Nuclear Physics B (Proc. Suppl.) 99A (1994) 1–1
North-Holland

PHASES OF SIMPLICIAL QUANTUM GRAVITY

Herbert W. Hamber

Department of Physics
University of California at Irvine
Irvine, CA 92717, USA

Recent results for simplicial quantum gravity are reviewed. Models are considered in which the topology is fixed, and the edge lengths are varied while the coordination number is held fixed ('Regge calculus'). As expected on the basis of universality, in two dimensions the results for pure gravity critical exponents are found to be in agreement with the conformal field theory predictions. The effects of both higher derivative terms and gravitational measure contributions are investigated in detail, as well as the inclusion of a D-component scalar field. For sufficiently large D a phase transition is found, leading from the Liouville phase into a branched polymer phase. The results suggest universal critical behavior, and are in good agreement with recent results obtained with the dynamical triangulation models. While no phase transition is found for pure gravity in two dimensions, a phase transition between a 'rough' and a 'smooth' phase of spacetime is found in both three and four dimensions, in agreement with the results of the $2+\epsilon$-expansion in the continuum. In both cases the transition appears to be continuous (suggesting therefore the existence of a lattice continuum limit), and the critical exponents can be estimated. While fluctuations in the local volumes are responsible for critical behavior in two dimensions, in three and four dimensions it is instead the fluctuations in the local curvature that diverge at the critical point.

1. Introduction

In this review I will concentrate on the simplicial formulation of quantum gravity, also known as Regge calculus [1] (for reviews, see refs. [2–5]; some earlier reviews can be found in [6,7]). One of the advantages of the simplicial approach lies in the fact that it can be formulated in any space-time dimension (including the physically relevant case of four dimensions), and that it can be shown to be classically completely equivalent to general relativity. This correspondence is particularly transparent in the usual weak field expansion [8], with the edge lengths on the lattice playing the role of the components of the metric field in the continuum. Furthermore the correspondence between lattice and continuum quantities is clear, and the interpretation of the terms in the action, as well as the identification and separation of the measure contribution, is unambiguous. The weak field expansion has not been worked out yet for dynamically triangulated models for random surfaces [22,21,23], which lack reparametrization invariance as well (for example only one triangulation of flat space exists, a regular triangular lattice). On the other hand reparametrization invariance seems to be recovered in all models at the quantum level, which represents a rather remarkable result. Of course, since gravity is not well defined in the continuum, a number of difficulties, related for example to the gravitational measure factor [18], the unboundedness of the euclidean gravitational action above two dimensions [19], and the lack of perturbative renormalizability in greater than two dimensions [20], persist in the lattice formulation as well. For a more complete list of early references on simplicial gravity, the reader is referred to ref. [2].

Some important issues that need to be addressed in lattice quantum gravity include the following:

- •Restoration of coordinate invariance
- •Possible phases of the theory
- •Universality in lattice gravity

In this review I will in sequence discuss the following topics, in the context of the simplicial quantum gravity model to be discussed below,

- •Pure lattice gravity in d = 2
- •Gravity coupled to D scalar fields
- •Universality and phases of 2 − d gravity
- •Dependence on the measure
- •Phases of pure gravity in d = 3
- •Critical behavior in d = 3
- •Phases of pure gravity in d = 4
- •Critical behavior in d = 4

A detailed description of the construction of the action for simplicial lattice gravity without and with matter fields can be found in the literature [14,16,17,2]. In a number of cases the correspondence with the continuum answer can be established quite rigorously and in great generality [8–10,13]. Here we will only recall the geometric correspondence between continuum and lattice quantities in *two* dimensions, and which can be verified explicitly within the lattice and continuum weak field expansion [17],

$$\int d^2x \, \sqrt{g(x)} \qquad \rightarrow \quad \textstyle\sum_i A_i$$

$$\int d^2x \, \sqrt{g(x)} \; R(x) \qquad \rightarrow \; 2 \, \textstyle\sum_i \delta_i$$

$$\int d^2x \, \sqrt{g(x)} \; R^2(x) \;\; \rightarrow \; 4 \, \textstyle\sum_i \delta_i^2 / A_i \tag{1}$$

where δ_i is the deficit angle at the vertex i, and A_i is the area associated with the site i (which is not necessarily unique, since the lattice can be subdivided geometrically in more than one way). Similar formulae hold for higher dimensions; given reasonable geometric and positivity properties, universality is expected to lead to the same results for universal quantities (like physical observables, exponents, anomalous dimensions etc.) in some continuum limit. (This is known to be the case in ordinary lattice field theories (lattice $\lambda\phi^4$, lattice QCD, lattice QED, ...), where the physical particle spectrum is expected to be independent of specific details of the ultraviolet lattice cutoff and the specific form of the lattice action, as long as it preserves the basic symmetries; it has also been verified explicitly to some extent in the case of two-dimensional gravity [26], where good agreement is found between the lattice gravity results and the conformal field theory predictions for critical exponents. As in the continuum, in the simplicial formulation the local curvature $R(x) \sim 2\delta_i/A_i$ is a continuous function of the relevant edge lengths; the straightforward geometric correspondence of lattice and continuum quantities is one nice feature of the Regge calculus approach, and allows one for example to distinguish clearly between various higher dimensional curvature invariants [16].

While simplicial quantum gravity can be formulated on a random lattice [27], it appears advantageous at least initially to adopt a regular lattice, which is much easier to work with from a computational point of view. The continuous diffeomorphism invariance discussed in [1,8], as well as in [26], is of course not lost by going to such a regular lattice. On the other hand the random lattice might appear more satisfactory from a conceptual point of view, since it incorporates, for smooth manifolds, the invariance under 'large' lattice diffeomorphisms, whereas in the regular lattice only 'small' lattice diffeomorphisms are allowed. Thus the two different lattices induce quite a different cutoff structure in orbit space. Eventually one hopes to redo all the calculations for such a random lattice. Most of the numerical simulations of lattice gravity in four dimensions have been done for such regular lattices [14–17,28]. In most of the following we will discuss results for a simplicial complex topologically equivalent to the torus (in two, three and four dimensions), and, in one instance, to the two-sphere. This is not because one attaches any specific significance to the torus, but because it is easier to work with. In the end though, one expects ultraviolet renormalization effects to be independent of the boundary conditions, and therefore of the topology.

One further issue that needs to be addressed is the problem of the gravitational lattice measure. In the continuum the form of the measure for the $g_{\mu\nu}$ fields appears not to be unique [18,29,34]. The reason for the ambiguity appears to be a lack of a clear definition of what is meant by $\prod_x$ in the gravitational functional measure. In spite of some recurrent claims to the contrary, it would seem that such an ambiguity persist in *all* known lattice formulations of quantum gravity. However the difference among the measures seems to be in the power of $\sqrt{g}$ in the prefactor, which corresponds to some product of volume factors on the lattice. These volume factors do not give rise on the lattice to edge-edge coupling terms (derivatives of the metric in the continuum), and are therefore strictly local.

DeWitt [32,18] has argued that the gravitational measure should be constructed by first introducing a super-metric over metric deformations, which in its simplest local form then leads to a functional measure for pure gravity in d dimensions of the type

$$d\mu[g] = \prod_x g^{(d-4)(d+1)/8} \prod_{\mu \geq \nu} dg_{\mu\nu} \tag{2}$$

Another popular (pure) gravitational measure in the continuum is the Misner scale-invariant measure [29]

$$d\mu[g] = \prod_x \; g^{-(d+1)/2} \prod_{\mu \geq \nu} dg_{\mu\nu} \tag{3}$$

It is unique if the product over x is interpreted as one over 'physical' points, and coordinate invariance is imposed at one and the same 'physical' point. Other forms of the measure for the gravitational field have also been suggested [33], some inspired by the canonical quantization approach to gravity [34]. If matter fields are present, then the gravitational measure has to be further modified [32] (see discussion below).

On the simplicial lattice it is clear that the edge lengths are the elementary degrees of freedom, which uniquely specify the geometry for a given incidence matrix, and over which one should perform the functional integral [14,15,3]. Indeed the induced metric at a simplex is related to the edge lengths squared within that simplex, via the expression for the invariant line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. The correspondence between the induced metric field defined within a simplex s, and the lengths of the edges of that simplex [6], is given explicitly by

$$g_{ij}(s) = \frac{1}{2} \left[\, l^2_{s,s+i} + l^2_{s,s+j} - l^2_{s+i,s+j} \, \right] \tag{4}$$

One notices that in general each edge is shared between several contiguous simplices, and that an integration over the edges is not simply related to an integration over the metric (even though there are $d(d+1)/2$ edges for each simplex, just as there are $d(d+1)/2$ independent components for the metric tensor in d dimensions). Thus originally the pure gravity measure

$$\int d\mu_\epsilon[l] = \prod_{\text{edges ij}} \int_0^\infty dl_{ij}^2 \, l_{ij}^q \; F_\epsilon[l] \qquad (5)$$

was suggested. Here $q = 0, -2$, and $F_\epsilon[l]$ is a function of the edge lengths, with the property that it is equal to one whenever the triangle inequalities and their higher dimensional analogues are satisfied, and zero otherwise. $q = 0$ then corresponds to the DeWitt measure in four dimensions, while $q = -2$ corresponds to a scale invariant measure like the continuum Misner measure. The parameter ϵ can be introduced as an ultraviolet cutoff at small edge lengths: the function $F_\epsilon[l]$ is zero if any of the edges are equal or less than ϵ (the introduction of such a cutoff seems to be necessary in four dimensions [15], but not in two [17]. The above measure is of course correct in the weak field limit, where all continuum measures agree as well, and integrates over coordinate independent quantities, the invariant lengths of the edges.

But since we have argued that it is not entirely clear what the continuum measure should be, it would seem of interest to explore the sensitivity of the results to the type of gravitational measure employed. One would expect on the basis of universality that different measures might lead to the same continuum limit. Another class of pure gravity measures which can be written down on the lattice is obtained by considering the 'volume associated with an edge' V_{ij} (which corresponds to the quantity $\sqrt{g(x)}$ in the continuum), and writing

$$\int d\mu_\epsilon[l] = \prod_{\text{edges ij}} \int_0^\infty V_{ij}^{2\sigma} \; dl_{ij}^2 \; F_\epsilon[l] \qquad (6)$$

with $\sigma = -1/d$ for the lattice analogue of the Misner measure, and $\sigma = (d-4)/4d$ for the lattice analogue of the DeWitt measure.

Eventually one would like to see how the results depend on the form of the measure and on the measure parameter σ (or q). In the case of two dimensional gravity, which will be discussed below, extensive studies can be done, and compared to exact results known from conformal field theory, There the numerical results show that different measures, within a certain universality class, will give the same results for infrared sensitive quantities, like correlation functions at large distances, and critical exponents. The lattice path integral might not be meaningful though for certain values of σ; in particular if the measure parameter σ is too negative in two dimensions, then the measure factor tends to favor configurations of triangles which are long and thin, with a small area and a large perimeter. The lattice tends to degenerate into a lower-dimensional manifold, a situation far from the desired continuum limit, and which one would like to avoid.

2. Pure 2-d Gravity

A suitable action for pure lattice gravity in two dimensions was given in ref. [17], in the form

$$I = \sum_{\text{sites i}} \left[\lambda \, A_i - k \, \delta_i + a \, \frac{\delta_i^2}{A_i} \right] \qquad (7)$$

In the limit of small fluctuations around a smooth background, the above lattice action was shown to correspond to the continuum action

$$I = \int d^2x \, \sqrt{g} \, \left[\lambda - \frac{k}{2} \, R + \frac{a}{4} \, R^2 \right] \qquad (8)$$

up to higher order lattice corrections. For a manifold of fixed topology the term proportional to k of course can be dropped. The higher derivative term proportional to a was introduced since it can be useful in controlling the fluctuations in the intrinsic local curvature, although it is expected to be irrelevant as far as critical properties are concerned (its presence prevents the appearance of conical singularities, where the Gaussian curvature might become very large, but it does not of course prevent 'folding' singularities, corresponding to singular structures in the manifold which appear in embedding space only; to control the latter an extrinsic curvature term is required). For $a \to \infty$ the manifold approaches a flat limit, whereas for $a \to 0$ local fluctuations in the curvature become more pronounced. It was further shown in [17] that no sensible ground state exists for $a < 0$, while the path integral is well behaved for $a > 0$ (and positive λ), and for this reason in the following we will only discuss the case $a \geq 0$. For $a > 0$, $\lambda > 0$ no phase transition was found in two dimensions, but long range excitations are clearly present due to the Liouville mode (see discussion below).

Recently the numerical simulations results have been significantly extended [26,35] to cover a variety of critical exponents, which can be compared to the conformal field theory predictions (for $D < 1$), and also in order to explore the phase structure for $D \geq 1$, where no exact results are available. In the numerical simulations the configurations of edge lengths are generated by the standard Monte Carlo algorithm, as discussed in refs. [17], using as a background space a network of unit squares divided into triangles by drawing in parallel sets of diagonals, with periodic boundary conditions to minimize edge effects.

2.1. Critical Exponent γ_χ

In our previous work [17] the canonical ensemble (with fixed bare cosmological constant λ) was considered. But in order to compare with the exact conformal field theory results obtained by KPZ [38–40], one should instead consider an ensemble where the total area A is kept fixed. In such an ensemble the limit $A \to \infty$ corresponds to $\lambda \to \lambda_c$ in the canonical ensemble. One approaches the limit $A \to \infty$ by letting the number of sites $N \to \infty$, for fixed elementary triangle areas. The lattice analogue of

$$Z\,[A] = \int d\mu[g]\; \delta\Big(\int \sqrt{g} - A\Big)\; \exp\,(-I[g]) \qquad (9)$$

is then expected for large area to behave like

$$Z\,[A] \underset{A\to\infty}{\sim} A^{-3+\gamma_\chi}\; \exp\,(-(\lambda - \lambda_0)A) \qquad (10)$$

with $\gamma_\chi = \chi(\gamma - 2)/2 + 2$. Here χ is the Euler characteristic, and

$$\gamma = \frac{1}{12}(D - 1 - \sqrt{(D-1)(D-25)}) \qquad (11)$$

the string susceptibility exponent [38]. Pure gravity without matter fields then corresponds to the case $D = 0$, and in particular on the torus one has the prediction $\gamma_\chi = 2$, independent of D [39,40].

It is easy to see that the exponent γ_χ can be related to a finite size correction [26],

$$\frac{a}{4}\, \frac{< \int \sqrt{g}\; R^2 >_A}{A} \underset{A\to\infty}{\sim} \mathrm{c} - \frac{2-\gamma_\chi}{A} + \cdots \qquad (12)$$

One can compute the coefficient of the finite size correction to R^2 by investigating both the torus and the sphere. In both cases the lattices considered contained from 48 to 12288 edges (corresponding to lattices with between 4^2 and 64^2 sites); the coupling a was set $= 1$, and the measure dl/l of eqn. (5) was used. More details can be found in [26].

In the case of the torus the results for the coefficient of the $1/A$ term are quite accurate, and one finds $2 - \gamma_\chi = 0.025 \pm 0.022$, consistent with the exact result for the torus $\gamma_\chi = 2$. For the sphere the results for γ_χ are not quite as accurate as for the torus, but one still gets the estimate $\gamma = -0.55 \pm 0.22$, to be compared to the expected KPZ answer of $\gamma = -1/2$. The above results, which are in good agreement with the continuum KPZ prediction, suggest a complete restoration of general coordinate invariance at large distances in the lattice theory.

According to Polyakov [36] the only degree of freedom of pure 2-d gravity is the Liouville mode, corresponding to local fluctuations in the area. It is therefore meaningful to investigate the critical properties of the area fluctuation or Liouville field $\varphi(x)$ on the simplicial lattice. One can define the discrete analogue of the continuum Liouville field $\varphi(x) = \ln \sqrt{g(x)}$ as $\varphi_i = \ln A_i$ (A_i is the area associated with the site i), and compute the Liouville field susceptibility on a finite lattice

$$\chi_\varphi(L) = A \, [< \varphi^2 > - < \varphi >^2] \tag{13}$$

where $L = \sqrt{A}$, and we have defined

$$\varphi = \frac{1}{A} \sum_i \ln A_i \tag{14}$$

Indeed (for the measure dl/l at $a = 0$) one finds, using finite size scaling on tori of sizes $L = \sqrt{A} = 8 - 128$ (see fig. 1),

$$\ln \chi_\varphi(L) \underset{L \to \infty}{\sim} c + (2 - \eta_\varphi) \, \ln L \tag{15}$$

with a critical exponent $2 - \eta_\varphi = 2.08 \pm 0.12$, in good agreement with the Liouville prediction $\eta_\varphi = 0$. Therefore the results suggest again a complete restoration of reparametrization invariance in the quantum theory. This conclusion is further supported by the results obtained when a massless scalar field is coupled to gravity.

2.2. Scalar Fields Coupled to Gravity

One can couple to gravity a D-component scalar field ϕ_i^a, $a = 1, ..., D$, with the scalar field defined at the vertices of the lattice triangles. A suitable action for a massless field is then

Fig. 1. Area fluctuation or Liouville susceptibility, as a function of lattice size. The circles correspond to pure gravity and $a = 0$ ($D = 0$). The straight line on the logarithmic scale indicates a growth proportional to L^2, in agreement with the Liouville theory prediction.

$$I\,[\phi] = \frac{1}{2} \sum_{\text{edges ij}} V_{ij} \, \left(\frac{\phi_i^a - \phi_j^a}{l_{ij}} \right)^2 \tag{16}$$

where V_{ij} is the volume associated with the edge ij. The above lattice action then corresponds to the continuum expression

$$\frac{1}{2} \int d^2x \, \sqrt{g} \; g^{\mu\nu} \; \partial_\mu \phi^a \; \partial_\nu \phi^a \tag{17}$$

The scalar fields can also be considered as coordinates in a D-dimensional euclidean embedding space, and the results can be compared to predictions from random surface models [36–38].

It appears that the gravitational measure has to be modified in the presence of the matter fields. The DeWitt argument (involving now the norm of the scalar field deformation) suggests, for a D-component scalar field, the invariant measure [32]

$$d\mu[\phi] = \prod_x g^{D/4}(x) \; \prod_a d\phi^a(x) \tag{18}$$

By combining it with the pure gravitational measure, it gives the combined measure for the gravitational and scalar degrees of freedom in two dimensions

$$d\mu[g]d\mu[\phi] = \prod_{x,\mu\geq\nu} g^{\sigma}(x) dg_{\mu\nu}(x) \prod_{x,a} d\phi^a(x) \quad (19)$$

with $\sigma = (D-3)/12$ for the DeWitt measure, and $\sigma = (D-6)/12$ for the Misner measure. Going over to the lattice, one observes that the various measures discussed up to now (including the dl/l measure) can be recast in the form

$$\prod_{\text{edges ij}} \int_0^\infty \frac{dl_{ij}^2}{l_{ij}^2} F_\epsilon[l] \times \prod_{\text{sites i}} \prod_a \int_{-\infty}^{\infty} d\phi_i^a \times \exp\Big(\sum_{ij} \ln [\, l_{ij}^{2\alpha} V_{ij}^{2\sigma} \,] \Big) \quad (20)$$

with $\alpha = \sigma = 0$ for the dl/l measure, and $\alpha = 1$ and σ equal to the above values for the lattice analogues of the DeWitt and Misner measure. As will be discussed below, one can explore the phases of two-dimensional gravity as a function of the measure parameters α and σ, and one finds that some degree of universality with respect to the gravitational measure seems to hold.

In order to study the properties of the scalar field coupled to gravity, and attempt to compare with recent related work in the continuum [40], one can measure the discrete analogue of the coordinate invariant quantity

$$< \phi^2 >= \frac{1}{D} \frac{< \int \sqrt{g}\, (\phi^a - \bar{\phi}^a)^2 >}{< \int \sqrt{g} >} \quad (21)$$

with the average defined as

$$\bar{\phi}^a = \frac{\int \sqrt{g}\; \phi^a}{\int \sqrt{g}} \quad (22)$$

Since one is working in an ensemble in which the total area is fixed, and equal to the number of sites, one has $\int \sqrt{g} = \sum_i A_i = N = A$. On the lattice one should measure therefore [26]

$$< \phi^2 >= \frac{1}{DN} < \sum_i A_i \; (\phi_i^a - \bar{\phi}^a)^2 > \quad (23)$$

with

$$\bar{\phi}^a = \frac{1}{N} \sum_i A_i \; \phi_i^a \quad (24)$$

Initially the case $D = 0$ was considered (no feedback of the scalar field on the geometry), together with $D = 1$ and $D = 2$. For the coefficient of the R^2-term in the action $a = 0.1$ and $a = 0.001$ was chosen, motivated by the intention to explore the sensitivity of the results to what is expected to be an irrelevant term. (The lattices ranged in size from $8^2 = 64$ to $512^2 = 262144$ sites, and the scalar field updates were performed both by Metropolis Monte Carlo, as well as by a heat bath).

As is illustrated in fig. 2 for $D = 1$, the results clearly suggest a linear behavior of $< \phi^2 >$ in $\ln A$ for the values of D (0,1,2) and a considered

$$< \phi^2 > \underset{A\to\infty}{\sim} c_0 + c_1 \; \ln A \quad (25)$$

implying an infinite Hausdorff dimension in embedding space. For the pure $\ln A$ fit there seems to be only a weak dependence of the fitted coefficients on D and a, for small D.

If one attempts to fit $< \phi^2 >$ to a power of L instead (as suggested for example by some authors [40], namely

$$\ln < \phi^2 > \underset{A\to\infty}{\sim} c_0 + c_1 \ln A \quad (26)$$

one finds values for $c_1 = 2/d_H$, where d_H is the fractal dimension of the surface, in the range 0.15 ± 0.02, or d_H about 13. On the other hand the χ^2 parameter for the fits is always at least an order of magnitude larger than in the previously discussed case, and furthermore d_H shows a clear trend towards an increase with lattice size (on a log-log plot there is a clear curvature in the data). From this one concludes that the

Fig. 2. Scalar field average $< \phi^2 >$ for gravity coupled to a massless scalar field ($D = 1$), with $a = 0.1$ (circles) and $a = 0.001$ (squares). The diamonds indicate results for the DeWitt measure (dl^2), with $a = 0$. The straight line corresponds to a logarithmic divergence, or $d_H = \infty$.

fractal dimension is always infinite in the model for $D = 0, 1, 2$ and for the dl/l measure. Furthermore for $D = 1$ there is no indication of a $(\ln A)^2$-term [40], and one can obtain a stringent bound on its coefficient, $c_2 \leq 0.002(2)$.

It is of interest then to explore how some of the above results depend on the gravitational measure. For $D = 1$ and $a = 0$ the simulations were repeated using the lattice analogue of the DeWitt measure. For $D = 1$ this particular measure becomes quite simple, $\prod dl^2$, and all volume factors cancel out. One finds that the modification of the measure changes the non-universal coefficients of the constant and lnA terms, but leaves the functional dependence on A unchanged, and in particular the result $d_H \equiv \infty$ [26].

2.3. Phases of 2-d Gravity

It is of interest to investigate further the dependence of physical results, like the embedding fractal dimension d_H, on both the gravitational measure (i.e. the measure parameter σ) and the number of components of the scalar field D, perhaps in some limits, like large D, and more singular measures (negative values of σ). To this end we have redone the calculations of $< \phi^2 >$ for larger values of D, namely $D = 4$, $D = 8$ and $D = 12$, and on lattices varying in size between 64 and 4096 sites with $a = 0.001$, using the 'flat' (D-independent) measure dl/l.

From the numerical results [35] one can clearly see that as D increases, the coefficient of the $\ln A$ term in $\frac{1}{D} < \phi^2 >$ increases, until for $D = 12$ the behavior is more consistent with a power law behavior in the area. Indeed in this last case a power law fit gives $d_H \approx 4$, which is close the fractal dimension for branched polymers (trees), $d_H = 4$. (The large error bars in the data stem from the fact that for $D = 12$ the model has entered into a new phase in which relaxation times are extremely long. Indeed the step size in the simulation has to be decreased by four orders of magnitude to keep the acceptances of order one, which suggests that the model we are considering is probably not even appropriate for this phase. Another indication that this is the case comes from the fact that a number of edges start to become quite long, while others get very short; regions develop where the curvature is very large in magnitude, and it becomes increasingly difficult to get rid of these 'defects', especially on the larger lattices).

There are a number of ways by which one can try to locate more accurately the transition, as a function of D. Indeed on a finite lattice a sharp transition between a phase in which $d_H = \infty$ and $d_H = 4$ will be somewhat broadened. One way is

Fig. 3. Effective Hausdorff dimension as a function of D, for the flat dl/l measure (a=0.001 (diamonds), a=0.1 (squares) and a=1.0 (stars)), the DeWitt-type measure $dl^2 V_l^{(D-1)/2}$ (a=0.001, circles), and the $dl/lV_l^{D/2}$ measure (a=0.001, diagonal crosses. All results are from an 8^2 lattice.

Fig. 4. Effective Hausdorff dimension as a function of the measure parameter σ, with measure $dl/lV_l^{2\sigma}$, for a=0.001 (circles) and a=1.0 (squares), and D=0. All results are from an 8^2 lattice.

to try to determine where the coefficient of the $\ln A$-term diverges

$$< \phi^2 > \underset{A\to\infty}{\sim} c_0 + c_1 \ \ln A$$
$$c_1 \sim \frac{c}{D - D_c} \tag{27}$$

which gives from our data $D_c \approx 13$. Alternatively one can fit the data to a finite power of A, and estimate D_c from where the effective power start to become very small. One finds $D_c \approx 14$, roughly consistent with the previous estimate. In conclusion for the measure dl/l the transition is close to $D = 13$. (One can also try to extract a value for the susceptibility exponent γ_χ; one finds for $D = 4, 8, 12$ $\gamma_\chi \sim 1.8, 1.7, -3.8$, which is on the one hand consistent for small D with the previous results for $D = 0, 1, 2$ (γ_χ=2), and for larger D with the fact that γ should eventually become negative [41,42]. On the other hand we do not believe that our results are accurate for $D > D_c$, for the reasons mentioned above).

One would like to understand how the previous results for the transition depend on the gravitational measure, the R^2 term in the action, and D. Since performing a full exploration of the phase diagram using a set of many different lattice sizes is quite time consuming, one can restrict oneself to a fixed lattice (of 64 sites), and define an effective fractal dimension $d_H(A)$ via

$$d_H(A) = 2 \left[\frac{\ln < \phi^2 >_A}{\ln A} \right]^{-1} \underset{A\to\infty}{\sim} d_H \tag{28}$$

Then we have explored the measure dl/l for $a = 0.001, 0.1, 1.0$, the measure $dl/l \times A_l^{D/2}$ for $a = 0.001$, and the lattice analogue of the DeWitt measure, $dl^2 \times A_l^{(D-1)/2}$, also with $a = 0.001$; D was varied between 0 and 32. As can be seen from fig. 3, the transition at finite D seems to disappear when the correct DeWitt weighting factor for the scalar field measure $\sqrt{g}^{D/2}$ is taken

Fig. 5. Phase diagram for 2-d gravity coupled to D-component scalar field; σ is the gravitational measure parameter, defined in the text. Upper curve: higher derivative coupling a=0.001, lower curve: a=1.0. The dashed line corresponds to the DeWitt measure.

into account for large D (for small D its effect appears to be negligible). On the other hand, if the coefficient of the R^2 term a is varied for the dl/l measure, then it seems that D_c can be shifted by one or two units. In other words the location of the transition in D seems to be non-universal, and even appears to depend on a.

If instead of varying D one fixes D and varies the measure parameter σ, a similar transition is encountered for sufficiently negative σ. In this case one can write the measure as $dl/l \times A_l^{2\sigma}$, and vary 2σ between 0 and -12, setting $a = 0.001$ and $a = 1.0$. The results one obtains in this case are shown in fig. 4.

As the measure becomes increasingly singular (large negative σ), one again encounters a transition to the branched polymer (tree) phase, with d_H approaching four. (Due to the smallness of the lattice one observes some substantial rounding, which presumably will sharpen as one goes to larger lattices. For $a = 0.001$ the location of the transition can be estimated at $2\sigma_c \approx -12$ by comparing to the analogous behavior for the transition in D discussed above. If a is larger, then it seems that the transition moves to even more negative values for σ, as expected from the effect of the R^2 term which tends to suppress singularities in the curvature.) The above results can then be summarized in the phase diagram of fig. 5.

One concludes that there is a whole line of phase transitions for all D's considered here ($0 \geq D \geq 32$), which crosses the $\sigma = 0$ axis only for larger (~ 12) values of D. For the DeWitt measure we find no transition in the region considered, but one cannot exclude one for even larger values of D ($D > 32$), even though we are more inclined to believe that such a transition never takes place for the DeWitt measure (which is the preferred gravitational measure in the presence of scalar fields).

Our results are in qualitative agreement with the results of the authors of ref. [41], who investigated in detail the phase diagram for dynamically triangulated random surfaces based on equilateral triangles, and employed the 'flat' ($\sigma = 0$) measure. They found a transition for $D_c \approx 4$. A similar result was also found by a different group [42], using the same model as the previous authors, but the dependence of their results on the (measure) parameter α seems to be just the opposite of what is observed here. In particular they find a transition for the lattice analogue of the DeWitt measure, and we don't. Due to the lack of universality in D_c discussed above, some of these discrepancies are not surprising. One can ascribe this situation to the difficulty in deciding in the DTRS model what is the analogue of the continuum measure contribution, and what is the analogue of a curvature squared term, since the correspondence with continuum operators has to be obtained by some suitable local averaging. For example the minimal area associated with a point is proportional to q=3 for equilateral triangles, while it is zero in our formulation of simplicial gravity, just as in the continuum. This accounts for rather different weighting properties in the path integral. Our results seem also to indicate that the approach to the continuum is perhaps faster in the Regge calculus approach, which deals with continuously varying curvatures, and can approximate a curved manifold more accurately with less triangles.

3. Gravity in Three Dimensions

While the physical interesting case is four dimensions, it appears worthwhile to investigate the intermediate case of three dimensions, which is less trivial than two-dimensional gravity since the Einstein action is no longer a topological invariant. It is also far less complex (as far as the lattice interactions are concerned) than the four-dimensional problem, but shares some of the same problems, namely the unboundedness of the pure gravitational action, as well as the lack of perturbative renormalizability.

The pure gravity action on the lattice will be chosen to be

$$I = \sum_{\text{edges h}} \Big[\lambda \, V_h - k \, l_h \delta_h + a \, \frac{l_h^2 \delta_h^2}{V_h} \Big] \tag{29}$$

as discussed in refs. [16,2]. In the classical continuum limit the above action is equivalent to the continuum action

$$I = \int d^3x \, \sqrt{g} \, \Big[\; \lambda - \frac{k}{2} \, R + \frac{a}{4} \, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \Big] \tag{30}$$

with a cosmological constant term (proportional to λ), an Einstein-Hilbert term and a higher derivative term. One could consider the Regge-Einstein action by itself ($a = 0$), but then the euclidean action would be unbounded from below, and problems might arise, depending on the choice of measure. More details about the formulation and results in three-dimensional gravity can be found in ref. [47], where the weak field expansion is also worked out in detail. Recently there also has been some work on three dimensional generalizations of the DTRS model [48], which might represent an alternative approach to what is being discussed here (even though the classical limit of that model still needs to be worked out, and shown to be equivalent to classical gravity; this appears to be non-trivial since the local curvatures are discrete).

In the numerical simulations the cubic lattice was employed, with three face diagonals and a body diagonal introduced to make the cube rigid. Lattices of size between 4^3 (with 448 edges) and 32^3 (with 229376 edges) were considered;

Fig. 6. Average curvature $\mathcal{R}$ as a function of k, for $\lambda = 1$ and $a = 0.005$ (dl^2 measure).

Fig. 7. Curvature fluctuation $\chi_{\mathcal{R}}$ as a function of k, for the same parameters as in fig. 6.

some shorter runs with lattices of size 64^3 (with 1835008 edges) were also done. The measure was chosen to be dl^2, in order to compare directly with results with the same measure in d=2 and d=4 (see following section). (The lengths of our runs typically vary between 100k Metropolis Monte Carlo iterations on the 4^3 lattice, 40k on the 8^3 lattice, 10k on the 16^3 lattice, and 1-2k iterations on the 32^3 lattice. As starting configurations on the larger lattices duplicated copies of the smaller lattices are used for each k, allowing for additional equilibration sweeps after duplicating the lattice in all directions.) One should emphasize that at this point the nature of the results is still rather preliminary. A more detailed description of the results can be found in ref. [47].

Quantities of physical interest which have been computed include the average curvature $\mathcal{R}$

$$\mathcal{R} \;=\; < l^2 > \frac{< 2 \sum_h \delta_h l_h >}{< \sum_h V_h >}$$

$$\sim \; \frac{< \int \sqrt{g}\; R >}{< \int \sqrt{g} >} \tag{31}$$

and the average curvature squared $\mathcal{R}^2$

$$\begin{aligned} \mathcal{R}^2 \;&=\; < l^2 >^2 \frac{< 4 \sum_h \delta_h^2 l_h^2 / V_h >}{< \sum_h V_h >} \\ &\sim \; \frac{< \int \sqrt{g}\; R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} >}{< \int \sqrt{g} >} \end{aligned} \tag{32}$$

Here the sum over hinges h is simply a sum over the edges in the simplicial lattice. In four dimensions one knows that there is a continuous transition between a 'smooth' (small negative average curvature) and a 'rough' (very large positive average curvature) phase of spacetime [15,44,45]. Besides $\mathcal{R}$ and $\mathcal{R}^2$, one can also estimate the lattice analogues of the fluctuations in the local curvatures

$$\begin{aligned} \chi_{\mathcal{R}} \;&=\; \frac{1}{< \sum_h V_h >} \\ &\times \Big[< (2 \sum_h \delta_h l_h)^2 > - < 2 \sum_h \delta_h l_h >^2 \Big] \end{aligned} \tag{33}$$

and of the fluctuations in the local volumes

$$\chi_V = \frac{1}{< \sum_h V_h >} \times \left[< (\sum_h V_h)^2 > - < \sum_h V_h >^2 \right] \quad (34)$$

These definitions are completely analogous to the ones previously used in four dimensions [15,16,44,45]. A divergence in the fluctuation is then indicative of long range correlations (a massless particle).

The results obtained for the average curvature $\mathcal{R}$ are shown for the case $a = 0.005$ in fig. 6. (Up to now two values of a, 0.0 and 0.005, have been studied, and $\lambda = 1$ was fixed (λ sets the overall scale in the action). The more accurate results are for $a = 0.005$, since the runs were longer, as described above. The statistical errors in $\mathcal{R}$ are estimated by the usual binning procedure, and represent one standard deviation). One notices that as k is varied, the curvature is negative for sufficiently small k, and appears to go to zero continuously at some finite value k_c. For $k \geq k_c$ the curvature is really infinite (or very large on a large lattice), and the simplices tend to collapse into degenerate configurations with very small volumes ($< V_h > / < l^2 >^2 \sim 0$) (this is the region of the usual weak field expansion ($G \to 0$). For k close to, but less than, k_c (and $\lambda = 1$) one can write

$$\mathcal{R}(k) \sim_{k \to k_c} A_{\mathcal{R}} \, (k_c - k)^{\delta}$$
$$\chi_{\mathcal{R}}(k) \sim_{k \to k_c} A_{\chi} \, (k_c - k)^{\delta - 1} \quad (35)$$

where δ is a universal exponent characteristic of the transition. After performing a simultaneous fit to $\mathcal{R}$ in $A_{\mathcal{R}}$, k_c and the exponent δ, and using close to k_c the data on the largest lattice available, one finds the results summarized in Table I. A weighted average of all the lattice results gives the preliminary estimate $\delta = 0.82 \pm 0.05$

For different values of a the curvature vanishes

a	$A_{\mathcal{R}}$	k_c	δ
0.000	-15.3(12)	0.115(5)	0.82(3)
0.005	-5.10(16)	0.162(3)	0.80(3)

Table 1
A summary of estimates in 3-d simplicial quantum gravity, of the critical amplitude $A_{\mathcal{R}}$, the critical point k_c and the critical exponent δ, for two values of the higher derivative coupling a.

along some line in the (k, a)-plane, and for sufficiently negative $a = a_0 \approx -0.0025$ the ground state ceases to exist. This not unexpected, since for sufficiently negative a the higher derivative term completely cancels the higher order lattice correction (proportional to Riemann squared) present in the Regge action, which is only an approximation to the pure Einstein action for small curvatures. It is not any different from the relationship between ordinary derivatives and finite difference formulae, and has been discussed in detail in the quantum field theory context by Symanzik [49,50]. This phenomenon is already seen in the weak field expansion, which gives the correct sign and order of magnitude of a_0 (the leading higher order correction has a large coefficient $O(100)$ for the Regge action) [47]. On the other hand the higher order lattice corrections to the pure Regge action ($a = 0$) stabilize the theory, at least for the dl^2 measure.

From the analysis of the curvature fluctuation $\chi_{\mathcal{R}}$ one obtains similar values for δ and k_c. In fig. 7 the curvature susceptibility is shown, again for $a = 0.005$. A more careful analysis [47] shows how the curvature fluctuation at the critical point grows with the size of the system, as expected from finite size scaling at a continuous phase transition,

$$\ln \chi_{\mathcal{R}} \underset{L \to \infty}{\sim} c + \frac{\alpha}{\nu} \ln L \quad (36)$$

with $\alpha/\nu = d(1-\delta)/(1+\delta)$. The estimate for δ

obtained from finite size scaling (0.91(15)) is in good agreement with the results quoted above, from $\mathcal{R}$. If on the other hand one computes the volume susceptibility χ_V, one finds that it approaches a finite value at k_c, suggesting the absence of critical volume fluctuations. This situation should be contrasted to the two-dimensional case, where the volume fluctuations (the Liouville mode) are found to be massless, as expected from continuum arguments [26], and are similar to the four-dimensional results, to be discussed in the next section.

In conclusion, the results are consistent with the picture of a vanishing curvature and a divergent curvature fluctuation, at the same value of k_c, and with a well-defined critical exponent. The existence of a continuous phase transition suggests the existence of a well defined lattice continuum limit in the neighborhood of the critical point at k_c.

4. Gravity in Four Dimensions

Classically, simplicial quantum gravity is known to converge to the continuum theory of gravity as the lattice spacing is reduced [8–10,16]. In the quantum theory the correspondence is not so obvious, since there are a number of ambiguities both in the continuum and in the lattice formulation (such as the problem of the measure discussed before). Furthermore no exact results are available to compare with, as in two dimensions. In four dimensions the lattice calculations are made difficult by the fact that the interactions among edges are complex, since there are a number of possible terms both in the pure gravity action and in the measure contribution. In addition there is a conceptual issue of what physical quantities (correlations) should be measured, and for what class of boundary conditions [3]. Only a small set of these questions have been addressed up to now, mostly pertaining to an investigation of the phase diagram and the location of possible phase transition points, as first discussed in [15,16,2].

One of the advantages of the simplicial approach is the fact that the correspondence between lattice and continuum operators is rather straightforward. In particular a clear distinction exists in the discrete functional integral between action and measure contributions. Furthermore the presence of classical gravitational waves (gravitons) can be shown explicitly in the context of the weak field expansion in four dimensions [8]. Here the emphasis will be on non-perturbative aspects of simplicial quantum gravity.

4.1. Action and Gravitational Measure

The discrete analog of the euclidean higher derivative action in four dimensions reads [14,16]

$$
\begin{aligned}
I = & \sum_{\text{hinges h}} \left[\lambda \, V_h - k \, \delta_h A_h + 4b \, \frac{A_h^2 \delta_h^2}{V_h} \right] \\
& + \frac{1}{3}(a - 4b) \sum_{\text{sites p}} V_p \sum_{\text{hinges h,h}' \supset \text{p}} \epsilon_{h,h'} \\
& \times \left[\omega_h \frac{A_h \delta_h}{V_h} - \omega_{h'} \frac{A_{h'} \delta_{h'}}{V_{h'}} \right]^2
\end{aligned}
\tag{37}
$$

Here δ_h is the deficit angle at a hinge (triangle), A_h is the area of the hinge, and V_h is the volume associated with that hinge. The numerical factor $\epsilon_{h,h'}$ is equal to 1 if the two hinges h, h' have one edge in common and -2 if they do not. The motivations leading to the above discrete action have been discussed in detail elsewhere [16], and will not be repeated here. The higher derivative term proportional to b can be considered as a regulator, and allows one to establish contact with results for continuum higher derivative theories. Matter fields can also be included in a rather straightforward way, except possibly for fermions.

In the classical continuum limit the above action is equivalent to the continuum euclidean higher derivative action

$$I = \int d^4x \sqrt{g} \left[\lambda - \frac{k}{2} R + b R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \frac{1}{2}(a - 4b) C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right] \quad (38)$$

with a cosmological constant term (proportional to λ), the Einstein-Hilbert term ($k/2 = 1/16\pi G$ here, where G is the bare Newton constant), and two higher derivative terms with dimensionless coupling constants a^{-1} and b^{-1}. One could consider the Einstein action by itself (which is *not* quite the same as the Regge action, due to the higher order lattice corrections [8,16,47], but then the euclidean action would be unbounded from below, unless the edge lengths are restricted in some way. Furthermore for the pure Einstein theory the higher derivative terms are generated anyhow by radiative corrections at one loop [51], so it seems reasonable to include them from the start. See ref. [52] for some interesting conjectures about perturbatively non-renormalizable interactions. It is known from perturbation theory that the above extended higher derivative action leads to a renormalizable and asymptotically free theory (in a^{-1} and b^{-1}) in four dimensions [54–56]. For sufficiently large $\lambda > 3k^2/16b$ the euclidean action is also bounded from below, leading to a convergent functional integral. Due to the complexity of the Weyl term, up to now only the case $a = 4b$ was considered, which corresponds to a pure Riemann squared higher derivative contribution.

In four dimensions the classical continuum limit is taken by requiring that the local curvature be small on the scale of the local lattice spacing, which is equivalent to imposing

$$\left| \frac{A_h \delta_h}{V_h} \right| \ll \frac{1}{\sqrt{V_h}} \quad (39)$$

This condition can be met by having the coefficient of the curvature squared terms large; otherwise the results are expected to depend on such details as the lattice structure and the lattice transcription of the continuum action. In the classical continuum limit the above action is equivalent to the continuum higher derivative action of eqn. (38). Quantum mechanically it is known that the continuum limit has to be taken at a phase transition point, i.e. at a non-trivial fixed point of the renormalization group, if one can be found. At the fixed point the correlation length diverges and the relevant excitations then extend over many lattice spacings, making the specific lattice details fade away. To find the quantum mechanical continuum limit, it might or might not be necessary to add a large higher derivative contribution; if a continuous transition is found already for small a and even $a = 0$, as seems to be the case, then the quantum continuum limit might already exist in the pure Regge theory ($a = 0$). Adding the higher derivative term proportional to a would then not affect in any way the continuum limit, which might very well be independent of a (like in lattice QCD, where physical low-energy observables are independent of the gauge coupling g in the continuum limit).

As in two and three dimensions, an important issue that needs to be addressed is the problem of the gravitational measure. Here we will consider the four-dimensional pure gravity measure (discussed in the Introduction)

$$\int d\mu_\epsilon[\, l \,] = \prod_{\text{edges ij}} \int_0^\infty V_{ij}^{2\sigma} \, dl_{ij}^2 \, F_\epsilon[\, l \,] \quad (40)$$

with $2\sigma = (d-4)/2d = 0$ for the lattice analogue of the DeWitt measure. (In our previous work [14–16] we had considered the scale invariant measure dl/l; on the basis of universality one would expect that physical results should not depend on the specific form of the prefactors of the measure, nor on specific short distance details of the action. This is certainly supported by the results in two dimensions [26]). On the other hand, the lattice path integral might not be meaningful though for certain values of σ, for which the measure becomes too singular.

While the cosmological constant term prevents any of the edge lengths from becoming too long, the measure plays an important role in avoiding configurations in which some of the edge lengths could become very short. In turn this affects the fluctuations of the action, since for example one has for the Regge action contribution the bound

$$2\pi(1-q_h)l_0^2 \ \leq \ < \delta_h A_h > \ \leq \ 2\pi l_0^2 \tag{41}$$

where q_h denotes the number of simplices meeting at the hinge h, and l_0 is of the order of the average edge length. The presence of an average edge length induces effectively a cutoff in the curvatures, and thus mimics the effect of a curvature squared term.

A useful identity can be obtained by considering the scaling properties of the euclidean path integral (here for the case $a = 4b$),

$$Z(\lambda, k, a) = \int d\mu_\epsilon[\, l \,] \; e^{-I\,[\, l \,]} \tag{42}$$

Since for the dl^2 measure one has

$$Z(\lambda, k, a) = (\frac{k}{\lambda})^{N_1} \; Z(\frac{k^2}{\lambda}, \frac{k^2}{\lambda}, a) \tag{43}$$

one obtains the identity

$$k < \delta_h A_h > - \, 2\lambda < V_h > + \, \frac{N_1}{N_0} = 0 \tag{44}$$

where N_0 and N_1 denote the number of sites and edges in the lattice, respectively. This identity can be useful in verifying the correctness of the numerical results.

4.2. Phases of 4-d Gravity

It is of interest to explore by non-perturbative methods the phase diagram of the lattice theory described above. In the simulations to be discussed below the simplicial lattice was chosen to be regular and built out of rigid hypercubes, which can be subdivided into simplices by introducing face diagonals, body diagonal and hyperbody diagonals. This choice is clearly not unique, and is dictated by a criterion of simplicity, with the advantage that such a lattice can be used to study rather large systems with little modification. The edges are then individually varied (by moving at random through the lattice), and a new trial edge length is accepted as usual with probability $min(1, exp(-\Delta I))$, where ΔI is the variation of the action under the change in edge length. If the triangle inequalities or their higher dimensional analogues are violated, the new edge length is rejected. In order to compute the variation in the action under the change of one edge length, a large number of adjoining triangles and their deficit angles have to be considered [57].

Mostly lattices of up to size 8^4 (with 61440 edges) were considered; we have also done some short runs on 16^4 lattices (with 983040 edges), but the results will not be discussed here in detail since the statistics is at this point still rather low. (The lengths of the runs typically vary between 30k Metropolis Monte Carlo iterations on the 2^4 lattice, 10k iterations on the 4^4 lattice, and 1-2k iterations on the 8^4 lattice. On the 16^4 lattice the results are from a few hundred sweeps. As starting configurations on the larger lattices duplicated copies of the smaller lattices are used for each k, allowing for additional equilibration sweeps after duplicating the lattice). One should emphasize that at this point the nature of the results is still rather preliminary. A more detailed description of the results can be found elsewhere

[44,46].

As in three dimensions, quantities of physical interest which can be computed include the average curvature $\mathcal{R}$

$$\mathcal{R} \;=\; < l^2 > \frac{< 2 \sum_h \delta_h A_h >}{< \sum_h V_h >}$$
$$\sim \;\; \frac{< \int \sqrt{g}\; R >}{< \int \sqrt{g} >} \tag{45}$$

and the average curvature squared $\mathcal{R}^2$

$$\mathcal{R}^2 \;=\; < l^2 >^2 \frac{< 4 \sum_h \delta_h^2 A_h^2 / V_h >}{< \sum_h V_h >}$$
$$\sim \;\; \frac{< \int \sqrt{g}\; R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} >}{< \int \sqrt{g} >} \tag{46}$$

Previous studies indicated the presence of a transition between a 'smooth' (small negative average curvature) and a 'rough' (very large positive average curvature) phase of spacetime. It is of interest to investigate in greater detail this transition between the 'smooth' and the 'rough' phases of spacetime. Besides $\mathcal{R}$ and $\mathcal{R}^2$, one can also estimate the lattice analogues of the fluctuations in the local curvatures

$$\chi_{\mathcal{R}} \;=\; \frac{1}{< \sum_h V_h >}$$
$$\times \Big[< (2 \sum_h \delta_h A_h)^2 > - < 2 \sum_h \delta_h A_h >^2 \Big] \tag{47}$$

and of the fluctuations in the local volumes

$$\chi_V \;=\; \frac{1}{< \sum_h V_h >}$$
$$\times \Big[< (\sum_h V_h)^2 > - < \sum_h V_h >^2 \Big] \tag{48}$$

A divergence in the fluctuation is then indicative of long range correlations (a massless particle) in the relevant spin channel.

The results obtained for the average curvature $\mathcal{R}$ are shown, for $a = 0.005$, $\lambda = 1$ and for different lattice sizes, in fig. 8. (The statistical errors in $\mathcal{R}$ are estimated by the usual binning procedure, and represent one standard deviation; one finds that as long as one does not move too close to k_c, the autocorrelations are contained).

Fig. 8. Average curvature $\mathcal{R}$ as a function of k, for $\lambda = 1$ and $a = 0.005$ (dl^2 measure). The squares refer to $L = 4$, and the diamonds to $L = 8$.

One notices that as k is varied, the curvature appears to go to zero at some finite value k_c. Since k is really a microscopic parameter that gets renormalized by quantum fluctuations, one should consider also negative values of k (which do not necessarily imply an attractive renormalized, effective long-distance coupling constant). For k close to, but less than, k_c (and $\lambda = 1$) one can write

$$\mathcal{R}(k) \quad \sim_{k \to k_c} A_{\mathcal{R}} \; (k_c - k)^{\delta}$$
$$\chi_{\mathcal{R}}(k) \; \sim_{k \to k_c} A_{\chi} \; (k_c - k)^{\delta - 1} \tag{49}$$

where δ is a universal exponent characteristic of the transition. Performing a simultaneous fit to $\mathcal{R}$ in $A_{\mathcal{R}}$, k_c and the exponent δ, one finds the results summarized in Table II.

In general the quality of the fits appears to

a	$A_{\mathcal{R}}$	k_c	δ
0.000	-40.9(53)	0.083(9)	0.57(8)
0.005	-3.76(8)	0.242(4)	0.62(2)
0.020	-0.725(11)	0.422(7)	0.61(2)
0.100	-0.077(5)	1.084(20)	0.60(21)

Table 2
Estimates, for different values of a, of the critical amplitude $A_{\mathcal{R}}$, the critical point k_c and the critical curvature exponent δ.

be quite good. Altogether four values of a were considered, with five to six values of k for each a. (The most accurate results with the smallest errors are for $a = 0.005$ and 0.02, since for $a = 0.1$ the curvature is quite small and more difficult to measure accurately, while for $a = 0$ the fluctuations are significant due to the fact that one is quite close to the singularity at $k \approx 0$, $a \approx -0.0011$, where the curvature becomes infinite (see below)). A weighted average of all the 4^4 lattice results gives $\delta = 0.70 \pm 0.13$, while a weighted average of all the 8^4 lattice results gives $\delta = 0.59 \pm 0.09$. If one combines the results for the curvature from both lattice sizes before attempting the fits, one obtains $\delta = 0.60 \pm 0.06$, and presumably also independent of a as expected from universality. For different values of a the curvature vanishes along a line in the (k, a)-plane which resembles quite closely a parabola

$$a(k_c) \;=\; a_0 + a_1 \; k_c^2 \tag{50}$$

Assuming this form one finds $a_0 = -0.0011(9)$ and $a_1 = 0.083(12)$. Alternatively one can try to determine a_0 by assuming that the average curvature amplitude (close to k_c) diverges for small a like

$$A_{\mathcal{R}}(a) \;\sim\; A_0 \; (a - a_0)^{-\sigma} \tag{51}$$

One finds from the results in Table II $a_0 = -0.0011(4)$ and $\sigma = 1.38(5)$, in good agreement with the previous estimate for the critical value a_0.

The function $k_c(a)$ from eqn. (50) appears to have two branches, and along the negative k branch one finds that the curvature diverges; this then would seem to suggest that the curvature has a sharp discontinuity at $a = a_0$ and $k = 0$, where it jumps from zero to infinity. In the continuum the presence of such a phase transition line is inferred from the domain of boundedness of the euclidean action. For comparison, in the continuum one has from the classical action $a_0 = 0$ and $a_1 = 3/4$, whereas for example from the regular tessellation of the sphere α_5 one gets to lowest order $a_0 = 0$ and $a_1 = 0.4710$. Here instead a_0 is non-vanishing and negative, as a result of higher order classical, as well as radiative, corrections, which seem to induce an effective positive R^2-type term, and therefore lead to a stabilization of the pure Regge action (which corresponds to a=0). (This latter action is known to be equivalent to the euclidean Einstein action only up to higher order lattice corrections, which in principle can be computed within the framework of the lattice weak field expansion [47]). One can argue furthermore that since the lattice higher derivative theories we have considered here ($a > 0$) and the reflection positive [59,58] (and therefore unitary) pure Regge theory ($a = 0$) appear to belong to the same phase, at least for large enough G, they should give rise to the same, unitary, lattice quantum continuum limit.

From the analysis of the curvature fluctuation $\chi_{\mathcal{R}}$ one obtains similar and consistent values for δ and k_c, but as expected with larger errors, since the fluctuation is more difficult to compute accurately. In fig. 9 the curvature susceptibility is shown, again for $a = 0.005$; other values of a show a rather similar behavior. The results are consistent with the picture of a vanishing cur-

Fig. 9. Curvature fluctuation $\chi_{\mathcal{R}}$ as a function of k, for the same parameters as in fig. 8.

Fig. 10. Volume density fluctuation χ_V as a function of k, for the same parameters as in fig. 8.

vature and a divergent curvature fluctuation, at the same value of k_c. (At this point one cannot entirely exclude a discontinuous (first-order) transition at k_c, with a rather small discontinuity. But from our results there is clearly no evidence for such a discontinuous transition). The results presented above give correctly a negative sign for the average curvature, which is needed for $k < k_c$ in order to have a positive fluctuation $\chi_{\mathcal{R}}$. Furthermore the average curvature becomes complex for $k > k_c$, a reflection of the fact that the theory becomes unstable in that regime. The curvature is really infinite (or very large on a large lattice) in this phase, and the simplices collapse into degenerate configurations with very small volumes ($< V_h > / < l^2 >^2 \sim 0$). This is the region of the usual weak field expansion ($G \rightarrow 0$), and it is therefore not surprising that such an expansion has difficulties in extending to the region where a sensible path integral for pure gravity can be defined. A qualitative picture of the phase diagram for pure gravity with the above lattice action is sketched in fig. 11. The dependence of the critical point k_c (for $a = 0$) and the critical curvature exponent δ on the dimension is sketched in figs. 12 and 13, respectively.

If one computes the volume susceptibility χ_V instead, one finds that it approaches a finite value at k_c, suggesting the *absence* of critical volume fluctuations. These appear to be desirable properties in a quantum theory of gravity, where the excitations in the continuum are expected to be massless gravitons, without massless scalar states and without massless volume density fluctuations. This situation resembles the three-dimensional case discussed before, and should be contrasted to the two-dimensional case, where the volume fluctuations (the Liouville mode) are found to be massless, as expected from continuum arguments [26].

The results for the average curvature $\mathcal{R}$ are not inconsistent with known results within the weak field expansion in the continuum (for small a). Substituting $k^{-1} = 8\pi G$, and setting $k_c = c\Lambda^2$, where c is a constant independent of k, and Λ the ultraviolet cutoff (here of the order of the average inverse lattice spacing $\sim < l^2 >^{-1/2}$), one obtains from eqn. (49)

$$\begin{aligned} \mathcal{R}(G) \sim \; & A_{\mathcal{R}} \left(\frac{-1}{8\pi G} \right)^{\delta} \left[1 - c\Lambda^2 8\pi G \right]^{\delta} \\ \sim \; & A_{\mathcal{R}} \left(\frac{-1}{8\pi G} \right)^{\delta} \Big[1 + \delta \, c\Lambda^2 (-8\pi G) \\ & + \frac{\delta(\delta-1)}{2} \, (c\Lambda^2)^2 \, (-8\pi G)^2 + \cdots \Big] \end{aligned} \quad (52)$$

One can see that $\mathcal{R}$ is possibly not analytic at $G = 0$, and an expansion in powers of G involves increasingly higher powers of the ultraviolet cutoff Λ, as expected from a theory which is not perturbatively renormalizable in G.

If one assumes that the curvature R has scaling dimension d_R, $R \sim m^{d_R}$, then one obtains from eqn. (49) and from the definition of the curvature fluctuation in eqn. (47)

$$\begin{aligned} \mathcal{R}(k) \; & \sim_{k \to k_c} (\Delta k)^{\delta} \sim m^{d_R} \\ \chi_{\mathcal{R}}(k) \; & \sim_{k \to k_c} (\Delta k)^{\delta - 1} \sim m^{2 d_R - d} \end{aligned} \quad (53)$$

with $\Delta k = k_c - k$. Classically one would expect $d_R = 2$; here instead $d_R = d\delta/(1+\delta)$ and $m \sim (\Delta k)^{\nu}$ with $\nu = (\delta + 1)/d = 0.40 \pm 0.02$. (The same result is found alternatively by setting the action fluctuation, or specific heat, exponent $\alpha \equiv 2 - d\nu = \delta - 1$, since the average curvature represents one of the contributions to the average action). Therefore in this model the average curvature appears to be related to a dynamical 'graviton mass' via

$$\mathcal{R}(k) \sim m^{d\delta/(1+\delta)} \quad (54)$$

Only at k_c do both the curvature and the 'graviton mass' vanish. Away from k_c the size of this mass is related to the average curvature of the 'universe' via the above equation.

It would appear that close to the transition one is dealing with two rather different length scales. One can define first a length scale R_0 associated exclusively with the space-time average of the curvature, and therefore related to some average curvature radius,

$$\frac{< 2 \sum_h \delta_h A_h >}{< \sum_h V_h >} \; \equiv \; \pm \frac{1}{R_0^2} \sim \left(\frac{4\lambda}{k} \right)_{eff} \quad (55)$$

and the choice of sign depends on whether the average curvature is positive or negative. As one approaches the fixed point at k_c, this length scale becomes very large. Naturally another length scale M_0^{-1} can then be associated with the average volume (per hinge or per site)

$$\frac{N}{\sum_h V_h} \; \equiv \; \frac{M_0}{R_0^3} \sim \lambda_{eff} \quad (56)$$

Close to and below the critical point this particular combination approaches some constant value. Therefore the results suggest that one is dealing simultaneously with a very large (R_0) and a very small (M_0^{-1}) length scale, associated with the curvature and density of the 'universe', respectively. On the lattice, which has a finite cutoff built in, it makes only sense to compute *dimensionless* ratios like

$$\frac{1}{R_0 M_0} \sim \left(\frac{4\lambda}{k^2} \right)_{eff} \underset{k \to k_c}{\sim} (k_c - k)^{2\delta} \quad (57)$$

As k approaches the fixed point k_c, this quantity vanishes with a power 2δ (in agreement with the observed fact that the renormalized cosmological constant is very small, once it is expressed in the appropriate units). The renormalized cosmological constant can be made very small, and here it is simply a consequence of the fact that the curvature is small in units of the average volume per site, and as long as one does not cross over into the 'rough' phase of gravity ($k > k_c$). There no sensible ground state seems to exist, at least within the context of this model.

Still it would be more desirable if one could compute the effective, renormalized Newton's

Fig. 11. Phase diagram for pure 4-d simplicial quantum gravity with a higher derivative coupling a (for fixed $\lambda = 1$). The curve represents an estimate for the phase transition line $a(k_c)$, where the curvature fluctuations diverge, and which separates the 'smooth' from the 'rough' phase of gravity.

constant directly. In order to achieve this goal some correlations at fixed geodesic distance have to be computed, a notoriously difficult task [16,2]. As a step in this direction one can compute both the edge-edge and curvature-curvature connected correlation functions at fixed geodesic distance d, and close to the critical point at k_c, where one expects the power-law decay

$$\begin{aligned} G^{(l)}_{\alpha\beta}(d) &= < l^2_\alpha(d)\; l^2_\beta(0) >_c \\ &\underset{d\to\infty}{\sim} < l^2 >^2 \; T^{(l)}_{\alpha\beta} \; \frac{C_l}{4\pi^2 d^2} \\ G^{(R)}_{\alpha\beta}(d) &= < \delta_\alpha A_\alpha(d)\; \delta_\beta A_\beta(0) >_c \\ &\underset{d\to\infty}{\sim} < l^2 >^2 \; T^{(R)}_{\alpha\beta} \; \frac{C_R}{4\pi^2 d^8} \end{aligned} \tag{58}$$

Here the indices α,β label the edges and hinges (triangles) within a hypercube respectively; on the specific simplicial lattice we are considering $G^{(l)}_{\alpha\beta}(d)$ is a 15×15 matrix for each d, while $G^{(R)}_{\alpha\beta}(d)$ is a 50×50 matrix.

In general the above correlations will contain possibly particles of different spin (0,1,2,..), but only the lightest (massless) state with spin two should dominate at large distances. Therefore one expects that the largest eigenvalue $\lambda^{(l)}_{max}(d)$ of the edge-edge correlation matrix $G^{(l)}_{\alpha\beta}(d)$ will decay like $1/d^2$ for large geodesic distance d. The quantity $C_l = 4\pi^2 d^2 \lambda^{(l)}_{max}(d)$ should approach a constant, which can be taken as a possible definition of the effective Newton's constant in units of the ultraviolet cutoff, $1/k_{eff} \equiv 8\pi G_{eff} = C_l$.

It would seem from our results that this quantity tends to a finite value as k tends to k_c. For example, from the edge-edge correlations up to about geodesic distance 8 on the 16^4 lattice, and for $a = 0.005$ and $k = k_c = 0.239$, one finds $1/k_{eff} \sim C_l \approx 3.85$, which is quite close to the bare value $1/k \approx 4.18$. More generally it would seem that the average curvature in units of the ultraviolet cutoff tends to zero (in the way described before) as one approaches the fixed point, while the effective Newton's constant ap-

Fig. 12. Critical point $k_c = 1/8\pi G_c$, as a function of the space-time dimension d, for the pure Regge action ($a = 0$), and for $\lambda = 1$. The numerical lattice results for $d = 3$ and $d = 4$ are shown, together with the result $k_c^{-1} = 0$ for $d = 2$.

Fig. 13. Critical exponent δ as a function of the space-time dimension d. The numerical lattice results for $d = 3$ and 4 are shown, together with the $2 + \epsilon$ result [56], $\delta \sim 2/\epsilon$.

proaches some finite value which is of the same order as the cutoff. Alternatively one could determine $1/k_{eff}$ by computing the analogues of Wilson loops, which involve the determination of the dependence of the deficit angle associated with a large loop on its physical perimeter length.

It is of interest to explore other correlations, which are of a purely geometric nature. One finds that some of the geometric properties of the discrete simplicial manifold are close to being euclidean. As an example we have considered how the number of points within geodesic distance d and $d + \Delta d$ scales with the geodesic distance itself. This quantity is equivalent, up to a constant which depends on the average lattice spacing, to the physical three-volume within geodesics distance d and $d + \Delta d$. In practice the geodesic distance between two arbitrary points on the simplicial lattice can be determined from a fixed configuration of edge lengths by selecting, among all the possible random walks between the two points which are less than some cutoff length ($\approx 2L$), the one with the shortest length. One finds for the small distances considered up to now ($d \leq 8\sqrt{< l^2 >} \approx 16$), which correspond therefore to only a few lattice spacings,

$$N(d) \underset{d \to \infty}{\sim} d^{\, d_v} \tag{59}$$

with $d_v \approx 3.4$, which is roughly consistent with the flat space value of three.

Many questions have remained open. It would be of interest to investigate how our results depend on a for even larger values, and complete the picture for the phase diagram for pure gravity. The sensitivity of the results on the measure should be explored further. It would also be of interest to investigate these questions in the presence of matter fields, as well as for surfaces with boundaries. While there are indications that the excitations which become massless at the critical point correspond to shear waves, it would seem important to un-ambiguously determine the nature the spin of the lowest state, which should correspond to a graviton (work in this direction is in progress). Finally it would be of interest to reproduce some of the above results in the context of a model for lattice gravity based not on a regular lattice with fixed coordination number as we have done here, but instead on a random simplicial lattice. This would provide further evidence that one is dealing with the correct continuum theory. Other models for quantum gravity in four dimensions based on regular lattices are discussed in [60–62] and in [63].

5. Conclusions

In the preceding sections we have discussed results relevant for a model of simplicial quantum gravity, based on Regge's model. It is characteristic of our model that the variations in the geometry of space are described by fluctuating edge lengths on a lattice with fixed coordination number.

In *two dimensions* we have investigated in detail how the results for critical properties depend on what are expected to be irrelevant (R^2-type) terms, as well as on the form of the gravitational measure. In the case of pure gravity we have computed the string susceptibility exponent γ_χ for both the torus and the sphere, and found good agreement with the expected exact answers from conformal field theory. For the torus we have computed critical properties of the Liouville field, which corresponds to area density fluctuations on the lattice, and found them to be in complete agreement with the expectation that the Liouville field behaves like a free massless field for $\lambda \to \lambda_c$. By adding a D-component scalar field to the model, we have been able to compute the average $< \phi^2 >$, and therefore make contact with results on models of random surfaces. These predict an infinite fractal dimension for the surface, at least in the absence of extrinsic curvature terms. We found that for the torus the fractal dimension is indeed infinite for $D = 0, 1$ and 2. For small D our results seem to be insensitive, within some range of parameters, to the presence of an R^2-type term in the action or to the detailed form of the measure. We have argued that for sufficiently singular measures though, the triangulation will tend to collapse into a degenerate configuration of edges, and this phenomenon is clearly seen for a sufficiently negative measure parameter σ, even at $D = 0$ (pure gravity). For larger D one sometimes finds a phase transitions to a branched polymer phase (with a correct fractal dimension of four), depending on the specific form of the gravitational measure used. We have sketched what we believe is the phase diagram for our model of 2-d gravity.

In *three dimensions* the action becomes less trivial due to the influence of the Regge-Einstein term. A higher derivative term was also included as a regulator, together with the cosmological constant term. A transition between a 'smooth' and a 'rough' phase of pure gravity was found. The transition is a continuous one if approached from the smooth phase (small k or large G), which has negative average curvature. The critical exponents were estimated, and it appears that at the point where the average curvature vanishes, the curvature fluctuations diverge, leading to a well defined lattice continuum limit. The results are very different from two dimensions (where fluctuations in the volume diverge instead) and resemble somewhat more the four-dimensional case.

In *four dimensions* we have studied in some detail a model for pure lattice gravity, with a higher derivative term acting as a regulator. In our earlier work we had found on small lattices a sharp transition between a 'smooth' and a 'rough' phase of gravity. The detailed nature of the transition had remained unclear. Arguments in the continuum suggest that such a transition should appear above two dimensions. The results on the phase transition have now been extended to the 'DeWitt' measure (as well as to much larger lattices), and it appears that the transition is a continuous one if approached from the smooth phase (small k or large G), which has a negative average curvature. We have argued that the usual weak field expansion on the other hand starts out in what has been described here as the 'rough' phase, and for which there appears to be no sensible definition for the path integral, in the sense that the curvatures are infinite and the volumes are zero, in units of the average edge length.

We have computed critical exponents associated with what appears to be a non-trivial fixed point for euclidean gravity, and have shown that at the point where the average curvature vanishes, the curvature fluctuations diverge. This implies that a quantum continuum limit exists in the neighborhood of the critical point. It is a non-trivial result that the curvature, in spite of the fact that it is in general non-zero, vanishes at the critical point (instead of approaching some finite value), and suggests that in the quantum continuum limit spacetime becomes close to flat on the average in this model.

Many questions have remained open. It would be of interest to investigate how the results depend on the coupling a and the choice of measure, and complete the picture for the phase diagram for pure gravity. For larger a one expects that the transition will move to larger values of k. Universality of the continuum limit would suggest that the results for exponents and other infrared sensitive quantities should not be affected. It would also be of interest to investigate these questions in the presence of matter fields, as well as for surfaces with boundaries.

Acknowledgements

The work described here for 2-d gravity was done in collaboration with M. Gross, and the one on 3-d gravity in collaboration with R. Williams. The numerical computations were performed at the NSF-sponsored San Diego and Pittsburgh Supercomputer Centers. This research was supported in part by the National Science Foundation under grant NSF-PHY-8906641.

References

[1] T. Regge, *Nuovo Cimento* **19** (1961) 558.

[2] H.W. Hamber, in the proceedings of the *1984 Les Houches Summer School*, Session XLIII, edited by K. Osterwalder and R. Stora (North Holland, 1986) pp. 375-439, and references therein.

[3] J. B. Hartle, in the *Proceedings of the Third Moscow Seminar on Quantum Gravity*, M. Markov, V. Berezin and V. Frolov eds., World Scientific 1985; J.B. Hartle, in the *Proceedings of the 14th Yamada Conference on Gravitational Collapse and Relativity*, H. Sato and T. Nakamura eds. (World Scientific, Singapore) p. 329.

[4] H.W. Hamber, in the proceedings of the *1989 Cargèse* NATO workshop on *Probabilistic Methods in Quantum Field Theory and Quantum Gravity*, (Plenum Press, New York 1990) pp. 77-87.

[5] P. Menotti, *Nucl. Phys. B* **Proc. Suppl.** 17 (1990) 29.

[6] J.A. Wheeler, in *'Relativity, Groups and Topology'*, edited by B. DeWitt and C. DeWitt (Gordon and Breach, New York, 1964).

[7] C.W. Misner, K.S. Thorn and J.A. Wheeler, *'Gravitation'*. (Freeman, San Francisco, 1973).

[8] M. Roček and R. M. Williams, *Phys. Lett.* **104B** (1981) 31, and *Z. Phys.* **C21** (1984) 371.

[9] J. Cheeger, W. Müller and R. Schrader, in the *Heisenberg Symposium*, München 1981, Springer Lecture Notes in Physics, P. Breitlohner and H. P. Dürr eds., New York 1982;
J. Cheeger, W. Müller and R. Schrader, *Comm. Math. Phys.* **92** (1984) 405.

[10] R. Friedberg and T.D. Lee, *Nucl. Phys.* **B242** (1984) 145;
G. Feinberg, R. Friedberg, T.D. Lee and H.C. Ren, *Nucl. Phys.* **B245** (1984) 343.

[11] J. B. Hartle, *J. Math. Phys.* **26** (1985) 804, and **27** (1985) 287;
J. B. Hartle, *Class. Quantum Grav.* **2** (1985) 707.

[12] M. Bander and C. Itzykson, *Nucl. Phys.* **B257** **[FS14]** (1985) 531;
A. Jevicki and M. Ninomiya, *Phys. Lett.* **150B** (1985) 115 and *Phys. Rev.* **D33** (1986) 1634.

[13] D.A. Eliezer, *Nucl. Phys.* **B319** (1989) 667.

[14] H.W. Hamber and R. M. Williams, *Nucl. Phys.* **B248** (1984) 392.

[15] H.W. Hamber and R. M. Williams, *Phys. Lett.* **157B** (1985) 368.

[16] H.W. Hamber and R. M. Williams, *Nucl. Phys.* **B269** (1986) 712.

[17] H.W. Hamber and R. M. Williams, *Nucl. Phys.* **B267** (1986) 482.

[18] B. DeWitt, in *'General Relativity - An Einstein centenary survey'*, edited by S.W. Hawking and W. Israel, Cambridge University press (1979), and further references therein.

[19] S.W. Hawking, in *'General Relativity - An Einstein Centenary Survey'*, edited by S.W. Hawking and W. Israel, Cambridge University press (1979), and references therein.

[20] S. Weinberg, in *'General Relativity - An Einstein Centenary Survey'*, edited by S.W. Hawking and W. Israel, Cambridge University press (1979), and references therein.

[21] F. David, *Nucl. Phys.* **B257** **[FS14]** (1985) 45 and 543;
A. Billoire and F. David, *Nucl. Phys.* **B275** (1986) 617;
F. David, J. Jurkiewicz, A. Krzywicki and B. Petersen, *Nucl. Phys.* **B290** (1987) 218.

[22] J. Ambjørn, B. Durhuus and J. Fröhlich, *Nucl. Phys.* **B257** **[FS14]** (1985) 433;
J. Ambjørn, B. Durhuus, J. Frölich and P. Orland, *Nucl. Phys.* **B270** **[FS16]** (1986) 457;
J. Ambjørn, B. Durhuus and J. Frölich, *Nucl. Phys.* **B275** **[FS17]** (1986) 161.

[23] V.A. Kazakov, *Phys. Lett.* **150B** (1985) 282;
V.A. Kazakov, I.K. Kostov and A.A. Migdal, *Phys. Lett.* **157B** (1985) 295;
D.V. Boulatov, V.A. Kazakov, I.K. Kostov and A.A. Migdal, *Nucl. Phys.* **B275** (1986) 641;
D.V. Boulatov and V.A. Kazakov, *Phys. Lett.* **186B** (1987) 379;
V.A. Kazakov and A.A. Migdal, *Nucl. Phys.* **B311** (1988) 171.
M. Agishtein and A.A. Migdal, Princeton preprint PUPT-1171 and 1180 (1990).

[24] V.A. Kazakov, *Phys. Lett.* **119A** (1986) 140;
D.V. Boulatov and V.A. Kazakov, *Phys. Lett.* **186B** (1987) 379.

[25] M. Douglas and S. Shenker, *Nucl. Phys.* **B335** (1989) 635;
E. Brezin and V.A. Kazakov, *Phys. Lett.* **119A** (1986) 140;
D. Gross and A. Migdal, *Phys. Rev. Lett.* **64** (1990) 127;
G. Parisi, Rome preprint 89/30 (1989);
F. David, Saclay preprint SPhTh/90-090 (1990).

[26] M. Gross and H.W. Hamber, *Nucl. Phys.* **B** (1991).

[27] N. Christ, R. Friedberg and T.D. Lee, *Nucl. Phys.* **B202** (1982) 89 and *Nucl. Phys.* **B210** **[FS6]** (1982) 310,337;
C. Itzykson, *Cargèse Lecture Notes 1983*, published in the proceedings (Plenum Press, New York, 1985).

[28] B. Berg, *Phys. Rev. Lett.* **55** (1985) 904 and *Phys. Lett.* **176B** (1986) 39.

[29] C.W. Misner, *Rev. Mod. Phys.* **29** (1957) 497;

[30] L. Fadeev and V. Popov, *Sov. Phys. Usp.* **16** (1974) 777;

[31] N. P. Konopleva and V. N. Popov, *'Gauge Fields'*, Harwood Academic Publishers, New York 1979, and references therein.

[32] B. DeWitt, in *'Dynamical Theory of Groups and Fields'*, (Gordon and Breach, New York, 1965);
B. DeWitt, *Phys. Rev.* **160** (1967) 1113;
K. Fujikawa, *Nucl. Phys.* **B226** (1983) 437;
K. Fujikawa and O. Yasuda, *Nucl. Phys.* **B245** (1984)

446.

[33] V. De Alfaro, S. Fubini and G. Furlan, *Nuovo Cim.* **B57** (1981) 227;
K. Fujikawa and P. van Nieuwenhuizen, *Phys. Rev.* **D37** (1988) 391.

[34] H. Leutwyler, *Phys. Rev.* **134** (1964) 1155;
E. S. Fradkin and G. A. Vilkovisky, *Phys. Rev.* **D8** (1973) 4241;
E. S. Fradkin and G. A. Vilkovisky, CERN preprint TH-2332 (1977), unpublished.

[35] H.W. Hamber, to appear in the proceedings of the 1990 *Elba International Conference on Monte Carlo Methods*, published by Editrice Compositori (UC Irvine preprint UCI-Th-92-60, November 1990).

[36] A. M. Polyakov, *Phys. Lett.* **103B** (1981) 207.

[37] A. M. Polyakov, *Mod. Phys. Lett.* **A2** (1987) 899.

[38] V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, *Mod. Phys. Lett.* **A3** (1988) 819.

[39] F. David, *Mod. Phys. Lett.* **A3** (1988) 1651; 207;
F. David and E Guitter, *Europhys. Lett.* **3** (1987) 1169, and *Nucl. Phys.* **B293** (1988) 332.

[40] J. Distler and H. Kawai, *Nucl. Phys.* **B321** (1989) 509;
J. Distler, Z. Hlousek and H. Kawai, *Int. J. Mod. Phys.* **A5** (1990) 1093.

[41] J. Ambjørn, P. De Forcrand, F. Koukiou and D. Petritis, *Phys. Lett.* **197B** (1987) 548.

[42] F. David, J. Jurkiewicz, A. Krzywicki and B. Peterson, *Nucl. Phys.* **B290 [FS20]** (1987) 218;
J. Jurkiewicz, A. Krzywicki, B. Peterson and B. Söderberg, *Phys. Lett.* **213B** (1988) 511.

[43] J. Ambjørn, *Nucl. Phys.* (Proc. Suppl.) **B20**, 701-710, and these proceedings.

[44] H.W. Hamber, UC Irvine preprint UCI-Th-90-60 (November 1990), submitted to Nucl. Phys. **B**.

[45] H.W. Hamber, *Nucl. Phys.* **B20** (Conf. Proc. Supp.) (1991) 728.

[46] H.W. Hamber, UC Irvine preprint UCI-Th-91-11 (June 1991), submitted to *Phys. Rev. D.*.

[47] H.W. Hamber and R.M. Williams, CERN preprint in preparation.

[48] M. Gross, *Nucl. Phys.* **B20** (Conf. Proc. Supp.) (1991) 724;
J. Ambjørn and S. Varsted, Nordita preprint NBI-HE-91-17 (January 1991), and these proceedings;
M.E. Agisthein and A.A. Migdal, Princeton preprint (March 1991).

[49] K. Symanzik, in *'Mathematical Problems in Theoretical Physics'*, edited by R. Schrader et al. (Springer, Berlin, 1982) (Lecture Notes in Physics vol. 153).

[50] M. Lüscher, Les Houches lecture notes 1984 (North Holland, 1984), and references therein.

[51] G. 't Hooft and M. Veltman, *Ann. Inst. Poincaré* **20** (1974) 69;
S. Deser and P. van Nieuwenhuizen, *Phys. Rev.* **D10** (1974) 401;
M. Goroff and A. Sagnotti, *Nucl. Phys.* **B266** (1986) 709.

[52] G. Parisi, *Nucl. Phys.* **B 100** (1975) 368.

[53] H. Kawai and M. Ninomiya, *Nucl. Phys.* **B336** (1990) 115.

[54] B. DeWitt and R. Utiyama, *J. Math. Phys.* **3** (1962) 608.

[55] K.S. Stelle, *Phys. Rev.* **D16** (1977) 953.

[56] J. Julve and M. Tonin, *Nuovo Cimento* **46B** (1978) 137;
E.S. Fradkin and A.A. Tseytlin, *Phys. Lett.* **104B** (1981) 377, and *Nucl. Phys.* **B201** (1982) 469;
E.S. Fradkin and A.A. Tseytlin, *Phys. Lett.* **106B** (1981) 63;
I.G. Avramidy and A.O. Baravinsky, *Phys. Lett.* **159B** (1985) 269;
D. Boulware and D. Gross, *Nucl. Phys.* **B233** (1984) 1.

[57] H.W. Hamber, *Int. Jour. Sup. Appl.* (1991), in press (preprint UCI-TH-91-7, March 1991).

[58] J. Fröhlich, *'Regge Calculus and Discretized Gravitational Functional Integrals'*, I.H.E.S. preprint 1981 (unpublished).

[59] K. Osterwalder and R. Schrader, *Comm. Math. Phys.* **31** (1973) 83, and ibid. **42** (1975) 281;

[60] L. Smolin, *Nucl. Phys.* **B148** (1979) 333;
A. Das, M. Kaku and P. K. Townsend, *Phys. Lett.* **81B** (1979) 11;
C. Mannion and J. G. Taylor, *Phys. Lett.* **100B** (1981) 261;
M. Kaku, *Phys. Rev.* **D27** (1983) 2819;
E. T. Tomboulis, *Phys. Rev. Lett.* **52** (1984) 1173;
K. Kondo, *Prog. Theor. Phys.* **72** (1984) 841;
P. Menotti and A. Pelissetto, *Ann. Phys. (NY)* **170** (1986) 287; and *Phys. Rev.* **D35** (1987) 1194.

[61] P. Menotti and A. Pelissetto, *Nucl. Phys.* **B288** (1987) 813.

[62] S. Caracciolo and A. Pelissetto, *Phys. Lett.* **193B** (1987) 237, *Nucl. Phys.* **B299** (1988) 693, and *Phys. Lett.* **207B** (1988) 468.

[63] B. DeWitt, in the proceedings of the *'International School of Cosmology and Gravitation'*, (Erice 1989).