1 + c_\xi\left(\frac{t}{\xi}\right)^{1/\nu} + \dots\right] \longrightarrow G(t) \simeq G\left[1 + c_\xi\left(\frac{t^2}{t^2 + \xi^2}\right)^{\frac{1}{2\nu}} + \dots\right]$$

IR regulator

Modified cosmological expansion rate





Λ-dominated expansion at *later times*

Standard FRW expansion at *early times*

Running G effects are maximal *“now”*

[T. D’Amour 2007]

Static Isotropic Solution

Start again from *fully covariant* effective field equations

$$\boxed{R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G \, (1 + A(\Box)) \, T_{\mu\nu}}$$

$$A(\Box) = a_0 \left(\frac{m^2}{-\Box + m^2} \right)^{1/2\nu}$$

General *static isotropic* metric

$$\lambda \simeq 1/\xi^2 \longrightarrow 0$$

$$ds^2 = -B(r)\, dt^2 + A(r)\, dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

$$A(r)^{-1} = 1 - \frac{2MG}{r} + \frac{\sigma(r)}{r}$$

$$B(r) = 1 - \frac{2MG}{r} + \frac{\theta(r)}{r}$$

a_0 small

$r \gg 2MG$

Search solution for a point source, or *vacuum solution* for r$\neq$0.

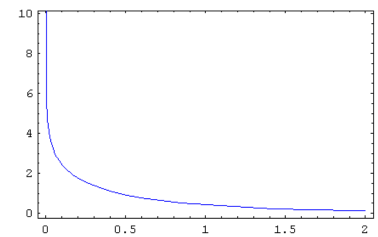
$$T_{\mu\nu} = \mathrm{diag}\,[\, B(r)\, \rho(r), \; A(r)\, p(r), \; r^2\, p(r), \; r^2 \sin^2\theta \, p(r) \,]$$

H. & Williams, PLB 2006; PRD 2007

Static Isotropic Solution

Non-relativistic solution can be obtained from *vacuum density:*

$$\rho_m(r) \;=\; \frac{1}{8\pi}\, c_\nu\, a_0\, M\, m^3\, (m\,r)^{-\frac{1}{2}(3-\frac{1}{\nu})}\, K_{\frac{1}{2}(3-\frac{1}{\nu})}(m\,r) \qquad m = 1/\xi$$



$$4\,\pi \int_0^\infty r^2\, dr\, \rho_m(r) \;=\; a_0\, M$$

Promote $\varrho(r)$ to a covariantly conserved, relativistic perfect fluid, with a *p(r):*

$$T_{\mu\nu} \;=\; \mathrm{diag}\,[\, B(r)\,\rho(r),\; A(r)\,p(r),\; r^2\,p(r),\; r^2\,\sin^2\theta\,p(r)\,] \qquad \boxed{\nabla^\mu\, T_{\mu\nu} = 0}$$

Relativistic field equations become:

$$-\lambda B(r) + \frac{A'(r)B(r)}{rA(r)^2} - \frac{B(r)}{r^2A(r)} + \frac{B(r)}{r^2} \;=\; 8\pi G B(r)\rho(r)$$

$$\lambda A(r) - \frac{A(r)}{r^2} + \frac{B'(r)}{rB(r)} + \frac{1}{r^2} \;=\; 8\pi G A(r)p(r)$$

$$[\,p(r) + \rho(r)\,]\,\frac{B'(r)}{2B(r)} + p'(r) \;=\; 0$$

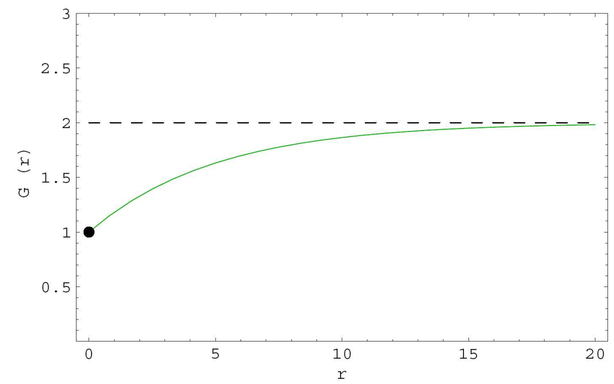
$$-\frac{B'(r)^2r^2}{4A(r)B(r)^2} + \lambda r^2 - \frac{A'(r)B'(r)r^2}{4A(r)^2B(r)} + \frac{B''(r)r^2}{2A(r)B(r)} - \frac{A'(r)r}{2A(r)^2} + \frac{B'(r)r}{2A(r)B(r)} \;=\; 8G\pi r^2p(r)$$

Relativistic Fluid cont'd

And finally …

$$B(r) = 1 - \frac{2\,M\,G}{r} + \frac{4\,a_0\,M\,G\,m^3}{3\,\pi}\,r^2\,\ln(m\,r) + ..$$

$$A^{-1}(r) = 1 - \frac{2\,M\,G}{r} + \frac{4\,a_0\,M\,G\,m^3}{3\,\pi}\,r^2\,\ln(m\,r) + ..$$



…which can be *consistently* interpreted as a ***G(r):***

$$\boxed{G \;\rightarrow\; G(r) = G\left(1 + \frac{a_0}{3\,\pi}\,m^3\,r^3\,\ln\frac{1}{m^2\,r^2} + \ldots\right)}$$

$$m = 1/\xi$$

Reminiscent of QED (Uehling) answer:

$a_0 \simeq 42.$

$$Q(r) = 1 + \frac{\alpha}{3\,\pi}\ln\frac{1}{m^2\,r^2} + \ldots \qquad m\,r \ll 1$$

Outlook

- More Work is Needed
  - $2 + \varepsilon$ expansion to three loops is a clear, feasible goal.
  - Systematic careful investigation of 4d s. gravity should be pursued
  - Status of weak coupling phase unclear
  - Connection with other lattice models, eg hypercubic?

- Covariant Effective Field Equations
  - Formulation of fractional operators.
  - Further investigation on nature of solutions (horizons).
  - Possible Cosmological (observable) ramifications.

The End

“Herb, as far as I know you are the only one that still believes in this non-trivial ultraviolet fixed point scenario [for gravity] ☺ ”

Howard Georgi, January 30, 2008

Large D and Strong Coupling

- Strong coupling expansion for gravity,

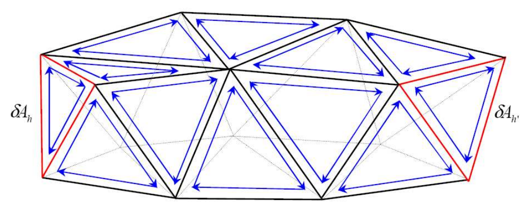
$$Z_{latt}(k) = \int d\mu(l^2)\; e^{k \sum_h \delta_h A_h} = \sum_{n=0}^{\infty} \frac{1}{n!} k^n \int d\mu(l^2) \left(\sum_h \delta_h A_h \right)^n$$

$$k = 1/(8\pi G)$$

$$< \delta A > = \frac{\sum_{n=0}^{\infty} \frac{1}{n!} k^n \int d\mu(l^2)\, \delta A \left(\sum_h \delta_h A_h \right)^n}{\sum_{n=0}^{\infty} \frac{1}{n!} k^n \int d\mu(l^2) \left(\sum_h \delta_h A_h \right)^n}$$

$$\mathcal{R}(k) \underset{k \to k_c}{\sim} A_{\mathcal{R}} \, (k_c - k)^{\delta}$$

$$(-1)^n A_{\mathcal{R}} \frac{(\delta - n + 1)(\delta - n + 2) \ldots \delta}{n! \, k_c^{n-\delta}} k^n$$



At large d, strong coupling (large G) expansion simplifies considerably, as *excluded volume* effects can be neglected in this limit …

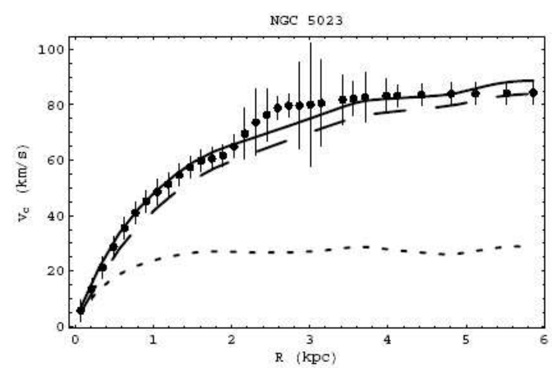
Galactic Rotation Curves

- Straightforward, calculable relationship between *potential modification* and deviations in *galactic rotation curves*

$$\Phi(r) = -\frac{Gm}{r}\left[1 + \left(\frac{r}{r_c}\right)^{\beta}\right]$$

$$v_c^2(r) = \frac{Gm}{r}\left[1 + (1-\beta)\left(\frac{r}{r_c}\right)^{\beta}\right]$$

Capoziello, Cardone,Troisi
astro-ph0602349 (2006)



Very large values of rc (ξ) make effects tiny on kp scales.

Relation to R^n Gravity Models

- *Superficial* resemblance of running $G(r)$ model to R^n & scalar-tensor gravity theories (within *FRW* cosmology framework).

S. Capoziello, A.Troisi et al
S. Carroll et al ; E. Flanagan

- Obtained - from running $G(r)$ models - by simply replacing scale factor *a(t)* with scalar curvature R :

$$R = 6\left(k + \dot{a}^2(t) + a(t)\,\ddot{a}(t)\right)/a^2(t)$$

$$I_{eff} \simeq \frac{1}{16\pi G}\int dx\,\sqrt{g}\left(R + \frac{f\,\xi^{-\frac{1}{\nu}}}{(R)^{\frac{1}{2\nu}-1}} - 2\,\lambda\right)$$

$$\lambda \simeq 1/\xi^2$$

$F(R)$ models generally lack justification as to why only Ricci scalar R should be considered in action.

Static Isotropic Solution in d Dim's

Covariant effective field equations in d space-time dimensions

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G \left(1 + A(\Box)\right) T_{\mu\nu}$$

$$\begin{cases} A(\Box) = a_0 \left(\dfrac{m^2}{-\Box + m^2} \right)^{1/2\nu} \\ \lambda \simeq 1/\xi^2 \longrightarrow 0 \end{cases}$$

In the absence of a running G, static isotropic solutions in d dimensions are given by:

Myers and Perry, Ann. Phys 1986
Xu, Class Q Grav 1988

$$A^{-1}(r) = B(r) = 1 - 2MGc_d\, r^{3-d} - \frac{2\lambda}{(d-2)(d-1)}\, r^2$$

$$c_d = 4\pi\Gamma(\tfrac{d-1}{2})/(d-2)\pi^{\frac{d-1}{2}}$$

Based on same arguments as in d = 4 would expect solution to exists only if $\boxed{\nu = 1/(d-1)}$ consistent with the result $\nu = 0$ found on the lattice at d = ∞. Problem not fully worked out yet.

Effects of small gauge breaking

- "Dynamical stability of local gauge symmetry"

D. Foerster, H. B. Nielsen, M. Ninomiya , Physics Letters B 94, 135 (1980)

*"We show that the large distance behavior of gauge theories is **stable**, within certain limits, with respect to addition of gauge non-invariant interactions at small distances."*

- G. Parisi argument - for compact groups (ca. 1996, unpublished):

"The effects of *small* gauge-breaking terms average to zero"

Proof based on *Elitzur's theorem* on the Impossibility Of Spontaneously Breaking Local Symmetries (PRD 12, 3978,1975), for compact gauge groups.

$$Z = \int D\Omega\, DA\, e^{-S_{sym}(A_\Omega) - \delta S(A_\Omega)} = \int D\Omega\, DA\, e^{-S_{sym}(A) - \delta S(A_\Omega)}$$

$$\int D\Omega\, e^{-\,\delta S(A_\Omega)} = e^{-\delta S_{sym}(A)} \qquad \delta S_{sym}(A) = \sum_x \delta\mathcal{L}_{sym}(A, x)$$

Quantum "Gravity" in two dimensions ?

- KPZ formula predicts ν=3/2 etc. (non-Onsager exponents) for c=1/2 (Ising spins) coupled to gravity (*"gravitational dressing of exponents"*).
- *A flat space* realization of same KPZ exponents is found instead:

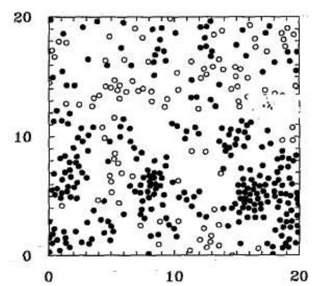
The change in the exponents appears due to the *randomness* of the interaction.

Vekic, Liu & H, PLB 1994; PRD 1994

$$I[x,S] \;=\; -\sum_{i<j} J_{ij}(x_i,x_j)\, W_{ij}\, S_i S_j - h \sum_i W_i\, S_i$$

$$J_{ij}(x_i,x_j) = \begin{cases} 0 & \text{if } |x_i - x_j| > R \\ J & \text{if } r < |x_i - x_j| < R \end{cases}$$

$$Z = \prod_{i=1}^{N} \sum_{S_i=\pm 1} \left(\prod_{a=1}^{d} \int_0^L dx_i^a\right) \exp(-I[x,S])$$



PHYSICAL REVIEW D VOLUME 51, NUMBER 8 15 APRIL 1995

Random Ising spins in two dimensions: A flat space realization of the Knizhnik-Polyakov-Zamolodchikov exponents

Marco Vekić
Institute for Theoretical Physics, University of California Santa Barbara, Santa Barbara, California 93106

Shao Liu
Department of Physics, University of California Irvine, Irvine, California 92717

Herbert W. Hamber
Theory Division, CERN, CH-1211 Genève 23, Switzerland

(Received 25 July 1994)

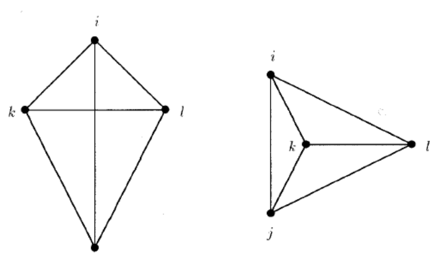
A model describing Ising spins with short range interactions moving randomly in a plane is considered. In the presence of a hard-core repulsion, which prevents the Ising spins from overlapping, the model is analogous to a dynamically triangulated Ising model with spins constrained to move on a flat surface. As a function of coupling strength and hard-core repulsion the model exhibits multicritical behavior, with first- and second-order transition lines terminating at a tricritical point. The thermal and magnetic exponents computed at the tricritical point are consistent with the KPZ values associated with Ising spins, and with the exact two-matrix model solution of the random Ising model, introduced previously to describe the effects of fluctuating geometries.

Table 1: Critical exponents of random and non-random Ising models.

	γ/ν	β/ν	α/ν	α	ν
Onsager solution on regular flat lattice	1.75	0.125	0	0	1
Ising spins coupled to gravity[30, 33]	1.73(2)	0.124(3)	-0.06(11)	-	0.98(1)
Matrix model and CFT [57, 58]	1.333...	0.333...	-0.666...	-1.0	1.5
Random Ising spins in *flat* space[60]	1.32(3)	0.31(4)	-0.65(4)	-0.98(4)	1.46(8)

Is there Q. Gravity in two dimensions ?

Follow up: dynamically triangulated Ising spins (c=1/2) on *fixed curved geometry* (sphere) also give KPZ exponents.



Some lattice re-linkings

ELSEVIER   Nuclear Physics B 452 (1995) 415–428

Is there quantum gravity in two dimensions? ⋆

Wolfgang Beirl [a,1], Bernd A. Berg [b,c,2]

[a] *Institut für Kernphysik, Technische Universität Wien, A-1040 Vienna, Austria*
[b] *Department of Physics, The Florida State University, Tallahassee, FL 32306, USA*
[c] *Supercomputer Computations Research Institute, Tallahassee, FL 32306, USA*

Received 17 May 1995; accepted 7 July 1995

Abstract

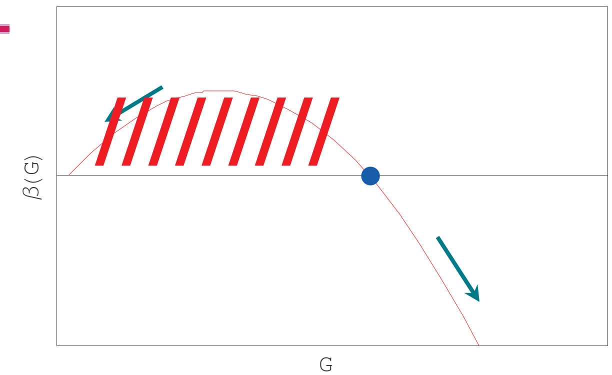
A hybrid model that allows one to interpolate between the (original) Regge approach and dynamical triangulation is introduced. The gained flexibility in the measure is exploited to study dynamical triangulation in a *fixed* geometry. Our numerical results support KPZ exponents. A critical assessment concerning the apparent lack of gravitational effects in two dimensions follows.

Beirl and Berg, NPB 1995

Three Approaches Compared

◊ Continuum 2+ε expansion (1 loop):

$$G(k^2) \; \simeq \; G_c \left[1 \, + \, \left(\frac{m^2}{k^2} \right)^{(d-2)/2} \; + \; \dots \right]$$



◊ Simplicial Lattice QG:

$$G(\Box) \; = \; G \left[1 \, + \, a_0 \left(\frac{m^2}{\Box} \right)^{\frac{1}{2\nu}} \; + \; \dots \right]$$

$$\begin{cases} G(r) \; = \; G \left(1 \, + \, \frac{a_0}{3\,\pi} \, m^3 \, r^3 \, \ln \, \frac{1}{m^2 \, r^2} \, + \, \dots \right) \\ \phi(r) \underset{r \to \infty}{\sim} - \frac{M\,G}{r} \left[1 \, + \, a_0 \, \left(1 \, - \, c_l \, (m\,r)^{\frac{1}{2\nu} - 1} \, e^{-mr} \right) \right] \end{cases}$$

$$\boxed{\nu = 1/3}$$

◊ E-H “truncation” (Reuter/Litim):

$$G(k) = \frac{G_0}{1 + \omega \, G_0 \, k^2} \longrightarrow G(r) = \frac{G_0 \, r^3}{r^3 + \tilde{\omega} \, G_0 \, [r + \gamma G_0 M]}$$

- ω generally complex
- Scale ξ ? Phases ?
- Formulation of G(r) *not* covariant
- Effects fitted to g. rotation curves