Renormalization-group calculations for U(1)-symmetric spin systems and gauge theories

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We present a real-space renormalization-group analysis for the U(1) spin model in two and three dimensions and for compact electrodynamics in three and four dimensions. The Hamiltonian formulation of the theory defined on a hypercubic spatial lattice is used. The nonperturbative renormalization-group transformations we derive give continuous phase transitions for the spin models, in two and three dimensions. In three dimensions the gap exponent agrees remarkably well with series-expansion results. For the gauge theory we find, in agreement with Polyakov, that compact electrodynamics in three dimensions confines for all values of the coupling constant. A calculation for the string tension shows that it vanishes with an essential singularity at zero coupling. In four dimensions for weak enough coupling confinement is lost and the theory shows a phase transition between a charge-confining and a free-photon phase.

I. INTRODUCTION

Considerable attention has recently been devoted to spin systems and gauge theories on a lattice invariant under a compact continuous-symmetry group.[1,2] A conjecture by Migdal in fact suggests that the critical properties of four-dimensional gauge theories are closely related to those of two-dimensional spin systems.[3] In the lattice gauge theory the fields are defined on compact manifolds, and one can argue that this situation would arise naturally for electrodynamics if the gauge group were an unbroken remnant of a larger unifying compact group.

In this paper we derive some simple real-space renormalization-group transformations that allow us to investigate correlations in Abelian [U(1)] spin and gauge systems for all values of the coupling constant. The renormalization method we adopt is essentially of the block-spin type and is therefore nonperturbative in nature.[4] It has been previously used in several contexts.[5-10] For the gauge theory the method we employ ensures that gauge invariance is preserved for the renormalized theory defined on the rescaled lattice. Although the block Hamiltonian is not fully gauge invariant, the new renormalized Hamiltonian describing the physics at twice the original length scale possesses the original gauge symmetry with respect to the new lattice sites. The only other approach we know of that preserves gauge invariance is the one employed by Migdal and Kadanoff. We regard this work also as a step towards the goal of understanding confinement and related problems in non-Abelian gauge theories using nonperturbative techniques. The methods we use can in fact be applied to non-Abelian spin and gauge theories with limited additional complications.

In Sec. II we study the planar model in the Hamiltonian (transfer matrix) formulation in $1+1$ and $2+1$ dimensions.[11] The truncation procedure we adopt is such that at every site we keep an arbitrarily large number of eigenstates of the strong-coupling limit of the Hamiltonian. The truncation to a finite number of states is expected not to change the critical properties of the model. But because of our choice of approximating theory, we also do not in general expect to be able to give a satisfactory description of the weak-coupling region beyond the critical point in the untruncated model. In $1+1$ dimensions we find a phase transition between a massless phase and a massive one in the finite-spin theory. At the critical point the mass gap goes to zero algebraically. In $2+1$ dimensions we find a critical point at some finite coupling and we compute the critical exponents. Because of the variational character of our renormalization method, the exponents for "timelike" quantities, like the mass gap, are in much better agreement (less than one tenth of a percent) with known series-expansion results than the exponents for "spacelike" quantities, like the correlation length and the order parameter.

Section III is devoted to the study of the U(1) gauge theory in the Hamiltonian formulation in two and three spatial dimensions. In $2+1$ dimensions we are able to confirm the results by Polyakov about the absence of a free-photon phase in the compact version of the theory.[2] Our method seems capable of reproducing the qualitative features that arise in the Euclidean action formulation from evaluating the path integral around topologically nontrivial solutions. Our method can be regarded as complementary to the one employed in related work[12] (where the weak-coupling region was studied in detail), in the sense that some of our approximations become exact in the limit of strong coupling. In particular, our calculations suggest that the string tension vanishes with an essential singularity at zero coupling. In $3+1$ dimensions our renormalization-group analysis gives a phase transition at finite couplings. When the coupling is sufficiently strong we find that static charges are confined as a consequence of the compactness of the fields. On the other hand, for weak enough coupling confinement is lost and at large distances Maxwell's electrodynamics is recovered. At present we do not know of any other renormalization-group calculation that gives these results, although they are not unexpected. The picture we have described is in agreement with a conjecture by Migdal that the four-dimensional U(1) gauge theory should undergo a phase transition.[3]

II. U(1)-SYMMETRIC SPIN SYSTEMS

We start from the statistical-mechanics Hamiltonian

$$\mathcal{H} = -J\sum_{\vec r,\hat\mu} \vec n(\vec r)\cdot\vec n(\vec r+\hat\mu), \tag{2.1}$$

where the $\vec n$'s are unit vectors defined on the sites of a square lattice of spacing a and $\hat\mu$ labels the directions on the lattice. A useful parametrization for $\vec n$ is

$$\vec n = (\cos\theta, \sin\theta). \tag{2.2}$$

In the continuum limit we can write (up to a constant)

$$\mathcal{H} = \frac{1}{2g^2}\int d^dx(\nabla\vec n)^2 \tag{2.3}$$

with $J = 1/g^2$. The relativistic field theory corresponding to (2.3) is obtained by performing a Wick rotation $x_0 \to it$ and replacing the statistical-mechanics Hamiltonian by a classical action

$$S = \frac{1}{2g^2}\int dt\, d^{d-1}x(\partial_\mu\vec n)^2. \tag{2.4}$$

The integrand is the Lagrangian density and the momentum conjugate to the field θ is therefore

$$\frac{\partial\mathcal{L}}{\partial\dot\theta} = \frac{1}{g^2}\dot\theta. \tag{2.5}$$

For a theory with a continuous time axis and a discrete spatial axis, the Hamiltonian becomes

$$H = \frac{g^2}{2a}\sum_{\vec r,\hat\nu}\left[J_z{}^2(\vec r) - \frac{2}{y}\vec n(\vec r)\cdot\vec n(\vec r+\hat\nu)\right]. \tag{2.6}$$

We have also set $y = g^2$. The canonical commutation relations between the J's and the $\vec n$'s can be stated as

$$[J_z(\vec r), e^{\pm i\theta(\vec r\,')}] = \pm e^{\pm i\theta(\vec r)}\delta_{\vec r,\vec r\,'}. \tag{2.7}$$

It is useful to introduce

$$\begin{aligned}\varphi &= e^{i\theta} = n_1 + in_2,\\ \varphi^\dagger &= e^{-i\theta} = n_1 - in_2\end{aligned} \tag{2.8}$$

and rewrite H in the slightly more general form

$$\frac{2a}{g^2}H = \sum_{\vec r,\hat\nu}[\epsilon J_z{}^2(\vec r) - \Delta(\varphi(\vec r)\varphi^\dagger(\vec r+\hat\nu) + \mathrm{H.c.})] \tag{2.9}$$

with $\epsilon/\Delta = y$. The φ's are now simply ladder operators on eigenstates of J_z.

To perform a real-space renormalization-group analysis, we group the lattice sites into blocks. In $1+1$ dimensions the blocks consist of two sites (see Fig. 1). At each site we set up a basis spanned by the eigenstates $|m\rangle$ of J_z,

$$\begin{aligned}J_z|m\rangle &= m|m\rangle,\\ \varphi^\dagger|m\rangle &= |m+1\rangle,\\ \varphi|m\rangle &= |m-1\rangle,\end{aligned} \tag{2.10}$$

and limit ourselves to $|m| \le s$, with s some fixed large number. This approximation is justified for large y [this can be seen from the form of H (2.6) for large y] and is consistent with our block-spin method since the coupling between blocks is of order $1/y$. Because of our choice of finite-s Hamiltonian, the operators $\psi^\dagger$ and ψ have become nilpotent and do not possess nonzero eigenvalues, which forces the order parameter $\langle\psi\rangle$ to be zero

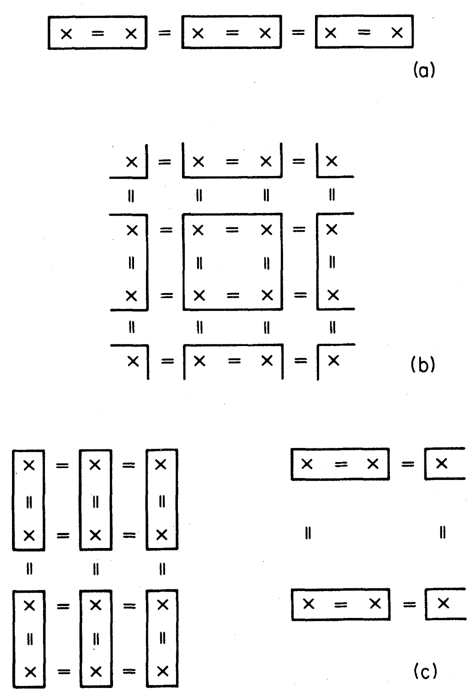

FIG. 1. Blocks of spins for the planar model: (a) in $1+1$ dimensions, (b), (c) in $2+1$ dimensions.

for all couplings (although we get nontrivial Δ renormalizations). On the other hand, the infinite spin and the truncated Hamiltonian give exactly the same answers to nth order in perturbation theory in $1/y$, as long as $n \leq s$. We expect therefore to get reliable answers for quantities that have an expansion around $1/y = 0$, as long as we keep s large. Furthermore, the universality hypotheses would suggest that critical properties (like exponents) should actually be independent of s. The states of the block can be labeled by the total angular momentum (since $\hat{H}$ does not mix these different sectors) and the block Hamiltonian can be diagonalized in each of these sectors independently. We choose the renormalized parameter ϵ to be

$$\epsilon_R = \lambda_{\pm 1} - \lambda_0, \tag{2.11}$$

where λ_0 is the lowest eigenvalue in the $J_{z\,\mathrm{tot}} = 0$ sector and $\lambda_{\pm 1}$ is the lowest eigenvalue in the $J_{z\,\mathrm{tot}} = \pm 1$ sectors. With this prescription the block Hamiltonian in fact renormalizes into one of exactly the same form for weak coupling ($y \ll 1$). The renormalized intersite coupling is defined as

$$\Delta_R = K^2 \Delta, \quad K = \langle +1 | \varphi^\dagger | 0 \rangle \tag{2.12}$$

and the renormalized coupling is therefore

$$y_R = \epsilon_R / \Delta_R. \tag{2.13}$$

It is useful to define the function

$$R(y) = y_R - y, \tag{2.14}$$

which has the property that it vanishes at a fixed point. The graph of this function for the (1+1)-dimensional planar model is shown in Fig. 2. The critical point is (for large s) at

$$y_c = 2.202\,953\ldots. \tag{2.15}$$

Series expansions suggest $y_c \simeq 1.2$.[11] In the large-s limit one can show that for large y

$$\begin{aligned} \epsilon_R &= \epsilon\left(1 - \frac{1}{y} + \frac{3}{4}\frac{1}{y^2} + \cdots\right), \\ \Delta_R &= \frac{\Delta}{2}\left(1 + \frac{1}{y} - \frac{1}{16}\frac{1}{y^2} + \cdots\right), \\ R(y) &= y - 4 + \frac{6}{y} + \cdots, \end{aligned} \tag{2.16}$$

and for small y

$$R(y) = -\tfrac{1}{2}y + \tfrac{1}{4}(y/2)^{3/2} + O(y^2). \tag{2.17}$$

It is easy in this scheme to compute the mass gap μ and the order-parameter renormalization constant M. In the high-temperature phase we have

$$\mu \propto \lim_{n\to\infty} \epsilon^{(n)}, \quad M \propto \lim_{n\to\infty} (\Delta^{(n)})^{1/2}, \quad (y > y_c) \tag{2.18}$$

whereas in the low-temperature phase

$$\mu \propto \lim_{n\to\infty} \Delta^{(n)}, \quad M \propto \lim_{n\to\infty} (\Delta^{(n)})^{1/2}. \quad (y < y_c) \tag{2.19}$$

Here $\epsilon^{(n)}$ and $\Delta^{(n)}$ are the nth iterates of ϵ and Δ under the renormalization-group transformation. A numerical study of the recursion relations shows that $\Delta^{(n)}$ iterates to zero both above and below y_c, whereas $\epsilon^{(n)}$ iterates to a finite value above y_c and to zero below y_c. We can therefore conclude that a massive phase exists for $y > y_c$. Below y_c the Hamiltonian iterates to a constant, which has no gap. The behavior of the mass gap in the finite-spin model for all couplings is shown in Fig. 3. We should mention the fact that it appears that our block-spin renormalization procedure is unable to reproduce an essential singularity in the mass gap at y_c. We do not know at present how to circumvent this problem without substantially altering the simple structure of our renormaliza-

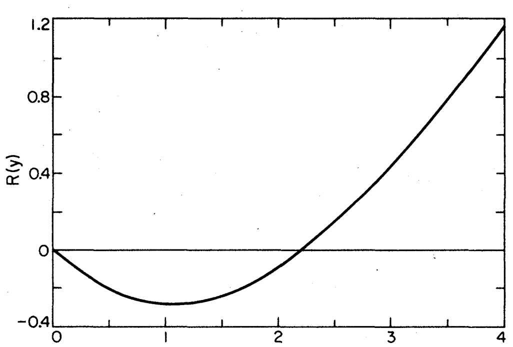

FIG. 2. The function $R(y)$ for the planar model in 1+1 dimensions, using 21 states at every site.

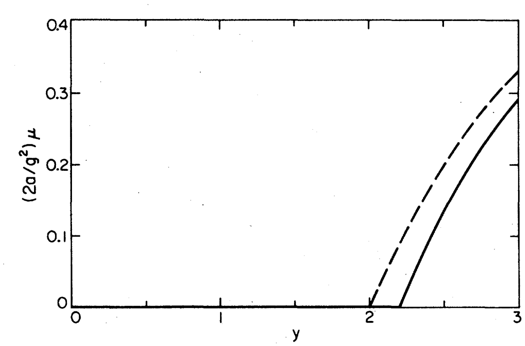

FIG. 3. The mass gap for the finite-spin planar model in 1+1 dimensions. The dashed line is the lowest-order perturbative expansion $(1 - 2/y)$.

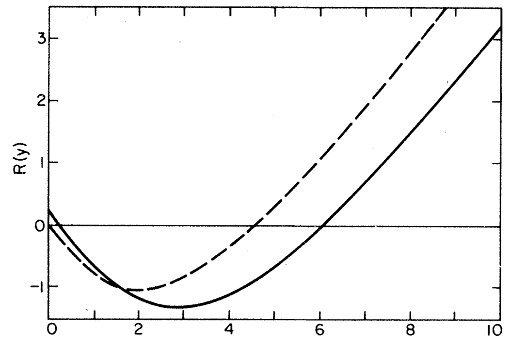

FIG. 4. The functions $R(y)$ for the planar model in $2+1$ dimensions. The continuous line is for four sites in a block, keeping three states at every site. The dashed line is for blocks of two sites and 21 states at every site.

tion transformation.

The planar model in $2+1$ dimensions is studied in analogous fashion. We now group the lattice sites into blocks of four (see Fig. 1) and again diagonalize the Hamiltonian in each sector characterized by different $J_{z\,\mathrm{tot}}$. When we keep three states at each site and use the same renormalization prescription as for the $(1+1)$-dimensional case, we obtain the function $R(y)$ shown in Fig. 4. The critical point is at

$$y_c = 6.024\,318\ldots\,, \tag{2.20}$$

corresponding to a second-order phase transition.

In general, we can introduce two functions λ_ϵ and λ_Δ that describe how ϵ and Δ change under the renormalization group:

$$\begin{aligned} \epsilon_R &= \lambda_\epsilon(y)\epsilon\,, \\ \Delta_R &= \lambda_\Delta(y)\Delta\,. \end{aligned} \tag{2.21}$$

We find that in the large-y phase, $\epsilon^{(n)}$ iterates to a constant and $\Delta^{(n)}$ to zero. In the other phase ($y<y_c$) both $\epsilon^{(n)}$ and $\Delta^{(n)}$ iterate to larger and larger values. Since we know that the three-dimensional (∞-spin) xy model has a Goldstone phase with no mass gap and a nonvanishing order parameter, we interpret these results as a breakdown of the finite-spin (finite-s) Hamiltonian in the $y<y_c$ phase. This is not surprising since the order parameter, for example, does not possess an expansion around $1/y=0$. At the fixed point y_c, the two functions λ_ϵ, λ_Δ take the same value $\lambda = 0.702\,20$ and using the value for $R'(y_c) = 0.693\,83$ we obtain, as described in the Appendix, the exponent for the mass gap

$$\nu_G = 0.670\,84\,, \tag{2.22}$$

which should be compared with the series-expansion result $\nu = 0.670 \pm 0.006$.[13] For the correlation length exponent (obtained by linearizing the recursion relations at the critical point) we get

$$\nu_\xi = \frac{1}{y_T} = 1.3153 \tag{2.23}$$

and we can define an "order-parameter" exponent (see the Appendix):

$$\beta = 0.9931\,. \tag{2.24}$$

The fact that our variational renormalization-group method tends to give much better exponents for energylike quantities had previously been noted in other contexts.[7,8] Under the renormalization group lengths get doubled whereas energies are rescaled by λ and therefore time by $1/\lambda$. Owing to this asymmetric scaling of space and time under the renormalization group, we find $\nu_G \ll \nu_\xi$, whereas for an exact transformation they should turn out to be equal.

In order to investigate how our results change when we modify the renormalization scheme, we can also group lattice points into blocks of two and use the one-dimensional recursion relation twice, as shown in Fig. 1. It would appear that in this way we might generate anisotropic couplings, but this is not the case. Again, using a large (~ 21) number of states at every site, we get a critical point at

$$y_c = 4.548\,198 \tag{2.25}$$

with $\lambda = 0.699\,41$ and $R'(y_c) = 0.705\,68$, from which we compute

$$\nu_G = 0.669\,55\,, \tag{2.26}$$

which agrees to 0.2% with the previous value. In this approximation the exponent ν_G seems to be independent of s, the number of states retained at every site. For large y one can show that

$$R(y) = y - 7 + \frac{73}{8}\,\frac{1}{y} + \cdots\,. \tag{2.27}$$

This function is also shown in Fig. 4.

III. U(1) GAUGE THEORY

The compact Abelian gauge theory we will consider is defined by the Euclidean lattice action[1]

$$I = \frac{1}{2g^2} \sum_p (1 - \cos\theta_{\mu\nu})\,. \tag{3.1}$$

The sum extends over all plaquettes on the hypercubic lattice of spacing a. $\theta_{\mu\nu}$ is the field-strength tensor on the lattice

$$\begin{aligned} \theta_{\mu\nu} &= \Delta_\mu \theta_\nu - \Delta_\nu \theta_\mu\,, \\ \theta_\mu &= agA_\mu \end{aligned} \tag{3.2}$$

and Δ_μ is the lattice derivative in the direction μ.

The fields θ_μ are defined now on the links of the lattice. Again we can define a quantum-mechanical Hamiltonian for the theory, in a way that is analogous to the derivation in the case of the planar model.[14,15] The result is

$$\frac{2a}{g^2}H=\sum_L J_z^2(L)-\frac{1}{y}\sum_p(\varphi\varphi\varphi^\dagger\varphi^\dagger+\text{H.c.}). \tag{3.3}$$

Here L is an index that labels the links on the lattice and the operators J_z and φ have the same meaning as in the spin system. The product of φ's is taken around the border of a plaquette and $y=g^4$. The electric-field operator is

$$E(L)=\frac{g}{a^2}J_z(L) \tag{3.4}$$

and because of the compactness of the fields it can take only integer values, in units of g/a^2. The commutation relations between $A(L)$ and $E(L)$ are

$$[A(L),E(L')]=\frac{i}{a^3}\delta_{L,L'}. \tag{3.5}$$

For small g the cosine in (3.3) can be expanded and the Hamiltonian becomes (in $3+1$ dimensions)

$$H=a^3\sum_L \tfrac{1}{2}E^2(L)+a^3\sum_p\tfrac{1}{2}\Big(\sum_{\partial p}A\Big)^2+O(g^2a^4). \tag{3.6}$$

In the continuum limit we replace the sums by integrals and the lattice curl by the corresponding differential operator and obtain

$$H=\tfrac{1}{2}\int d^3x[\vec{E}^2+(\vec\nabla\times\vec{A})^2]+O(g^2a^4), \tag{3.7}$$

which is the Hamiltonian for Maxwell's electrodynamics. The Hamiltonian (3.3) commutes with the unitary operator that rotates the phase angles of the two-dimensional variables that live on the links emanating from the site i by $\chi(i)$, for every site i:

$$G(\chi)=\exp\Big[\sum_i J_z(L_i)\chi(i)\Big]. \tag{3.8}$$

Local gauge invariance means that $\chi(i)$ can be different from site to site. Under $G(\chi)$ the field transforms as

$$G(\chi)\theta_\mu(L)G^{-1}(\chi)=\theta_\mu(L)+\Delta_\mu\chi. \tag{3.9}$$

We require the ground state to be gauge invariant for all χ,

$$G(\chi)|0\rangle=|0\rangle, \tag{3.10}$$

and this implies that gauge-noninvariant quantities like the order parameter and φ-φ correlation functions vanish identically. A gauge-invariant correlation function is the Wilson loop

$$W(\Gamma)=\Big\langle 0\Big|\prod_{L\in}\varphi(L)\Big|0\Big\rangle, \tag{3.11}$$

where the product is around a closed loop on the lattice. The different behavior of this function for large loops can be used to characterize the phases of the theory. Alternatively, we can introduce two widely separated static charges and compute the energy of this configuration with respect to the vacuum. This quantity divided by the distance between the two charges defines the string tension.

It is known that for strong coupling ($y\gg1$) the theory exhibits linear confinement in any dimension. To lowest order in perturbation theory the electric string tension is given by

$$T_E=\frac{g^2}{2a^2}\Big(1-\frac{d-1}{3}\frac{1}{y^2}+\cdots\Big) \tag{3.12}$$

with d the dimensionality of space. This is just a consequence of the nonlinearities in the dynamics that are generated by the compactness of the fields. In this phase the gauge-invariant excitations are closed loops of electric flux and the lowest-lying excitations are just simple boxes (see Fig. 5).

As the coupling strength decreases, the vacuum becomes populated with loops of arbitrary size. For $y\sim1$ the size of these loops is comparable to the distance between them. We want to answer the question of whether the string tension vanishes at some finite coupling. It would be rather disconcerting if the Abelian lattice gauge theory in four dimensions were to lead to confinement for all couplings.

We will first consider the case of $2+1$ dimensions. The renormalization-group transformation we will construct will have the property of preserving gauge invariance.[7,9] In order to perform a real-space renormalization-group analysis, we group links and plaquettes into disjoint blocks. We write

$$\frac{2a}{g^2}H=\sum_b H_b, \tag{3.13}$$

where H_b is a block Hamiltonian containing eight links and four plaquettes,

$$H_b=\alpha\sum_{L=1}^{8}J_z^2(L)-\beta\sum_{P_1P_2P_3P_4}(\varphi\varphi\varphi^\dagger\varphi^\dagger+\text{H.c.}). \tag{3.14}$$

We have allowed again for $\alpha\neq1$ and define now the coupling to be $y=\alpha/\beta$. A new theory is now

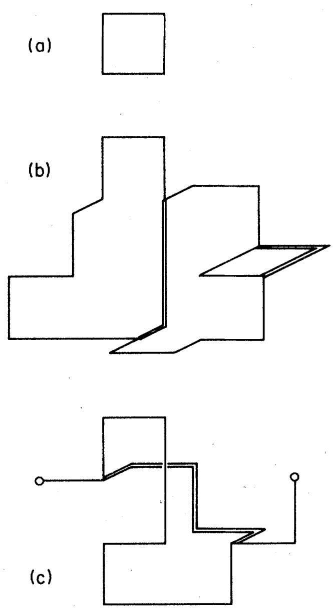

FIG. 5. (a) A single plaquette on the lattice. (b) A closed string of electric flux. On the single lines $E^2=1$ and on the double lines $E^2=4$ in units of g^2/a^4. (c) A string connecting two static charges.

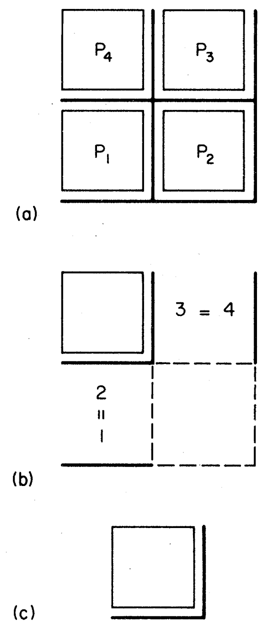

FIG. 6. Blocks for the gauge theory in 2 + 1 dimensions: (a) the original four plaquettes and eight links, (b) the plaquette and the two pairs of coupled links that survive after the first decimation, (c) the new block on the rescaled lattice.

defined through the Hamiltonian

$$\frac{2a}{g^2} H = \sum_b H_b' \tag{3.15}$$

with (see Fig. 6)

$$H_b' = 2\alpha \sum_{L=1}^{4} J_z^{\,2}(L) - \gamma[\varphi(1)\varphi^\dagger(2) + \varphi(3)\varphi^\dagger(4) + \text{H.c.}] - \beta[\varphi\varphi\varphi^\dagger\varphi^\dagger(P_4) + \text{H.c.}]. \tag{3.16}$$

H_b' contains only four links and one plaquette. Furthermore, the first two terms of H_b' do not contain operators associated with links of neighboring plaquettes. The factor 2α that multiplies the first term is chosen in order that excitations caused by the second term have the same energy as box excitations in H_b. γ is a parameter that is determined by the requirement that the ground-state energy of H_b and H_b' are the same. Since, however, we are unable to obtain them in closed form, we resort to perturbation theory in $1/y$ and find

$$\frac{\gamma^2}{\alpha^2} = \frac{3}{2}\,\frac{1}{y^2}\left[1 + \frac{379}{5760}\,\frac{1}{y^2} + O\!\left(\frac{1}{y^4}\right)\right]. \tag{3.17}$$

The second step now consists in reducing further the degrees of freedom by replacing the variables associated with the links 1 and 2 (and 3 and 4) by a single set of operators. Given the links 1 and 2, for example, we proceed to truncate to the lowest-lying states of the Hamiltonian

$$H'' = 2\alpha[J_z^{\,2}(1) + J_z^{\,2}(2) - \gamma(\varphi(1)\varphi^\dagger(2) + \text{H.c.})]. \tag{3.18}$$

But this is just the problem we solved in the preceding section with

$$\epsilon = 2\alpha, \quad \Delta = \gamma. \tag{3.19}$$

Here again we truncate to the lowest-lying states in each angular momentum sector. After this step the renormalized Hamiltonian is defined on a length scale that is twice the original one and can be expressed in the original form as

$$\left(\frac{2a}{g^2}H\right)_R = \alpha_R \sum_L J_z^{\,2} - \beta_R \sum_P (\varphi\varphi\varphi^\dagger\varphi^\dagger + \text{H.c.}) + \text{const.} \tag{3.20}$$

The renormalized parameters are, in terms of the quantities computed with the planar block Hamiltonian,

$$\alpha_R = \epsilon_R = \lambda_\pm - \lambda_0,$$
$$\beta_R = K^4 \beta, \tag{3.21}$$
$$y_R = \frac{\alpha_R}{\beta_R}.$$

We now wish to briefly comment on the question of gauge invariance. If we consider the gauge-transformation operators acting on the sites at the corners of the lower-right square in the block of Fig. 5(a), we see that H_b is invariant under each of these generators separately, while H_b' is only invariant under their product. But if we define the gauge generator for the block to be the sum of the four gauge generators on the corners, then we see that H_b' is invariant. A state with zero charge on the coarse lattice corresponds to those sectors on the finer lattice for which the sum of the four generators gives zero.

Again we can define the function

$$R(y) = y_R - y \tag{3.22}$$

and find, for large y,

$$\alpha_R = 2\alpha\left(1 - \sqrt{3/8}\,\frac{1}{y} + \cdots\right),$$
$$\beta_R = \tfrac{1}{4}\beta\left(1 + \sqrt{3/2}\,\frac{1}{y} + \cdots\right), \tag{3.23}$$
$$R(y) = 7y - \sqrt{216} + \frac{75}{4}\,\frac{1}{y} + \cdots .$$

For small y, using a large number of states and the lowest-order result for γ, and exploiting known results from the theory of Mathieu functions, we get

$$\alpha_R = \alpha(1 + c_1 y^{-3/4} e^{-c_2/y^{1/2}} + \cdots),$$
$$\beta_R = \beta(1 - c_3 y^{1/2} + \cdots) \tag{3.24}$$

and, therefore,

$$R(y) = c_3 y^{3/2} + O(y^3) \tag{3.25}$$

with

$$c_1 = (2^{33}3^3)^{1/8}/\sqrt{\pi}, \quad c_2 = 4(\tfrac{3}{2})^{1/4}, \quad c_3 = \tfrac{1}{2}(\tfrac{2}{3})^{1/4}.$$

The function $R(y)$ is shown in Fig. 7. Using the first- or the second-order result for λ does not significantly affect the shape of $R(y)$. (At $y = \frac{1}{20}$ to lowest order we get 0.0056 and at the next order 0.0023.) We find no indication of a phase transition for finite values of the coupling constant.

In terms of $g = y^{1/4}$, the β function $\beta(g) = R(y)/(4g^3)$ is given by

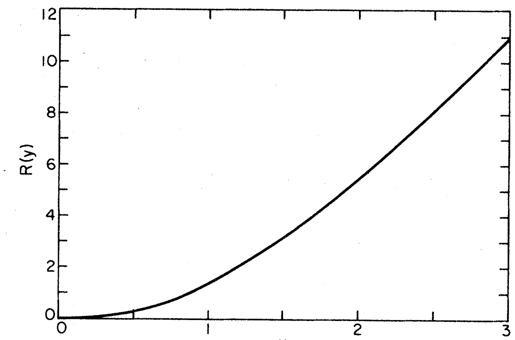

FIG. 7. The function $R(y)$ for the gauge theory in 2 + 1 dimensions.

$$\frac{\beta(g)}{g} = \frac{1}{g}\,\frac{\partial g}{\partial \ln a} = \frac{c_3}{4} g^2(1 + c_1 g e^{-c_2/g^2} + \cdots)$$

and it has an essential singularity at $g = 0$.

We can define electric and magnetic string tensions, which are the energies per unit length of an electric (magnetic) flux tube connecting two static sources. In our renormalization scheme the string tensions are given by

$$T_E = \frac{g^2}{2a^2} \lim_{n\to\infty} \frac{\alpha^{(n)}}{2^n},$$
$$T_M = \frac{g^2}{2a^2} \lim_{n\to\infty} \frac{\beta^{(n)}}{2^n}. \tag{3.26}$$

In this way we do not need to actually introduce static sources to compute the tension. If we start out with a coupling of order 1, we iterate as many times as necessary to make the coupling either weak ($y \ll 1$) or strong ($y \gg 1$), after which we can apply perturbation theory to compute the quantities of interest. Depending on the dimension of these quantities, we have then to divide by the appropriate power of the lattice-spacing rescaling factor.

In 2 + 1 dimensions we find that the magnetic tension is always zero, whereas the electric tension vanishes with an essential singularity at the origin. To lowest order this is a direct consequence of the structure of the recursion relation (3.24), as shown in the Appendix. We find that our results are consistent with a behavior of the electric string tension of the type

$$\frac{2a^2}{g^2} T_E \underset{y\to 0}{\sim} A e^{-b/y^\sigma} \tag{3.27}$$

with $b = 1.53419$ and $\sigma = \frac{1}{2}$. A graph of the string tension is shown in Fig. 8.

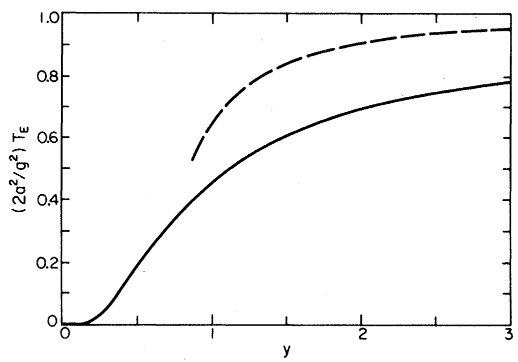

FIG. 8. The electric string tension for the gauge theory in 2 + 1 dimensions.

It seems therefore that our method is capable of reproducing correctly also some of the qualitative features of the weak-coupling region, using techniques that are significantly different from previous works.[2,12] In particular, our analysis suggests a behavior for the static confining potential that closely resembles the results obtained by Polyakov. His method is rather different and consists in evaluating the Euclidean path integral for compact QED around nontrivial topological configurations. In fact, the situation we encounter is not dissimilar from what one expects in four-dimensional non-Abelian gauge theories.

The analysis for the U(1) gauge theory in 3 + 1 dimensions follows along the same lines. Again we write

$$\frac{2a}{g^2}H=\sum_b H_b \tag{3.28}$$

and H_b now contains 24 plaquettes and 24 links. These are shown in Fig. 9. The 12 links and 6 plaquettes that survive after the first decimation are also shown. The links in H_b' are now coupled four at a time through a nearest-neighbor spin coupling of strength γ with

$$\frac{\gamma^2}{\alpha^2}=\frac{3}{2}\frac{1}{y^2}+O\left(\frac{1}{y^4}\right). \tag{3.29}$$

Now we are led to the solution of the problem of a four-site block of planar spins. Using the results of the preceding section we can write

$$\begin{aligned}\alpha_R&=\epsilon_R=\lambda_{\pm1}-\lambda_0,\\ \beta_R&=2K^4\beta,\\ y_R&=\frac{\alpha_R}{\beta_R}.\end{aligned} \tag{3.30}$$

The factor of 2 in the definition of β_R stems from the fact that now two plaquettes couple the blocks

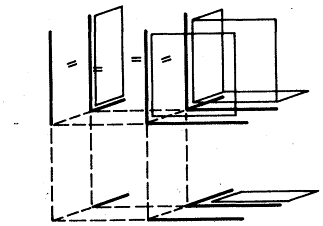

FIG. 9. The block for the gauge theory in 3 + 1 dimensions.

in each spatial direction (see Fig. 6).

For large y we find

$$\begin{aligned}\alpha_R&=2\alpha\left(1-\sqrt{3/2}\,\frac{1}{y}+\cdots\right),\\ \beta_R&=\tfrac{1}{8}\beta\left(1+\sqrt{27/8}\,\frac{1}{y}+\cdots\right),\\ R(y)&=15y-\sqrt{2400}+\frac{387}{4}\frac{1}{y}+\cdots.\end{aligned} \tag{3.31}$$

The function $R(y)$ is shown in Fig. 10. The fixed point is at

$$y_c=0.849\,749\,8\ldots \tag{3.32}$$

and there the theory undergoes a phase transition between an electric confining and a nonconfining phase. Our result compares favorably with an estimate based on an energy-entropy balancing for large monopole current loops in the Euclidean lattice formulation, which gives $y_c\simeq 0.93$.[16] Monte Carlo simulations give $y_c=1.02\pm0.05$.[17]

Our renormalization-group transformation (which because of the approximations should be

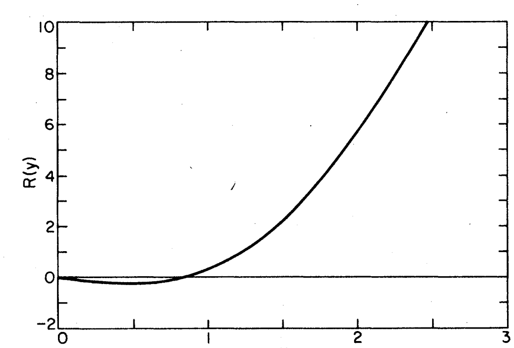

FIG. 10. The function $R(y)$ for the gauge theory in 3 + 1 dimensions.

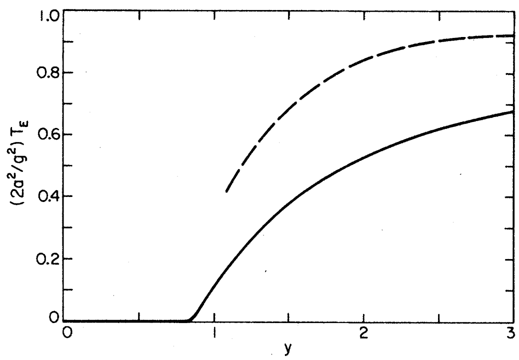

FIG. 11. The electric string tension for the gauge theory in 3 + 1 dimensions. The dashed line is the result from lowest-order perturbation theory.

fully trusted only for $y > y_c$) indicates that the magnetic string tension vanishes for all couplings, whereas the electric string tension converges to a finite value for $y > y_c$ (see Fig. 11). At y_c it goes to zero continuously, with $\nu_T = 1.1805$. The mass-gap exponent is $\nu_G = 0.4007$. When the bare coupling is small enough ($y < y_c$) then at large distances the theory is equivalent to free electrodynamics. On the other hand, for large bare couplings ($y > y_c$) static charges are confined as a consequence of the self-interactions of the photon field, inherent in the compact formulation of the theory. Our results agree nicely with a conjecture by Migdal that four-dimensional compact QED should undergo a phase transition at finite couplings, whereas non-Abelian gauge theories should not. In the continuum limit this would prevent electric charges from being confined (for $y < y_c$, but the same would not happen for colored charges. It remains still an open question of how the string tension vanishes at the critical point. Our analysis at the present time does not seem sophisticated enough to reproduce an essential singularity in the strong tension at finite coupling.

IV. CONCLUSIONS

We have shown how simple renormalization-group methods can be effective in describing the large-distance properties of spin systems and gauge theories in the Hamiltonian formulation. The "block-spin" renormalization prescription we adopt seems to be capable of reproducing well many of the qualitative and quantitative features of the phase transitions we encounter. Because of the variational truncation procedure we adopt, space and time scale in an asymmetric way under the renormalization group and we see that our results for timelike quantities appear to be more reliable. Our simple variational truncation method on the other hand does not seem well suited for searching for essential singularities at finite couplings. We regard our results in the case of the U(1) gauge theories encouraging and see as a next application of our methods non-Abelian gauge theories in four dimensions.

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APPENDIX: SCALING RELATIONS AND EXPONENTS

We have defined the function $R(y)$ to be

$$R(y) = y_R - y. \tag{A1}$$

Close to y_c we can expand $R(y)$ in a Taylor series,

$$R(y) = R_c'(y - y_c) + \tfrac{1}{2}R_c''(y - y_c)^2 + \cdots, \tag{A2}$$

and therefore close to y_c we have

$$y_R - y_c = (1 + R_c')(y - y_c) + \tfrac{1}{2}R_c''(y - y_c)^2 + \cdots. \tag{A3}$$

For definiteness we shall consider the mass gap for a spin system, given by

$$\mu \propto \lim_{n\to\infty} \epsilon^{(n)} \quad (y > y_c),$$
$$\mu \propto \lim_{n\to\infty} \Delta^{(n)} \quad (y < y_c), \tag{A4}$$

where $\epsilon^{(n)}$ and $\Delta^{(n)}$ are the nth iterates of ϵ and Δ under the renormalization-group transformation. Close to y_c the mass gap behaves like

$$\mu \underset{y \to y_{c+}}{\sim} (y - y_c)^{\nu_G}. \tag{A5}$$

If we choose y_1 and y_2 close to y_c and related by

$$y_2 = y_1 + R(y_1), \tag{A6}$$

then we have

$$\frac{\mu_2}{\mu_1} = \left(\frac{y_2 - y_c}{y_1 - y_c}\right)^{\nu_G} = (1 + R_c')^{\nu_G}. \tag{A7}$$

But we also know how μ gets rescaled under renormalization, namely

$$\frac{\mu_1}{\mu_2} = \lambda(y_c), \tag{A8}$$

sufficiently close to y_c, and therefore

$$\nu_G = \frac{\ln(1/\lambda)}{\ln(1+R_c')}. \tag{A9}$$

For the string tension we would have gotten

$$\nu_T = \frac{\ln(2/\lambda)}{\ln(1+R_c')}. \tag{A10}$$

Now it can happen that $R_c' = 0$. If we assume

$$\mu \underset{y\to y_{c+}}{\sim} Ae^{-b/(y-y_c)^\sigma} \tag{A11}$$

then we obtain for $R_c'' \neq 0$, $\sigma = 1$ and

$$b = \frac{\ln(1/\lambda)}{\frac{1}{2}R_c''}. \tag{A12}$$

In general if we have, close to y_c,

$$R(y) \underset{y\to y_{c+}}{\sim} C(y-y_c)^{1+\sigma}, \tag{A13}$$

then

$$\mu \underset{y\to y_{c+}}{\sim} Ae^{-b/(y-y_c)^\sigma} \tag{A14}$$

with b given by

$$b = \frac{1}{c}\ln\frac{1}{\lambda}. \tag{A15}$$

For the string tension the corresponding relation is

$$b = \frac{1}{c}\ln\frac{2}{\lambda}. \tag{A16}$$

The correlation length exponent ν_ξ on the other hand is given by the inverse of the thermal exponent y_T,

$$\nu_\xi = \frac{1}{y_T} \tag{A17}$$

and y_T is obtained by linearizing the recursion relations at the critical point:

$$y_R - y_c = 2^{y_T}(y-y_c) \tag{A18}$$

and therefore

$$\nu_\xi = \frac{\ln 2}{\ln(1+R_c')}. \tag{A19}$$

If $1/\lambda = 2$ were true, then we would have $\nu_\xi = \nu_G$. The magnetization is defined by

$$M = \lim_{n\to\infty} K^{(n)} \tag{A20}$$

and by the analogous scaling argument we can show that

$$\beta = \frac{\ln(1/K_c)}{\ln(1+R_c')}, \tag{A21}$$

where K_c is defined by how the operator φ renormalizes at y_c,

$$[\varphi]_R = K_c\varphi. \tag{A22}$$

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