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Constraints on Gravitational Scaling Dimensions from Non-Local Effective Field Equations

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ABSTRACT

Quantum corrections to the classical field equations, induced by a scale dependent gravitational constant, are analyzed in the case of the static isotropic metric. The requirement of general covariance for the resulting non-local effective field equations puts severe restrictions on the nature of the solutions that can be obtained. In general the existence of vacuum solutions to the effective field equations restricts the value of the gravitational scaling exponent ν^{-1} to be a positive integer greater than one. We give further arguments suggesting that in fact only for $\nu^{-1} = 3$ consistent solutions seem to exist in four dimensions.

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Over the last few years evidence has been increasing to suggest that quantum gravitation, even though plagued by uncontrollable divergences in standard weak coupling perturbation theory [1], might actually make sense, and lead to testable predictions at the non-perturbative level. These new results in general arise from the non-trivial scaling properties of the gravitational coupling constants in the vicinity of a non-trivial ultraviolet fixed point in four dimensions. As is often the case in physics, the best arguments do not come from often incomplete and partial results in a single model, but more appropriately from the level of consistency that various, often quite unrelated, field theoretic approaches provide.

The main aspect we wish to investigate in this paper is the nature of the specific predictions about the running of Newton's constant G, as they apply to the standard static isotropic metric. Our starting point will be the solution of the non-relativistic Poisson equation, which for a localized point source can be investigated for various values of the gravitational scaling exponent ν. But a more appropriate setting will be a relativistic, generally covariant framework, wherein the effects of the leading quantum correction can be studied systematically, and for which we will show that the existence of vacuum solutions severely restricts the possible values for the exponent ν. Specifically, we will show that no consistent solution to the effective non-local field equations can be found unless ν^{-1} is an integer greater than one. To check the overall consistency of the results, a different approach to the solution of the covariant effective field equations for the static isotropic metric will be pursued, in terms of an effective vacuum density and pressure. In this case one finds that unless the exponent ν is equal to $1/3$, a consistent solution cannot be obtained.

The starting point for our discussion is the form of the running gravitational coupling in the vicinity of the ultraviolet fixed point at G_c, as obtained from the lattice theory of gravity, and given in [3]

$$G(k^2) \;=\; G_c \left[1 \,+\, a_0 \left(\frac{m^2}{k^2} \right)^{\frac{1}{2\nu}} \,+\, O(\, (m^2/k^2)^{\frac{1}{\nu}}) \right] \tag{1}$$

with $m = 1/\xi$, $a_0 > 0$ and $\nu \simeq 1/3$. Usually the quantity G_c in the above expression is identified with the laboratory scale value, $\sqrt{G_c} \sim \sqrt{G_{phys}} \sim 1.6 \times 10^{-33} cm$, the reason being that the scale ξ can be very large, roughly of the same order as the scaled cosmological constant λ. Quantum corrections on the r.h.s are therefore quite small as long as $k^2 \gg m^2$, which in real space corresponds to the "short distance" regime $r \ll \xi$. [3] For more details the reader is referred to the recent papers [3-5], and further references therein.

[3]The result of Eq. (1) is in fact quite similar what one finds for gravity in $2+\epsilon$ dimensions [6, 7, 8], if one allows for a different value of exponent ν as one transitions from two to four dimensions, $G(k^2) \simeq G_c \left(1 + \left(m^2/k^2 \right)^{(d-2)/2} + \dots \right)$. See also the recent results discussed in [9, 10].

For $k^2 \to 0$ the quantum correction proportional to a_0 diverges, and the infrared divergence needs to be regulated. A natural infrared regulator exists in the form of $m = 1/\xi$, and therefore a properly infrared regulated version of the previous expression is

$$G(k^2) \;\simeq\; G_c \left[1 \,+\, a_0 \left(\frac{m^2}{k^2 \,+\, m^2} \right)^{\frac{1}{2\nu}} \,+\, \dots \right] \tag{2}$$

with $m = 1/\xi$ the (tiny) infrared cutoff. Thus the gravitational coupling approaches the finite value $G_\infty = (1 + a_0 + \dots)\, G_c$, independent of $m = 1/\xi$, at very large distances $r \gg \xi$. In this work we will be concerned with the static limit, where the non-relativistic Newtonian potential can be defined as

$$\phi(r) \;=\; (-M) \int \frac{d^3\mathbf{k}}{(2\pi)^3} \, e^{i\,\mathbf{k}\cdot\mathbf{x}} \, G(\mathbf{k^2}) \, \frac{4\pi}{\mathbf{k^2}} \tag{3}$$

The static potential $\phi(r)$ can be obtained from Eq. (2) directly by Fourier transform, or equivalently from the solution of Poisson's equation with a point source at the origin. In the limit of weak fields, the relativistic field equations give for the ϕ field (with $g_{00}(x) \simeq -(1 + 2\phi(x)\,)$

$$\Delta \, \phi(x) \;=\; 4\,\pi \, G \, \rho(x) \tag{4}$$

and for a point source at the origin the first term on the r.h.s is just $4\pi M G \, \delta^{(3)}(\mathbf{x})$. The solution for $\phi(r)$, obtained by Fourier transforming back to real space Eq. (3), gives in the large r limit

$$\phi(r) \;\underset{r \to \infty}{\sim}\; - \frac{M\,G}{r} \, \left[1 \,+\, a_0 \left(1 \,-\, c_l \, (m\,r)^{\frac{1}{2\nu} - 1} \, e^{-mr} \right) \right] \tag{5}$$

with $c_l = 1/(\nu \, 2^{\frac{1}{2\nu}} \, \Gamma(\frac{1}{2\nu}))$. The part in $G(k^2)$ proportional to a_0 can equivalently be represented as a source term ρ_m in Poisson's equation, the latter determined from the inverse Fourier transform of the correction term in Eq. (2),

$$a_0 \, M \left(\frac{m^2}{\mathbf{k^2} \,+\, m^2} \right)^{\frac{1}{2\nu}} \tag{6}$$

One finds

$$\rho_m(r) \;=\; \frac{1}{8\pi} \, c_\nu \, a_0 \, M \, m^3 \, (m\,r)^{-\frac{1}{2}(3 - \frac{1}{\nu})} \, K_{\frac{1}{2}(3 - \frac{1}{\nu})}(m\,r) \tag{7}$$

with $c_\nu \;\equiv\; 2^{\frac{1}{2}(5 - \frac{1}{\nu})} / \sqrt{\pi} \, \Gamma(\frac{1}{2\,\nu})$. Note that the vacuum polarization density $\rho_m(r)$ has the normalization property

$$4\,\pi \int_0^\infty r^2 \, dr \, \rho_m(r) \;=\; a_0 \, M \tag{8}$$

and that $\rho_m(r)$ diverges at small r for $\nu \geq 1/3$. In the small r limit and for general $\nu > \frac{1}{3}$, one then finds from Poisson's equation, using the expansion of the modified Bessel function $K_n(x)$ for small arguments,

$$\phi(r) \;\underset{r \to 0}{\sim}\; - \frac{M\,G}{r} \,+\, a_0 \, M \, G \, c_s \, m^{\frac{1}{\nu}} \, r^{\frac{1}{\nu} - 1} \,+\, \dots \tag{9}$$

with $c_s = \nu | \sec\left(\frac{\pi}{2\nu}\right) | / \Gamma(\frac{1}{\nu})$.

Solutions to Poisson's equation with a running G provide some useful insights into the structure of quantum corrections, but a complete analysis requires a study of the full relativistic field equations, which will be discussed next. A set of effective field equations incorporating the running of G is obtained from the replacement [2]

$$G \; \rightarrow \; G(\Box) \; = \; G \left[1 \, + \, a_0 \left(\frac{m^2}{\Box} \right)^{\frac{1}{2\nu}} \, + \, \dots \right] \; \equiv \; G \left(1 \, + \, A(\Box) \right) \tag{10}$$

with the d'Alembertian $\Box$ expressing the running of G as in either Eqs. (1) or (2). The non-local effective field equations then read

$$R_{\mu\nu} \, - \, \tfrac{1}{2} \, g_{\mu\nu} \, R \, + \, \lambda \, g_{\mu\nu} \; = \; 8 \, \pi \, G \; (1 + A(\Box)) \; T_{\mu\nu} \tag{11}$$

with $A(\Box)$ given by Eq. (10), and $\lambda \simeq 1/\xi^2$. The use of the d'Alembertian $\Box$ to describe the running of couplings in gauge theories and quantum gravity was discussed in some detail, for example, in [11]. The corresponding trace equation is

$$R \, - \, 4 \, \lambda \; = \; - \, 8 \, \pi \, G \; (1 + A(\Box)) \; T \tag{12}$$

Being manifestly covariant, these expressions at least satisfy some of the requirements for a set of consistent field equations incorporating the running of G. The d'Alembertian $\Box$ operator is defined here through the appropriate combination of covariant derivatives

$$\Box \; \equiv \; g^{\mu\nu} \, \nabla_\mu \nabla_\nu \tag{13}$$

and its explicit form depends on the specific tensor nature of the object it is acting on. In general the operator $A(\Box)$ has to be defined by a suitable analytic continuation from positive integer powers, which is usually done by computing $\Box^n$ for positive integer n, and then analytically continuing to $n \rightarrow -1/2\nu$. Let us set for now the cosmological constant $\lambda = 0$, since its contribution can always be added at a later stage. As long as one is interested in static isotropic solutions, one can take for the metric the most general form

$$ds^2 \; = \; - \, B(r) \, dt^2 \, + \, A(r) \, dr^2 \, + \, r^2 \, (d\theta^2 \, + \, \sin^2 \theta \, d\varphi^2) \tag{14}$$

For the energy momentum tensor we will take the perfect fluid form

$$T_{\mu\nu} \; = \; \mathrm{diag} \, [\, B(r) \, \rho(r), \; A(r) \, p(r), \; r^2 \, p(r), \; r^2 \, \sin^2 \theta \, p(r) \,] \tag{15}$$

and a point source as the origin is simply represented as

$$T_{\mu\nu}(r) \;=\; \mathrm{diag}[\,B(r)\,\rho(r),\,0,\,0,\,0\,] \tag{16}$$

with the source proportional to a $3-d$ delta function.

Consider first the trace equation

$$R \;=\; -8\,\pi\,G(\Box)\,T \;\equiv\; -\,8\,\pi\,G\;(1\,+\,A(\Box))\;T \;=\; +8\,\pi\,G\;(1\,+\,A(\Box))\;\rho \tag{17}$$

where we have used the fact that the point source at the origin is described just by the density term. One then computes the repeated action of the invariant d'Alembertian on T,

$$\Box\,(-8\pi GT) \;=\; \Box(+8\pi G\rho) \;=\; \frac{16\,G\,\pi\,\rho'}{r\,A} - \frac{4\,G\,\pi\,A'\,\rho'}{A^2} + \frac{4\,G\,\pi\,B'\,\rho'}{A\,B} + \frac{8\,G\,\pi\,\rho''}{A} \tag{18}$$

In view of the rapidly escalating complexity of the problem, it seems sensible to expand around the Schwarzschild solution, and set

$$A(r)^{-1} \;=\; 1\,-\,\frac{2\,M\,G}{r}\,+\,\frac{\sigma(r)}{r} \qquad\quad B(r) \;=\; 1\,-\,\frac{2\,M\,G}{r}\,+\,\frac{\theta(r)}{r} \tag{19}$$

where the correction to the standard solution are parametrized here by the two functions $\sigma(r)$ and $\theta(r)$, both assumed to be "small", i.e. proportional to a_0 as in Eq. (10), with a_0 considered a small parameter. To simplify the problem even further, we will assume that for $2MG \ll r \ll \xi$ (the "physical" regime) one can set

$$\sigma(r) \;=\; -\,a_0\,M\,G\,c_\sigma\,r^\alpha \qquad\quad \theta(r) \;=\; -\,a_0\,M\,G\,c_\theta\,r^\beta \tag{20}$$

This assumption is in part justified by the form of the non-relativistic correction of Eqs. (9). Then for $\alpha = \beta$ (the equations seem impossible to satisfy if α and β are different) one obtains for the scalar curvature

$$R \;=\; 0\,+\,\alpha\;(2\,c_\sigma + (\alpha-1)\,c_\theta)\;a_0\,M\,G\,r^{\alpha-3}\;+\;+O(a_0^2) \tag{21}$$

A first result can be obtained in the following way. Since in the ordinary Einstein case one has for a perfect fluid $R = -8\pi GT = +8\pi G(\rho - 3p)$, and since $\rho_m(r) \sim r^{\frac{1}{\nu}-3}$ from Eq. (7) in the same regime, one concludes that a solution is given by

$$\alpha \;=\; \frac{1}{\nu} \tag{22}$$

which is also consistent with the Poisson equation result of Eq. (9).

The next step up would be the consideration of the action of $\Box$ on the point source, as it appears in the full effective field equations of Eq. (11), with again $T_{\mu\nu}$ described by Eq. (16). One perhaps surprising fact is the generation of an effective pressure term by the action of $\Box$, suggesting that both terms should arise in the correct description of vacuum polarization effects,

$$\begin{aligned} (\Box\, T_{\mu\nu})_{tt} &= -\frac{\rho\, B'^2}{2\, A\, B} + \frac{2\, B\, \rho'}{r\, A} - \frac{B\, A'\, \rho'}{2\, A^2} + \frac{B'\, \rho'}{2\, A} + \frac{B\, \rho''}{A} \\ (\Box\, T_{\mu\nu})_{rr} &= -\frac{\rho\, B'^2}{2\, B^2} \end{aligned} \tag{23}$$

and $(\Box\, T_{\mu\nu})_{\theta\theta} = (\Box\, T_{\mu\nu})_{\varphi\varphi} = 0$. A similar effect, namely the generation of an effective vacuum pressure term in the field equations by the action of $\Box$, was seen already in the case of the Robertson-Walker [2].

To check the overall consistency of the approach, consider next the set of effective field equations that are obtained when the operator $(1 + A(\Box))$ appearing in Eqs. (11) and (12) is moved over to the gravitational side. Since the r.h.s of the field equations then vanishes for $r \neq 0$, one has apparently reduced the problem to one of finding vacuum solutions of a modified, non-local field equation. Let us first look at the relatively simple trace equation. If we denote by δR the lowest order variation (that is, of order a_0) in the scalar curvature over the ordinary vacuum solution $R = 0$, then one has

$$\frac{1}{8\pi G\, A(\Box)}\, \delta R \;=\; 0 \tag{24}$$

On a generic scalar function $F(r)$ one has the following action of the covariant d'Alembertian $\Box$:

$$\Box\, F(r) \;=\; -\frac{A'F'}{2A^2} + \frac{B'F'}{2AB} + \frac{2F'}{rA} + \frac{F''}{A} \tag{25}$$

Assuming a power law correction, as in Eq. (20), with $\alpha = \beta$, as in Eq. (21), one then finds

$$\begin{aligned} \Box\;\; R \;&\rightarrow\; a_0\, M\, G\, r^{\alpha-5}\; (\, 2\, c_\sigma + c_\theta\, (\alpha - 1)\,)\; \alpha\, (\alpha - 2)\, (\alpha - 3) \\ \Box^2\, R \;&\rightarrow\; a_0\, M\, G\, r^{\alpha-7}\; (\, 2\, c_\sigma + c_\theta\, (\alpha - 1)\,)\; \alpha\, (\alpha - 2)\, (\alpha - 3)\, (\alpha - 4)\, (\alpha - 5) \end{aligned} \tag{26}$$

and so on, and for general $n \rightarrow +\frac{1}{2\nu}$

$$\Box^n\, R \;\rightarrow\; a_0\, M\, G\; (\, 2\, c_\sigma + c_\theta\, (\alpha - 1)\,)\; \frac{\alpha\, \Gamma(2 + \frac{1}{\nu} - \alpha)}{\Gamma(2 - \alpha)}\; r^{\alpha - \frac{1}{\nu} - 3} \tag{27}$$

Therefore the only possible power solution for $r \gg MG$ is $\alpha = 0, 2 \ldots \frac{1}{\nu} + 1$, with c_σ and c_θ unconstrained to this order.

Next we examine the full effective field equations (as opposed to just their trace part) as in Eq. (11) with $\lambda = 0$, If one denotes by $\delta G_{\mu\nu} \equiv \delta \left(R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R \right)$ the lowest order variation (that is, of order a_0) in the Einstein tensor over the ordinary vacuum solution $G_{\mu\nu} = 0$, then one has

$$\frac{1}{8\pi G\, A(\Box)}\; \delta \left(R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}\, R \right) \;=\; 0 \tag{28}$$

again for $r \neq 0$. Here the covariant d'Alembertian operator $\Box$ acts on a second rank tensor and would thus seem to require the calculation of as many as 1920 terms, of which many fortunately vanish by symmetry. In the static isotropic case the components of the Einstein tensor are given by

$$\begin{aligned}
G_{tt} &= \frac{A'B}{rA^2} - \frac{B}{r^2A} + \frac{B}{r^2} \\
G_{rr} &= -\frac{A}{r^2} + \frac{B'}{rB} + \frac{1}{r^2} \\
G_{\theta\theta} &= -\frac{B'^2r^2}{4AB^2} - \frac{A'B'r^2}{4A^2B} + \frac{B''r^2}{2AB} - \frac{A'r}{2A^2} + \frac{B'r}{2AB} \\
G_{\varphi\varphi} &= \sin^2\theta\, G_{\theta\theta}
\end{aligned} \tag{29}$$

After acting with $\Box$ on this expression one finds a rather complicated result. Here we will list only $(\Box G)_{tt}$:

$$\begin{aligned}
&\frac{6BA'^3}{rA^5} + \frac{2BA'^2}{r^2A^4} - \frac{4B'A'^2}{rA^4} - \frac{2BA'}{r^3A^3} - \frac{6BA''A'}{rA^4} + \frac{B''A'}{rA^3} + \frac{6B}{r^4A} \\
&-\frac{6B}{r^4A^2} - \frac{4B'}{r^3A} + \frac{4B'}{r^3A^2} - \frac{BA''}{r^2A^3} + \frac{2B'A''}{rA^3} + \frac{B''}{r^2A} - \frac{B''}{r^2A^2} + \frac{BA^{(3)}}{rA^3}
\end{aligned} \tag{30}$$

If one again assumes that the corrections are given by a power, as in Eq. (20), with $\alpha = \beta$, then one has to lowest order

$$\begin{aligned}
G_{tt} &= a_0\, M\, G\, c_\sigma\, \alpha\, r^{\alpha-3} \\
G_{rr} &= -\, a_0\, M\, G\, (c_\sigma + c_\theta(\alpha-1))\, r^{\alpha-3} \\
G_{\theta\theta} &= -\frac{1}{2}\, a_0\, M\, G\, (c_\sigma + c_\theta(\alpha-1))\, (\alpha-1)\, r^{\alpha-1}
\end{aligned} \tag{31}$$

with the $\varphi\varphi$ component again proportional to the $\theta\theta$ component. Applying $\Box$ on the above Einstein tensor one then gets

$$\begin{aligned}
(\Box\, G)_{tt} &= a_0\, G\, M\, c_\sigma\, \alpha(\alpha-2)\,(\alpha-3)\, r^{\alpha-5} \\
(\Box\, G)_{rr} &= -\, a_0\, G\, M\, \left(c_\sigma + c_\theta\,(\alpha-1)\right)\, \alpha\,(\alpha-3)\, r^{\alpha-5} \\
(\Box\, G)_{\theta\theta} &= -\frac{1}{2}\, a_0\, G\, M\, \left(c_\sigma + c_\theta\,(\alpha-1)\right)\, \alpha\,(\alpha-3)^2\, r^{\alpha-3}
\end{aligned} \tag{32}$$

(with the $\varphi\varphi$ component proportional to the $\theta\theta$ component), and so on. One then has for general $n \to +\frac{1}{2\nu}$

$$\begin{aligned}
(\Box^n\, G)_{tt} &\to a_0\, G\, M\, c_\sigma\, \frac{\Gamma(2+\frac{1}{\nu}-\alpha)}{(\alpha-1)\,\Gamma(-\alpha)}\, r^{\alpha-3-\frac{1}{\nu}} \\
(\Box^n\, G)_{rr} &\to -a_0\, G\, M\, \left(c_\sigma + c_\theta\,(\alpha-1)\right)\, \frac{\Gamma(2+\frac{1}{\nu}-\alpha)}{(\alpha-1)\left(\alpha-\frac{1}{\nu}\right)\Gamma(-\alpha)}\, r^{\alpha-3-\frac{1}{\nu}}
\end{aligned}$$

$$(\Box^n G)_{\theta\theta} \quad \rightarrow \quad -\frac{1}{2} a_0 \, G \, M \, \left(c_\sigma + c_\theta \, (\alpha - 1) \right) \frac{\left(\alpha - 1 - \frac{1}{\nu}\right) \Gamma(2 + \frac{1}{\nu} - \alpha)}{(\alpha - 1) \left(\alpha - \frac{1}{\nu}\right) \Gamma(-\alpha)} \, r^{\alpha - 1 - \frac{1}{\nu}} \tag{33}$$

Inspection of the above results reveals a common factor $1/\Gamma(-\alpha)$, which would allow only integer powers $\alpha = 0, 1, 2 \ldots$, but the additional factor of $1/(\alpha - 1)$ excludes $\alpha = 1$ from being a solution. Even for α close to $1/\nu$ (as expected on the basis of the non-relativistic expression of Eq. (9), as well as from Eq. (22)) $\nu \sim 1/\alpha - \epsilon$ only integer values $\alpha = 2, 3, 4 \ldots$ are allowed. In general the problem of finding a complete general solution to the effective field equations by this method lies in the difficulty of computing arbitrarily high powers of $\Box$ on general functions such as $\sigma(r)$ and $\theta(r)$, which eventually involve a large number of derivatives. Assuming for these functions a power law dependence on r simplifies the problem considerably, but also restricts the kind of solutions that one is likely to find. More specifically, if the solution involves (say for small r, but still with $r \gg 2MG$) a term of the type $r^\alpha \ln mr$, as in Eqs. (9), (48) and (51) for $\nu \rightarrow 1/3$, then this method will have to be dealt with very carefully. This is presumably the reason why in some of the Γ-function coefficients encountered here one finds a power solution (in fact $\alpha = 3$) for ν close to a third, but one gets indeterminate expression if one sets exactly $\alpha = 1/\nu = 3$.

The earlier discussion of the non-relativistic case suggests that the quantum correction due to the running of G can be approximately described by Poisson's equation, with a source term related to a vacuum energy density $\rho_m(r)$, distributed around the static source of strength M in accordance with the result of Eqs. (7) and Eq. (8). These expressions, in turn, were obtained by Fourier transforming back to real space the original result for $G(k^2)$ of Eq. (2). Furthermore, in the preceding discussion of the relativistic case it was found (as in [2] for the Robertson-Walker metric case) that a manifestly covariant implementation of the running of G, via the $G(\Box)$ given in Eq. (10), will induce a non-vanishing effective pressure term in the field equations. This result can be seen clearly, in the case of the static isotropic metric, for example from the result of Eq. (23). We will therefore now consider a relativistic perfect fluid, with energy-momentum tensor, which in the static isotropic case reduces to Eq. (15). The tt, rr and $\theta\theta$ components of the field equations then read

$$\begin{aligned}
-\lambda B + \frac{A'B}{rA^2} - \frac{B}{r^2 A} + \frac{B}{r^2} &= 8\pi G \, B \, \rho \\
\lambda A - \frac{A}{r^2} + \frac{B'}{rB} + \frac{1}{r^2} &= 8\pi G \, A \, p \\
\lambda r^2 - \frac{B'^2 r^2}{4AB^2} - \frac{A'B'r^2}{4A^2 B} + \frac{B'' r^2}{2AB} - \frac{A' r}{2A^2} + \frac{B' r}{2AB} &= 8\pi G \, r^2 \, p
\end{aligned} \tag{34}$$

with the $\varphi\varphi$ component equal to $\sin^2\theta$ times the $\theta\theta$ component. Energy conservation $\nabla^\mu T_{\mu\nu} = 0$

implies

$$(p+\rho)\,\frac{B'}{2B}+p' \;=\; 0 \tag{35}$$

and forces a definite relationship between $B(r)$, $\rho(r)$ and $p(r)$. The three field equations and the energy conservation equation are, as usual, not independent, because of the Bianchi identity.

It seems reasonable to attempt to solve the above equations (usually considered in the context of relativistic stellar structure) with the density given by the $\rho_m(r)$ of Eqs. (7). This of course raises the question of how the relativistic pressure $p(r)$ should be chosen, an issue that the non-relativistic calculation did not have to address. We will argue below that covariant energy conservation completely determines the pressure in the static case, leading to consistent equations and solutions (note that in particular it would not be consistent to take $p(r)=0$). Since the function $B(r)$ drops out of the tt field equation, the latter can be integrated immediately, giving

$$A(r)^{-1} \;=\; 1+\frac{c_1}{r}-\frac{\lambda}{3}\,r^2-\frac{8\pi G}{r}\int_0^r dx\,x^2\,\rho(x) \tag{36}$$

It also seems natural here to identify $c_1 = -2MG$, which of course corresponds to the correct solution for $a_0=0$ ($p=\rho=0$). Next, the rr field equation can be solved for $B(r)$,

$$B(r) \;=\; \exp\left\{c_2-\int_{r_0}^{r} dy\,\frac{1+A(y)\,(\lambda\,y^2-8\pi G\,y^2\,p(y)-1)}{y}\right\} \tag{37}$$

with the constant c_2 again determined by the requirement that the above expression for $B(r)$ reduce to the standard Schwarzschild solution for $a_0=0$ ($p=\rho=0$), giving $c_2=\ln(1-2MG/r_0-\lambda r_0^2/3)$. The last task left therefore is the determination of the pressure $p(r)$. Using the rr field equation, $B'(r)/B(r)$ can be expressed in term of $A(r)$ in the energy conservation equation. Inserting then the explicit expression for $A(r)$, from Eq. (36), one obtains

$$p'(r)+\frac{\left(8\pi G\,r^3\,p(r)\;+\;2MG\;-\;\frac{2}{3}\lambda r^3\;+\;8\pi G\,\int_{r_0}^{r} dx\,x^2\rho(x)\right)\,(p(r)+\rho(r))}{2\,r\left(r\;-\;2MG\;-\;\frac{\lambda}{3}\,r^3-8\pi G\,\int_0^r dx\,x^2\,\rho(x)\right)} \;=\; 0 \tag{38}$$

which is usually referred to as the equation of hydrostatic equilibrium. From now on we will focus only the case $\lambda=0$. The last equation, a non-linear differential equation for $p(r)$, can be solved to give the desired solution $p(r)$, which then, by equation Eq. (37), determines the remaining function $B(r)$. In our case though it will be sufficient to solve the above equation for small a_0, where a_0 (see Eqs. (2) and (7)) is the dimensionless parameter which, when set to zero, makes the solution revert back to the classical one. It will also be convenient to pull out of $A(r)$ and $B(r)$ the Schwarzschild solution part, by introducing the small corrections $\sigma(r)$ and $\theta(r)$, as defined in Eq. (19), both of which are expected to be proportional to the parameter a_0. One then has

$$\theta(r) \;=\; \exp\left\{c_2+\int_{r_0}^{r} dy\,\frac{1+8\pi G\,y^2\,p(y)}{y-2MG-8\pi G\int_0^y dx\,x^2\,\rho(x)}\right\}+2MG\;-\;r \tag{39}$$

Again, the integration constant c_2 needs to be chosen here so that the normal Schwarzschild solution is recovered for $p = \rho = 0$. To order a_0 the resulting equation for $p(r)$, from Eq. (38), is

$$\frac{MG\,(p(r)\,+\,\rho(r))}{r\,(r\,-\,2MG)}\,+\,p'(r)\;\simeq\;0 \tag{40}$$

Note that in regions where $p(r)$ is slowly varying, $p'(r) \simeq 0$, one has $p \simeq -\rho$, i.e. the fluid contribution is acting like a cosmological constant term with $\sigma(r) \sim \theta(r) \sim -(\rho/3)\,r^3$. The last differential equation can then be solved for $p(r)$,

$$p_m(r)\;=\;\frac{1}{\sqrt{1-\frac{2MG}{r}}}\left(c_3-\int_{r_0}^{r}dz\,\frac{MG\rho(z)}{z^2\sqrt{1-\frac{2MG}{z}}}\right) \tag{41}$$

where the constant of integration has to be chosen so that when $\rho(r) = 0$ (no quantum correction) one has $p(r) = 0$ as well. Because of the singularity in the integrand at $r = 2MG$, we will take the lower limit in the integral to be $r_0 = 2MG + \epsilon$, with $\epsilon \to 0$. To proceed further, one needs the explicit form for $\rho_m(r)$, which was given in Eqs. (7) and (8). The required integrands involve for general ν the modified Bessel function $K_n(x)$, and can be therefore a bit complicated. Here we will limit our investigation to the small r ($mr \ll 1$) and large r ($mr \gg 1$) behavior. Since $m = 1/\xi$ is very small, the first limit appears to be of greater physical interest. For small r the density $\rho_m(r)$ has the following behavior (see Eq. (7)),

$$\rho_m(r)\;\underset{r\to 0}{\sim}\;A_0\,r^{\frac{1}{\nu}-3} \tag{42}$$

for $\nu > 1/3$, with the constant

$$A_0\;=\;\frac{|\sec\left(\frac{\pi}{2\nu}\right)|}{4\pi\,\Gamma\left(\frac{1}{\nu}-1\right)}\,a_0\,M\,m^{\frac{1}{\nu}} \tag{43}$$

determined from the small x behavior of the modified Bessel function $K_n(x)$. For $\nu < 1/3$ $\rho_m(r) \sim$ const. $a_0\,M\,m^3$, independent of r. For $\nu = 1/3$ the expression for $\rho_m(r)$ is given later in Eq. (46). Therefore in this limit, with $\frac{1}{3} < \nu < 1$, one has

$$A^{-1}(r)\;=\;1\,-\,\frac{2\,M\,G}{r}\,-\,2\,a_0\,M\,G\,c_s\,m^{\frac{1}{\nu}}\,r^{\frac{1}{\nu}-1}\,+\,\ldots \tag{44}$$

with the constant $c_s = \nu|\sec\left(\frac{\pi}{2\nu}\right)|/\Gamma(\frac{1}{\nu}-1)$. For $\nu = 1/3$ the last contribution is indistinguishable from a cosmological constant term $-\frac{\lambda}{3}r^2$, except for the fact that the coefficient here is quite different, being proportional to $\sim a_0\,M\,G\,m^3$. To determine the pressure, we suppose that it as well has a power dependence on r in the regime under consideration, $p_m(r) = c_p\,A_0\,r^{\gamma}$, where c_p is a numerical constant, and then substitute $p_m(r)$ into the pressure equation Eq. (40). This gives, past the horizon $r \gg 2MG$,

$$(2\gamma - 1)\, c_p\, M\, G\, r^{\gamma-1} - c_p\, \gamma\, r^{\gamma} - M\, G\, r^{1/\nu-4} \simeq 0 \tag{45}$$

giving the same power $\gamma = 1/\nu - 3$ as for $\rho(r)$, $c_p = -1$ and surprisingly also $\gamma = 0$, implying that in this regime only $\nu = 1/3$ gives a consistent solution. Again, the resulting correction is quite similar to what one would expect from a cosmological term, with an effective $\lambda_m/3 \simeq 8\pi\, \nu\, a_0\, M\, G\, m^{\frac{1}{\nu}}$.

The case $\nu = 1/3$ requires special treatment, and one needs to go back to the expression for $\rho_m(r)$ for $\nu = 1/3$,

$$\rho_m(r) = \frac{1}{2\pi^2}\, a_0\, M\, m^3\, K_0(m\, r) \tag{46}$$

For small r one then has

$$\rho_m(r) = -\frac{a_0}{2\pi^2}\, M\, m^3 \left(\ln \frac{m\, r}{2} + \gamma \right) + \dots \tag{47}$$

and consequently from Eq. (36),

$$A^{-1}(r) = 1 - \frac{2\, M\, G}{r} + \frac{4\, a_0\, M\, G\, m^3}{3\, \pi}\, r^2\, \ln(m\, r) + \dots \tag{48}$$

From Eq. (40) one then obtains an expression for the pressure $p_m(r)$, and one finds

$$p_m(r) = \frac{a_0 M m^3 \log(mr)}{2\pi^2} - \frac{a_0 M m^3 \log\left(r + r\sqrt{1 - \frac{2MG}{r}} - MG \right)}{2\pi^2 \sqrt{1 - \frac{2MG}{r}}} + \frac{a_0 M m^3}{\pi^2} + \frac{a_0 M m^3 c_3}{2\pi^2 \sqrt{1 - \frac{2MG}{r}}} \tag{49}$$

where c_3 is again an integration constant. Here we will be content with the $r \gg 2MG$ limit of the above expression, which we shall write therefore as

$$p_m(r) = \frac{a_0}{2\pi^2}\, M\, m^3\, \ln(m\, r) + \dots \tag{50}$$

After performing the required r integral in Eq. (39), and evaluating the resulting expression in the limit $r \gg 2MG$, one obtains an expression for $\theta(r)$, and from it

$$B(r) = 1 - \frac{2\, M\, G}{r} + \frac{4\, a_0\, M\, G\, m^3}{3\, \pi}\, r^2\, \ln(m\, r) + \dots \tag{51}$$

The expressions for $A(r)$ and $B(r)$ are, for $r \gg 2\, M\, G$, consistent with a gradual slow increase in G in accordance with the formula

$$G \to G(r) = G \left(1 + \frac{a_0}{3\, \pi}\, m^3\, r^3\, \ln \frac{1}{m^2\, r^2} + \dots \right) \tag{52}$$

and therefore consistent as well with the original result of Eqs. (1) or (2), namely that the classical laboratory value of G is obtained for $r \ll \xi$. In fact it is quite reassuring that the renormalization properties of $G(r)$ as inferred from $A(r)$ are the same as what one finds from $B(r)$. For large r one has instead, from Eq. (7) for $\rho_m(r)$,

$$\rho_m(r) \underset{r \to \infty}{\sim} A_0 \, r^{\frac{1}{2\nu} - 2} \, e^{-m r} \tag{53}$$

with $A_0 = 1/\sqrt{128\pi} \, c_\nu \, a_0 \, M \, m^{1+\frac{1}{2\nu}}$. In the same limit, the integration constants is chosen so that the solution for $A(r)$ and $B(r)$ at large r corresponds to a mass $M' = (1 + a_0) M$ (see the expression for the integrated density in Eq. (8)), or equivalently

$$\sigma(r) \;\sim\; \theta(r) \underset{r \to \infty}{\sim} -2 \, a_0 \, M \, G \tag{54}$$

One then recovers a result similar to the non-relativistic expression of Eq. (5), with $G(r)$ approaching the constant value $G_\infty = (1 + a_0) \, G$, up to an exponentially small correction in mr at large r.

In conclusion, it appears that a solution to the relativistic static isotropic problem of the running gravitational constant can be found, provided that the exponent ν in either Eq. (2) or Eq. (11) is close to one third. This last result seems to be linked with the fact that the running coupling term acts in some way like a local cosmological constant term, for which the r dependence of the vacuum solution for small r is fixed by the nature of the Schwarzschild solution with a cosmological constant term. Furthermore, in $d \geq 4$ dimensions the Schwarzschild solution to Einstein gravity with a cosmological term is given by [12]

$$A^{-1}(r) \;=\; B(r) \;=\; 1 - \frac{8 \, M \, G \, \pi \, \Gamma(\frac{d-1}{2})}{(d-2) \, \pi^{\frac{d-1}{2}}} \, r^{3-d} - \frac{2\lambda}{(d-2)(d-1)} \, r^2 \tag{55}$$

which would suggest, in analogy with the results for $d = 4$ given previously, that in $d \geq 4$ dimensions only $\nu = 1/(d-1)$ is possible, if the correction again behaves locally like a cosmological constant term. This last result would also be in agreement with the exact value $\nu = 0$ found at $d = \infty$ [5].

To summarize, the starting point for our discussion of the renormalization group running of G is Eq. (1), valid at short distances $k \gg m$, or its improved infrared regulated version of Eq. (2). While a solution to the non-relativistic Poisson equation can be given for various values of the exponent ν, the scale dependence of G can also be consistently embedded in a relativistic covariant framework using the d'Alembertian $\Box$ operator. This then leads to a set of nonlocal effective field equations, whose consequences can be worked out for the static isotropic metric, at least in a regime where $2MG \ll r \ll \xi$, and under the assumption of a power law correction. We have found that the structure of the leading quantum correction is such that it severely restricts the possible values for the exponent ν, in the sense that no consistent solution to the effective non-local field equations, incorporating the running of G, can be found unless ν^{-1} is an integer. A somewhat different approach to the solution of the static isotropic metric was pursued in terms of an effective vacuum density of Eq. (7), and a vacuum pressure chosen so as to satisfy a covariant energy conservation for the vacuum polarization contribution. The main result there is the derivation, from the relativistic field equations, of an expression for the metric coefficients $A(r)$ and $B(r)$, given for $2MG \ll r \ll \xi$ in Eqs. (48), (51) and (52). From the nature of the solution for $A(r)$ and $B(r)$ one finds again that unless the exponent ν is close to 1/3, a consistent solution to the field equations cannot be found.

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