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CRITICAL BEHAVIOR IN SIMPLICIAL QUANTUM GRAVITY *

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Recent results for simplicial quantum gravity in four dimensions are reviewed. Effects of both higher derivative terms and gravitational measure contributions are investigated. Prospects for solving numerically quantized gravity in four dimensions are discussed.

1. INTRODUCTION

Four-dimensional quantum gravity is a complex theory due to the unboundedness of the pure gravitational action, its non-renormalizability and non-polynomial nature[1]. We will concentrate here on the simplicial formulation of quantum gravity also known as Regge calculus (for a review, see refs. [2,3]). One of the advantages of the approach lies in the fact that it can be formulated in any space-time dimension (including the physically relevant case of four dimensions), and that it can be shown to be classically equivalent to general relativity. Furthermore the correspondence between lattice and continuum quantities is clear, and the interpretation of the terms in the action as well as the identification and separation of, for example, the measure contribution is unambiguous[4-10]. For a more complete list of early references, see refs. [2,3].

One important issue that needs to be addressed in lattice gravity is the problem of the gravitational measure. In the continuum the form of the measure for the $g_{\mu\nu}$ fields appears not to be unique [16-18], and it would seem that such an ambiguity persist in *all* known lattice formulation of quantum gravity. However the difference among the measures seems to be in the power of $\sqrt{g}$ in the prefactor, which corresponds to some product of volume factors on the lattice. DeWitt has argued that the gravitational measure should be constructed by first introducing a super-metric over metric deformations, which leads then to the functional measure for pure gravity in d dimensions form

$$d\mu[g] = \prod_x g^{(d-4)(d+1)/8} \prod_{\mu \geq \nu} dg_{\mu\nu} \qquad (1.1)$$

Other forms of the measure for the gravitational field have also been suggested, inspired by the canonical quantization approach to gravity[18]. If matter fields are present, then the gravitational measure has to be further modified[16].

On the simplicial lattice the edge lengths are the elementary degrees of freedom, which uniquely specify the geometry for a given incidence matrix, and over which one should perform the functional integral[2,8,10]. A class of pure gravity measures which can be written down on the lattice is obtained by considering the 'volume associated with an edge' V_{ij}, and writing for example in two dimensions

$$\int d\mu_\epsilon[l] = \prod_{edges\ ij} \int_0^\infty V_{ij}^{2\sigma}\, dl_{ij}^2\, F_\epsilon[l] \qquad (1.2)$$

with $\sigma = -1/2$ for the lattice analogue of the Misner measure, and $\sigma = -1/4$ for a lattice analogue of the DeWitt measure (note that the 'Misner' and dl/l measure share the property of being scale invariant). One would like to see how the results depend on the form of the measure and on σ. Our simulations suggest that, at least in two dimensions, different measures, within a certain universality class, will give the same results for infrared sensitive quantities, like correlation functions at large distances and critical exponents. We believe though that the lattice path integral might not be meaningful for certain values of σ. We have found in particular that if σ is too negative in two dimensions, then the measure factors tend to favor configurations of triangles which are long and thin, with a small area and a large perimeter.

2. GRAVITY IN FOUR DIMENSIONS

The four-dimensional case is substantially more complex than the two-dimensional one for a number of reasons, which include the fact that there are more terms in the pure gravity action, there are no exact results in the full theory to compare with, and finally that the lattice structure and the interactions are more complex and therefore the lattices that have been studied up to know are quite small. Furthermore there is a conceptual issue of what physical quantities should be measured, and for what boundary conditions. Only a small set of these questions have been addressed up to know.

In our numerical studiies we have employed the discrete analog of the higher derivative action in the form [2,8,10]

$$I = \sum_{hinges\ h} \left[\lambda\, V_h - k\, \delta_h A_h + 4b\, \frac{A_h^2 \delta_h^2}{V_h} \right]$$

$$+ \frac{1}{3}(a - 4b) \sum_{sites\ p} V_p \sum_{hinges\ h,h' \supset p}$$

$$\epsilon_{h,h'} \left[\omega_h \frac{A_h \delta_h}{V_h} - \omega_{h'} \frac{A_{h'} \delta_{h'}}{V_{h'}} \right]^2 \qquad (2.1)$$

(the numerical factor $\epsilon_{h,h'}$ is equal to 1 if the two hinges h, h' have one edge in common and -2 if they do not). The motivations leading to the above action are discussed in detail in [2,8,10] and will not be repeated here. The introduction of the higher derivative terms is motivated by the fact that the resulting extended theory of gravity is renormalizable in four dimensions[19–23].

In the classical continuum limit the above action is equivalent to the continuum higher derivative action

$$I = \int d^4x \sqrt{g} \left[\lambda - \frac{k}{2} R + b\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right.$$

$$\left. + \frac{1}{2}(a - 4b)\, C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right] \qquad (2.2)$$

with a cosmological constant term (proportional to λ), the Einstein term ($k = 1/8\pi G$, where G is the bare Newton constant), and two higher derivative terms with additional dimensionless coupling constants a^{-1} and b^{-1}.

Non-perturbative studies of quantum gravity were thus motivated by the search for a non-trivial fixed point in four dimensions[20,2]. The first numerical simulations for lattice gravity, with the above action, were discussed in [2,10]. As in previous two-dimensional studies, the lattice was chosen to be regular and built out of rigid hypercubes. This choice is not unique, and is dictated mostly by a criterion of simplicity, with the advantage that such a lattice can be used to study rather large systems with little algorithm modification. Also for reasons of simplicity up to now only the action with $a = 4b$ only (no Weyl term) was considered. In the numerical simulations which were done the lattice was chosen of size LxLxLxL with $15L^4$ edges, and only the cases $L = 2$ (240 edges), $L = 4$ (3840 edges), and exceptionally $L = 8$ (61440 edges), were considered. Periodic boundary conditions were used, and the topology was therefore restricted to a hypertorus; other topologies can in principle be studied by changing the boundary conditions.

In the case in which all the couplings are zero ($a = b = k = \lambda = 0$) the total action is zero, and variations in the edge lengths are only constrained by the non-trivial gravitational measure of eq. (1. 2). Quantities of interest which have been computed include the average curvature $\mathcal{R}$

$$\mathcal{R} = \langle l^2 \rangle \frac{\langle 2 \sum_h \delta_h A_h \rangle}{\langle \sum_h V_h \rangle} \qquad (2.3)$$

and the average curvature squared $\mathcal{R}^2$

$$\mathcal{R}^2 = \langle l^2 \rangle^2 \frac{\langle 4 \sum_h \delta_h^2 A_h^2 / V_h \rangle}{\langle \sum_h V_h \rangle} \qquad (2.4)$$

which are here both dimensionless quantities, since they have been expressed in units of the average edge length. Remarkably one finds that at strong coupling the system tends to develop an average negative curvature.

When the parameters λ, k and a in eq. (2.1) take non-zero values, then one finds a sudden very sharp transition between a state in which the curvature is small and one in which it is very large, presumably a reflection of the unbounded fluctuations in the conformal modes found in the continuum. For example for the dl/l measure, $\lambda = 1$ and for small higher derivative coupling ($a = 4b = 0.005$), the average curvature $\mathcal{R}$ jumps from a small negative value to a large positive one for $k \simeq 2.0$ or greater. A similar type of transition is found if k is kept fixed and λ is varied[2,10]. The jump in $\mathcal{R}$ is so

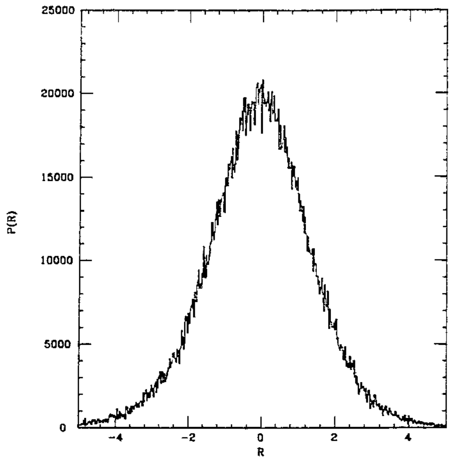

Figure 1 : Histogram of the distribution of curvatures $\delta_h A_h \sim \frac{1}{2}\sqrt{g}\, R$ on a lattice with $15 \times 16^4 = 983040$ edges, and for $\lambda = 1$, $k = 0.2$ and $a = 0.005$ (dl^2 measure).

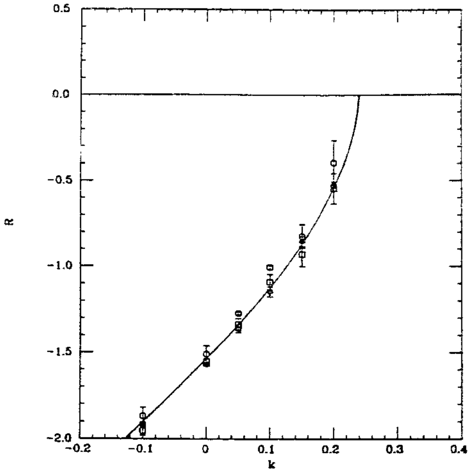

Figure 2 : Average curvature $\mathcal{R}$ as a function of k, for $\lambda = 1$ and $a = 0.005$ (dl^2 measure). The circles refer to $L = 2$, the squares to $L = 4$, and the diamonds to $L = 8$.

large, that it appears to be indicative of perhaps a discontinuous transition. Indications of a discontinuous transition were later found also for the pure Einstein action in a fixed volume ensemble[13], as well as in the hypercubic lattice formulation of gravity

Recently we have performed a number of large scale simulations of pure gravity using the dl^2 ('DeWitt') measure, and employing lattices of size up to 8^4 (with 61440 edges) (we have also done some short runs on 16^4 lattices (with 983040 edges), but we will not discuss the results here since the statistics is at this point still too low). Let us emphasize that at this point the nature of the results is still rather preliminary. The lengths of our runs typically vary between 18k iterations on the 2^4 lattice, 6k iterations on the 4^4 lattice, and at least 800 iterations on the 8^4 lattice. In all cases the starting lattices were duplicated copies of the smaller lattice, for each k, and as usual a number of additional thermalization sweeps were performed for each lattice and value of k. In Fig. 1 we show the distribution of curvatures $\delta_h A_h \sim \sqrt{g}R(x)$ on the 16^4 lattice, for $\lambda = 1$, $k = 0.2$ and $a = 0.005$.

Our intention was to explore the dependence of the results on the measure, and to investigate in more detail the transition between the 'smooth' and the 'rough' phase of spacetime described above.

Besides the quantities $\mathcal{R}$ and $\mathcal{R}^2$ discussed before, we have also computed the lattice analogues of the fluctuations in the curvatures

$$\chi_{\mathcal{R}} = \frac{1}{<\sum_h V_h>} \left[< (2\sum_h \delta_h A_h)^2 > - < 2\sum_h \delta_h A_h >^2\right] \quad (2.5)$$

and of the fluctuations in the volumes

$$\chi_V = \frac{1}{<\sum_h V_h>} \left[< (\sum_h V_h)^2 > - < \sum_h V_h >^2\right] \quad (2.6)$$

The dimensionless average curvature $\mathcal{R}$ is shown, for different lattice sizes, in Fig. 2. One notes that as k is varied, the curvature goes to zero at some value k_c. In this particular case, namely for $\lambda = 1$ and $a = 0.005$, one finds $k_c = 0.239 \pm 0.012$ on the largest lattice ($L = 8$), with values compatible within errors on the smaller lattices. For k close to, but less than, k_c one can write

$$\begin{aligned} \mathcal{R} \underset{k\to k_c}{\sim} A_{\mathcal{R}}\,(k_c - k)^{\delta} \\ \chi_{\mathcal{R}} \underset{k\to k_c}{\sim} A_{\chi}\,(k_c - k)^{\delta - 1} \end{aligned} \quad (2.7)$$

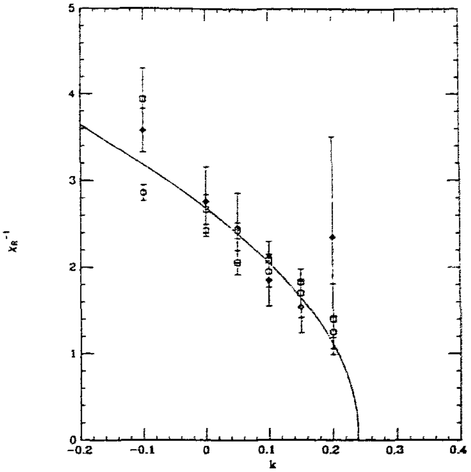

Figure 3 : Curvature fluctuation $\chi_{\mathcal{R}}$ as a function of k, for the same parameters as in Fig. 2.

and performing a simultaneous fit in A, k_c and the exponents, one finds $\delta = 0.61 \pm 0.04$ ($A_{\mathcal{R}} = -3.68 \pm 0.07$) from the average curvature, and $\delta - 1 = -0.35 \pm 0.22$ from the curvature susceptibility. From the curvature susceptibility (see also Fig. 3) we estimate $k_c = 0.24 \pm 0.06$, in agreement with the previous estimate from the average curvature, but with a much larger error due to the difficulty of estimating the susceptibility accurately on the larger lattice, since our runs are still a bit too short.

The above results are consistent with the picture of a vanishing curvature and a divergent curvature fluctuation, at the same value of k_c. If we compute the volume susceptibility, we find on the other hand that it approaches a finite value at k_c, suggesting the absence of critical volume fluctuations (see Fig. 4). These appear to be desirable properties in a theory supposedly describing gravity, where the excitations in the continuum are expected to be massless gravitons, without any massless scalar particles and no massless volume density fluctuations.

On the other side of the transition ($k \geq k_c$) one has that the curvatures are infinite since the simplices collapse into degenerate configurations with very small volumes ($< V_h > / < l^2 >^2 \sim 0$). This is the region of the weak field expansion ($G \to 0$), and it is therefore not surprising that it has difficulties in extending to the region where

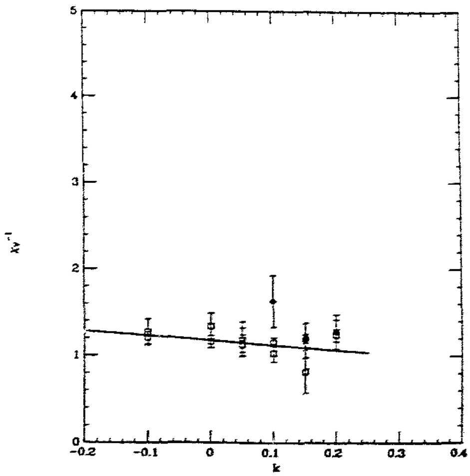

Figure 4 : Volume density fluctuation χ_V as a function of k, for the same parameters as in Fig. 2.

a sensible path integral for pure gravity can be defined. A qualitative picture of the phase diagram for pure gravity with our action is sketched in Fig. 5.

The result presented above for the average curvature $\mathcal{R}$ is not inconsistent with known results within the weak field expansion. Substituting $k^{-1} = 8\pi G$, and setting $k_c = c\Lambda^2$, where c is a constant independent of k, and Λ the ultraviolet cutoff, here of the order of the average inverse lattice spacing $\sim < l^2 >^{-1/2}$, one obtains

$$\begin{aligned} \mathcal{R} \sim & A_{\mathcal{R}} \left(\frac{-1}{8\pi G}\right)^{\delta} (1 - c\Lambda^2 8\pi G)^{\delta} \\ \sim & A_{\mathcal{R}} \left(\frac{-1}{8\pi G}\right)^{\delta} \Big[1 + \delta\, c\Lambda^2(-8\pi G) \\ & + \frac{\delta(\delta-1)}{2} (c\Lambda^2)^2 (-8\pi G)^2 + \cdots \Big] \end{aligned} \tag{2.8}$$

One can see that, besides $\mathcal{R}$ possibly not being analytic at $G = 0$, an expansion in powers of G involves increasingly higher powers of the ultraviolet cutoff Λ, as expected from a theory which is not perturbatively renormalizable in G. The above results are therefore not inconsistent with what is known in the continuum. More detailed and careful computations are needed to better understand these interesting issues, and it seems that correlations should be computed in order to determine the excitation spectrum.

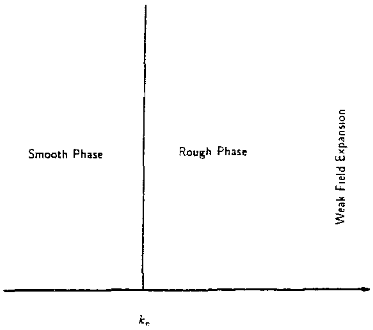

Figure 5 : Phase diagram for pure 4-d gravity.

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