Simplicial gravity coupled to scalar matter

Herbert W. Hamber [1]

Department of Physics, University of California at Irvine, Irvine, CA 92717, USA

Ruth M. Williams

Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge CB3 9EW, UK

A model for quantized gravity coupled to matter in the form of a single scalar field is investigated in four dimensions. For the metric degrees of freedom we employ Regge's simplicial discretization, with the scalar field defined at the vertices of the four-simplices. We examine how the continuous phase transition found earlier, separating the smooth from the rough phase of quantized gravity, is influenced by the presence of scalar matter. A determination of the critical exponents seems to indicate that the effects of matter are rather small, unless the number of scalar flavors is large. Close to the critical point where the average curvature approaches zero, the coupling of matter to gravity is found to be weak. The nature of the phase diagram and the values for the critical exponents suggest that gravitational interactions increase with distance.

1. Introduction

Any serious attempt at understanding the ground-state properties of quantized gravity has to include at some stage the consideration of the effects of matter fields. While there are many choices for the matter fields and for their interactions, the simplest actions to deal with in the framework of a lattice model for gravity are the ones that represent one (or more) scalar fields. In this paper we will discuss a first attempt at determining those effects.

Regge's model is the natural discretization for quantized gravity [1]. At the classical level, it is completely equivalent to general relativity, and the correspondence is particularly transparent in the lattice weak field expansion, with the invariant edge lengths playing the role of infinitesimal geodesics in the continuum. In the limit of smooth manifolds with small curvatures, the continuous diffeomorphism invariance of the continuum theory is recovered [2,3]. But in contrast to ordinary lattice gauge theories, the model is formulated entirely in terms of

coordinate invariant quantities, the edge lengths, which form the elementary degrees of freedom in the theory [4,5].

Recent work based on Regge's simplicial formulation of gravity has shown, in pure gravity without matter, the appearance in four dimensions of a phase transition in the bare Newton constant, separating a smooth phase with small negative average curvature from a rough phase with large positive curvature [6,7]. While the fractal dimension is rather small in the rough phase, indicating a tree-like geometry for the ground state, it is very close to four in the smooth phase close to the critical point. Furthermore, a calculation of the critical exponents in the smooth phase close to the critical point indicates that the transition is apparently second order with divergent curvature fluctuations, and that a lattice continuum can be constructed.

Very similar results have recently been obtained in the dynamical triangulation model for gravity, in the sense that a similar phase transition was found separating what appear to be the same type of phases [8]. This development represents an alternative and complementary approach to what is being discussed here. However it has not been possible yet in these models to extract the critical exponents, and it is therefore not clear yet whether a continuum limit really exists. In particular it appears that close to the transition, the dynamical triangulation model does not give rise to the correct scaling properties for the curvature, which are necessary to define a lattice continuum limit really exists. In particular it appears that close to the transition, the dynamical triangulation model does not give rise to the correct scaling properties for the curvature, which are necessary to define a lattice continuum limit. It is therefore unclear whether the transition is first order as a consequence of the discreteness of the curvatures, with no continuum limit (as one finds for example in lattice gauge theories based on discrete subgroups of $SU(N)$ [9]). While in two dimensions both lattice models lead to similar results both in the absence and presence of scalar matter [10–12] in three dimensions the dynamical triangulation model has no continuum limit [13], in apparent disagreement with the continuum expectations [14,15], and the simplicial Regge gravity results [3], which suggest instead that a well-defined continuum limit exists (albeit trivial in the absence of matter, with the scalar curvature playing the role of a scalar field). These results are rather disappointing, since it would be desirable to have two rather different, independent discretizations for gravity, with the same lattice continuum limit. It is not clear yet at this point whether these results indicate a fundamental flaw in the model (lack of restoration of broken diffeomorphism invariance), or simply a perhaps surmountable technical difficulty in determining exponents. For a clear recent review of some of these aspects in the dynamically triangulated models we refer the reader to the last reference in [8].

In this paper we will present some first result on the properties of Regge's simplicial gravity coupled to a scalar field, as derived from numerical studies on lattices of up to $24 \times 16^4 = 1\,572\,864$ simplices. The paper is organized as follows.

First we discuss in sect. 2 the simplicial action and measure for the combined gravitational and scalar degrees of freedom. Then we digress in sect. 3 on what is known about the effects of scalar matter fields in the continuum, to the extent that the results will be relevant for out later calculations. We then present in sect. 4 the definition of physical observables which can be measured when scalar fields are present, besides the purely gravitational ones introduced previously, and how these can be related to effective low energy couplings. In sect. 5 we present our results and their interpretation, and in sect. 6 we give a discussion on how other quantities such as the curvature and volume distributions can be obtained close to the critical point. Sect. 7 then contains our conclusions.

2. Action and measure for the scalar field

Following ref. [17], the four-dimensional pure gravity action on the lattice is written as

$$I_g[l] = \sum_{\text{hinges } h} \left[\lambda V_h - kA_h\delta_h + a\frac{A_h^2\delta_h^2}{V_h} \right], \tag{2.1}$$

where V_h is the volume per hinge (represented by a triangle in four dimensions), A_h is the area of the hinge and δ_h the corresponding deficit angle, proportional to the curvature at h. The term proportional to k is the original Regge action. In the lattice weak field expansion, the last two terms both contain higher derivative contributions [2,3] (in the last term it is the leading contribution). This is a simple consequence of the fact that on the lattice finite differences give rise, when Fourier transformed, to terms involving trigonometric functions of the lattice momenta. The higher-order corrections are in general expected to be irrelevant in the continuum limit, if one can be found, and unless the coefficient a is taken to be very large in this limit. Whenever systematic studies have been done, there are indications that this is indeed the case [12,3], as one would expect from the experience gained in other, simpler model field theories. The results of ref. [7] in four dimensions also suggest that the corrections are negligible in the lattice continuum limit $(k \to k_c)$, and that the "ghost mass" associated with the higher derivative corrections remains of the order of the ultraviolet cutoff, of the order of the inverse average lattice spacing, $m_{\text{ghost}} \sim \pi/l_0$ (for a general discussion of some of these points in simpler field theory models, see e.g. ref. [16]). In the context of the present work the higher derivative terms will be considered as convenient invariant regulators, in addition to the usual lattice cutoff.

In the classical continuum limit the above action is equivalent [2,3,17–19] to

$$I_g[g] = \int d^4x \sqrt{g} \left[\lambda - \tfrac{1}{2}kR + \tfrac{1}{4}aR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \ldots \right], \tag{2.2}$$

with a cosmological constant term (proportional to λ), the Einstein–Hilbert term ($k = 1/(8\pi G)$), and a higher derivative term, and with the dots indicating higher-order lattice corrections. In the following we will follow the convention of choosing the fundamental lattice spacing to be equal to one; the correct power of the lattice spacing needed to convert lattice to continuum quantities can always be restored by invoking dimensional arguments (but we have to remember that due to the dynamical nature of the lattice, the average distance between sites, in units of the fundamental lattice spacing, will still depend on the bare couplings and the measure). For an appropriate choice of bare couplings, the above lattice action is bounded below for a regular lattice, even for $a = 0$, due to the presence of the lattice momentum cutoff [2]. For non-singular measures and in the presence of the λ-term such a regular lattice can be shown to arise naturally. The higher derivative terms can be set to zero ($a = 0$), but they nevertheless seem to be necessary for reaching the lattice continuum limit, and are in any case generated by radiative corrections already in weak coupling perturbation theory. When scalar fields are introduced, higher derivative terms are generated as well by the quantum fluctuations of the scalar field. Renormalization group arguments then suggest that in general the continuum limit should be explored in this enlarged multi-parameter space. Some very interesting suggestions regarding properties of non-renormalizable theories beyond perturbation theory have been put forward in ref. [20].

Next a scalar field is introduced, as the simplest type of dynamical matter that can be coupled to gravity. Consider an n_f-component field ϕ_i^a, $a = 1, \ldots, n_f$, and define this field at the vertices of the simplices. Introduce finite lattice differences defined in the usual way,

$$\left(\Delta_\mu \phi^a\right)_i = \frac{\phi_{i+\mu}^a - \phi_i^a}{l_{i,i+\mu}}. \tag{2.3}$$

The index μ labels the possible directions in which one can move from a point in a given triangle, and $l_{i,i+\mu}$ is the length of the edge connecting the two points. For simplicity let us consider for now the case $n_f = 1$. Then add to the above discrete pure gravitational action the contribution

$$I_\phi[l, \phi] = \tfrac{1}{2} \sum_{\langle ij \rangle} V_{ij} \left(\frac{\phi_i - \phi_j}{l_{ij}}\right)^2 + \tfrac{1}{2} \sum_i V_i \left(m^2 + \xi R_i\right) \phi_i^2 + \sum_i V_i U(\phi_i) + \ldots, \tag{2.4}$$

where $U(\phi)$ is a potential for the scalar field, and the term containing the discrete analog of the scalar curvature involves

$$V_i R_i \equiv \sum_{h \supset i} \delta_h A_h \sim \sqrt{g}\, R. \tag{2.5}$$

In the expression for the scalar action, V_{ij} is the volume associated with the edge l_{ij}, while V_i is associated with the site i. There is more than one way to define such a volume [17,21,22], but under reasonable assumptions, such as positivity, one should get equivalent results in the continuum. The agreement between different lattice actions in the smooth limit can be shown explicitly in the lattice weak field expansion, but the calculations can be rather tedious and we will present the results elsewhere. Here we will restrict ourselves to the baricentric volume subdivision [17] which is the simplest to deal with. The above lattice action then corresponds to the continuum expression

$$I_\phi[g, \phi] = \tfrac{1}{2} \int \sqrt{g} \left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + (m^2 + \xi R)\phi^2 \right] + \int \sqrt{g}\, U(\phi) + \dots, \quad (2.6)$$

with the induced metric related in the usual way to the edge lengths [2,3]. As is already the case for the purely gravitational action, the correspondence between lattice and continuum operators is true classically only up to higher derivative corrections. But such higher derivative corrections in the scalar field action are expected to be irrelevant and we will not consider them here any further. The scalar field potential $U(\phi)$ could contain quartic contributions, whose effects are of interest in the context of cosmological models where spontaneously broken symmetries play an important role. For the moment we will be considering a scalar field without direct self-interactions, and will set $U = 0$.

The lattice scalar action contains a mass parameter m, which has to be tuned to zero in lattice units to achieve the lattice continuum limit for scalar correlations. The dimensionless coupling ξ is arbitrary; two special cases are the minimal ($\xi = 0$) and the conformal ($\xi = \tfrac{1}{6}$) coupling case. As an extreme case one could consider a situation in which the matter action by itself is the only action contribution, without any kinetic term for the gravitational field, but still with a non-trivial gravitational measure; integration over the scalar field would then give rise to an effective non-local gravitational action.

Having discussed the action, let us turn now to the measure. The discretized partition function can be written as

$$Z = \int d\mu[l]\, d\mu[\phi]\, \det\{-I_g[l] - I_\phi[l, \phi]\}. \quad (2.7)$$

It is well known that the continuum gravitational measure is not unique, and different regularizations will lead to different forms for the measure. DeWitt has argued that the gravitational measure should have the form [23,24]

$$\int d\mu[g] = \int \prod_x g^{(d-4)(d+1)/8} \prod_{\mu \geq \nu} dg_{\mu\nu}. \quad (2.8)$$

The main difference between various euclidean measures seems to be in the power of $\sqrt{g}$ in the prefactor, which on the lattice corresponds to some product of volume factors. On the lattice these volume factors do not give rise to coupling terms, and are therefore strictly *local*. It should also be clear that since diffeomorphism invariance is lost in *all* lattice models of gravity, at least away from smooth manifolds (the very definition of a lattice breaks local Poincaré invariance), there is no clear criterion at this point to help one decide which measure should be singled out. We have argued before that the power appearing in the measure should be considered as an additional, hopefully irrelevant, bare parameter [17].

On the simplicial lattice the invariant edge lengths represent the elementary degrees of freedom, which uniquely specify the geometry for a given incidence matrix. Since the induced metric at a simplex is linearly related to the edge lengths squared within that simplex, one would expect the lattice analog of the DeWitt metric to simply correspond to dl^2 [4]. We will therefore write the lattice measure as [6,17,25]

$$\int d\mu_\epsilon[l] = \prod_{\text{edges } ij} \int_0^\infty V_{ij}^{2\sigma} \, dl_{ij}^2 F_\epsilon[l], \qquad (2.9)$$

where V_{ij} is the "volume per edge", $F_\epsilon[l]$ is a function of the edge lengths which enforces the higher-dimensional analogs of the triangle inequalities, and $\sigma = 0$ for the lattice analog of the DeWitt measure for pure gravity. The parameter ϵ is introduced as an ultraviolet cutoff at small edge lengths: the function $F_\epsilon[l]$ is zero if any of the edges are equal to or less than ϵ. In general it is needed for sufficiently singular measures; for the $\sigma = 0$ measure such a parameter is not necessary since the triangle inequalities already strongly suppress small edge lengths [7], and so we will set it to zero. Note therefore that no cutoff is imposed on small or large edge lengths, if a non-singular measure such as dl^2 is used. This fact is essential for the recovery of diffeomorphism invariance close to the critical point, where on a large lattice a few rather long edges, as well as some rather short ones, start to appear [1]. Eventually it is of interest to systematically explore the sensitivity of the results to the type of gravitational measure employed. This has been done to a certain extent in two [12] and three [3] dimensions. The conclusion seems to be that for non-singular measures the results relevant for the lattice continuum limit (i.e. the long-distance properties of the theory, as characterized for example by the critical exponents) appear to be independent of σ. From a general point of view it is difficult to see how local volume factors, which involve no gradient terms, can possibly affect the nature of the continuum limit, which is expected to be dominated by shear-wave-like distortions of the geometry of space-time. The experience gained so far seems to indicate that the volume factors coming from the measure will only affect the overall lattice scale and the shape of the distribution for the edge lengths, and will lead therefore to different renormal-

izations of the cosmological constant, but will leave the long-wavelength excitation spectrum, which is determined by the relatively small fluctuations in the edge lengths about the lattice equilibrium position, unaffected. But of course these arguments cannot be taken as a substitute for a systematic investigation of this issue in four dimensions.

In the presence of matter, similar considerations apply. If an n_f-component scalar field is coupled to gravity the power σ appearing in the measure has to be changed, due to an additional factor of $\prod_x (\sqrt{g})^{n_f/2}$ in the continuum gravitational measure. On the lattice one then has $\sigma = n_f/30$, since with our discretization of space-time based on hypercubes there are $2^d - 1 = 15$ edges emanating from each lattice vertex. The additional measure factor insures that

$$\int \prod_x \left\{ d\phi (\sqrt{g})^{n_f/2} \right\} \exp\left(-\tfrac{1}{2} m^2 \int \sqrt{g}\, \phi^2 \right) = \left[\left(\frac{2\pi}{m^2} \right)^{n_f/2} \right]^V = \text{const.}, \quad (2.10)$$

or that for large mass, the scalar field completely decouples, leaving only the dynamics of the pure gravitational field.

3. Effects of matter fields

As long as the scalar action is quadratic, one can formally integrate out the matter fields and obtain an effective lagrangian contribution written entirely in terms of the metric field,

$$\int d\mu[\phi] \exp\left(-\tfrac{1}{2} \int \sqrt{g}\, \phi M[g] \phi \right)$$

$$= \int \prod_x \left\{ d\phi (\sqrt{g})^{n_f/2} \right\} \exp\left\{ -\tfrac{1}{2} \int \sqrt{g}\, \phi M[g] \phi \right\}$$

$$\sim \{\det M[g]\}^{-n_f/2} \sim e^{-I_{\text{eff}}[g]}. \quad (3.1)$$

Here we have from the scalar field action

$$\langle x | M[g] | y \rangle \equiv \left(-\partial^2 + \xi R + m^2 \right) \delta(x - y), \quad (3.2)$$

where ∂^2 is the usual covariant laplacian,

$$\partial^2 \phi \equiv \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g}\, g^{\mu\nu} \partial_\nu \phi. \quad (3.3)$$

The full effective action, with terms from eq. (2.2) included, can be obtained from

the results of ref. [27] (after introducing a proper-time short-distance cutoff of the order of $s_0 \sim 1/\Lambda^2$). One finds then

$$I_{\text{eff}}[g] = \int \sqrt{g} \left[\lambda' - \tfrac{1}{2}k'R + \tfrac{1}{4}a'R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \ldots \right], \tag{3.4}$$

with effective couplings (for one flavor, $n_f = 1$)

$$\lambda' = \lambda + \frac{1}{64\pi^2}\Lambda^4 - \frac{1}{32\pi^2}m^2\lambda^2 + \frac{1}{64\pi^2}m^4 \ln \Lambda^2 + \ldots,$$

$$k' = k + \frac{1}{16\pi^2}\left(\xi - \tfrac{1}{6}\right)\Lambda^2 + \frac{1}{16\pi^2}\left(\xi + \tfrac{1}{6}\right)m^2 \ln \Lambda^2 + \ldots,$$

$$a' = a + \frac{1}{1920\,\pi^2}\ln \Lambda^2 + \ldots. \tag{3.5}$$

For a fixed cutoff these corrections are quite small in magnitude compared to the corresponding gravitational radiative corrections computed in the $2 + \epsilon$ expansion [14,15] or in higher derivative theories [28]. We will see later that this is also clearly the case for the lattice results. As in ordinary gauge theories, matter vacuum polarization effects are small unless one has a large number of matter fields (in which case even a new phase might appear). To the extent that the lattice scalar action is equivalent in the lattice continuum limit to the corresponding continuum scalar action, the above perturbative results, valid for small curvatures, should be relevant for the lattice model as well.

The effects of matter fields are small also from the point of view of the $2 + \epsilon$ perturbative expansion for gravity [14,15]. One analytically continues in the space-time dimension by using dimensional regularization, and applies perturbation theory around $d = 2$, where Newton's constant is dimensionless (it is not entirely clear if this approach makes sense beyond perturbation theory). In this expansion the dimensionful bare coupling is written as $G_0 = \Lambda^{2-d}G$, where Λ is an ultraviolet cutoff (corresponding on the lattice to a momentum cutoff of the order of the inverse average lattice spacing, $\Lambda \sim \pi/\langle l^2\rangle^{1/2}$) and G a dimensionless bare coupling constant. A double expansion in G and ϵ then leads to a nontrivial fixed point in G above two dimensions, where some local averages and their fluctuations are expected to develop an algebraic singularity in G. Close to two dimensions the gravitational beta functions is given to one loop by

$$\beta(G) \equiv \frac{\partial G}{\partial \log \Lambda} = \epsilon G - \tfrac{2}{3}(25 - n_f)G^2 + \ldots, \tag{3.6}$$

where n_f is the number of massless scalar fields. To lowest order the ultraviolet fixed point is at

$$G^* = \frac{3\epsilon}{2(25 - n_f)} + O(\epsilon^2). \tag{3.7}$$

Integrating eq. (3.6) close to the non-trivial fixed point in $2 + \epsilon$ dimensions we obtain

$$\mu_0 = \Lambda \exp\left(-\int^G \frac{dG'}{\beta(G')}\right) \underset{G \to G^*}{\sim} \Lambda \, |G - G^*|^{-1/\beta'(G^*)} \sim \Lambda \, |G - G^*|^{1/\epsilon}, \tag{3.8}$$

where μ_0 is an arbitrary integration constant, with dimension of a mass, and which should be identified with some physical scale. The derivative of the beta function at the fixed point defines the critical exponent ν, which to this order is independent on n_f,

$$\beta'(G^*) = -\epsilon = -1/\nu. \tag{3.9}$$

The possibility of algebraic singularities in the neighborhood of the fixed point, appearing in vacuum expectation values such as the average curvature and its derivatives, is then a natural one, at least from the point of view of the $2 + \epsilon$ expansion.

The previous results also illustrate how in principle the lattice continuum limit should be taken [16]. It corresponds to $\Lambda \to \infty$, $G \to G^*$ with μ_0 held constant; for fixed lattice cutoff the continuum limit is approached by tuning G to G^*. Alternatively, we can choose to compute dimensionless ratios directly, and determine their limiting value as we approach the critical point (we will show examples of this later). Away from G^* one will in general expect to encounter some lattice artifacts, which reflect the non-uniqueness of the lattice transcription of the continuum action and measure, as well as its reduced symmetry properties.

Let us conclude this section by mentioning that the Nielsen–Hughes formula [29] for the one-loop beta function associated with a spin-s particle in four dimensions provides for a physical interpretation of the fact that the matter contribution is so small compared to the gravitational one. It appears that this result is related to the fact that the spin of the graviton is not a small number. Considering only spin 0 and 2, the formula gives the lowest-order result for the beta function coefficient as

$$16\pi^2\beta_0 = -\sum_s (-1)^{2s}\left[(2s)^2 - \tfrac{1}{3}\right] = -\tfrac{1}{3}(47 - n_f), \tag{3.10}$$

making the matter contribution quite negligible unless the number of flavors is large. In higher derivative theories one finds similar large coefficients. It is

encouraging that similar results are found from the lattice calculations to be described below. Furthermore, for a sufficiently large number of flavors one would expect eventually a phase transition (if these lowest-order results are taken seriously), due to the change of sign in the beta function.

4. Observables

When we consider gravity coupled to a scalar field, we can distinguish two types of observables, those involving the metric field (the edge lengths) only, and those involving also the scalar field. Quantities such as the expectation value of the scalar curvature, the fluctuations in the curvatures or the curvature correlations belong to the first class, while scalar field averages and scalar correlations belong to the second class.

Following ref. [6], we define the following gravitational physical observables, such as the average curvature:

$$\mathcal{R}(\lambda, k, a) \sim \frac{\left\langle \int \sqrt{g}\, R \right\rangle}{\left\langle \int \sqrt{g} \right\rangle}, \tag{4.1}$$

and the fluctuation in the local curvatures,

$$\chi_{\mathcal{R}}(\lambda, k, a) \sim \frac{\left\langle \left(\int \sqrt{g}\, R\right)^2 \right\rangle - \left\langle \int \sqrt{g}\, R \right\rangle^2}{\left\langle \int \sqrt{g} \right\rangle} \sim \frac{1}{V}\frac{\partial^2}{\partial k^2}\ln Z. \tag{4.2}$$

The lattice analogs of these expressions are readily written down by making use of the correspondences [17,25]

$$\int d^4x \sqrt{g} \rightarrow \sum_{\text{hinges } h} V_h, \tag{4.3}$$

$$\int d^4x \sqrt{g}\, R \rightarrow 2 \sum_{\text{hinges } h} \delta_h A_h, \tag{4.4}$$

$$\int d^4x \sqrt{g}\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \rightarrow 4 \sum_{\text{hinges } h} V_h \left(\delta_h^2 A_h^2 / V_h^2\right). \tag{4.5}$$

On the lattice we prefer to define quantities in such a way that variations in the average lattice spacing $\sqrt{\langle l^2 \rangle}$ are compensated by the appropriate factor as

determined from dimensional considerations. In the case of the average curvature we define therefore the lattice quantity $\mathscr{R}$ as

$$\mathscr{R} = \langle l^2 \rangle \frac{\left\langle 2 \sum_h \delta_h A_h \right\rangle}{\left\langle \sum_h V_h \right\rangle}, \tag{4.6}$$

and similarly for the curvature fluctuation. The curvature fluctuation is related to the (connected) scalar curvature correlator at zero momentum,

$$\chi_{\mathscr{R}} \sim \frac{\int \mathrm{d}^4 x \int \mathrm{d}^4 y \langle \sqrt{g}\, R(x) \sqrt{g}\, R(y) \rangle_c}{\left\langle \int \mathrm{d}^4 x \sqrt{g} \right\rangle}. \tag{4.7}$$

A divergence in the fluctuation is then indicative of long-range correlations (a massless particle). Close to the critical point one expects for large separations a power law decay in the geodesic distance,

$$\langle \sqrt{g}\, R(x) \sqrt{g}\, R(y) \rangle \underset{|x-y| \to \infty}{\sim} \frac{1}{|x-y|^{2n}}, \tag{4.8}$$

which in turn leads to the expectation $\chi_{\mathscr{R}} \sim L^{d-2n}$, where $L \sim V^{1/d}$ is the linear size of the system. In refs. [6,7] it was found that $\chi_{\mathscr{R}}$ diverges close to the critical point as

$$\chi_{\mathscr{R}} \underset{k=k_c, L \to \infty}{\sim} L^{d(1-\delta)/(1+\delta)}, \tag{4.9}$$

where δ is the curvature critical exponent introduced in ref. [6], and therefore $n = \delta d/(1+\delta) = d - 1/\nu$, with the exponent ν defined as $\nu = (1+\delta)/d$. Note that for a *scalar* field in four dimensions one would expect $\nu = 1/2$, whereas we find $\delta \approx 0.63$ and therefore $\nu \approx 0.41$.

It is of interest to contrast the behavior of the preceding quantities, associated with the curvature, with the analogous quantities involving the local volumes (or the square root of the determinant of the metric in the continuum) only. We can consider therefore the average volume $\langle V \rangle$, and its fluctuation defined as

$$\chi_V(\lambda, k, a) \sim \frac{\left\langle \left(\int \sqrt{g} \right)^2 \right\rangle - \left\langle \int \sqrt{g} \right\rangle^2}{\left\langle \int \sqrt{g} \right\rangle} \sim \frac{1}{V} \frac{\partial^2}{\partial \lambda^2} \ln Z. \tag{4.10}$$

The latter is then related to the connected volume correlator at zero momentum,

$$\chi_V \sim \frac{\int d^4x \int d^4y \langle \sqrt{g(x)} \sqrt{g(y)} \rangle_c}{\left\langle \int d^4x \sqrt{g} \right\rangle}. \tag{4.11}$$

We have argued before [6] that fluctuations in the curvature are sensitive to the presence of a spin-2 massless particle, while fluctuations in the volume probe only the correlations in the scalar channel. In the case of gravity a dramatic difference is therefore expected in the two type of correlations. Indeed the numerical simulations show clearly a divergence in the curvature fluctuations, but at the same time no divergence in the volume fluctuations. Other, more complex invariant correlation functions at fixed geodesic distance can be written down and measured [7].

Let us turn now to the observables involving the scalar field. Due to the form of the action, the average of the scalar field is always zero, but one can compute the discrete analog of the following coordinate invariant fluctuation:

$$\chi_\phi = \frac{\left\langle \int d^4x \int d^4y \sqrt{g(x)} \, \phi(x) \sqrt{g(y)} \, \phi(y) \right\rangle}{\left\langle \int d^4x \sqrt{g(x)} \right\rangle}$$

$$- \frac{\left\langle \int d^4x \sqrt{g(x)} \, \phi(x) \right\rangle \left\langle \int d^4y \sqrt{g(y)} \, \phi(y) \right\rangle}{\left\langle \int d^4x \sqrt{g(x)} \right\rangle} \tag{4.12}$$

(again, for the gaussian scalar action we will be considering, the second term on the r.h.s. will be zero). On the lattice such an expression can be written as

$$\chi_\phi \sim \frac{\left\langle \sum_{ij} V_i \phi_i V_j \phi_j \right\rangle}{\left\langle \sum_i V_i \right\rangle} - \frac{\left\langle \sum_i V_i \phi_i \right\rangle \left\langle \sum_j V_j \phi_j \right\rangle}{\left\langle \sum_i V_i \right\rangle}. \tag{4.13}$$

Since χ_ϕ is the zero-momentum component of the scalar particle propagator, it is expected to diverge like m^{-2} for small mass, up to anomalous dimensions. Also of

interest are the local coordinate invariant averages

$$\langle \phi^2 \rangle \equiv \frac{\left\langle \int d^4 x \sqrt{g}\, \phi^2 \right\rangle}{\left\langle \int d^4 x \sqrt{g} \right\rangle}$$

$$\langle \phi^4 \rangle \equiv \frac{\left\langle \int d^4 x \sqrt{g}\, \phi^4 \right\rangle}{\left\langle \int d^4 x \sqrt{g} \right\rangle}. \tag{4.14}$$

For *free* fields one expects the following dependence on the scalar field mass:

$$\langle \phi^2 \rangle = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + m^2} = \frac{1}{16\pi^2} \left[\Lambda^2 - m^2 \ln \frac{\Lambda^2 + m^2}{m^2} \right], \tag{4.15}$$

$$\langle \phi^4 \rangle = 2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 + m^2)^2} = \frac{1}{8\pi^2} \left[\ln \frac{\Lambda^2 + m^2}{m^2} + \frac{m^2}{\Lambda^2 + m^2} - 1 \right], \tag{4.16}$$

where Λ is the ultraviolet momentum cutoff. In the interacting case one antici-
pates, among other effects, a multiplicative renormalization of the mass parameter
m. In the presence of gravity, the behavior of these quantities will be discussed
below.

We can write schematically the propagator for the scalar field in a *fixed*
background geometry specified by some distribution of edge lengths as

$$G(d) = \left\langle y \left| \frac{1}{-\partial^2 + \xi R + m^2} \right| x \right\rangle, \tag{4.17}$$

where d is the geodesic distance between the two space-time points being
considered. Now fix one point at the origin 0, and use the discretized form of the
scalar field action of eq. (2.4). Then the discrete equation of motion for the field ϕ_i
in the presence of a δ-function source of unit strength localized at the origin gives
us the sought-after Green function. For $\xi = 0$ we write the equation as

$$\phi_i = \frac{1}{W_i} \left(\sum_{j \neq i} W_{ij} \phi_j + \delta_{i0} \right), \tag{4.18}$$

with the weights W given by

$$W_i = \sum_{j \neq i} \left(\frac{m^2}{2} + \frac{1}{l_{ij}^2} \right) V_{ij}, \quad W_{ij} = \frac{V_{ij}}{l_{ij}^2}. \tag{4.19}$$

Here the sums extend over nearest-neighbor points only, V_{ij} is the volume associated via a baricentric subdivision with the edge ij, and δ_{i0} is a delta-function source localized at the origin on site 0. The above equation for ϕ_i can be solved by an iterative procedure, taking $\phi_i = 0$ as an initial guess. After the solution ϕ_i has been determined by relaxation, at large distances from the origin one has

$$\phi_i \sim G(d_{i0}) \sim A\sqrt{m/d_{i0}^3}\ \exp(-md_{i0}),\qquad(4.20)$$

which determines the geodesic distance d_{i0} from lattice lattice point 0 to lattice point i. This method is more efficient and accurate than trying to determine the geodesic distance by sampling paths connecting the two points as was done in ref. [7], but is of course equivalent to it [30].

In quantum gravity it is of great interest to try to determine the value of the low-energy, renormalized coupling constants, and in particular the effective cosmological constant λ_{eff} and the effective Newton constant $G_{\text{eff}} = (8\pi k_{\text{eff}})^{-1}$. Equivalently, one would like to be able to determine the large-distance limiting value of a dimensionless ratio such as $\lambda_{\text{eff}}G_{\text{eff}}^2$, and its dependence on the linear size of the system $L = V^{1/4}$. (In the real world one knows that $G_{\text{eff}} = (1.6160 \times 10^{-33}\ \text{cm})^2$, while $\lambda_{\text{eff}}G_{\text{eff}}^2 \sim 10^{-120}$ is very small). The vacuum expectation value of the scalar curvature can be used as a definition of the effective, long-distance cosmological constant

$$\mathscr{R} \sim \frac{\left\langle \int \sqrt{g}\, R \right\rangle}{\left\langle \int \sqrt{g} \right\rangle} \sim \left(\frac{4\lambda}{k}\right)_{\text{eff}}.\qquad(4.21)$$

In the pure gravity case one finds that there is a critical point in k at which the curvature vanishes, and for $k < k_c$ one has

$$\mathscr{R} \underset{k \to k_c}{\sim} -A_{\text{R}}(k_c - k)^\delta \qquad(4.22)$$

and thus $(\lambda/k)_{\text{eff}} \to 0$ in lattice units. The location of the critical point k_c and the amplitude in general depend on the higher derivative coupling a and other non-universal parameters, but the exponent is expected to be universal, and was estimated previously to be about 0.63; more details can be found in refs. [6,7].

One immediate consequence of this result is that in the smooth phase with $k < k_c$ (or $G > G_c \equiv G^*$), the gravitational coupling constant G must increase with distance (anti-screening), at least for rather short distances. Introducing an arbitrary momentum scale μ, one has close to the ultraviolet fixed point the following short-distance behavior for Newton's constant:

$$G(\mu) - G^* = [G(\Lambda) - G^*]\left(\frac{\Lambda}{\mu}\right)^{1/\nu}\qquad(4.23)$$

with Λ the ultraviolet cutoff; the exponents δ and ν are calculable and are related to each other via the scaling relation $\nu = (1 + \delta)/4 \approx 0.41$. The opposite behavior (screening) would be true in the phase with $k > k_c$, but such a phase is known not to be stable and leads to no lattice continuum limit [7].

If the system is of finite extent, with linear dimensions $L = V^{1/4}$, then the scaling laws for $\mathscr{R}$ should also give the volume dependence of the effective cosmological constant at the fixed point. For pure gravity one finds at the critical point:

$$\mathscr{R} \underset{L \gg l_0}{\sim} \left(\frac{1}{L} \right)^{\delta/\nu}, \tag{4.24}$$

with l_0 of the order of the average lattice spacing, $l_0 = \sqrt{\langle l^2 \rangle}$, and $\delta/\nu \approx 1.52$. The critical point here is defined, as usual, as the point in bare coupling constant space where the curvature fluctuations diverge in the infinite volume limit. Similarly for the dimensionless coupling G in a finite volume, one expects the scaling behavior

$$G(\mu) \underset{L, 1/\mu \gg l_0}{\sim} G_c + \left(\frac{1}{\mu L} \right)^{1/\nu}. \tag{4.25}$$

These results are all direct consequences of the scaling laws and the values of the critical exponents [7]. An important issue is how these results are affected by the presence of dynamical matter. This will be addressed later in the paper.

The gravitational exponent δ determines the universal scaling behavior of a variety of observables. Among the simplest ones which are relevant for simple cosmological models one can mention the FRW scale factor $a(t)$, as it appears in the line element

$$ds^2 = -dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right\}, \tag{4.26}$$

and which we would expect to scale at short distances according to the equation

$$\frac{a^2(t)}{a^2(t_0)} \underset{t \gg t_0}{\sim} \left(\frac{t}{t_0} \right)^{\delta/\nu} \tag{4.27}$$

with $ct_0 = l_0$. It is amusing to note that in this model the scale factor cannot exhibit a singularity for short times, $t \sim t_0$. For such short distances the strong fluctuations in the metric field and the curvature prevent this from happening. We should add though that the scale factor itself is essentially a semiclassical quantity, linked to a specific ansatz for the (classical) metric at large distances. In the presence of strong metric fluctuations it is no longer clear that it remains a well-defined concept.

The bare Newton constant also describes the coupling of gravity to matter at scales comparable to the ultraviolet cutoff. Consider the classical equations of motion for pure Einstein gravity with a cosmological constant term,

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \qquad (4.28)$$

Here we have followed the usual conventions by defining $\Lambda = 8\pi G \lambda$ (not to be confused with the ultraviolet momentum cutoff introduced earlier). In the presence of higher derivative terms and higher-order lattice corrections this is of course not the right equation (the equations of motion for higher derivative gravity are substantially more complex), but at sufficiently large distances it should be the appropriate equation if the average curvature is small and a sensible continuum limit can be found in the lattice theory. If we have only one real scalar field, the energy–momentum tensor is given by

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \tfrac{1}{2} g_{\mu\nu} \left(\partial_\lambda \phi \partial^\lambda \phi + m^2 \phi^2 \right) \qquad (4.29)$$

(we will consider from now on only the case $\xi = 0$). Taking the trace we obtain

$$R = 4\Lambda - 8\pi G T_\mu^\mu = 4\Lambda + 8\pi G \left[(\partial \phi)^2 + 2m^2 \phi^2 \right]. \qquad (4.30)$$

Now consider the effects of quantum fluctuations, and separate the pure gravity and matter contributions to the scalar curvature, by writing for the average curvature $\langle R \rangle = \langle R_{\text{gravity}} \rangle + \langle R_{\text{matter}} \rangle$, where $\langle R \rangle$ is the average of the total scalar curvature in the presence of matter, and $\langle R_{\text{gravity}} \rangle$ is the same quantity in the absence of matter. More specifically, by the expectation value $\langle R_{\text{gravity}} \rangle$ we will simply mean the averages obtained in the absence of any matter fields, as computed in ref. [7]. We will see below that $\langle R_{\text{matter}} \rangle$ represents a rather small contribution, unless there are many scalar fields contributing to the vacuum polarization. In the presence of quantum fluctuations, we can write therefore for the matter correction

$$\langle R_{\text{matter}} \rangle = 8\pi G \langle (\partial \phi)^2 + 2m^2 \phi^2 \rangle = 8\pi G \left[2 \langle I_\phi \rangle + m^2 \langle \phi^2 \rangle \right]. \qquad (4.31)$$

In other words, the change in the average value of the scalar curvature that arises when matter fields are included is proportional to Newton's constant G, and it is expected to be positive. This is indeed what will be found in the numerical simulations discussed below, even though the magnitude of the correction is quite small (in agreement with the perturbative arguments presented in the previous section). To the extent that the feedback of the scalar degrees of freedom on the gravitational degrees of freedom appears to be rather small (almost to the point of being difficult to measure), we shall argue below that gravity is indeed 'weak', at least for the type of scalar action we have investigated here.

5. Numerical procedure

In order to explore the ground state of four-dimensional simplicial gravity coupled to matter beyond perturbation theory one has to resort to numerical methods. As in our previous work, the edge lengths and scalars are updated by a standard Metropolis algorithm, generating eventually an ensemble of configurations distributed according to the action of eqs. (2.1) and (2.4), with the inclusion of the appropriate generalized triangle inequality constraints arising from the nontrivial gravitational measure. Further detail of the method as applied to pure gravity are discussed in ref. [32], and will not be repeated here, since the scalar action contribution can be dealt with in essentially the same way.

We have not included here a term coupling the scalar field directly to the curvature ($\xi = 0$), since the continuum perturbative results discussed previously appear rather similar for different values of $\xi \neq \frac{1}{6}$, and the scalar action becomes significantly simpler for $\xi = 0$. Also we note that, in the absence of matter, $\langle R \rangle$ itself vanishes at the critical point [6,7]. In mean-field theory, we can replace the term $R\phi^2$ by $R\langle\phi^2\rangle$. Since $\langle\phi^2\rangle$ is finite at the critical point (see discussion below), we expect the inclusion of this term to mostly affect a renormalization of the critical coupling k_c (related to the critical value of Newton's constant by $k_c = 1/(8\pi G_c)$), which should not change the universal critical behavior.

Let us point out here only the fact that, while the scalar field action of eq. (2.4) looks rather innocuous, due to the simplicial nature of the lattice a large number of interaction terms are involved at each site: at each vertex there are 15 edges emanating in the positive lattice 'directions', and 15 in the negative lattice 'directions' [2]. In the update of the scalar field each of the 30 edge volumes V_{ij} has to be re-computed, by adding together the contributions from all the four-simplices that meet on that edge. For the edge volume one has

$$V_{ij} = \tfrac{1}{10} \sum_{\text{simplices } s \supset ij} V_s \tag{5.1}$$

since there are ten edges per simplex in four dimensions. Here the volume of an n-simplex with edge lengths l_{ij} is given as usual by the determinant

$$V_n = \frac{(-1)^{(n+1)/2}}{n! 2^{n/2}} \begin{vmatrix} 0 & 1 & 1 & \dots \\ 1 & 0 & l_{12}^2 & \dots \\ l & l_{21}^2 & 0 & \dots \\ 1 & l_{31}^2 & l_{32}^2 & \dots \\ \dots & \dots & \dots & \dots \\ 1 & l_{n1}^2 & l_{n2}^2 & \dots \\ 1 & l_{n+1,1}^2 & l_{n+1,2}^2 & \dots \end{vmatrix}^{1/2}, \tag{5.2}$$

and corresponds to the determinant of a 6×6 matrix in the case of a four-simplex; when expanded out it contains 130 distinct terms. Furthermore the number of four-simplices meeting on a given edge depends on the type of edge. With our simplicial subdivision of the four-dimensional hypercubes that make up the lattice, we have four body principals, six face diagonals, four body diagonals and one hyperbody diagonal per hypercube [2]. For a body principal or hyperbody diagonal there are 24 four-simplices meeting on it, while for a face or body diagonal there are 12 four-simplices meeting on it. When updating one scalar field by the multi-hit Monte Carlo or heat bath method, the 30 neighboring link contributions need to be computed once, with their associated link volumes, and special care has to be taken of the order of the edge lengths appearing in the simplex formulae. When updating a given edge length, all the scalar field action contributions involving that particular edge have to evaluated, in addition to the purely gravitational part. For a body principal and hyperbody diagonal there are 65 such contributions that have to be added up, while for a face or body diagonal 35 such contributions have to be added up. By assigning then special fixed values to the edge lengths, one can perform a number of checks against the expected analytical result to verify that the volumes are computed and added up correctly. Even though the program is quite computing intensive, it is well suited for a massively parallel machine. In the two parallel versions of the program we have written, a large number (64–256) of independent edge and scalar variables are all updated simultaneously in parallel.

We considered lattices of size between $4 \times 4 \times 4 \times 4$ (256 vertices, 3840 edges, 6144 simplices) and $16 \times 16 \times 16 \times 16$ (65 536 vertices, 983 040 edges, 1 572 864 simplices). Even though these lattices are not very large, one should keep in mind that due to the simplicial nature of the lattice there are many edges per hypercube with many interaction terms, and as a consequence the statistical fluctuations are comparatively small, unless one is very close to a critical point. In all cases the measure over the edge lengths was of the form $dl^2 V_l^{n_f/30}$ times the triangle inequality constraints (see eq. (2.9)). We shall restrict here our attention to the case $n_f = 1$; results for larger values of n_f will be presented elsewhere.

The topology was restricted to a four-torus (periodic boundary conditions), and it is expected that for this choice boundary effects on physical observables should be minimized. One could perform similar calculations with lattices of different topology, but the universal infrared scaling properties of the theory should be determined only by short-distance renormalization effects, independently of the specific choice of boundary conditions. This is a consequence of the fact that the renormalization group equations are *independent* of the boundary conditions, which enter only in their solution as it affects the correlation functions through the presence of a new dimensionful parameter L. Thus the four-torus should be as good as any other choice of topology, as long as we consider the universal long-distance properties.

Let us give here a few details about the runs performed to compute the averages. In the presence of matter fields, the lengths of the runs are much shorter than in the pure gravity case [7], since the scalar field update is rather time-consuming. The couplings λ and a in the gravitational action of eq. (2.1) were fixed, as in the pure gravity case, to 1 and 0.005, respectively. For pure gravity this choice leads to a well-defined ground state for $k \leqslant k_c \approx 0.244$ (the system then resides in the smooth phase, with a fractal dimension very close to four). In the presence of matter, we also restricted most of our runs to this physically more interesting phase, where the curvature is small and negative. We investigated five values of k (0.0, 0.05, 0.1, 0.15, 0.20), and for each value we looked at a scalar mass of 1.0, 0.5 and 0.2 in lattice units. In addition, we have accurate results for infinite mass from the previous *pure* gravity calculations. Besides the results on lattices with $L = 4$ for all the above values of k and m, we also have accurate results on lattices of size $L = 8$ and 16 for $m = 0.5$, and of size $L = 8$ for $m = 0.2$. For these values of the scalar mass, the scalar correlations only extend over a few lattice spacings, and finite-size effects should therefore be contained (we have checked that this is indeed the case for the quantities we have measured). In general we are interested in a regime in which the scalar mass is much larger than the infrared cutoff, but much smaller than the lattice ultraviolet cutoff, or

$$\sqrt{\langle l^2 \rangle} \ll m^{-1} \ll V^{1/4}, \tag{5.3}$$

in order to avoid finite lattice spacing and finite volume effects. Similarly, one should also impose the constraint that the scale of the curvature in magnitude should be much smaller than the average lattice spacing, but much larger than the size of the system, or

$$\langle l^2 \rangle \ll \langle l^2 \rangle |\mathscr{R}|^{-1} \ll V^{1/2}. \tag{5.4}$$

It is equivalent to the statement that in momentum space the physical scales should be much smaller that the ultraviolet cutoff, but much larger than the infrared one.

The lengths of the runs typically varied between $2-6k$ Monte Carlo iterations on the 4^4 lattice, $1-2k$ on the 8^4 lattice, and $0.6-0.9\ k$ on the 16^4 lattice. The runs are comparatively longer on the larger lattices, since it was possible in that case to use a fully parallel version of the program. As input configurations, we used the thoroughly thermalized configurations generated previously for pure gravity. These configurations are rather 'close' to the ones that include the effects of matter, since the feedback of matter turns out to be rather small. On the larger lattices duplicated copies of the smaller lattices are used as starting configurations for each k, allowing for additional equilibration sweeps after duplicating the lattice in all four directions. This allows for a substantial savings in time, since the

initial edge length configuration on the larger lattice is already quite close to a representative configuration. We have found that in the well behaved phase ($k < k_c$) the autocorrelation times are contained, of the order of at most about one hundred sweeps. When we duplicate the smaller lattice to a larger lattice, almost no drift in the averages is observed during later re-thermalization, which indicates that for our parameters the finite-size corrections are small. On the larger lattices, because there are so many variables to average over, the statistical fluctuations from configuration to configuration are of course much smaller.

6. Results

In the pure gravity case, one finds that for fixed positive a and λ (the latter can be set equal to one without loss of generality, since it determines the overall scale) and sufficiently small k, the curvature is small and negative (smooth phase), and goes to zero at the critical point $k_c(a)$, where the curvature fluctuation diverges. In the pure gravity case we write therefore, for k less than k_c

$$\mathscr{R}(k, a) \underset{k \to k_c(a)}{\sim} -A_{\mathscr{R}}(a)\big(k_c(a) - k\big)^{\delta}, \tag{6.1}$$

$$\chi_{\mathscr{R}}(k, a) \underset{k \to k_c(a)}{\sim} A_{\chi}(z)\big(k_c(a) - k\big)^{\delta - 1}, \tag{6.2}$$

where δ is a universal curvature critical exponent, characteristic of the gravitational transition [6]. Here we will only consider the case $a = 0.005$, for which the phase transition is second order, leading therefore to a well-defined continuum limit at least in the pure gravity case [7]. For $k \geqslant k_c$ the curvature is very large (rough phase), and the lattice tends to collapse into degenerate configurations with very long, elongated simplices (with $\langle V_h \rangle / \langle l^2 \rangle^2 \sim 0$). (In ref. [7] several values for a were studied, and it was found that the model actually exhibits multicritical behavior. While for $a = 0.005$ one finds a second-order phase transition, for $a = 0$ the singularity appears to be in fact logarithmic ($\delta = 0$), suggesting a first-order transition with no continuum limit for sufficiently small a, with a multicritical point separating the two transition lines.)

When including the effects of the scalar field, one finds that the largest changes are in the average volumes (which decrease by about three percent for a scalar mass $m = 0.5$) and the average edge lengths. But such changes are somewhat uninteresting, since they correspond effectively to a shift (here actually an increase) in the bare cosmological constant (also by about the same percentage, since $\delta V/V \sim -\delta\lambda/\lambda$). We note here incidentally that such a small effect is consistent with the perturbative result of eq. (3.5), which predicts an increase in the effective cosmological constant λ by about one percent, for a cutoff $\Lambda \sim \pi/l_0 \sim 1$. Indeed
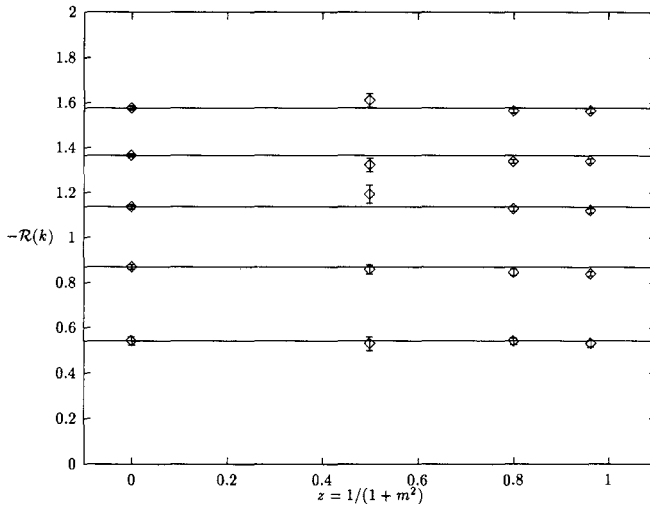
Fig. 1. Average curvature $\mathscr{R}$ as a function of the mass of the scalar field m, for different values of $k = 1/8\pi G$. From top to bottom $k = 0.0, 0.05, 0.1, 0.15, 0.2$. The values for pure gravity ($z = 0$) are included for comparison, and drawn also as lines of constant $\mathscr{R}$. The values for $m = 1.0$ ($z = 0.5$) and $m = 0.2$ ($z = 0.962$) are from a relatively small lattice with $L = 4$ and are therefore for reference only, while the values for $m = 0.5$ ($z = 0.80$) are averages from the $L = 8$ and $L = 16$ lattices, with much smaller uncertainties. The slight but clear decrease in the magnitude of the curvature in the presence of the scalar field should be noted.

before we have chosen to define observables in such a way that these effects are largely compensated, by rescaling by an appropriate power of the average lattice spacing, as in eq. (4.6). Physically more interesting are the results for the average curvature in the presence of the scalar field. As can be seen from fig. 1, the effects of the feedback of one scalar field on the curvature are quite small. It is useful to display the results as a function of $z = 1/(1 + m^2)$, since this allows us to put the results for infinite mass (no scalar feedback, from ref. [7]) on the same graph. The most accurate results in the presence of the scalar field are for $m = 0.5$, where we have relatively accurate results for three different lattice sizes ($L = 4, 8, 16$) and the highest statistics. The points for $m = 1.0$ are for reference only, since they are from an $L = 4$ lattice only. For $m = 0.5$ and $m = 0.2$ the results show a small but clear systematic decrease in the magnitude of the average curvature in the smooth phase for all values of k, at the level of a few percent; to see such a small effect long runs were needed. The results are in qualitative agreement with the expectation that the presence of the scalar field should give a positive contribution to the average curvature. In any case, for all values of the mass we have considered, the effects are rather small.

As should be clear from the discussion in sect. 5, we are interested in how the critical behavior of the theory is affected in the neighborhood of the critical point
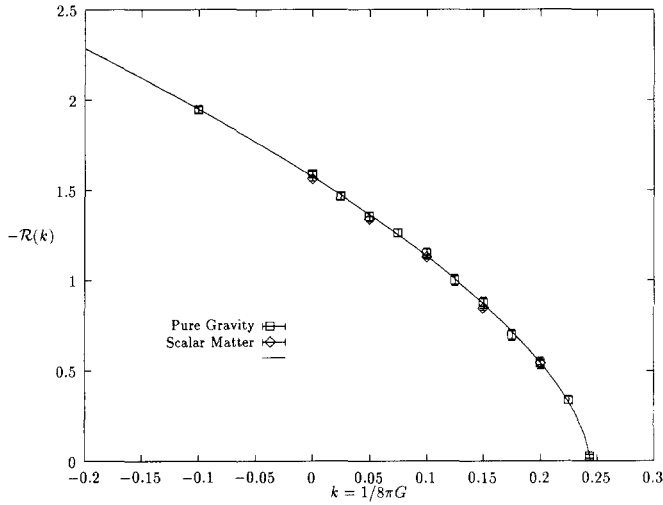
Fig. 2. Comparison of the average curvature $\mathscr{R}$ as a function of k in the presence ($\diamond$) and absence ($\square$) of the scalar field, with mass $m = 0.5$. The results for pure gravity are from ref. [7] on an $L = 16$ lattice. The line corresponds to a fit of the pure gravity results to an algebraic singularity, as discussed in the text.

by the presence of the scalar field. We will write therefore again for the average curvature, now in the presence of the scalar field,

$$\mathscr{R} \underset{k \to k_c}{\sim} -A_{\mathscr{R}}(k_c - k)^{\delta}, \tag{6.3}$$
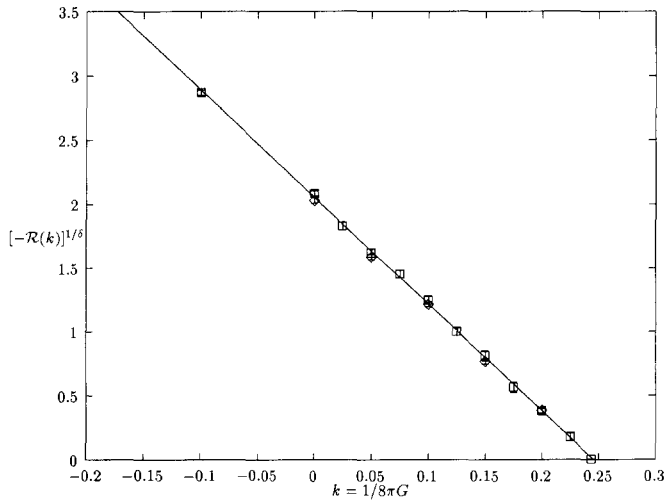
Fig. 3. Minus the average curvature $\mathscr{R}$ raised to the power $1/\delta = 1/0.63$. Parameters and data are the same as in fig. 2. The straight line is a fit to the pure gravity results. The linearity is now quite striking.

where now we expect $A_{\mathscr{R}}$, k_c, δ to depend also on the number of scalar flavors, n_f. In the presence of the scalars we have to look at the scaling limit $m \to 0$, which in practical terms corresponds to a mass much smaller than the inverse average lattice spacing. It is not clear if $m = 0.5$ (where we have our most accurate results) in our case corresponds already to such a scaling region, but our results should not be too far off, if the experience in other lattice models can be used here as a guide. If we adopt the same procedure as for pure gravity, and fit the average curvature for $m = 0.5$ to an algebraic singularity, we find $A_{\mathscr{R}} = 3.68(5)$, $k_c = 0.243(2)$ and $\delta = 0.61(6)$. This should be compared to the estimates for pure gravity (and for the same value of $a = 0.005$), $A_{\mathscr{R}} = 3.79(4)$, $k_c = 0.244(1)$ and $\delta = 0.63(3)$ [7]. In fig. 2 we compare the results for the average curvature $\mathscr{R}(k)$ with and without the presence of the scalar fields. In fig. 3 the same data is used to display $[-\mathscr{R}(k)]^{1/\delta}$ instead, which as can be seen from the graph deviates very little from a straight line behavior in k, if one uses $\delta = 0.63$.

We conclude therefore that, within our errors, switching on the scalar fields leaves the exponents almost unchanged, and the critical point moves very little; our results suggests that k_c decreases when we include the effects of the scalar field. Again we notice that such a small shift is not unexpected on the basis of the perturbative result of eq. (3.5), which also suggests a small decrease in the effective k, for a cutoff $\Lambda \sim \pi/l_0 \sim 1$. For small non-integer n_f we can expand the amplitude, critical value of k and the exponent in powers of the number of flavors n_f,

$$A_{\mathscr{R}} = A_0 + n_f A_1 + O(n_f^2)$$

$$k_c = k_0 + n_f k_1 + O(n_f^2)$$

$$\delta = \delta_0 + n_f \delta_1 + O(n_f^2), \tag{6.4}$$

and for the average curvature itself we get

$$\mathscr{R} \underset{n_f \to 0}{\sim} -A_0(k_0 - k)^{\delta_0} \left\{ 1 + n_f \left[\frac{A_1}{A_0} + \frac{\delta_0 k_1}{k_0 - k} + \delta_1 \ln(k_0 - k) \right] + O(n_f^2) \right\}, \tag{6.5}$$

which shows that the k_1 renormalization is dominant for very small n_f. Since the results for $n_f = 1$ indicate that the corrections due to the scalar field are quite small, we would tend to conclude that coefficients of the n_f terms must be rather small, and that the pure gravity theory is already a good approximation to the full theory including scalars, provided n_f is not too large.

Let us assume for the moment that k_1 and δ_1 are so small that they can be neglected to a first approximation when we consider a single scalar matter field (in the $2 + \epsilon$ expansion the matter corrections are certainly very small, and the exponent is independent of the number of matter fields to leading order in ϵ).
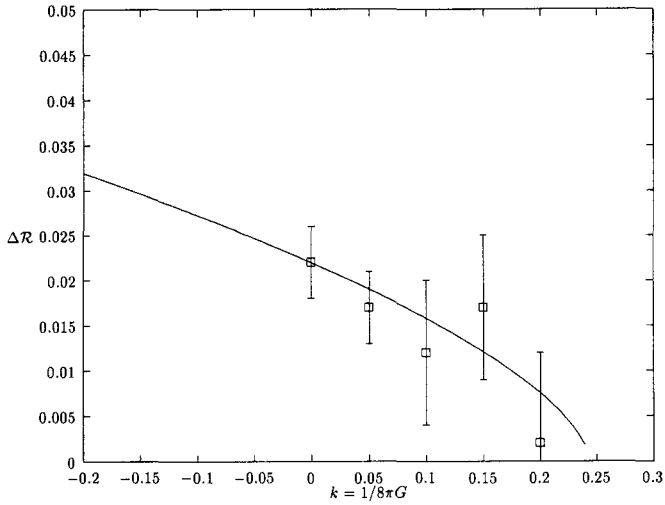
Fig. 4. Difference $\Delta\mathscr{R}(k)$ between the average curvature in the presence and absence of one scalar field, again for $m = 0.5$ and $L = 8$, 16. The difference is small and positive. The curve represents a behavior close to the critical point of the type $\Delta\mathscr{R}(k) \sim A(k_c - k)^\delta$, with $\delta \approx 0.63$ and $k_c \approx 0.244$ (the values for pure gravity).

Then the difference between the average curvature in the presence of the scalar field and in pure gravity determines the ratio of curvature amplitudes A_1/A_0,

$$\frac{\mathscr{R}_{\text{matter}}}{\mathscr{R}_{\text{gravity}}} = \frac{\mathscr{R}_{\text{gravity + matter}} - \mathscr{R}_{\text{gravity}}}{\mathscr{R}_{\text{gravity}}} \underset{k \to k_c}{\sim} \frac{A_1}{A_0}. \qquad (6.6)$$

The difference in the numerator is of course quite small, and requires a very accurate measurement of the average curvature in both cases. At the same time it provides a direct determination of the physical effects of dynamical matter fields, on a quantity that represents a direct physical observable, since the average curvature can in principle be measured by performing parallel transports of vectors around large closed loops. The calculated difference $\mathscr{R}_{\text{gravity + matter}} - \mathscr{R}_{\text{gravity}}$ is shown in fig. 4, together with a fit to a behavior $\sim (k_c - k)^\delta$, treating only the amplitude as a free parameter. To reduce any systematic effects coming from finite-volume corrections, it is advisable to subtract the average curvatures on the *same* lattice size. In addition, such a subtraction can be done without any assumption about the (singular) behavior of the curvature at k_c. One then estimates approximately for the ratio $A_1/A_0 \approx 0.053/3.79 = 0.014$; we will leave a more accurate quantitative determination of this ratio for future work. We note though that the sign of the matter correction to the curvature is consistent with the fact that the effective Newton constant gives rise to an attractive interaction ($G_{\text{eff}} > 0$), thereby adding a positive contribution to the pure gravity average curvature.

For an explanation for the smallness of such a ratio, we can look again at the formula (3.10). There the relative smallness of the matter contribution is simply a consequence of the particle's relative spin. For spin zero and spin two, as we have here, the ratio of the matter over gravity contributions is $\sim \frac{1}{3}/(4s^2 - \frac{1}{3}) = 0.021$, indeed of the same order as the ratio we computed. One can go perhaps as far as turning this argument around, and argue that the smallness of the vacuum polarization effects compared to the purely gravitational contribution is an indirect indication of the spin-two nature of the graviton (if we were to treat the value of the graviton spin as an unknown parameter, we would obtain a value very close to two, $s \sim 2.5$).

Let us turn now to a discussion of the renormalization properties of the couplings G and λ. It is clear from the preceding discussion that the effects of scalar matter are quite small. In the following we shall therefore not distinguish between the cases with and without matter fields, assuming that if there are only a few matter fields, the exponents will not change drastically.

As we indicated previously, using the methods of finite size scaling [33], one can translate the dependence of the curvature on $k - k_c$ into a statement about the *volume* dependence of the curvature at the critical point k_c. In a finite volume, of linear size L, finite-size scaling (from eqs. (4.21) and (4.24)) gives

$$(G\lambda)_{\text{eff}}(L) \underset{L, 1/\mu \gg l_0}{\sim} l_0^{-2}\left(\frac{l_0}{L}\right)^{4-1/\nu}, \tag{6.7}$$

since essentially the correlation length ξ saturates at the system size, $\xi \sim (k_c - k)^{-\nu} \sim L$. Combining this result with eq. (4.25), one obtains for the dimensionful Newton constant the following scale dependence, valid for short distances $1/\mu \ll L$:

$$G_{\text{eff}}(\mu) \underset{L, 1/\mu \gg l_0}{\sim} l_0^2 G_c + l_0^2\left(\frac{1}{\mu L}\right)^{1/\nu} \tag{6.8}$$

(with $1/\nu \approx 2.46$), and for the dimensionful cosmological constant

$$\lambda_{\text{eff}}(\mu) \underset{L, 1/\mu \gg l_0}{\sim} l_0^{-4}(\mu l_0)^{4-1/\nu}\left[G_c + \left(\frac{1}{\mu L}\right)^{1/\nu}\right]^{-1} \tag{6.9}$$

(with $4 - 1/\nu \approx 1.54$). Here again l_0 is of the order of the average lattice spacing, and we have restored the correct dimensions for G_{eff} (length squared) and λ_{eff} (inverse length to the fourth power). For the dimensionless ratio $G^2\lambda$ we then obtain the cutoff-independent result

$$(G^2\lambda)_{\text{eff}}(\mu) \underset{L, 1/\mu \gg l_0}{\sim} (\mu l_0)^{4-1/\nu}\left[G_c + \left(\frac{1}{\mu L}\right)^{1/\nu}\right]. \tag{6.10}$$

As a check, it is immediate to see that the exponent associated with G_{eff} is indeed what one would expect from the form of the Einstein part of the gravitational action in eq. (2.2) and the value of the curvature critical exponent δ, irrespective of whether matter fields are present or not (the specific values of δ and ν will depend of course on how many matter fields are present).

In conclusion, it seems that the dimensionless ratio $G^2\lambda$ can be made very small, provided the momentum scale μ is small enough, or, in other words, at sufficiently large distances. We should add also that the fixed point value for the dimensionless gravitational constant, G_c, is in general non-universal and cutoff-dependent, and depends on the specific way in which an ultraviolet cutoff is introduced (here via an average lattice spacing). In our model it is of order one for very small a, but for larger a it decreases in magnitude. One notices that the smaller G_c, the smaller the distance dependence of $G(r)$, since one has for the distance variation the result

$$\frac{\delta G(r)}{G(r)} = \frac{\nu^{-1}}{G_c(L/r)^{1/\nu} + 1} \frac{\delta r}{r} \tag{6.11}$$

(we have set $r = 1/\mu$), so in practice G_c cannot be too large. For small G_c, l_0^2 becomes substantially larger than the Planck length. It should be pointed out here that there is apparently no reason why in this model the effective coupling G_{eff} should turn out to be of the same order as the ultraviolet cutoff l_0^{-1}, and indeed it does not. The previous results seem to indicate that the situation is more subtle. Let us add also that we do not expect the results to depend significantly on the form of the lattice scalar action we have used. In particular the presence of additional higher derivative terms involving the scalar fields should not affect the results close to the continuum limit, since the corrections should be suppressed by inverse powers of the ultraviolet cutoff.

Another simple way of interpreting the results related to the scalar field is as follows. Close to the critical point, the average curvature approaches zero, and at large distances it is therefore legitimate to write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the flat metric, and $h_{\mu\nu}$ is a small correction. Then the scalar field action of eq. (2.4) is, again at large distances, close to the action describing a free scalar, and its coupling to gravity is correspondingly weak. At short distances the geometry fluctuates wildly, and the coupling between gravity and matter is strong, while at large distances the fluctuations eventually average out to zero, effectively reducing the coupling.

Turning to the behavior of the scalar field itself, we show in fig. 5 the results for $\langle \phi^2 \rangle$, in fig. 6 those for $\langle \phi^4 \rangle$ (see eq. (4.14)), and in fig. 7 for χ_ϕ (defined in eq. (4.13)). The behavior of these three quantities is qualitatively rather similar to their free field behavior (eqs. (4.15) and (4.16)), and is not too sensitive, at the level of our accuracy, to the value of k. We note in particular that $\langle \phi^2 \rangle$ approaches a
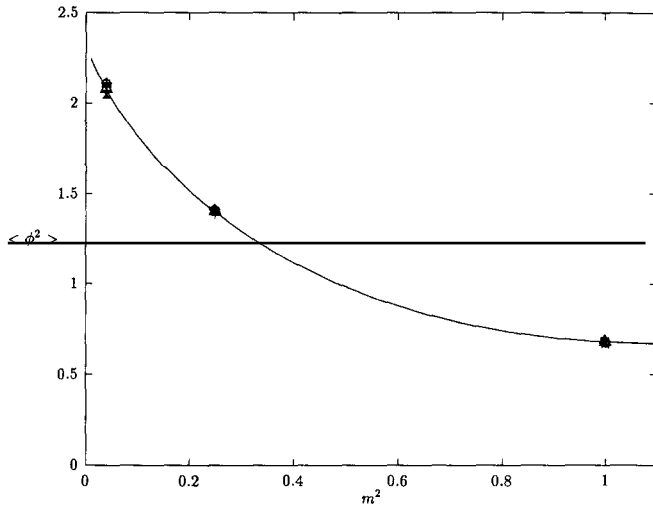
Fig. 5. The scalar field average $\langle \phi^2 \rangle$ as a function of m, and for different values of the bare gravitational coupling k ($k = 0.0$, 0.05, 0.10, 0.15, 0.20). The data for $m = 1.0$ and $m = 0.2$ is from a lattice with $L = 4$, while data for $m = 0.5$ from lattices with $L = 8$ and 16. The line is a fit assuming the free-field dependence on the mass m.

constant at $m = 0$, while both $\langle \phi^4 \rangle$ and χ_ϕ diverge at $m = 0$, in agreement with a multiplicative mass renormalization (no shift in the critical point for the field ϕ, which remains at $m = 0$).

Let us conclude this section with a brief, qualitative discussion of the phase diagram, reconsidered in light of the results obtained in the presence of scalar matter. In the case of pure gravity, the phase diagram shows a line of critical points
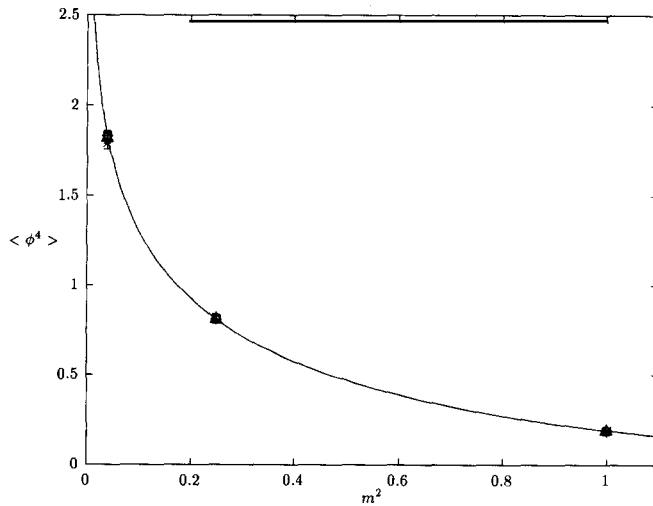

Fig. 6. Same as in fig. 5, but for the scalar field average $\langle \phi^4 \rangle$.
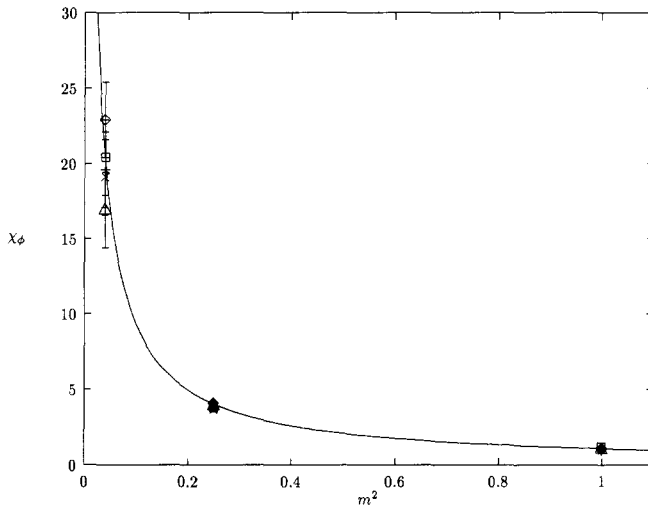
Fig. 7. Same as in fig. 5, but for the scalar field fluctuation χ_ϕ.

in the (a, k) plane separating the smooth from the rough (or collapsed) phase of gravity. The curvature vanishes along this line when it is approached from the smooth phase, and for some sufficiently negative $a < a_0 < 0$ a stable ground state ceases to exist entirely. For $a = 0$ or very small positive a, the transition from one phase to the other is first order, with no continuum limit, while for larger a is becomes second order, with a well-defined lattice continuum limit, as we indicated previously. These findings in particular would seem to indicate the presence of a multicritical point, where the two transition lines intersect [7].

In the presence of scalar matter fields, and for sufficiently large a, our new results presented here seem to suggest that a continuum limit still exists. In addition, we have found that in the smooth phase the average curvature decreases in magnitude by a small but calculable relative amount. A quantitative estimate for the amount of this decrease gives $\Delta\mathscr{R}/\mathscr{R} \sim A_1/A_0 \approx 0.014$. As the number of (degenerate) scalar fields increases, we expect this trend to continue, until $\Delta\mathscr{R}/\mathscr{R} \sim n_f A_1/A_0 \sim 1$, at which point a new phase transition might take place, in the sense that the smooth phase disappears altogether (we expect that the critical value k_c will continue to decrease, and might even become negative at some point). The appearance of a new phase in the presence of matter, with the geometry resembling branched polymers, is a well-known fact in two dimensions [34]. In fig. 8 we have sketched what a possible phase diagram in the (k, n_f) plane might look like. Presumably this new phase is nothing but the rough phase found for $n_f = 0$ and sufficiently large k. It is characterized by very long elongated simplices, with very small volumes, and a fractal dimension much smaller than four, reminiscent of a tree-like structure of space-time. Given our rather limited results, a crude estimate for the critical number of flavors at which this is expected
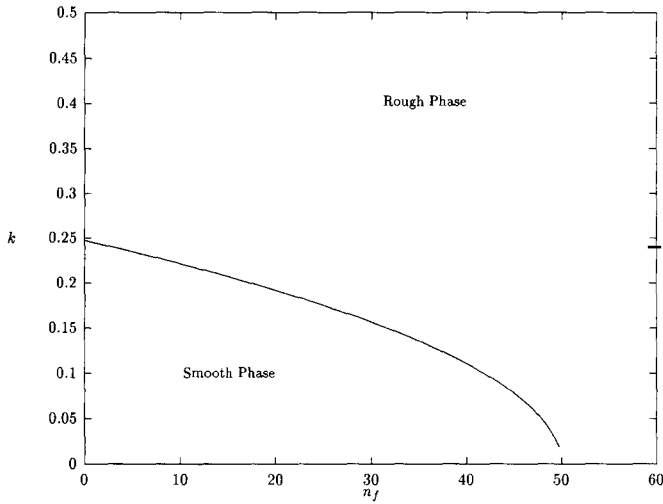
Fig. 8. A possible schematic phase diagram for gravity coupled to n_f scalar fields. The presence of the scalar fields shifts the critical point $k_c = 1/8\pi G_c$ towards smaller values as the number of scalar flavors is increased, until the smooth phase disappears entirely for some large number of flavors.

to happen would be $n_f \sim 71$, a rather large number. But such an estimate is not inconsistent with the perturbative estimates of eqs. (3.6) and (3.10), which also give such large numbers (24 and 47, respectively). And of course for such large values, we expect deviations from linearity in n_f, and we will have to leave a direct investigation of this issue for future work. Finally let us remark that since the effects of fermions can be mimicked by having scalars with negative n_f, the above conclusions would be rather different in that case, and their presence should rather impede the appearance of this new phase transitions. While scalars tend to make the geometry rougher, fermions should make it smoother.

7. Volume and curvature distributions

In this section we will discuss the properties of volume and curvature distributions, and how their behavior close to the critical point, which defines the lattice continuum limit, can be related largely to the critical exponents discussed previously. Let us assume that close to the critical point λ_c one has for the average volume a singularity of the type

$$\langle V \rangle \equiv \left\langle \int \sqrt{g} \right\rangle \sim -\frac{\partial}{\partial \lambda} \ln Z \underset{\lambda \to \lambda_c}{\sim} \frac{V_0}{(\lambda - \lambda_e)^\omega} + \text{reg} \qquad (7.1)$$

with $\omega \neq 1$, and "reg" denotes the regular part. For the volume fluctuation one then expects close to λ_c

$$\langle V^2 \rangle - \langle V \rangle^2 \sim \frac{\partial^2}{\partial \lambda^2} \ln Z \underset{\lambda \to \lambda_c}{\sim} \frac{\omega V_0}{(\lambda - \lambda_c)^{\omega+1}} + \text{reg}, \tag{7.2}$$

and it follows that the partition function close to the singularity is given by

$$Z_{\text{sing}}(\lambda) \sim \exp\left\{ -\int^\lambda d\lambda' \frac{V_0}{(\lambda' - \lambda_c)^\omega} + \text{reg} \right\}. \tag{7.3}$$

Now let us introduce the quantity $N(V)$ defined by

$$N(V) = \int d\mu[g] \, \delta\left(\int \sqrt{g} - V \right) e^{-I[g]}. \tag{7.4}$$

It can be evaluated from

$$N(V) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\lambda \, Z(\lambda) \, e^{\lambda V}, \tag{7.5}$$

to give, in the saddle-point approximation, the following expression for the density of states:

$$N(V) \sim V^{\gamma-3} \exp\{\lambda_c V(1 + b/V^{1/\omega})\}, \tag{7.6}$$

where b is a constant involving ω, V_0 and λ_c, and the exponent γ parameterizes a possible power law correction. Let us denote by $\langle \dots \rangle_V$ the averages obtained in the fixed volume ensemble. Then it is easy to see, from the transformation properties of the fixed-volume partition function under a change of scale, that one has

$$\frac{\partial \ln N(V)}{\partial V} = -\frac{1}{V} + \frac{\sigma}{4} + \frac{k}{2} \frac{\left\langle \int \sqrt{g} R \right\rangle_V}{V}, \tag{7.7}$$

which can be combined with the previous equation to give the result, valid for large volumes and in the fixed volume ensemble [6],

$$\frac{\left\langle \int \sqrt{g} R \right\rangle_V}{V} \underset{V \to \infty}{\sim} c_0 - \frac{2-\gamma}{V} + \frac{c_1}{V^{1/\omega}} + \dots. \tag{7.8}$$

We have not calculated the above average in the fixed volume ensemble, but in the *canonical* ensemble, where the volume is allowed to fluctuate, one finds the following result close to the critical point [7]:

$$\frac{\left\langle \int \sqrt{g}\, R \right\rangle}{\left\langle \int \sqrt{g} \right\rangle} \underset{V \to \infty}{\sim} \frac{1}{V^{\delta/(1+\delta)}}, \tag{7.9}$$

with $\delta \approx 0.63$. It is reasonable to assume that the exponent ω is the same in the two ensembles, in which case one gets $\omega \approx 2.60$. But this result then implies that the volume fluctuations cannot drive a continuous phase transitions. If this were the case, then the specific heat exponent $\alpha \equiv 2 - 4\nu = 1 + \omega$ would have to be $\alpha < 1$ or $\nu > 1/d = 1/4$, otherwise the transition is expected to be first order [35], in which case one would not be able to define a lattice continuum limit. Indeed a direct determination of the volume fluctuations shows that they are always finite, and in particular do not diverge at the critical point at k_c, indicating that the mass associated with the volume fluctuations (the conformal mode) is of the order of the ultraviolet cutoff [6,7].

Let us look for completeness at the analogous result for the curvature distribution. Again the exponents appearing in this case can be related to the curvature critical exponent δ. Let us assume, as seems to be the case, that close to the critical point k_c one has

$$\mathscr{R}(k) \equiv \frac{\left\langle \int \sqrt{g}\, R \right\rangle}{\left\langle \int \sqrt{g} \right\rangle} \sim + \frac{1}{V} \frac{\partial}{\partial k} \ln Z \underset{k \to k_c}{\sim} -A_{\mathscr{R}}(k_c - k)^{\delta}. \tag{7.10}$$

(see eq. (6.1)). Then for the curvature fluctuation one expects close to k_c

$$\chi_{\mathscr{R}} \sim \frac{1}{V} \frac{\partial^2}{\partial k^2} \ln Z \underset{k \to k_c}{\sim} \frac{\delta A_{\mathscr{R}}}{(k_c - k)^{1-\delta}}. \tag{7.11}$$

Here we are interested in the singular part of the free energy. Close to the singularity the partition function is then given by

$$Z_{\text{sing}}(k) \sim \exp\left\{ -V\int^k \mathrm{d}k'\, A_{\mathscr{R}}(k_c - k')^{\delta} + \text{reg} \right\}. \tag{7.12}$$

Now let us introduce the quantity $N(R)$ defined by

$$N(R) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}k\, Z(k)\, e^{kR}, \tag{7.13}$$

with $R = -V\mathscr{R}$ (R is therefore a positive quantity, related to the magnitude of the curvature, in the smooth phase where $\mathscr{R} < 0$). In the saddle-point approximation the density of states is given by

$$N(R) \sim \exp\left\{ k_c R - \frac{\delta}{1+\delta} R[R/(VA_\mathscr{R})]^{1/\delta} \right\}. \tag{7.14}$$

We find therefore that the full probability distribution for R has an algebraic singularity close to $R = 0$ of the type

$$\ln P(R) \equiv -kR + \ln N(R) \sim (k_c - k)R - \frac{\delta}{1+\delta} R[R/(VA_\mathscr{R})]^{1/\delta}, \tag{7.15}$$

Again there will also be a regular part, which we have omitted here. One can verify that the stationary point of the distribution $P(R)$ gives indeed the singular behavior of eq. (6.1).

8. Conclusions

In the previous sections we have presented some first results regarding the effects of scalar matter on quantized gravity, in the context of a quantum gravity model based on Regge's simplicial formulation. It was found that the feedback of the scalar fields on the geometry is quite small on purely gravitational quantities such as the average curvature, in agreement with some of the perturbative predictions in the continuum, which also seem to suggest that the scalar vacuum polarization effects should be rather small. The qualitative features of the phase diagram for gravity, and in particular the appearance of a smooth and a rough phase, seem unchanged, at least as long as one does not have too many matter fields. It appears therefore that the approximation in which matter internal loops are neglected (quenched approximation) could be considered a reasonable one, and that quantities such as the critical exponents should not be too far off in this case. To the extent that the coupling between the scalar and metric degrees of freedom is weak close to the critical point, we have argued that gravity is indeed weak, and have presented a procedure by which the effective low-energy Newton constant can be estimated independently of the renormalized cosmological constant, which is determined from the scaling behavior of the average curvature close to the critical point. Our results suggest that in this model the effective gravitational coupling close to the ultraviolet fixed point grows with distance, and is expected to depend in a non-trivial way on the overall linear size of the system. For the gravitational coupling we have found an infrared growth away from the fixed point of the type $G(\mu) \sim \mu^{-1/\nu}$, while for the cosmological constant we have found

a decrease in the infrared, $\Lambda(\mu) \sim \mu^{4-1/\nu}$, with an exponent ν given approximately by $\nu \simeq 0.41$ and only weekly dependent on the matter content.

Finally let us add that our results bear some similarity with the results obtained recently from the dynamical triangulation model in four dimensions [36], where the scalar field also seems to give a rather small contribution. On the other hand the matter contribution does not seem to improve on the fact that in these models, which only allow discrete local curvatures, the average curvature does not show the correct scaling behavior close to the critical point, which is a necessary condition for defining a lattice continuum limit (in these models at the critical point the curvature diverges in physical units). Clearly more work is needed in both models to further clarify these issues.

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