Renormalization group running of Newton's constant G: The static isotropic case

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Corrections are computed to the classical static isotropic solution of general relativity, arising from nonperturbative quantum gravity effects. A slow rise of the effective gravitational coupling with distance is shown to involve a genuinely nonperturbative scale, closely connected with the gravitational vacuum condensate, and thereby related to the observed effective cosmological constant. We argue that in contrast to phenomenological approaches, the underlying functional integral formulation of the theory severely constrains possible scenarios for the renormalization group evolution of couplings. The general analysis is extended here to a set of covariant nonlocal effective field equations, intended to incorporate the full scale dependence of G, and examined in the case of the static isotropic metric. We find that the existence of vacuum solutions to the effective field equations in general severely restricts the possible values of the scaling exponent ν.

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I. INTRODUCTION

Over the last few years evidence has mounted to suggest that quantum gravitation, even though plagued by meaningless infinities in standard weak coupling perturbation theory, might actually make sense, and lead to a consistent theory at the nonperturbative level. As is often the case in physics, the best evidence does not come from often incomplete and partial results in a single model, but more appropriately from the level of consistency that various, often quite unrelated, field theoretic approaches provide. While it would certainly seem desirable to obtain a closed form analytical solution for the Euclidean path integral of quantum gravity, experience with other field theories suggests that this goal might remain unrealistic in the foreseeable future, and that one might have to rely in the interim on partial results and reasoned analogies to obtain a partially consistent picture of what the true nature of the ground state of nonperturbative gravity might be.

One aspect of quantum gravitation that has stood out for some time is the rather strident contrast between the naive picture one gains from perturbation theory, namely, the possibility of an infinite set of counterterms, uncontrollable divergences in the vacuum energy of just about any field including the graviton itself, and typical curvature scales comparable to the Planck mass [1–3], and, on the other hand, the new insights gained from nonperturbative approaches, which avoid reliance on an expansion in a small parameter (which does not exist in the case of gravity) and which would suggest instead a surprisingly rich phase structure, nontrivial ultraviolet fixed points [4–8] and genuinely nonperturbative effects such as the appearance of a gravitational condensate. The existence of nonperturbative vacuum condensates does not necessarily invalidate the wide range of semiclassical results [9–11] obtained in gravity so far, but reinterprets the gravitational background fields as suitable quantum averages, and further adds to the effective gravitational Lagrangian the effects of the (finite) scale dependence of the gravitational coupling, in a spirit similar to the Euler-Heisenberg corrections to electromagnetism.

Perhaps the goals that are sometimes set for quantum gravity and related extensions, that is, to explain and derive, from first principles, the values of Newton's constant and the cosmological constant, are placed unrealistically high. After all, in other well understood quantum field theories like QED and QCD the renormalized parameters (α, α_S, ...) are fixed by experiments, and no really compelling reason exists yet as to why they should take on the actual values observed in laboratory experiments. More specifically in the case of gravity, Feynman has given elaborate arguments as to why quantities such as Newton's constant (and therefore the Planck length) might have cosmological origin, and therefore unrelated to any known particle physics phenomenon [1].

In this paper we will examine a number of issues connected with the renormalization group running of gravitational couplings. We will refrain from considering more general frameworks (higher derivative couplings, matter fields etc.), and will focus instead on basic aspects of the pure gravity theory by itself. Our presentation is heavily influenced by the numerical and analytical results from the lattice theory of quantum gravity (LQG), which have, in our opinion, helped elucidate numerous details of the nonperturbative phase structure of quantum gravity, and allowed a first determination of the scaling dimensions directly in $d = 4$. The lattice provides a well defined ultraviolet regulator, reduces the continuum functional integral to a finite set of convergent integrals, and allows statistical field theory methods, including numerical ones, to be used to explore the nature of ground state averages and correlations.

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The scope of this paper is therefore to explore the overall consistency of the picture obtained from the lattice, by considering a number of core issues, one of which touches the analogy with a much better understood class of theories, QED and non-Abelian gauge theories (Sec. II). We will argue that, once one takes for granted a set of basic lattice results, it is possible to discuss a number of general features without having to explicitly resort to specific aspects of the lattice cutoff or the lattice action. For example, it is often sufficient to assume that a cutoff Λ is operative at very short distances, without having to involve in the discussion specific aspects of its implementation. In fact the use of continuum language, in spite of its occasional ambiguities when it comes to the proper, regulated definition of quantum entities, provides a more transparent language for presenting and discussing basic results.

The second aspect we wish to investigate in this paper is the nature of the rather specific predictions about the running of Newton's constant G. A natural starting point is the solution of the nonrelativistic Poisson equation (Sec. III), whose solutions for a point source can be investigated for various values of the exponent ν. We will then show that a scale dependence of G can be consistently embedded in a relativistic covariant framework, whose consequences can then be worked out in detail for specific choices of metrics (Sec. IV). For the static isotropic metric, we then derive the leading quantum correction and show that, unexpectedly, it seems to restrict the possible values for the exponent ν, in the sense that in some instances no consistent solution to the effective nonlocal field equations can be found unless ν^{-1} is an integer.

To check the overall consistency of the results, a slightly different approach to the solution of the static isotropic metric is discussed in Sec. V, in terms of an effective vacuum density and pressure. Again it appears that unless the exponent ν is close to $1/3$, a consistent solution cannot be obtained. At the end of the paper we add some general comments on two subjects we discussed previously. We first make the rather simple observation that a running of Newton's constant will slightly distort the gravitational wave spectrum at very long wavelengths (Sec. VI). We then return to the problem (Sec. VII) of finding solutions of the effective nonlocal field equations in a cosmological context [12], wherein quantum corrections to the Robertson-Walker metric and the basic Friedmann equations are worked out, and discuss some of the simplest and more plausible scenarios for the growth (or lack thereof) of the coupling at very large distances, past the de Sitter horizon. Sec. VIII contains our conclusions.

II. VACUUM CONDENSATE PICTURE OF QUANTUM GRAVITATION

The lattice theory of quantum gravity provides a well defined and regularized framework in which nonperturbative quantum aspects can be systematically investigated in a controlled fashion. Let us recall here some of the main results of the lattice quantum gravity (LQG) approach, and their relationship to related nonperturbative approaches.

(i) The theory is formulated via a discretized Feynman functional integral [13–27]. Convergence of the Euclidean lattice path integral requires in dimensions $d > 2$ a positive bare cosmological constant $\lambda_0 > 0$ [20]. The need for a bare cosmological constant is in line with renormalization group results in the continuum, which also imply that radiative corrections will inevitably generate a nonvanishing λ_0 term.

(ii) The lattice theory in four dimensions is characterized by *two phases*, one of which appears for G less than some critical value G_c, and can be shown to be physically unacceptable as it describes a collapsed manifold with dimension $d \simeq 2$. The quantum gravity phase for which $G > G_c$ can be shown instead to describe smooth four-dimensional manifolds at large distances, and remains therefore physically viable. The continuum limit is taken in the standard way, by having the bare coupling G approach G_c. The two phase structure persists in three dimensions [24], and even at $d = \infty$ [15], whereas in two dimensions one finds, as expected, only one phase [23].

(iii) The presence of two distinct phases in the lattice theory is consistent with the continuum $2 + \epsilon$ expansion result, which also predicts the existence of two phases above dimensions $d = 2$ [28–31]. The presence of a nontrivial ultraviolet fixed point in the continuum above $d = 2$, with nontrivial scaling dimensions, relates to the existence of a phase transition in the lattice theory [32–37]. The lattice results further suggest that the weakly coupled phase is in fact nonperturbatively *unstable*, with the manifold collapsing into a two-dimensional degenerate geometry. The latter phase, if it had existed, would have described gravitational screening.

(iv) One key nonperturbative quantity, the critical exponent ν, characterizing the nonanalyticity in the vacuum condensates at G_c, is naturally related to the derivative of the beta function at G_c in the $2 + \epsilon$ expansion. The value $\nu \simeq 1/3$ in four dimensions, found by numerical evaluation of the lattice path integral, is close but somewhat smaller than the lowest order ϵ expansion result $\nu = 1/(d - 2)$. An analysis of the strongly coupled phase of the lattice theory further gives $\nu = 0$ at $d = \infty$ [15].

(v) The genuinely nonperturbative scale ξ, specific to the strongly coupled phase of gravity for which $G > G_c$, can be shown to be related to the vacuum expectation value of the curvature via $\langle \mathcal{R} \rangle \sim 1/\xi^2$, and is therefore presumably macroscopic [27]. It is naturally identified with the physical (scaled) cosmological constant λ; ξ therefore appears to play a role analogous to the nonperturbative scaling violation parameter $\Lambda_{\overline{MS}}$ of QCD.

(vi) The existence of a nontrivial ultraviolet fixed point (a phase transition in statistical mechanics language) implies a scale dependence for Newton's constant in the physical, strongly coupled phase $G > G_c$. To leading order in the vicinity of the fixed point the scale dependence is determined by the exponent ν, and the overall size of the corrections is set by the condensate scale ξ. Thus in the strongly coupled phase, gravitational vacuum polarization effects should cause the physical Newton's constant to grow slowly with distances.

Nontrivial fixed point and scale dependence of $G(\mu^2)$

This section will establish basic notation and provide some key results and formulas to be used later. For more details the reader is referred to the recent papers [12,13,15] and references therein.

For the running gravitational coupling we will assume in the vicinity of the ultraviolet fixed point the behavior

$$G(k^2) = G_c\left[1 + a_0\left(\frac{m^2}{k^2}\right)^{1/2\nu} + O((m^2/k^2)^{1/\nu})\right] \tag{2.1}$$

with $m = 1/\xi$, $a_0 > 0$, and $\nu \simeq 1/3$ [13]. We have argued previously that the quantity G_c in the above expression should in fact be identified with the laboratory scale value, $\sqrt{G_c} \sim \sqrt{G_{\text{phys}}} \sim 1.6 \times 10^{-33}$ cm, the reason being that the scale ξ can be very large. Indeed in the work of [12,14,27] it was discussed that ξ should be of the same order as the scaled cosmological constant λ. Quantum corrections on the right-hand side are therefore quite small as long as $k^2 \gg m^2$, which in real space corresponds to the "short distance" regime $r \ll \xi$.

The above expression diverges as $k^2 \to 0$, and the infrared divergence needs to be regulated. But a natural infrared regulator exists in the form of the dynamically generated scale $m = 1/\xi$, and therefore a properly infrared regulated version of the above expression is

$$G(k^2) \simeq G_c\left[1 + a_0\left(\frac{m^2}{k^2 + m^2}\right)^{1/2\nu} + \dots\right] \tag{2.2}$$

with $m = 1/\xi$ the (tiny) infrared cutoff. While certainly not unique, it can be considered as one of the simplest means by which one can regulate the unphysical infrared divergence of Eq. (2.1). A less elegant, but equivalent, procedure would consist in cutting off momentum integrals at $k_{\min} = m$, but we shall not pursue such an approach here. Then in the limit of large k^2 (small distances) the correction to $G(k^2)$ reduces to the expression in Eq. (2.1), namely

$$G(k^2) \underset{k^2/m^2\to\infty}{\sim} G_c\left[1 + a_0\left(\frac{m^2}{k^2}\right)^{1/2\nu}\left(1 - \frac{1}{2\nu}\frac{m^2}{k^2} + \dots\right) + \dots\right]. \tag{2.3}$$

Thus the gravitational coupling approaches the ultraviolet (UV) fixed point value G_c at "short distances" $r \ll \xi$. On the other hand its limiting behavior for small k^2 (large distances) we will take, from Eq. (2.2), to be given by

$$G(k^2) \underset{k^2/m^2\to 0}{\sim} G_\infty\left[1 - \left(\frac{a_0}{2\nu(1 + a_0)} + \dots\right)\frac{k^2}{m^2} + O(k^4/m^4)\right] \tag{2.4}$$

implying that the gravitational coupling approaches the finite value $G_\infty = (1 + a_0 + \dots)G_c$, independent of $m = 1/\xi$, at very large distances $r \gg \xi$. We should emphasize though that the main results of the paper will apply to the "short distance" regime, and thus will only make use of Eq. (2.1). Since the theory is formulated with an explicit ultraviolet cutoff Λ, the latter must appear somewhere, and indeed $G_c = \Lambda^{-2}\tilde{G}_c$, with the UV cutoff of the order of the Planck length $\Lambda^{-1} \sim 1.6 \times 10^{-33}$ cm, and $\tilde{G}_c$ a dimensionless number of order one. In Eqs. (2.1) or (2.2) the cutoff does not appear explicitly, it is absorbed into the definition of G_c.

The nonrelativistic, static Newtonian potential is then defined as

$$\phi(r) = (-M)\int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} G(\mathbf{k}^2)\frac{4\pi}{\mathbf{k}^2} \tag{2.5}$$

and therefore proportional to the $3 - d$ Fourier transform of

$$\frac{4\pi}{\mathbf{k}^2} \to \frac{4\pi}{\mathbf{k}^2}\left[1 + a_0\left(\frac{m^2}{\mathbf{k}^2}\right)^{1/2\nu} + \dots\right]. \tag{2.6}$$

As mentioned before, proper care has to be exercised in providing a properly infrared regulated version of the above expression, which, from Eq. (2.2), reads

$$\frac{4\pi}{(\mathbf{k}^2 + \mu^2)} \to \frac{4\pi}{(\mathbf{k}^2 + \mu^2)}\left[1 + a_0\left(\frac{m^2}{\mathbf{k}^2 + m^2}\right)^{1/2\nu} + \dots\right], \tag{2.7}$$

where the limit $\mu \to 0$ should be taken at the end of the calculation. We wish to emphasize here that the regulators $\mu \to 0$ and m are quite distinct. The distinction originates in the condition that m arises due to strong infrared effects and renormalization group properties in the quantum regime, while μ has nothing to do with quantum effects: it is required to make the Fourier transform of the classical Newtonian $4\pi/\mathbf{k}^2$ well defined. This is an important issue to keep in mind, and to which we will return later.

In practice the exponent ν appearing in Eqs. (2.1) and (2.2) (as well as in Eq. (2.10) appearing below) is determined from the singularities in G that arise in the Euclidean path integral for pure quantum gravity Z defined as

$$Z = \int [dg_{\mu\nu}] e^{-I_E[g]}, \tag{2.8}$$

with Euclidean action given by

$$I_E[g] = \lambda_0 \Lambda^d \int dx \sqrt{g} - \frac{1}{16\pi G_0} \Lambda^{d-2} \int dx \sqrt{g} R, \tag{2.9}$$

in the vicinity of the nontrivial ultraviolet fixed point at G_c. Here λ_0 is the bare cosmological constant and G_0 the bare Newton's constant, both measured here in units of the cutoff (we follow here customary notation used in cutoff field theories, and denote by Λ the ultraviolet cutoff, not to be confused with the scaled cosmological constant).[1]

There are several correlation functions one can compute to extract ν and a_0 directly, either through the decay of Euclidean invariant correlations at fixed geodesic distance [25], or, equivalently, from the correlations of Wilson lines associated with the propagation of heavy spinless particles [27]. In either case one expects the following scaling result close to the fixed point

$$\xi^{-1} = m \underset{G(\Lambda)\to G_c}{\sim} \Lambda \left[\frac{G(\Lambda) - G_c}{a_0 G_c} \right]^{\nu}, \tag{2.10}$$

where Λ is the ultraviolet cutoff (the inverse lattice spacing) and a_0 a numerical constant. The continuum limit is approached in the standard way by having $G \to G_c$ and Λ large, with m kept fixed. Detailed knowledge of $m(G)$ allows one to independently estimate the exponent ν. As far as the quantity a_0 is concerned, one can estimate $a_0 \simeq 42$ [25,27].

At first it might appear that in pure gravity one has two independent couplings (λ_0 and G), but in reality a simple scaling argument shows that there can only be one, which can be taken to be a suitable dimensionless ratio. Indeed in the functional integral of Eq. (2.8) one can suitably rescale the metric so as the obtain a unit coefficient for the cosmological constant term

$$g'_{\mu\nu} = \lambda_0^{2/d} g_{\mu\nu}, \qquad g'^{\mu\nu} = \lambda_0^{-2/d} g^{\mu\nu}. \tag{2.11}$$

Then the nontrivial part of the gravitational functional integral over metrics can only depend on λ_0 and G_0 through the combination [20]

$$\tilde{G} \equiv G_0 \lambda_0^{(d-2)/d}. \tag{2.12}$$

The existence of an ultraviolet fixed point is then entirely controlled by this (naturally dimensionless) parameter only, both on the lattice [13] and in the continuum [29]. The individual scaling dimensions of the cosmological constant and of the gravitational coupling constant therefore do not have separate physical meaning.

The question that remains open is then the following: which coupling should be allowed to run within the renormalization group framework? Since the path integral in four dimensions only depends on the ratio $\tilde{G}^2 = G_0^2 \lambda_0$ (which is expected to be scale dependent), one has several choices; for example G runs and the cosmological constant λ_0 is fixed. Alternatively, G runs and the *scaled* cosmological constant $\lambda \equiv G\lambda_0$ is kept fixed; or G is fixed and λ runs etc. In our opinion the correct answer is that the combination $\lambda \equiv 8\pi G \cdot \lambda_0$, corresponding to the *scaled* cosmological constant (which has dimensions of mass squared), should be kept *fixed*, while Newton's constant is allowed to run in accordance to the scale dependence obtained from $\tilde{G}$. The reasons for this choice are threefold. First, in the weak field expansion it is the combination $\lambda \equiv G\lambda_0$ that appears as a masslike term (and not λ_0 or G separately). A similar conclusion is reached if one just compares the appearance of the field equations for gravity to say QED (massive via the Higgs mechanism), or a self-interacting scalar field. Secondly, the scaled cosmological constant represents a measure of physical curvature, as should be clear from how the scaled cosmological constant relates, for example, to the expectation values of the scalar curvature at short distances (i.e. for infinitesimally small loops, whose size is comparable to the cutoff scale). A third argument involves the consideration of the gravitational analogue of the Wilson loop [15], defined here as a path-ordered exponential of the affine connection $\Gamma^{\lambda}_{\mu\nu}$ around a closed planar loop

$$W(\Gamma) \sim \left\langle \mathrm{Tr}\, \mathcal{P} \exp\left[\int_C \Gamma^{\lambda}_{\cdot\cdot} dx_{\lambda} \right] \right\rangle. \tag{2.13}$$

Borrowing from the well-established results in non-Abelian lattice gauge theories with compact groups [38,39], one would expect that the expected decay of near-planar Wilson loops with area A would be given by

$$W(\Gamma) \sim \exp\left[\int_{S(C)} R^{\cdot}{}_{\cdot\mu\nu} A_C^{\mu\nu} \right] \sim \exp(-A/\xi^2), \tag{2.14}$$

where A is the minimal physical area spanned by the near-planar loop. The rapid decay of the Wilson loop as a function of the area is seen simply as a general and direct consequence of the disorder in the fluctuations of the local $O(4)$ rotation matrices $\mathbf{R}$ at strong coupling. One concludes therefore that the Wilson loop in gravity provides a measure of the magnitude of the large-scale, averaged curvature, operationally determined by the process of parallel-transporting test vectors around very large loops, and which therefore, from the above expression, is computed to be of the order $R \sim 1/\xi^2$. We will therefore assume in the following for the physical scaled cosmological constant

[1] We slightly deviate in this paper from the convention used in our previous work [12]. Because of ubiquitous ultraviolet cutoff Λ, we reserve here the symbol λ_0 for the cosmological constant, and λ for the *scaled* cosmological constant $\lambda \equiv 8\pi G \cdot \lambda_0$.

$$\lambda_{\text{phys}} \simeq \frac{1}{\xi^2}. \tag{2.15}$$

This relationship, taken at face value, implies a very large, cosmological value for $\xi \sim 10^{28}$ cm, given the present observational bounds on λ_{phys}.

In conclusion, the modified Einstein equations, incorporating the proposed quantum running of G, read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G(\Box) T_{\mu\nu} \tag{2.16}$$

with $\lambda \simeq \frac{1}{\xi^2}$ is the scaled cosmological constant, and only $G(\mu^2)$ on the right-hand side scale dependent. The precise mathematical meaning of $G(\Box)$ [12] will be given later in Sec. IV.

III. POISSON'S EQUATION WITH RUNNING G

Given the running of G from either Eq. (2.2), or Eq. (2.1) in the large $\mathbf{k}$ limit, the next step is naturally a solution of Poisson's equation with a point source at the origin, in order to determine the structure of the quantum corrections to the gravitational potential in real space. The more complex solution of the fully relativistic problem will then be addressed in the following sections. In the limit of weak fields the relativistic field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.1}$$

give for the ϕ field (with $g_{00}(x) \simeq -(1 + 2\phi(x))$

$$(\Delta - \lambda)\phi(x) = 4\pi G \rho(x) - \lambda, \tag{3.2}$$

which would suggest that the scaled cosmological constant λ acts like a mass term $m = \sqrt{\lambda}$. For a point source at the origin, the first term on the right-hand side is just $4\pi M G \delta^{(3)}(\mathbf{x})$. The solution for $\phi(r)$ can then be obtained simply by Fourier transforming back to real space Eq. (2.5), and, up to an additive constant, one has

$$\phi(r) = -MG\frac{e^{-mr}}{r} - \frac{a_0 m M G}{2^{1/2((1/\nu)-1)}\Gamma(1 + \frac{1}{2\nu})\sqrt{\pi}} \times (mr)^{1/2((1/\nu)-1)} K_{1/2((1/\nu)-1)}(mr), \tag{3.3}$$

where $K_n(x)$ is the modified Bessel function of the second type. The behavior of $\phi(r)$ would then be Yukawa-like $\phi(r) \sim \text{const}\, e^{-mr}/r$ and thus rapidly decreasing for large r.

But the reason why both of the above results are in fact *incorrect* (assuming of course the validity of general coordinate invariance at very large distances $r \gg 1/\sqrt{\lambda})$ is that the exact solution to the field equations in the static isotropic case with a λ term gives

$$-g_{00} = B(r) = 1 - \frac{2MG}{r} - \frac{\lambda}{3} r^2 \tag{3.4}$$

showing that the λ term definitely does *not* act like a mass term in this context.

Therefore the zeroth order contribution to the potential should be taken to be proportional to $4\pi/(\mathbf{k}^2 + \mu^2)$ with $\mu \to 0$, as already indicated in fact in Eq. (2.7). Also, proper care has to be exercised in providing an appropriate infrared regulated version of $G(\mathbf{k}^2)$, and therefore $V(\mathbf{k}^2)$, which from Eq. (2.7) reads

$$\frac{4\pi}{(\mathbf{k}^2 + \mu^2)}\left[1 + a_0\left(\frac{m^2}{\mathbf{k}^2 + m^2}\right)^{1/2\nu}\right] \tag{3.5}$$

and where the limit $\mu \to 0$ is intended to be taken at the end of the calculation.

There are in principle two equivalent ways to compute the potential $\phi(r)$, either by inverse Fourier transform of the above expression, or by solving Poisson's equation $\Delta\phi = 4\pi\rho$ with $\rho(r)$ given by the inverse Fourier transform of the correction to $G(k^2)$, as given later in Eq. (3.17). Here we will first use the first, direct method.

A. Large r limit

The zeroth order term gives the standard Newtonian $-MG/r$ term, while the correction in general is given by a rather complicated hypergeometric function. But for the special case $\nu = 1/2$ one has for the Fourier transform of the correction to $\phi(r)$

$$a_0 m^{1/\nu} \frac{4\pi}{\mathbf{k}^2 + \mu^2} \frac{1}{(\mathbf{k}^2 + m^2)^{1/2\nu}} \to a_0 m^2 \frac{e^{-\mu r} - e^{-mr}}{r(m^2 - \mu^2)} \sim_{\mu\to 0} a_0 m^2 \frac{1 - e^{-mr}}{m^2 r} \tag{3.6}$$

giving for the complete quantum-corrected potential

$$\phi(r) = -\frac{MG}{r}[1 + a_0(1 - e^{-mr})]. \tag{3.7}$$

For this special case the running of $G(r)$ is particularly transparent

$$G(r) = G_\infty\left(1 - \frac{a_0}{1 + a_0} e^{-mr}\right) \tag{3.8}$$

with $G_\infty \equiv (1 + a_0)G$ and $G \equiv G(0)$. G therefore increases slowly from its value G at small r to the larger value $(1 + a_0)G$ at infinity. Figure 1. provides a schematic illustration of the behavior of G as a function of r.

Returning to the general ν case, one can expand for small $\mathbf{k}$ to get the correct large r behavior

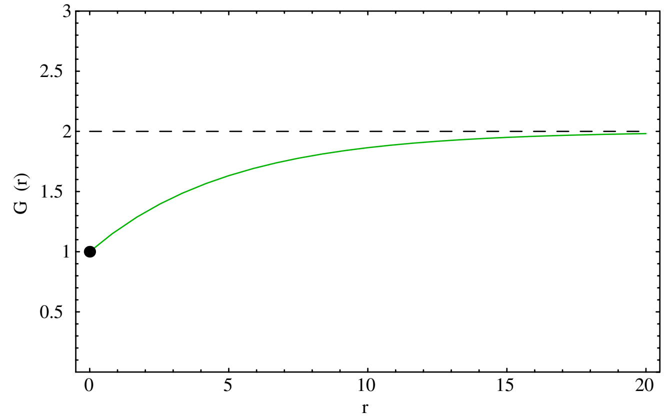

FIG. 1 (color online). Schematic scale dependence of the gravitational coupling $G(r)$ from Eq. (3.8), here for $\nu = 1/2$. The gravitational coupling rises initially like a power of r, and then approaches the asymptotic value $G_\infty = (1 + a_0)G$ for large r. The behavior for other values of $\nu > 1/3$ is similar.

$$\frac{1}{(\mathbf{k}^2 + \mu^2)(\mathbf{k}^2 + m^2)^{1/2\nu}} \simeq \frac{1}{m^{1/\nu}} \frac{1}{(\mathbf{k}^2 + \mu^2)} - \frac{1}{2\nu m^2} \times \frac{1}{(\mathbf{k}^2 + m^2)^{1/2\nu}} + \ldots. \tag{3.9}$$

After Fourier transform, one obtains the previous answer for $\nu = 1/2$, whereas for $\nu = 1/3$ one finds

$$-\frac{MG}{r}\left[1 + a_0\left(1 - \frac{3mr}{\pi} K_0(mr)\right)\right] \tag{3.10}$$

and for general ν

$$-\frac{MG}{r}\left[1 + a_0\left(1 - \frac{2^{(1/2)(3-(1/\nu))} mr}{2\nu\sqrt{\pi}\Gamma(\frac{1}{2\nu})} \times (mr)^{-(1/2)(3-(1/\nu))} K_{(1/2)(3-(1/\nu))}(mr)\right)\right]. \tag{3.11}$$

Using the asymptotic expansion of the modified Bessel function $K_n(x)$ for large arguments, $K_n(z) \sim \sqrt{\pi/2} z^{-1/2} e^{-z}(1 + O(1/z))$, one finally obtains in the large r limit

$$\phi(r) \underset{r\to\infty}{\sim} -\frac{MG}{r}[1 + a_0(1 - c_l (mr)^{(1/2\nu)-1} e^{-mr})] \tag{3.12}$$

with $c_l = 1/(\nu 2^{1/2\nu}\Gamma(\frac{1}{2\nu}))$.

B. Small r limit

In the small r limit one finds instead, using again Fourier transforms, for the correction for $\nu = 1/3$

$$(-MG)a_0 m^2 \frac{r^2}{3\pi}\left[\ln\left(\frac{mr}{2}\right) + \gamma - \frac{5}{6}\right] + O(r^3). \tag{3.13}$$

In the general case the complete leading correction to the potential $\phi(r)$ for small r (and $\nu > 1/3$) has the structure $(-\mathrm{const})(-MG)a_0 m^{1/\nu} r^{(1/\nu)-1}$. Note that the quantum correction always vanishes at short distances $r \to 0$, as expected from the original result of Eqs. (2.1) or (2.2) for $k^2 \to \infty$.[2]

The same result can be obtained via a different, but equivalent, procedure, in which one solves directly the radial Poisson equation for $\phi(r)$. First, for a point source at the origin, $4\pi MG\delta^{(3)}(\mathbf{x})$, with

$$\delta^{(3)}(\mathbf{x}) = \frac{1}{4\pi} 2\frac{\delta(r)}{r^2} \tag{3.14}$$

one sets $\Delta\phi(r) \to r^{-1} d^2/dr^2 [r\phi(r)]$ in radial coordinates. In the $a_0 \neq 0$ case one then needs to solve $\Delta\phi = 4\pi\rho$, or in the radial coordinate for $r > 0$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = 4\pi G\rho_m(r) \tag{3.15}$$

with the source term ρ_m determined from the inverse Fourier transform of the correction term in Eq. (2.2), namely

$$a_0 M\left(\frac{m^2}{\mathbf{k}^2 + m^2}\right)^{1/2\nu}. \tag{3.16}$$

One finds

$$\rho_m(r) = \frac{1}{8\pi} c_\nu a_0 M m^3 (mr)^{-(1/2)(3-(1/\nu))} K_{(1/2)(3-(1/\nu))}(mr) \tag{3.17}$$

with

$$c_\nu \equiv \frac{2^{(1/2)(5-(1/\nu))}}{\sqrt{\pi}\Gamma(\frac{1}{2\nu})}. \tag{3.18}$$

The vacuum polarization density ρ_m has the property

$$4\pi \int_0^\infty r^2 dr \rho_m(r) = a_0 M, \tag{3.19}$$

where the standard integral $\int_0^\infty dx x^{2-n} K_n(x) = 2^{-n}\sqrt{\pi}\Gamma(\frac{3}{2} - n)$ has been used. Note that the vacuum polarization distribution is singular close to $r = 0$, just as in QED.

The $r \to 0$ result for $\phi(r)$ (discussed in the following, as an example, for $\nu = 1/3$) can then be obtained by solving the radial equation for $\phi(r)$,

$$\frac{1}{r}\frac{d^2}{dr^2}[r\phi(r)] = \frac{2a_0 MGm^3}{\pi} K_0(mr), \tag{3.20}$$

where the (modified) Bessel function is expanded out to

[2] At very short distances $r \sim l_P$ other quantum corrections come into play, which are not properly encoded in Eq. (2.1), which after all is supposed to describe the universal running in the *scaling region* $l_P \ll r \ll \xi$. Furthermore, higher derivative terms could also have important effects at very short distances.

lowest order in r, $K_0(mr) = -\gamma - \ln(\frac{mr}{2}) + O(m^2 r^2)$, giving

$$\phi(r) = -\frac{MG}{r} + a_0 MGm^3 \frac{r^2}{3\pi}\left[-\ln\left(\frac{mr}{2}\right) - \gamma + \frac{5}{6}\right] + O(r^3), \tag{3.21}$$

where the two integration constants are matched to the large r solution of Eq. (3.11). Note again that the vacuum polarization density $\rho_m(r)$ has the expected normalization property

$$4\pi \int_0^\infty r^2 dr \frac{a_0 M m^3}{2\pi^2} K_0(mr) = \frac{2a_0 M m^3}{\pi} \cdot \frac{\pi}{2m^3} = a_0 M \tag{3.22}$$

so that the total enclosed additional "charge" is indeed just $a_0 M$, and $G_\infty = G_0(1 + a_0)$ [see for comparison also Eq. (3.11)]. Using then the same method for general $\nu > \frac{1}{3}$, one finds for small r [using the expansion of the modified Bessel function $K_n(x)$ for small arguments as given later in Eq. (5.24)]

$$\rho_m(r) \underset{r\to 0}{\sim} \frac{|\sec(\frac{\pi}{2\nu})|}{4\pi\Gamma(\frac{1}{\nu} - 1)} a_0 M m^{1/\nu} r^{(1/\nu)-3} \equiv A_0 r^{(1/\nu)-3} \tag{3.23}$$

and from it the general result

$$\phi(r) \underset{r\to 0}{\sim} -\frac{MG}{r} + a_0 MG c_s m^{1/\nu} r^{(1/\nu)-1} + \ldots \tag{3.24}$$

with $c_s = \nu|\sec(\frac{\pi}{2\nu})|/\Gamma(\frac{1}{\nu})$.

IV. RELATIVISTIC FIELD EQUATIONS WITH RUNNING G

Solutions to Poisson's equation with a running G provide some insights into the structure of the quantum corrections, but a complete analysis requires the study of the full relativistic field equations, which will be discussed next in this section. A set of relativistic field equations incorporating the running of G is obtained by doing the replacement [12]

$$G(k^2) \to G(\Box) \tag{4.1}$$

with the d'Alembertian $\Box$ intended to correctly represent invariant distances, and incorporating the running of G as expressed in either Eqs. (2.1) or (2.2),

$$G \to G(\Box) = G\left[1 + a_0\left(\frac{m^2}{-\Box + m^2}\right)^{1/2\nu} + \ldots\right]. \tag{4.2}$$

For the use of $\Box$ to express the running of couplings in gauge theories the reader is referred to the references in Ref. [40]. Here the $\Box$ operator is defined through the appropriate combination of covariant derivatives

$$\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu \tag{4.3}$$

and whose explicit form depends on the specific tensor nature of the object it is acting on, as in the case of the energy-momentum tensor

$$\Box T^{\alpha\beta\ldots}{}_{\gamma\delta\ldots} = g^{\mu\nu}\nabla_\mu(\nabla_\nu T^{\alpha\beta\ldots}{}_{\gamma\delta\ldots}). \tag{4.4}$$

Thus on scalar functions one obtains the fairly simple result

$$\Box S(x) = \frac{1}{\sqrt{g}}\partial_\mu g^{\mu\nu}\sqrt{g}\partial_\nu S(x), \tag{4.5}$$

whereas on second rank tensors one has the significantly more complicated expression $\Box T_{\alpha\beta} \equiv g^{\mu\nu}\nabla_\mu(\nabla_\nu T_{\alpha\beta})$. In general the invariant operator appearing in the above expression, namely

$$A(\Box) = a_0\left(\frac{m^2}{-\Box}\right)^{1/2\nu} \tag{4.6}$$

or its infrared regulated version

$$A(\Box) = a_0\left(\frac{m^2}{-\Box + m^2}\right)^{1/2\nu} \tag{4.7}$$

has to be suitably defined by analytic continuation from positive integer powers; the latter can be often be done by computing $\Box^n$ for positive integer n, and then analytically continuing to $n \to -1/2\nu$. In the following, the above analytic continuation from positive integer n will always be understood. Usually it is easier to work with the expression in Eq. (4.6), and then later amend the final result to include the infrared regulator, if needed.

One is therefore lead to consider the effective field equations of Eq. (2.16), namely

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = 8\pi G(1 + A(\Box))T_{\mu\nu} \tag{4.8}$$

with $A(\Box)$ given by Eq. (4.7) and $\lambda \simeq 1/\xi^2$, as well as the trace equation

$$R - 4\lambda = -8\pi G(1 + A(\Box))T. \tag{4.9}$$

Being manifestly covariant, these expressions at least satisfy some of the requirements for a set of consistent field equations incorporating the running of G, and can then be easily recast in a form similar to the classical field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = 8\pi G\tilde{T}_{\mu\nu} \tag{4.10}$$

with $\tilde{T}_{\mu\nu} = (1 + A(\Box))T_{\mu\nu}$ defined as an effective, gravitationally dressed, energy-momentum tensor. Just like the ordinary Einstein gravity case, in general $\tilde{T}_{\mu\nu}$ might not be covariantly conserved *a priori* $\nabla^\mu\tilde{T}_{\mu\nu} \neq 0$, but ultimately the consistency of the effective field equations demands that it be exactly conserved in consideration of the Bianchi identity satisfied by the Einstein tensor [12]. The ensuing new covariant conservation law

$$\nabla^\mu\tilde{T}_{\mu\nu} \equiv \nabla^\mu[(1 + A(\Box))T_{\mu\nu}] = 0 \tag{4.11}$$

can be then be viewed as a constraint on $\tilde{T}_{\mu\nu}$ (or $T_{\mu\nu}$) which, for example, in the specific case of a perfect fluid, implies a definite relationship between the density $\rho(t)$, the pressure $p(t)$ and the metric components [12].

From now on, we will set the cosmological constant $\lambda = 0$, and its contribution can then be added at a later stage. As long as one is interested in static isotropic solutions, one takes for the metric the most general form

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4.12}$$

and for the energy-momentum tensor the perfect fluid form

$$T_{\mu\nu} = \mathrm{diag}[B(r)\rho(r), A(r)p(r), r^2 p(r), r^2\sin^2\theta p(r)] \tag{4.13}$$

with trace $T = 3p - \rho$. The trace equation then only involves the (simpler) scalar d'Alembertian, acting on the trace of the energy-momentum tensor.

To the order one is working here, the above effective field equations should be equivalent to

$$\frac{1}{1+A(\Box)}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = (1 - A(\Box) + \ldots) \times \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 8\pi G T_{\mu\nu}, \tag{4.14}$$

where the running of G has been moved over to the ''gravitational'' side, and later expanded out, assuming the correction to be small. For the vacuum solutions, the right-hand side is zero for $r \neq 0$, and one can rewrite the last equation simply as

$$\frac{1}{8\pi G(1+A(\Box))}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 0. \tag{4.15}$$

A. Dirac delta function source

A mass point source is most suitably described by a Dirac delta function. The delta function at the origin can be represented, for example, as

$$\delta(r) = \lim_{\epsilon\to 0}\frac{\epsilon}{\pi(r^2+\epsilon^2)}. \tag{4.16}$$

As an example, the derivative operator $(d/dr)^n$ acting on the delta function would give, for small ϵ

$$(-1)^n\Gamma(n+2)\pi^{-1}\epsilon r^n(r^2+\epsilon^2)^{-n-1}, \tag{4.17}$$

which can be formally analytically continued to a fractional value $n = -1/\nu$

$$(-1)^{-1/\nu}\Gamma\left(2-\frac{1}{\nu}\right)\pi^{-1}\epsilon r^{-1/\nu}(r^2+\epsilon^2)^{(1/\nu)-1} \tag{4.18}$$

and then reexpressed as an overall factor in front of the original δ function

$$(-1)^{-1/\nu}r^{-1/\nu}(r^2+\epsilon^2)^{1/\nu}\Gamma\left(2-\frac{1}{\nu}\right)\delta(r) \to (-1)^{-1/\nu}r^{1/\nu}\Gamma\left(2-\frac{1}{\nu}\right)\delta(r). \tag{4.19}$$

The procedure achieves in this case the desired result, namely, the multiplication of the original delta function by a power $r^{1/\nu}$. Note also that if one just takes the limit $\epsilon \to 0$ at fixed r one always gets zero for any $r > 0$, so close to the origin r has to be sent to zero as well to get a nontrivial result.

The above considerations suggests that one should be able to write for the point source at the origin

$$T_{\mu\nu}(r) = \mathrm{diag}[B(r)\rho(r), 0, 0, 0] \tag{4.20}$$

with (in spherical coordinates)

$$\rho = M\delta^{(3)}(\mathbf{x}) \to M\frac{1}{4\pi}2\frac{\delta(r)}{r^2} \tag{4.21}$$

and the delta function defined as a suitable limit of a smooth function, and with vanishing pressure $p(r) = 0$.

Next we will consider as a warm-up the trace equation

$$R = -8\pi G(\Box)T \equiv -8\pi G(1+A(\Box))T = +8\pi G(1+A(\Box))\rho, \tag{4.22}$$

where we have used the fact that the point source at the origin is described just by the density term. One then computes the repeated action of the invariant d'Alembertian on T

$$\Box(-8\pi GT) = \Box(+8\pi G\rho) = \frac{16G\pi\rho'}{rA} - \frac{4G\pi A'\rho'}{A^2} + \frac{4G\pi B'\rho'}{AB} + \frac{8G\pi\rho''}{A}, \tag{4.23}$$

or, using the explicit form for $\rho(r)$

$$\Box(-8\pi GT) = -\frac{8GM\epsilon(-6AB - rA'B + rAB')}{\pi(r^2+\epsilon^2)^3A^2B} + O(\epsilon^3). \tag{4.24}$$

In view of the rapidly escalating complexity of the problem it seems sensible to expand around the Schwarzschild solution, and therefore set

$$A(r)^{-1} = 1 - \frac{2MG}{r} + \frac{\sigma(r)}{r} \tag{4.25}$$

and

$$B(r) = 1 - \frac{2MG}{r} + \frac{\theta(r)}{r}, \tag{4.26}$$

where the correction to the standard solution are parametrized here by the two functions $\sigma(r)$ and $\theta(r)$, both assumed to be ''small,'' i.e. proportional to a_0 as in

Eqs. (4.6) or (4.7), with a_0 considered a small parameter. Then for the scalar curvature, to lowest order in σ and θ, one has

$$\frac{GM(\sigma-\theta)-(2GM-r)(GM\theta'+(3GM-2r)\sigma'+(2GM-r)r\theta'')}{r^2(r-2MG)^2}. \tag{4.27}$$

To simplify the problem even further, we will assume that for $2MG \ll r \ll \xi$ (the "physical" regime) one can set

$$\sigma(r) = -a_0 MG c_\sigma r^\alpha \tag{4.28}$$

and

$$\theta(r) = -a_0 MG c_\theta r^\beta. \tag{4.29}$$

This assumption is in part justified by the form of the nonrelativistic correction of Eqs. (3.13). Then for $\alpha = \beta$ (the equations seem impossible to satisfy if α and β are different) one obtains for the scalar curvature

$$R = 0 + \alpha(2c_\sigma + (\alpha-1)c_\theta)a_0 MG r^{\alpha-3} + +O(a_0^2). \tag{4.30}$$

A first result can be obtained in the following way. Since in the ordinary Einstein case one has for a perfect fluid $R = -8\pi GT = +8\pi G(\rho - 3p)$, and since $\rho_m(r) \sim r^{(1/\nu)-3}$ from Eq. (3.17) in the same regime, one concludes that a solution is given by

$$\alpha = \frac{1}{\nu}, \tag{4.31}$$

which also seems consistent with the Poisson equation result of Eq. (3.24).

Next one needs the action of $\Box^n$ on the point source (here hidden in T). To lowest order one has for the source term

$$8\pi G\rho = MG\frac{4}{\pi}\frac{\epsilon}{r^2(r^2+\epsilon^2)}. \tag{4.32}$$

The d'Alembertian then acts on the source term and gives

$$\Box n(8\pi G\rho) \to \frac{(2n+2)!}{2}MG\frac{4}{\pi}r^{-2n-3}\left(\frac{\epsilon}{r}\right), \tag{4.33}$$

which can then be analytically continued to $n = -\frac{1}{2\nu}$, resulting in

$$(\Box)^{-(1/2\nu)}(8\pi G\rho) \to \frac{\Gamma(3-\frac{1}{\nu})}{2}MG\frac{4}{\pi}r^{(1/\nu)-3}\left(\frac{\epsilon}{r}\right). \tag{4.34}$$

After multiplying the above expression by a_0, consistency with left-hand side of the trace equation, Eq. (4.30), is achieved to lowest order in a_0 provided again $\alpha = 1/\nu$. To zeroth order in a_0, the correct solution is of course already built into the structure of Eqs. (4.25) and (4.26). Also note that setting $\epsilon = 0$ would give nonsensical results, and, in particular, the effective density would be zero for $r \neq 0$, in disagreement with the result of Eq. (3.17), $\rho_m(r) \sim r^{(1/\nu)-3}$ for small r.

The next step up would be the consideration of the action of $\Box$ on the point source, as it appears in the full effective field equations of Eq. (4.8), with again $T_{\mu\nu}$ described by Eq. (4.20). One perhaps surprising fact is the generation of an effective pressure term by the action of $\Box$, suggesting that both terms should arise in the correct description of vacuum polarization effects

$$(\Box T_{\mu\nu})_{tt} = -\frac{\rho B'^2}{2AB} + \frac{2B\rho'}{rA} - \frac{BA'\rho'}{2A^2} + \frac{B'\rho'}{2A} + \frac{B\rho''}{A},$$
$$(\Box T_{\mu\nu})_{rr} = -\frac{\rho B'^2}{2B^2}, \tag{4.35}$$

and $(\Box T_{\mu\nu})_{\theta\theta} = (\Box T_{\mu\nu})_{\varphi\varphi} = 0$. A similar effect, namely, the generation of an effective vacuum pressure term in the field equations by the action of $\Box$, is seen also in the Robertson-Walker metric case [12].

B. Effective trace equation

To check the overall consistency of the approach, consider the set of effective field equations that are obtained when the operator $(1 + A(\Box))$ appearing in Eqs. (4.8) and (4.9) is moved over to the gravitational side, as in Eq. (4.15). Since the right-hand side of the field equations then vanishes for $r \neq 0$, one has apparently reduced the problem to one of finding vacuum solutions of a modified, nonlocal field equation.

Let us first look at the simpler trace equation, valid again for $r \neq 0$. If we denote by δR the lowest order variation (that is, of order a_0) in the scalar curvature over the ordinary vacuum solution $R = 0$, then one needs to find solutions to

$$\frac{1}{8\pi G A(\Box)}\delta R = 0. \tag{4.36}$$

On a generic scalar function $F(r)$ one has the following action of the covariant d'Alembertian $\Box$:

$$\Box F(r) = -\frac{A'F'}{2A^2} + \frac{B'F'}{2AB} + \frac{2F'}{rA} + \frac{F''}{A}. \tag{4.37}$$

The Ricci scalar is complicated enough, even in this simple case, and equal to

$$\frac{B'^2}{2AB^2} + \frac{A'B'}{2A^2B} - \frac{2B'}{rAB} + \frac{2A'}{rA^2} - \frac{B''}{AB} - \frac{2}{r^2A} + \frac{2}{r^2}. \tag{4.38}$$

The action of the covariant d'Alembertian on it produces the rather formidable expression

$$\frac{5B'^4}{2A^2B^4}+\frac{3A'B'^3}{A^3B^3}-\frac{5B'^3}{rA^2B^3}+\frac{3A'^2B'^2}{A^4B^2}-\frac{6A'B'^2}{rA^3B^2}-\frac{5A''B'^2}{4A^3B^2}-\frac{6B''B'^2}{A^2B^3}+\frac{B'^2}{r^2A^2B^2}+\frac{7A'^3B'}{2A^5B}-\frac{9A'^2B'}{rA^4B}-\frac{A'B'}{r^2A^3B}-\frac{13A'A''B'}{4A^4B}$$
$$+\frac{4A''B'}{rA^3B}-\frac{23A'B''B'}{4A^3B^2}+\frac{9B''B'}{rA^2B^2}+\frac{A^{(3)}B'}{2A^3B}+\frac{5B^{(3)}B'}{2A^2B^2}-\frac{2B'}{r^3AB}+\frac{2B'}{r^3A^2B}+\frac{14A'^3}{rA^5}-\frac{4A'^2}{r^2A^4}+\frac{2B''^2}{A^2B^2}+\frac{2A'}{r^3A^2}-\frac{6A'}{r^3A^3}$$
$$+\frac{2A''}{r^2A^3}-\frac{19A'^2B''}{4A^4B}+\frac{8A'B''}{rA^3B}+\frac{2A''B''}{A^3B}+\frac{2A^{(3)}}{rA^3}+\frac{3A'B^{(3)}}{A^3B}-\frac{4B^{(3)}}{rA^2B}-\frac{B^{(4)}}{A^2B}+\frac{4}{r^4A}-\frac{4}{r^4A^2}. \tag{4.39}$$

To lowest order in the functions σ and θ, from Eq. (4.27), the scalar curvature is given by

$$(GM(\sigma-\theta)-(2GM-r)(GM\theta'+(3GM-2r)\sigma'+(2GM-r)r\theta''))/(r^2(r-2MG)^2). \tag{4.40}$$

After having the d'Alembertian $\Box$ act on this expression, one obtains the still formidable (here again to lowest order in σ and θ) result

$$(\theta^{(4)}r^7+2\sigma^{(3)}r^6-8GM\theta^{(4)}r^6-4\sigma''r^5+GM\theta^{(3)}r^5-15GM\sigma^{(3)}r^5+24G^2M^2\theta^{(4)}r^5+4\sigma'r^4+3GM\theta''r^4$$
$$+31GM\sigma''r^4-6G^2M^2\theta^{(3)}r^4+42G^2M^2\sigma^{(3)}r^4-32G^3M^3\theta^{(4)}r^4-12GM\theta'r^3-28GM\sigma'r^3-14G^2M^2\theta''r^3$$
$$-94G^2M^2\sigma''r^3+12G^3M^3\theta^{(3)}r^3-52G^3M^3\sigma^{(3)}r^3+16G^4M^4\theta^{(4)}r^3+12GM\theta r^2-12GM\sigma r^2+48G^2M^2\theta'r^2$$
$$+96G^2M^2\sigma'r^2+20G^3M^3\theta''r^2+132G^3M^3\sigma''r^2-8G^4M^4\theta^{(3)}r^2+24G^4M^4\sigma^{(3)}r^2-24G^2M^2\theta r+24G^2M^2\sigma r$$
$$-64G^3M^3\theta'r-160G^3M^3\sigma'r-8G^4M^4\theta''r-72G^4M^4\sigma''r+16G^3M^3\theta-16G^3M^3\sigma+32G^4M^4\theta'$$
$$+96G^4M^4\sigma')/((2GM-r)^3r^5). \tag{4.41}$$

Higher powers of the d'Alembertian $\Box$ then lead to even more complicated expressions, with increasingly higher derivatives of $\sigma(r)$ and $\theta(r)$. Assuming a power law correction, as in Eqs. (4.28) and (4.29), with $\alpha=\beta$, as in Eq. (4.30),

$$R=a_0MG\alpha(2c_\sigma+c_\theta(\alpha-1))r^{\alpha-3}+O(a_0^2) \tag{4.42}$$

and then

$$\Box R\rightarrow a_0MGr^{\alpha-5}(2c_\sigma+c_\theta(\alpha-1))\alpha(\alpha-2)(\alpha-3),$$
$$\Box^2R\rightarrow a_0MGr^{\alpha-7}(2c_\sigma+c_\theta(\alpha-1))\alpha(\alpha-2)\times(\alpha-3)(\alpha-4)(\alpha-5), \tag{4.43}$$

and so on, and for general $n\rightarrow+\frac{1}{2\nu}$

$$\Box^nR\rightarrow a_0MG(2c_\sigma+c_\theta(\alpha-1))\times\frac{\alpha\Gamma(2+\frac{1}{\nu}-\alpha)}{\Gamma(2-\alpha)}r^{\alpha-(1/\nu)-3}. \tag{4.44}$$

Therefore the only possible power solution for $r\gg MG$ is $\alpha=0,2\ldots\frac{1}{\nu}+1$, with c_σ and c_θ unconstrained to this order.

C. Full effective field equations

Next we examine the full effective field equations (as opposed to just their trace part) as in Eq. (4.15) with $\lambda=0$

$$\frac{1}{8\pi G(1+A(\Box))}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)=0, \tag{4.45}$$

valid for $r\neq0$. If one denotes by $\delta G_{\mu\nu}\equiv\delta(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R)$ the lowest order variation (that is, of order a_0) in the Einstein tensor over the ordinary vacuum solution $G_{\mu\nu}=0$, then one has

$$\frac{1}{8\pi GA(\Box)}\delta\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)=0 \tag{4.46}$$

again for $r\neq0$. Here the covariant d'Alembertian operator

$$\Box=g^{\mu\nu}\nabla_\mu\nabla_\nu \tag{4.47}$$

acts on a second rank tensor

$$\nabla_\nu T_{\alpha\beta}=\partial_\nu T_{\alpha\beta}-\Gamma^\lambda_{\alpha\nu}T_{\lambda\beta}-\Gamma^\lambda_{\beta\nu}T_{\alpha\lambda}\equiv I_{\nu\alpha\beta},$$
$$\nabla_\mu(\nabla_\nu T_{\alpha\beta})=\partial_\mu I_{\nu\alpha\beta}-\Gamma^\lambda_{\nu\mu}I_{\lambda\alpha\beta}-\Gamma^\lambda_{\alpha\mu}I_{\nu\lambda\beta}-\Gamma^\lambda_{\beta\mu}I_{\nu\alpha\lambda}, \tag{4.48}$$

and would thus seem to require the calculation of as many as 1920 terms, of which many fortunately vanish by symmetry. In the static isotropic case the components of the Einstein tensor are given by

$$G_{tt}=\frac{A'B}{rA^2}-\frac{B}{r^2A}+\frac{B}{r^2},$$
$$G_{rr}=-\frac{A}{r^2}+\frac{B'}{rB}+\frac{1}{r^2},$$
$$G_{\theta\theta}=-\frac{B'^2r^2}{4AB^2}-\frac{A'B'r^2}{4A^2B}+\frac{B''r^2}{2AB}-\frac{A'r}{2A^2}+\frac{B'r}{2AB},$$
$$G_{\varphi\varphi}=\sin^2\theta G_{\theta\theta}. \tag{4.49}$$

Using again the expansion of Eqs. (4.25) and (4.26) one obtains for the tt, rr and $\theta\theta$ components of the Einstein tensor, to lowest order in σ and θ,

$$G_{tt} \simeq \frac{(2GM - r)\sigma'}{r^3}, \qquad G_{rr} \simeq \frac{-\theta + \sigma + (-2GM + r)\theta'}{r(-2GM + r)^2},$$
$$G_{\theta\theta} \simeq \frac{(-GM + r)(\theta - \sigma + (2GM - r)(\theta' - \sigma')) + r(-2GM + r)^2\theta''}{2(-2GM + r)^2}, \qquad G_{\varphi\varphi} = \sin^2\theta G_{\theta\theta}. \tag{4.50}$$

After acting with $\Box$ on this expression one finds a rather complicated result. Here we will only list $(\Box G)_{tt}$:

$$\frac{6BA'^3}{rA^5} + \frac{2BA'^2}{r^2A^4} - \frac{4B'A'^2}{rA^4} - \frac{2BA'}{r^3A^3} - \frac{6BA''A'}{rA^4} + \frac{B''A'}{rA^3} + \frac{6B}{r^4A} - \frac{6B}{r^4A^2} - \frac{4B'}{r^3A} + \frac{4B'}{r^3A^2} - \frac{BA''}{r^2A^3} + \frac{2B'A''}{rA^3} + \frac{B''}{r^2A} - \frac{B''}{r^2A^2} + \frac{BA^{(3)}}{rA^3}. \tag{4.51}$$

To lowest order in $\sigma(r)$ and $\theta(r)$ one finds the slightly simpler expressions

$$\begin{aligned}(\Box G)_{tt} = {}& -(-2G^2M^2\theta + 2G^2M^2\sigma - 4G^3M^3\theta' + 2G^2M^2r\theta' - 28G^3M^3\sigma' + 38G^2M^2r\sigma' - 16GMr^2\sigma' + 2r^3\sigma' \\ & + 24G^3M^3r\sigma'' - 32G^2M^2r^2\sigma'' + 14GMr^3\sigma'' - 2r^4\sigma'' - 8G^3M^3r^2\sigma^{(3)} + 12G^2M^2r^3\sigma^{(3)} \\ & - 6GMr^4\sigma^{(3)} + r^5\sigma^{(3)})/(r^6(-2GM + r)),\end{aligned} \tag{4.52}$$

$$\begin{aligned}(\Box G)_{rr} = {}& (6G^2M^2\theta - 4GMr\theta - 6G^2M^2\sigma + 4GMr\sigma + 12G^3M^3\theta' - 14G^2M^2r\theta' + 4GMr^2\theta' - 12G^3M^3\sigma' \\ & + 6G^2M^2r\sigma' + 4GMr^2\sigma' - 2r^3\sigma' + 8G^3M^3r\theta'' - 12G^2M^2r^2\theta'' + 6GMr^3\theta'' - r^4\theta'' + 4G^2M^2r^2\sigma'' \\ & - 4GMr^3\sigma'' + r^4\sigma'' - 8G^3M^3r^2\theta^{(3)} + 12G^2M^2r^3\theta^{(3)} - 6GMr^4\theta^{(3)} + r^5\theta^{(3)})/(r^4(-2GM + r)^3),\end{aligned} \tag{4.53}$$

$$\begin{aligned}(\Box G)_{\theta\theta} = {}& (24G^3M^3\theta - 36G^2M^2r\theta + 16GMr^2\theta - 24G^3M^3\sigma + 36G^2M^2r\sigma - 16GMr^2\sigma + 48G^4M^4\theta' - 96G^3M^3r\theta' \\ & + 68G^2M^2r^2\theta' - 16GMr^3\theta' + 16G^4M^4\sigma' - 32G^3M^3r\sigma' + 28G^2M^2r^2\sigma' - 16GMr^3\sigma' + 4r^4\sigma' \\ & + 8G^4M^4r\theta'' - 12G^3M^3r^2\theta'' + 10G^2M^2r^3\theta'' - 5GMr^4\theta'' + r^5\theta'' - 24G^4M^4r\sigma'' + 52G^3M^3r^2\sigma'' \\ & - 46G^2M^2r^3\sigma'' + 19GMr^4\sigma'' - 3r^5\sigma'' - 24G^4M^4r^2\theta^{(3)} + 44G^3M^3r^3\theta^{(3)} - 30G^2M^2r^4\theta^{(3)} \\ & + 9GMr^5\theta^{(3)} - r^6\theta^{(3)} + 8G^4M^4r^2\sigma^{(3)} - 20G^3M^3r^3\sigma^{(3)} + 18G^2M^2r^4\sigma^{(3)} - 7GMr^5\sigma^{(3)} + r^6\sigma^{(3)} \\ & + 16G^4M^4r^3\theta^{(4)} - 32G^3M^3r^4\theta^{(4)} + 24G^2M^2r^5\theta^{(4)} - 8GMr^6\theta^{(4)} + r^7\theta^{(4)})/(2r^3(-2GM + r)^3),\end{aligned} \tag{4.54}$$

with the $\varphi\varphi$ component proportional to the $\theta\theta$ component. If one again assumes that the corrections are given by a power, as in Eqs. (4.28) and (4.29), with $\alpha = \beta$, then one has to zeroth order

$$\begin{aligned} G_{tt} &= a_0MGc_\sigma\alpha r^{\alpha-3}, \\ G_{rr} &= -a_0MG(c_\sigma + c_\theta(\alpha - 1))r^{\alpha-3}, \\ G_{\theta\theta} &= -\tfrac{1}{2}a_0MG(c_\sigma + c_\theta(\alpha - 1))(\alpha - 1)r^{\alpha-1}\end{aligned} \tag{4.55}$$

with the $\varphi\varphi$ component again proportional to the $\theta\theta$ component. Applying $\Box$ on the above Einstein tensor one then gets

$$\begin{aligned}(\Box G)_{tt} &= a_0GMc_\sigma\alpha(\alpha - 2)(\alpha - 3)r^{\alpha-5}, \\ (\Box G)_{rr} &= -a_0GM(c_\sigma + c_\theta(\alpha - 1))\alpha(\alpha - 3)r^{\alpha-5}, \\ (\Box G)_{\theta\theta} &= -\tfrac{1}{2}a_0GM(c_\sigma + c_\theta(\alpha - 1))\alpha(\alpha - 3)^2r^{\alpha-3}\end{aligned} \tag{4.56}$$

again with the $\varphi\varphi$ component proportional to the $\theta\theta$ component. Applying $\Box$ again one obtains

$$\begin{aligned}(\Box^2 G)_{tt} &= a_0GMc_\sigma\alpha(\alpha - 2)(\alpha - 3)(\alpha - 4)(\alpha - 5)r^{\alpha-7}, \\ (\Box^2 G)_{rr} &= -a_0GM(c_\sigma + c_\theta(\alpha - 1)) \\ &\quad \times \alpha(\alpha - 2)(\alpha - 3)(\alpha - 5)r^{\alpha-7}, \\ (\Box^2 G)_{\theta\theta} &= -\tfrac{1}{2}a_0GM(c_\sigma + c_\theta(\alpha - 1)) \\ &\quad \times \alpha(\alpha - 2)(\alpha - 3)(\alpha - 5)^2r^{\alpha-5},\end{aligned} \tag{4.57}$$

and so on, and for general $n \to +\frac{1}{2\nu}$

$$\begin{aligned}(\Box^n G)_{tt} &\to a_0GMc_\sigma\frac{\Gamma(2 + \frac{1}{\nu} - \alpha)}{(\alpha - 1)\Gamma(-\alpha)}r^{\alpha-3-(1/\nu)}, \\ (\Box^n G)_{rr} &\to -a_0GM(c_\sigma + c_\theta(\alpha - 1)) \\ &\quad \times \frac{\Gamma(2 + \frac{1}{\nu} - \alpha)}{(\alpha - 1)(\alpha - \frac{1}{\nu})\Gamma(-\alpha)}r^{\alpha-3-(1/\nu)}, \\ (\Box^n G)_{\theta\theta} &\to -\frac{1}{2}a_0GM(c_\sigma + c_\theta(\alpha - 1)) \\ &\quad \times \frac{(\alpha - 1 - \frac{1}{\nu})\Gamma(2 + \frac{1}{\nu} - \alpha)}{(\alpha - 1)(\alpha - \frac{1}{\nu})\Gamma(-\alpha)}r^{\alpha-1-(1/\nu)}.\end{aligned} \tag{4.58}$$

Inspection of the above results reveals a common factor

$1/\Gamma(-\alpha)$, which would allow only integer powers $\alpha = 0, 1, 2\ldots$, but the additional factor of $1/(\alpha - 1)$ excludes $\alpha = 1$ from being a solution. Even for α close to $1/\nu$ [as expected on the basis of the nonrelativistic expression of Eq. (3.24), as well as from Eq. (4.31)] $\nu \sim 1/\alpha - \epsilon$ only integer values $\alpha = 2, 3, 4\ldots$ are allowed. For the covariant divergences $\nabla^\mu(\Box^n G)_{\mu\nu}$ one has

$$\nabla^\mu(\delta G)_{\mu r} = 0 \tag{4.59}$$

and at the next order

$$\nabla^\mu(\Box G)_{\mu r} = 2a_0 G^2 M^2 \alpha(\alpha - 3)(c_\sigma + c_\theta(1-\alpha))r^{\alpha-7} \tag{4.60}$$

with the other components vanishing identically, and

$$\nabla^\mu(\Box^2 G)_{\mu r} = 4a_0 G^2 M^2 \alpha(\alpha - 3)(\alpha - 5)^2 \times (c_\sigma + c_\theta(1-\alpha))r^{\alpha-9} \tag{4.61}$$

again with the other components of the divergence vanishing identically.

In general the problem of finding a complete general solution to the effective field equations by this method lies in the difficulty of computing arbitrarily high powers of $\Box$ on general functions such as $\sigma(r)$ and $\theta(r)$, which eventually involve a large number of derivatives. Assuming for these functions a power law dependence on r simplifies the problem considerably, but also restricts the kind of solutions that one is likely to find. More specifically, if the solution involves (say for small r, but still with $r \gg 2MG$) a term of the type $r^\alpha \ln mr$, as in Eqs. (3.21), (5.35), and (5.38) for $\nu \to 1/3$, then this method will have to be dealt with very carefully. This is presumably the reason why in some of the Γ-function coefficients encountered here one finds a power solution (in fact $\alpha = 3$) for ν close to a third, but one gets indeterminate expression if one sets exactly $\alpha = 1/\nu = 3$.

V. THE QUANTUM VACUUM AS A FLUID

The discussion of Sec. III suggests that the quantum correction due to the running of G can be described, at least in the nonrelativistic limit of Eq. (2.2) as applied to Poisson's equation, in terms of a vacuum energy density $\rho_m(r)$, distributed around the static source of strength M in accordance with the result of Eqs. (3.17) and (3.19). These expressions, in turn, can be obtained by Fourier transforming back to real space the original result for $G(k^2)$ of Eq. (2.2).

Furthermore, it was shown in Sec. IV (and was discovered in [12] as well, see, for example, Eq. (7.8) later in this paper) that a manifestly covariant implementation of the running of G, via the $G(\Box)$ given in Eqs. (4.2) and (4.7), will induce a nonvanishing effective pressure term. This result can be seen clearly, in the case of the static isotropic metric, for example, from the result of Eq. (4.35).

We will therefore, in this section, consider a relativistic perfect fluid, with energy-momentum tensor

$$T_{\mu\nu} = [p + \rho]u_\mu u_\nu + g_{\mu\nu} p, \tag{5.1}$$

which in the static isotropic case reduces to Eq. (4.13),

$$T_{\mu\nu} = \mathrm{diag}[B(r)\rho(r), A(r)p(r), r^2 p(r), r^2 \sin^2\theta\, p(r)] \tag{5.2}$$

and gives a trace $T = 3p - \rho$.

The tt, rr, and $\theta\theta$ components of the field equations then read

$$-\lambda B(r) + \frac{A'(r)B(r)}{rA(r)^2} - \frac{B(r)}{r^2 A(r)} + \frac{B(r)}{r^2} = 8\pi G B(r)\rho(r), \tag{5.3}$$

$$\lambda A(r) - \frac{A(r)}{r^2} + \frac{B'(r)}{rB(r)} + \frac{1}{r^2} = 8\pi G A(r) p(r), \tag{5.4}$$

$$-\frac{B'(r)^2 r^2}{4A(r)B(r)^2} + \lambda r^2 - \frac{A'(r)B'(r)r^2}{4A(r)^2 B(r)} + \frac{B''(r)r^2}{2A(r)B(r)} - \frac{A'(r)r}{2A(r)^2} + \frac{B'(r)r}{2A(r)B(r)} = 8G\pi r^2 p(r), \tag{5.5}$$

with the $\varphi\varphi$ component equal to $\sin^2\theta$ times the $\theta\theta$ component.

Energy conservation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$[p(r) + \rho(r)]\frac{B'(r)}{2B(r)} + p'(r) = 0 \tag{5.6}$$

and forces a definite relationship between $B(r)$, $\rho(r)$, and $p(r)$. The three field equations and the energy conservation equation are, as usual, not independent, because of the Bianchi identity.

It seems reasonable to attempt to solve the above equations (usually considered in the context of relativistic stellar structure [41]) with the density $\rho(r)$ given by the $\rho_m(r)$ of Eqs. (3.17), (3.18), and (3.19).

This of course raises the question of how the relativistic pressure $p(r)$ should be chosen, an issue that the nonrelativistic calculation did not have to address. We will argue below that covariant energy conservation completely determines the pressure in the static case, leading to consistent equations and solutions (note that, in particular, it would not be consistent to take $p(r) = 0$).

Since the function $B(r)$ drops out of the tt field equation, the latter can be integrated immediately, giving

$$A(r)^{-1} = 1 + \frac{c_1}{r} - \frac{\lambda}{3}r^2 - \frac{8\pi G}{r}\int_0^r dx x^2 \rho(x), \tag{5.7}$$

which suggests the introduction of a function $m(r)$

$$m(r) \equiv 4\pi \int_0^r dx x^2 \rho(x). \tag{5.8}$$

It also seems natural in our case to identify $c_1 = -2MG$, which of course corresponds to the solution with $a_0 = 0$ ($p = \rho = 0$) (equivalently, the point source at the origin of strength M could be included as an additional δ-function contribution to $\rho(r)$).

Next, the rr field equation can be solved for $B(r)$

$$B(r) = \exp\left\{c_2 - \int_{r_0}^r dy \frac{1 + A(y)(\lambda y^2 - 8\pi G y^2 p(y) - 1)}{y}\right\}, \tag{5.9}$$

with the constant c_2 again determined by the requirement that the above expression for $B(r)$ reduce to the standard Schwarzschild solution for $a_0 = 0$ ($p = \rho = 0$), giving $c_2 = \ln(1 - 2MG/r_0 - \lambda r_0^2/3)$. The last task left therefore is the determination of the pressure $p(r)$.

Using the rr field equation, $B'(r)/B(r)$ can be expressed in term of $A(r)$ in the energy conservation equation, which results in

$$2rp'(r) - [1 + A(r)(\lambda r^2 - 8\pi G r^2 p(r) - 1)](p(r) + \rho(r)) = 0. \tag{5.10}$$

Inserting the explicit expression for $A(r)$, from Eq. (5.7), one obtains

$$p'(r) + \frac{(8\pi G r^3 p(r) + 2MG - \frac{2}{3}\lambda r^3 + 8\pi G \int_{r_0}^r dx x^2 \rho(x))(p(r) + \rho(r))}{2r(r - 2MG - \frac{\lambda}{3} r^3 - 8\pi G \int_0^r dx x^2 \rho(x))} = 0, \tag{5.11}$$

which is usually referred to as the equation of hydrostatic equilibrium. From now on we will focus only the case $\lambda = 0$. Then

$$p'(r) + \frac{(8\pi G r^3 p(r) + 2MG + 8\pi G \int_0^r dx x^2 \rho(x))(p(r) + \rho(r))}{2r(r - 2MG - 8\pi G \int_0^r dx x^2 \rho(x))} = 0. \tag{5.12}$$

The last equation, a nonlinear differential equation for $p(r)$, can be solved to give the desired solution $p(r)$, which then, by equation Eq. (5.9), determines the remaining function $B(r)$.

In our case though it will be sufficient to solve the above equation for small a_0, where a_0 [see Eq. (2.2) and (3.17)] is the dimensionless parameter which, when set to zero, makes the solution revert back to the classical one.

It will also be convenient to pull out of $A(r)$ and $B(r)$ the Schwarzschild solution part, by introducing the small corrections $\sigma(r)$ and $\theta(r)$ [already defined before in Eqs. (4.25) and (4.26)], both of which are expected to be proportional to the parameter a_0. One has

$$\sigma(r) = -8\pi G \int_0^r dx x^2 \rho(x) \equiv -2m(r)G \tag{5.13}$$

and

$$\theta(r) = \exp\left\{c_2 + \int_{r_0}^r dy \frac{1 + 8\pi G y^2 p(y)}{y - 2MG - 8\pi G \int_0^y dx x^2 \rho(x)}\right\} + 2MG - r. \tag{5.14}$$

Again, the integration constant c_2 needs to be chosen here so that the normal Schwarzschild solution is recovered for $p = \rho = 0$.

To order a_0 the resulting equation for $p(r)$, from Eq. (5.12), is

$$\frac{MG(p(r) + \rho(r))}{r(r - 2MG)} + p'(r) \simeq 0. \tag{5.15}$$

Note that in regions where $p(r)$ is slowly varying, $p'(r) \sim 0$, one has $p \simeq -\rho$, i.e. the fluid contribution is acting like a cosmological constant term with $\sigma(r) \sim \theta(r) \sim -(\rho/3)r^3$.

The last differential equation can then be solved for $p(r)$

$$p_m(r) = \frac{1}{\sqrt{1 - \frac{2MG}{r}}}\left(c_3 - \int_{r_0}^r dz \frac{MG\rho(z)}{z^2\sqrt{1 - \frac{2MG}{z}}}\right), \tag{5.16}$$

where the constant of integration has to be chosen so that when $\rho(r) = 0$ (no quantum correction) one has $p(r) = 0$ as well. Because of the singularity in the integrand at $r = 2MG$, we will take the lower limit in the integral to be $r_0 = 2MG + \epsilon$, with $\epsilon \to 0$.

To proceed further, one needs the explicit form for $\rho_m(r)$, which was given in Eqs. (3.17), (3.18), and (3.19). The required integrands involve for general ν the modified Bessel function $K_n(x)$, and can be therefore a bit complicated. But in some special cases the general form of the density ρ_m of Eq. (3.17)

$$\rho_m(r) = \frac{1}{8\pi} c_\nu a_0 M m^3 (mr)^{-(1/2)(3-(1/\nu))} K_{(1/2)(3-(1/\nu))}(mr) \tag{5.17}$$

reduces to a relatively simple expression, which we will list here. For $\nu = 1$ one has

$$\rho_m(r) = \frac{1}{2\pi^2} a_0 M m^3 \frac{1}{mr} K_1(mr), \tag{5.18}$$

whereas for $\nu = 1/2$ one has

$$\rho_m(r) = \frac{1}{4\pi} a_0 M m^3 \frac{1}{mr} e^{-mr} \tag{5.19}$$

and for $\nu = 1/3$

$$\rho_m(r) = \frac{1}{2\pi^2} a_0 M m^3 K_0(mr) \tag{5.20}$$

and finally for $\nu = 1/4$

$$\rho_m(r) = \frac{1}{8\pi} a_0 M m^3 e^{-mr}. \tag{5.21}$$

Note that $\rho_m(r)$ diverges at small r for $\nu \geq 1/3$

Here we will limit our investigation to the small r ($mr \ll 1$) and large r ($mr \gg 1$) behavior. Since $m = 1/\xi$ is very small, the first limit appears to be of greater physical interest.

A. Small r limit

For small r the density $\rho_m(r)$ has the following behavior [see Eq. (3.17)],

$$\rho_m(r) \underset{r\to 0}{\sim} A_0 r^{(1/\nu)-3} \tag{5.22}$$

for $\nu > 1/3$, with

$$A_0 \equiv \frac{c_k c_\nu}{8\pi} a_0 M m^{1/\nu} = \frac{|\sec(\frac{\pi}{2\nu})|}{4\pi\Gamma(\frac{1}{\nu} - 1)} a_0 M m^{1/\nu}, \tag{5.23}$$

where the dimensionless positive constant c_k is determined from the small x behavior of the modified Bessel function $K_n(x)$

$$x^{1/2((1/\nu)-3)} K_{1/2(3-(1/\nu))}(x) \underset{x\to 0}{\sim} -\frac{2^{1/2(1-(1/\nu))}\pi \sec(\frac{\pi}{2\nu})}{\Gamma(\frac{1}{2\nu} - \frac{1}{2})} x^{(1/\nu)-3} \equiv c_k x^{((1/\nu)-3)} \tag{5.24}$$

valid for $\nu > 1/3$, and c_ν is given in Eq. (3.18). For $\nu < 1/3$ $\rho_m(r) \sim \text{const}\, a_0 M m^3$, independent of r. For $\nu = 1/3$ the expression for $\rho_m(r)$ in Eq. (5.20) should be used instead.

Therefore in this limit, with $\frac{1}{3} < \nu < 1$, one has

$$m(r) \simeq 4\pi\nu A_0 r^{1/\nu} \tag{5.25}$$

and, from the definition of $\sigma(r)$,

$$\sigma(r) \simeq -2m(r)G = -8\pi\nu G A_0 r^{1/\nu} \tag{5.26}$$

and finally

$$A^{-1}(r) = 1 - \frac{2MG}{r} - 2a_0 MG c_s m^{1/\nu} r^{((1/\nu)-1)} + \ldots \tag{5.27}$$

with the constant $c_s = \nu|\sec(\frac{\pi}{2\nu})|/\Gamma(\frac{1}{\nu} - 1)$. For $\nu = 1/3$ the last contribution is indistinguishable from a cosmological constant term $-\frac{\lambda}{3}r^2$, except for the fact that the coefficient here is quite different, being proportional to $\sim a_0 M G m^3$.

To determine the pressure, we suppose that it as well has a power dependence on r in the regime under consideration, $p_m(r) = c_p A_0 r^\gamma$, where c_p is a numerical constant, and then substitute $p_m(r)$ into the pressure equation Eq. (5.15). This gives, past the horizon $r \gg 2MG$

$$(2\gamma - 1)c_p M G r^{\gamma-1} - c_p \gamma r^\gamma - M G r^{1/\nu - 4} \simeq 0 \tag{5.28}$$

giving the same power $\gamma = 1/\nu - 3$ as for $\rho(r)$, $c_p = -1$ and surprisingly also $\gamma = 0$, implying that in this regime only $\nu = 1/3$ gives a consistent solution. Again, the resulting correction is quite similar to what one would expect from a cosmological term, with an effective $\lambda_m/3 \simeq 8\pi\nu a_0 M G m^{1/\nu}$. One then has for ν near $1/3$

$$p_m(r) = A_0 c_p r^{((1/\nu)-3)} + \ldots \tag{5.29}$$

and thus from Eq. (5.14)

$$B(r) = 1 - \frac{2MG}{r} - 2a_0 MG c_s m^{1/\nu} r^{((1/\nu)-1)} + \ldots. \tag{5.30}$$

Both the result for $A(r)$ in Eq. (5.27), and the above result for $B(r)$ are, for $r \gg 2MG$, consistent with a gradual slow increase in G in accordance with the formula

$$G \to G(r) = G(1 + a_0 c_s m^{1/\nu} r^{1/\nu} + \ldots). \tag{5.31}$$

We note here that both expressions for $A(r)$ and $B(r)$ have some similarities with the approximate nonrelativistic (Poisson equation) result of Eq. (3.24), with the correction proportional to a_0 agreeing roughly in magnitude (but not in sign).

The case $\nu = 1/3$ requires a special treatment, since the coefficient c_k in Eq. (5.24) diverges as $\nu \to 1/3$. Starting from the expression for $\rho_m(r)$ for $\nu = 1/3$ in Eq. (5.20),

$$\rho_m(r) = \frac{1}{2\pi^2} a_0 M m^3 K_0(mr) \tag{5.32}$$

one has for small r

$$\rho_m(r) = -\frac{a_0}{2\pi^2} M m^3 \left(\ln\frac{mr}{2} + \gamma\right) + \ldots \tag{5.33}$$

and, therefore, from Eq. (5.14)

$$\sigma(r) = \frac{4a_0 M G m^3}{3\pi} r^3 \ln(mr) + \ldots \tag{5.34}$$

and consequently

$$A^{-1}(r) = 1 - \frac{2MG}{r} + \frac{4a_0MGm^3}{3\pi} r^2 \ln(mr) + \ldots. \tag{5.35}$$

From Eq. (5.15) one can then obtain an expression for the pressure $p_m(r)$, and one finds

$$p_m(r) = \frac{a_0 M m^3 \log(mr)}{2\pi^2} - \frac{a_0 M m^3 \log(r + r\sqrt{1 - \frac{2MG}{r}} - MG)}{2\pi^2 \sqrt{1 - \frac{2MG}{r}}} + \frac{a_0 M m^3}{\pi^2} + \frac{a_0 M m^3 c_3}{2\pi^2\sqrt{1 - \frac{2MG}{r}}} \tag{5.36}$$

where c_3 is again an integration constant. Here we will be content with the $r \gg 2MG$ limit of the above expression, which we shall write therefore as

$$p_m(r) = \frac{a_0}{2\pi^2} M m^3 \ln(mr) + \ldots. \tag{5.37}$$

After performing the required r integral in Eq. (5.14), and evaluating the resulting expression in the limit $r \gg 2MG$, one obtains an expression for $\theta(r)$, and consequently from it

$$B(r) = 1 - \frac{2MG}{r} + \frac{4a_0MGm^3}{3\pi} r^2 \ln(mr) + \ldots. \tag{5.38}$$

The expressions for $A(r)$ and $B(r)$ are, for $r \gg 2MG$, consistent with a gradual slow increase in G in accordance with the formula

$$G \rightarrow G(r) = G\left(1 + \frac{a_0}{3\pi} m^3 r^3 \ln\frac{1}{m^2 r^2} + \ldots\right), \tag{5.39}$$

and therefore consistent as well with the original result of Eqs. (2.1) and (2.2), namely, that the classical laboratory value of G is obtained for $r \ll \xi$. In fact it is reassuring that the renormalization properties of $G(r)$ as inferred from $A(r)$ are the same as what one finds from $B(r)$. Note that the correct relativistic small r correction of Eq. (5.39) agrees roughly in magnitude (but not in sign) with the approximate nonrelativistic, Poisson equation result of Eq. (3.21).

One further notices some similarities, as well as some rather substantial differences, with the corresponding QED result [42–44],

$$Q(r) = 1 + \frac{\alpha}{3\pi} \ln\frac{1}{m^2 r^2} + \ldots mr \ll 1. \tag{5.40}$$

In the gravity case, the correction vanishes as r goes to zero: in this limit one is probing the bare mass, unencumbered by its virtual graviton vacuum polarization cloud. On the other hand, in the QED case, as one approaches the source one is probing the bare charge, unscreened by the electron's vacuum polarization cloud, and whose magnitude diverges logarithmically for small r.

It should be recalled here that neither function $A(r)$ or $B(r)$ are directly related to the relativistic potential for particle orbits, which is given instead by the combination

$$V_{\text{eff}}(r) = \frac{1}{2A(r)}\left[\frac{l^2}{r^2} - \frac{1}{B(r)} + 1\right], \tag{5.41}$$

where l is proportional to the orbital angular momentum of the test particle [45].

Furthermore, from the metric of Eqs. (5.27) and (5.30) one finds for $\nu \rightarrow 1/3$ the following results for the curvature invariants

$$\begin{aligned} R^2 &= 1024 A_0^2 G^2 \pi^2, \\ R_{\mu\nu}R^{\mu\nu} &= 256 A_0^2 G^2 \pi^2, \\ R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} &= 16G^2\left(\frac{32\pi^2 A_0^2}{3} + \frac{3M^2}{r^6}\right), \end{aligned} \tag{5.42}$$

which are nonsingular at $r = 2MG$, and again consistent with an effective mass density around the source $m(r) \propto r^3$.

B. Large r limit

For large r one has instead, from Eq. (3.17) for $\rho_m(r)$

$$\rho_m(r) \underset{r\rightarrow\infty}{\sim} A_0 r^{((1/2\nu)-2)} e^{-mr}, \tag{5.43}$$

with $A_0 = 1/\sqrt{128\pi} c_\nu a_0 M m^{(1+(1/2\nu))}$. In the same limit, the integration constants is chosen so that the solution for $A(r)$ and $B(r)$ at large r corresponds to a mass $M' = (1 + a_0)M$ [see the expression for the integrated density in Eq. (3.19)], or equivalently

$$\sigma(r) \sim \theta(r) \underset{r\rightarrow\infty}{\sim} -2a_0 MG. \tag{5.44}$$

On then recovers a result similar to the nonrelativistic expression of Eqs. (3.7), (3.8), and (3.12), with $G(r)$ approaching the constant value $G_\infty = (1 + a_0)G$, up to exponentially small corrections in mr at large r.

In conclusion, it appears that a solution to relativistic static isotropic problem of the running gravitational constant can be found, provided that the exponent ν in either Eq. (2.2) or Eq. (4.8) is close to one third. This last result seems to be linked with the fact that the running coupling term acts in some way like a local cosmological constant term, for which the r dependence of the vacuum solution for small r is fixed by the nature of the Schwarzschild solution with a cosmological constant term.[3]

[3]In $d \geq 4$ dimensions the Schwarzschild solution to Einstein gravity with a cosmological term is Ref. [46] $A^{-1}(r) = B(r) = 1 - 2MGc_d r^{3-d} - \frac{2\lambda}{(d-2)(d-1)} r^2$, with $c_d = 4\pi\Gamma(\frac{d-1}{2})/(d-2)\pi^{(d-1)/2}$, which would suggest, in analogy with the results for $d = 4$ given in this section, that in $d \geq 4$ dimensions only $\nu = 1/(d-1)$ is possible. This last result would also be in agreement with the exact value $\nu = 0$ found at $d = \infty$ [15].

VI. DISTORTION OF THE GRAVITATIONAL WAVE SPECTRUM

A scale-dependent gravitational constant $G(k^2)$ will cause slight distortions in the spectrum of gravitational radiation at extremely low frequencies, to some extent irrespective of the nature of the perturbations that cause them. From the field equations with $\lambda = 0$

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \tag{6.1}$$

one obtains in the weak field limit with harmonic gauge condition

$$\Box h_{\mu\nu} = 8\pi G \bar{T}_{\mu\nu} \tag{6.2}$$

with as usual

$$\bar{T}_{\mu\nu} \equiv T_{\mu\nu} - \tfrac{1}{2} \eta_{\mu\nu} T_{\lambda}{}^{\lambda}. \tag{6.3}$$

Density perturbations $\delta\rho(\mathbf{x}, t)$ will enter the right-hand side of the field equations and give rise to gravitational waves with Fourier components

$$h_{\mu\nu}(k) = -8\pi G \frac{1}{k^2} \bar{T}_{\mu\nu}(p, \rho)(k) \tag{6.4}$$

giving for the power spectrum of transverse traceless (gravitational wave) modes

$$P_{TT}(k^2) \simeq k^3 |h_{TT}(k)|^2 = (8\pi)^2 G^2 \frac{1}{k} |\bar{T}(p, \rho)(k)|^2. \tag{6.5}$$

A scale-dependent gravitational constant, with variation in accordance with Eq. (2.2),

$$G \to G(k^2) \tag{6.6}$$

would affect the spectrum of very long wavelength modes via

$$P_{TT}(k^2) \simeq k^3 |h_{\mu\nu}(k)|^2 = (8\pi)^2 G^2(k^2) \frac{1}{k} |\bar{T}_{\mu\nu}(p, \rho)|^2. \tag{6.7}$$

Specifically, according to the expression in Eq. (2.2) for the running of the gravitational constant

$$\frac{G(k^2)}{G} \simeq 1 + a_0 \left(\frac{m^2}{k^2 + m^2} \right)^{1/2\nu} + \cdots \tag{6.8}$$

one has for the tensor power spectrum

$$P_{TT}(k^2) \simeq k^3 |h_{\mu\nu}(k)|^2 = (8\pi)^2 G^2 \frac{1}{k} \left[1 + a_0 \left(\frac{m^2}{k^2 + m^2} \right)^{1/2\nu} \right]^2 |\bar{T}_{\mu\nu}(p, \rho)|^2 \tag{6.9}$$

with the expression in square brackets varying perhaps by as much as an order of magnitude from short wavelengths $k \gg 1/\xi$, to very long wavelengths $k \sim 1/\xi$.

VII. QUANTUM COSMOLOGY—AN ADDENDUM

In this section we will discuss briefly what modifications are expected when one uses Eq. (2.2) instead of Eq. (2.1) in the effective field equations. In [12] cosmological solutions within the Friedmann-Robertson-Walker (FRW) framework were discussed, starting from the quantum effective field equations of Eq. (2.16),

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G (1 + A(\Box)) T_{\mu\nu} \tag{7.1}$$

with $A(\Box)$ defined in either Eq. (4.6) or Eq. (4.7), and applied to the standard Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \right\}. \tag{7.2}$$

It should be noted that there are *two* quantum contributions to this set of equations. The first one arises because of the presence of a nonvanishing cosmological constant $\lambda \simeq 1/\xi^2$, as in Eq. (2.15), originating in the nonperturbative vacuum condensate of the curvature. As in the case of standard FRW cosmology, this is the dominant contributions at large times t, and gives an exponential expansion of the scale factor.

The second contribution arises because of the running of G for $t \ll \xi$ in the effective field equations

$$G(\Box) = G(1 + A(\Box)) = G\left[1 + a_0 (\xi^2)^{-(1/2\nu)} + \ldots \right] \tag{7.3}$$

with $\nu \simeq 1/3$ and a_0 a calculable coefficient of order one (see Eqs. (2.1) and (2.2)).

In the simplest case, namely, for a universe filled with nonrelativistic matter ($p = 0$), the effective Friedmann equations then have the following appearance [12]

$$\frac{k}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G(t)}{3} \rho(t) + \frac{1}{3\xi^2} = \frac{8\pi G}{3} [1 + c_\xi (t/\xi)^{1/\nu} + \ldots] \rho(t) + \frac{1}{3\xi^2} \tag{7.4}$$

for the tt field equation, and

$$\frac{k}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)} + \frac{2\ddot{a}(t)}{a(t)} = -\frac{8\pi G}{3} [c_\xi (t/\xi)^{1/\nu} + \ldots] \rho(t) + \frac{1}{\xi^2} \tag{7.5}$$

for the rr field equation. The running of G appropriate for the Robertson-Walker metric (RW) metric, and appearing explicitly in the first equation, is described by

$$G(t) = G\left[1 + c_\xi \left(\frac{t}{\xi} \right)^{1/\nu} + \ldots \right] \tag{7.6}$$

(with c_ξ or the same order as a_0 of Eq. (2.1) [12]). Note that the running of $G(t)$ induces as well an effective pressure term in the second (rr) equation.[4]

One can therefore talk about an effective density

$$\rho_{\text{eff}}(t) = \frac{G(t)}{G}\rho(t) \tag{7.7}$$

and an effective pressure

$$p_{\text{eff}}(t) = \frac{1}{3}\left(\frac{G(t)}{G} - 1\right)\rho(t) \tag{7.8}$$

with $p_{\text{eff}}(t)/\rho_{\text{eff}}(t) = \frac{1}{3}(G(t) - G)/G(t)$.[5] Within the FRW framework, the gravitational vacuum polarization term behaves therefore in some ways (but not all) like a positive pressure term, with $p(t) = \omega\rho(t)$ and $\omega = 1/3$, which is therefore characteristic of radiation. One could therefore visualize the gravitational vacuum polarization contribution as behaving like ordinary radiation, in the form of a dilute virtual graviton gas: a radiative fluid with an equation of state $p = \frac{1}{3}\rho$. It should be emphasized though that the relationship between density $\rho(t)$ and scale factor $a(t)$ is very different from the classical case.

The running of $G(t)$ in the above equations follows directly from the basic result of Eq. (2.1) (with the dimensionless constant c_ξ proportional to a_0, with a numerical coefficient of order one given in magnitude in Ref. [12]), but transcribed, by explicitly computing the action of the covariant d'Alembertian $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ on $T_{\mu\nu}$, for the RW metric. In other words, following the more or less unambiguously defined sequence $G(k^2) \to G(\Box) \to G(t)$. At the same time, the discussion of Sec. I underscores the fact that for large times $t \gg \xi$ the form of Eq. (2.1), and therefore Eq. (7.6), is no longer appropriate, due to the spurious infrared divergence of Eq. (2.1) at small k^2. Indeed from Eq. (2.2), the infrared regulated version of the above expression should read instead

$$G(t) \simeq G\left[1 + c_\xi\left(\frac{t^2}{t^2 + \xi^2}\right)^{1/2\nu} + \ldots\right] \tag{7.9}$$

with $\xi = m^{-1}$ the (tiny) infrared cutoff. Of course it reduces to the expression in Eq. (7.6) in the limit of small times t, but for very large times $t \gg \xi$ the gravitational coupling, instead of unphysically diverging, approaches a constant, finite value $G_\infty = (1 + a_0 + \ldots)G_c$, independent of ξ. The modification of Eq. (7.9) should apply whenever one considers times for which $t \ll \xi$ is not valid. But since $\xi \sim 1\sqrt{\lambda}$ is of the order the size of the visible universe, the latter regime is largely of academic interest, and was therefore not discussed much in Ref. [12].

It should be noted that the effective Friedmann equations of Eqs. (7.4) and (7.5) also bear a superficial degree of resemblance to what might be obtained in some scalar-tensor theories of gravity, where the gravitational Lagrangian is postulated to be some singular function of the scalar curvature [47,48]. Indeed in the FRW case one has, for the scalar curvature in terms of the scale factor

$$R = 6(k + \dot{a}^2(t) + a(t)\ddot{a}(t))/a^2(t), \tag{7.10}$$

and for $k = 0$ and $a(t) \sim t^\alpha$ one has

$$R = \frac{6\alpha(2\alpha - 1)}{t^2}, \tag{7.11}$$

which suggests that the quantum correction in Eq. (7.4) is, at this level, nearly indistinguishable from an inverse curvature term of the type $(\xi^2 R)^{-1/2\nu}$, or $1/(1 + \xi^2 R)^{1/2\nu}$ if one uses the infrared regulated version. The former would then correspond the to an effective gravitational action

$$I_{\text{eff}} \simeq \frac{1}{16\pi G}\int dx\sqrt{g}\left(R + \frac{f\xi^{-(1/\nu)}}{(R)^{(1/2\nu)-1}} - 2\lambda\right) \tag{7.12}$$

with f a numerical constant of order one, and $\lambda \simeq 1/\xi^2$. But this superficial resemblance is seen here more as an artifact, due to the particularly simple form of the RW metric, with the coincidence of several curvature invariants not expected to be true in general.

VIII. CONCLUSIONS

In this paper we have examined a number of basic issues connected with the renormalization group running of the gravitational coupling. The scope of this paper was to explore the overall consistency of the picture obtained from the lattice, by considering a number of basic issues, one of which is the analogy, or contrast, with a much better understood class of theories such as QED and Yang-Mills theories.

The starting point for our discussion of the renormalization group running of G (Sec. II) is Eq. (2.1) (valid at short distances $k \gg m$, or, equivalently $r \ll \xi$), and its improved infrared regulated version of Eq. (2.2). The scale dependence for G obtained from the lattice is remarkably similar to the result of the $2 + \epsilon$ expansion in the continuum, with two important differences: only the strong coupling phase $G > G_c$ is physical, and for the exponent one has $\nu \simeq 1/3$ in four dimensions. The similarity between the two results in part also originates from the fact that in both cases the renormalization group properties of G are inferred (implicitly, in the $2 + \epsilon$ case) from the requirement that the nonperturbative scale of Eq. (2.10) be treated as an invariant.

[4]We wish to emphasize that we are *not* talking here about models with a time-dependent value of G. Thus, for example, the value of $G \simeq G_c$ at laboratory scales should be taken to be constant throughout most of the evolution of the universe.

[5]Strictly speaking, the above results can only be proven if one assumes that the pressure's time dependence is given by a power law, as discussed in detail in Ref. [12]. In the more general case, the solution of the above equations for various choices of ξ and a_0 has to be done numerically.

Inspection of the quantum gravitational functional integral Z of Eq. (2.8) reveals that its singular part can only depend of the dimensionless combination $\lambda_0 G^2$, up to an overall factor which cannot affect the nontrivial scaling behavior around the fixed point, since it is analytic in the couplings. This then leaves the question open of which coupling(s) run and which ones do not.

The answer in our opinion is possibly quite simple, and is perhaps best inferred from the nature of the Wilson loop of Eq. (2.14): the appropriate renormalization scheme for quantum gravity is one in which G runs with scale according to the prediction Eq. (2.2), and the scaled cosmological constant λ is kept fixed, as in Eqs. (2.15). Since the scale ξ is related to the observable curvature at large scales, it is an almost inescapable conclusion of these arguments that it must be macroscopic. Furthermore, it is genuinely nonperturbative and nonanalytic in G, and represents the effects of the gravitational vacuum condensate which makes its appearance in the strongly coupled phase $G > G_c$.

Another aspect we have investigated in this paper is the nature of the quantum corrections to the gravitational potential $\phi(r)$ in real space, arising from the scale dependence of Newton's constant G. The running is originally formulated in momentum space [see Eq. (2.2)], since it originates in the momentum dependence of G as it arises on the lattice, or in the equivalent renormalization group equations for G [26,27]. The solution $\phi(r)$ to the nonrelativistic Poisson equation for a point source is given in Eq. (3.21) of Sec. III for various values of the exponent ν. The solution is obtained by first computing the effective vacuum polarization density $\rho_m(r)$ of Eq. (3.17), and then using it as a source term in Poisson's equation. Already in the nonrelativistic case, the value $\nu = 1/3$ appears to stand out, since it leads to logarithmic corrections at short distances $r \ll \xi$.

A relativistic generalization of the previous results was worked out in Secs. IV and V. First it was shown that the scale dependence of G can be consistently embedded in a relativistic covariant framework using the d'Alembertian $\square$ operator, leading to a set of nonlocal effective field equations, Eq. (4.8). The consequences can then be worked out in some detail for the static isotropic metric (Sec. IV), at least in a regime where $2MG \ll r \ll \xi$, and under the assumption of a power law correction (otherwise the problem becomes close to intractable). One then finds that the structure of the leading quantum correction severely restricts the possible values for the exponent ν, in the sense that no consistent solution to the effective nonlocal field equations, incorporating the running of G, can be found unless ν^{-1} is an integer.

A somewhat different approach to the solution of the static isotropic metric was then discussed in Sec. V, in terms of the effective vacuum density of Eq. (3.17), and a vacuum pressure chosen so as to satisfy a covariant energy conservation for the vacuum polarization contribution. The main result is the derivation from the relativistic field equations of an expression for the metric coefficients $A(r)$ and $B(r)$, given in Eqs. (5.35) and (5.38). For $\nu = 1/3$ it implies for the running of G in the region $2MG \ll r \ll \xi$ the result of Eq. (5.39)

$$G(r) = G\left(1 + \frac{a_0}{3\pi} m^3 r^3 \ln\frac{1}{m^2 r^2} + \ldots\right), \tag{8.1}$$

indicating therefore a gradual, very slow increase in G from the "laboratory" value $G \equiv G(r = 0)$. For the actual values of the parameters appearing in the above expression one expects that m is related to the curvature on the largest scales, that $m^{-1} = \xi \sim 10^{28} cm$, and that $a_0 \sim O(10)$. From the nature of the solution for $A(r)$ and $B(r)$ one finds again that unless the exponent ν is close to $1/3$, a consistent solution of the field equations cannot be found. Note that for very large $r \gg \xi$ the growth in $G(r)$ saturates and the value $G_\infty = (1 + a_0)G$ is obtained, in accordance with the original formula of Eq. (2.2) for $k^2 \simeq 0$. A natural comparison is with the QED result of Eq. (5.40).

Even for general exponent ν the factor multiplying the static Newtonian potential $-MG/r$ is of the form $1 + a_0 c_s (mr)^{1/\nu}$, and therefore quite small unless one considers exceedingly large distances $r \sim 1/m$. So, for example, on solar system scales the question is whether the a_0 term can make an observable correction to classical relativistic effects, which are already very small (e.g. the bending of light by the sun is 1.75 seconds of arc). As an example, for $a_0 \sim 42$, $\nu \sim 1/3$ and $m^{-1} \sim 10^{28}$ cm, if one takes r to be comparable to the radius of the solar system ($\sim 10^{15}$ cm) then $(mr)^{1/\nu}$ is about 10^{-13} to a power of say 3. Since this term is very small compared with one, it is not likely to lead to observable effects on solar system scales in the near future.

At the end of the paper we have added some remarks on the solution of the gravitational wave equation with a running G. We find that a running Newton's constant will slightly distort the gravitational wave spectrum at very long wavelengths (Sec. VI), according to Eq. (6.9). Regarding the problem of finding solutions of the effective nonlocal field equations in a cosmological context [12], wherein quantum corrections to the Robertson-Walker metric and the basic Friedmann equations [Eqs. (7.4) and (7.5)] are worked out, we have discussed some of the simplest and more plausible scenarios for the growth (or lack thereof) of the coupling at very large distances, past the de Sitter horizon.

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