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Phases of four-dimensional simplicial quantum gravity

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The phase diagram and critical exponents for pure simplicial quantum gravity (Regge calculus) in four dimensions are discussed. In the small-G phase, where G is the bare Newton's constant, the simplices are collapsed and no continuum limit exists. In the large-G phase the ground state appears to be well behaved, and the curvature goes to zero continuously as the critical value of G is approached. Fluctuations in the curvature diverge at the critical point, while volume fluctuations remain finite. The critical exponents at the transition are estimated, and appear to be independent of the strength of the higher-derivative coupling a. With the lattice analogue of the DeWitt gravitational measure and for large enough G, the lattice higher-derivative theories ($a>0$) and the reflection-positive pure Regge theory ($a=0$) appear to belong to the same phase for large enough G, which would suggest a common, unitary quantum continuum limit.

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I. INTRODUCTION

Classical simplicial quantum gravity is known to converge to the continuum theory of gravity as the lattice spacing is reduced. In the quantum theory the correspondence is less clear since there are a number of ambiguities both in the continuum and lattice formulation (such as the problem of the measure [1–3]), and no exact results are available to compare with, as in two dimensions. In four dimensions the lattice calculations are made more difficult by the fact that there are a number of possible terms both in the pure gravity action and in the measure contribution. In addition there is a conceptual issue of what physical quantities should be measured, and for what class of boundary conditions. Only a small set of these questions has been addressed up to now, mostly pertaining to an investigation of the phase diagram and the location of possible phase-transition points [4].

One of the well-known advantages of the simplicial, geometric approach to quantum gravity lies in the fact that it can be formulated in any space-time dimension, including the physically relevant case of four dimensions. Furthermore, the correspondence between the lattice and continuum operators is rather straightforward. In particular a clear distinction exists in the discrete functional integral between action and measure contributions. The presence of classical gravitational waves (gravitons) can be shown explicitly in the context of the weak-field expansion in four dimensions [5–7]. In this paper nonperturbative aspects of simplicial quantum gravity will be explored, and previous results will be extended [4,8–11].

II. GRAVITATIONAL ACTION AND MEASURE

In this work the discrete analogue of the Euclidean higher-derivative action will be used, in the form [6,4]

$$I=\sum_{\text{hinges } h}\left[\lambda V_h-k\delta_h A_h+4b\frac{A_h^2\delta_h^2}{V_h}\right]+\frac{1}{3}(a-4b)\sum_{\text{sites } p}V^p\sum_{\text{hinges } h,h'\supset p}\epsilon_{h,h'}\left[\omega_h\frac{A_h\delta_h}{V_h}-\omega_{h'}\frac{A_{h'}\delta_{h'}}{V_{h'}}\right]^2 . \tag{2.1}$$

Here δ_h is the deficit angle at a hinge (triangle), A_h is the area of the hinge, and V_h is the volume associated with that hinge. The numerical factor $\epsilon_{h,h'}$ is equal to 1 if the two hinges h,h' have one edge in common and to -2 if they do not. The motivations leading to the above discrete action have been discussed in detail elsewhere [6], and will not be repeated here. The higher-derivative term proportional to b can be considered as a regulator, and allows one to establish contact with results for continuum higher-derivative theories. Matter fields can also be included in a rather straightforward way, except possibly for fermions.

In the classical continuum limit, the above action is equivalent to the continuum Euclidean higher-derivative

$$I=\int d^4x\sqrt{g}\,[\lambda-\tfrac{1}{2}kR+bR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}+\tfrac{1}{2}(a-4b)C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}] , \tag{2.2}$$

with a cosmological-constant term (proportional to λ), the Einstein-Hilbert term ($k/2=1/16\pi G$ here, where G is the bare Newton constant), and two higher-derivative terms with dimensionless coupling constants a^{-1} and b^{-1}. One could consider the Einstein action by itself (which is not quite the same as the Regge action due to higher-order lattice corrections [5]), but then the Euclidean action would be unbounded from below. Further-

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more, the higher-derivative terms would be generated anyhow by radiative corrections at one loop, so it seems reasonable to include them from the start. It is known that the above extended higher-derivative action leads to a renormalizable and asymptotically free theory (in a^{-1} and b^{-1}) in four dimensions [3]. For sufficiently large $\lambda > 3k^2/16b$ the Euclidean action is bounded from below, leading to a convergent functional integral. Due to the complexity of the Weyl term, in the following we will consider only the case $a=4b$, which corresponds to a pure Riemann-squared higher-derivative contribution.

An important issue that needs to be addressed in lattice gravity is the problem of the gravitational measure. In the continuum the form of the measure for the $g_{\mu\nu}$ fields is not unique [1]. On the simplicial lattice the edge lengths should be considered as the elementary degrees of freedom, which uniquely specify the geometry for a given incidence matrix, and over which one should perform the functional integration. Indeed the induced metric at a simplex is related to the edge lengths squared within that simplex via the expression for the invariant line element $ds^2=g_{\mu\nu}dx^\mu dx^\nu$. In this work we will consider a class of pure gravitational measures which can be written down on the simplicial lattice by considering the "volume associated with an edge" V_{ij}, and writing for the lattice integration measure

$$\int d\mu_\epsilon[l]=\prod_{\text{edges } ij}\int_0^\infty V_{ij}^{2\sigma} dl_{ij}^2 F_\epsilon[l] \, , \tag{2.3}$$

where $2\sigma=-1/d$ for the lattice analogue of the Misner measure, and $2\sigma=(d-4)/2d$ for a lattice analogue of the DeWitt measure in d dimensions. $F_\epsilon[l]$ is a function of the edge lengths with the property that it is equal to one whenever all the triangle inequalities and their higher-dimensional analogues are satisfied, and zero otherwise. The parameter ϵ can be introduced in general as a cutoff at small edge lengths; the function $F_\epsilon[l]$ vanishes if any of the edge lengths are less than ϵ.

In the following we will consider only the case $\sigma=0$ ("DeWitt measure"). In our previous work we had considered another measure of the form dl/l. Eventually one would like to show that physical results do not depend on the specific form of the measure and on σ, or on specific short-distance details of the action. Our results in two dimensions suggest that different measures, within a certain universality class, will give the same results for infrared-sensitive quantities, like correlation functions at large distances and critical exponents [10]. The lattice path integral might not be meaningful though for certain values of σ, for which the measure becomes too singular.

While the cosmological-constant term prevents any of the edge lengths from becoming too long, the measure plays an important role in avoiding configurations in which some of the edge lengths could become very short. In turn this affects the fluctuations of the action since, for example, one has for the Regge action contribution the bound

$$2\pi(1-q_h)l_0^2 \le \delta_h A_h \le 2\pi l_0^2 \, , \tag{2.4}$$

where q_h denotes the number of simplices meeting at the hinge h, and l_0 is of the order of the average edge length. The presence of an average edge length effectively induces a cutoff in the curvatures, and thus mimics the effect of a curvature-squared term.

A useful identity can be obtained by considering the scaling properties of the Euclidean path integral (here for the case $a=4b$)

$$Z(\lambda,k,a)=\int d\mu_\epsilon[l]e^{-I[l]} \, . \tag{2.5}$$

Since for the dl^2 measure one has

$$Z(\lambda,k,a)=\left[\frac{k}{\lambda}\right]^{N_1} Z\left[\frac{k^2}{\lambda},\frac{k^2}{\lambda},a\right] , \tag{2.6}$$

one obtains the identity

$$k\langle \delta_h A_h\rangle-2\lambda\langle V_h\rangle+\frac{N_1}{N_0}=0 \, , \tag{2.7}$$

where N_0 and N_1 denote the number of sites and edges in the lattice, respectively. This identity can be useful in verifying the correctness of the numerical results.

III. PHASES OF PURE GRAVITY

It is of interest to explore by nonperturbative methods the phase diagram of the lattice theory described above. In the simulations to be discussed below the simplicial lattice was chosen to be regular and built out of rigid hypercubes, which can be subdivided into simplices by introducing face diagonals, body diagonals, and hyperbody diagonals. This choice is clearly not unique and is dictated by a criterion of simplicity, with the advantage that such a lattice can be used to study rather large systems with little modification. The edges are then individually varied (by moving at random through the lattice), and a new trial edge length is accepted with probability $\min(1,\exp(-\Delta I))$, where ΔI is the variation of the action under the change in edge length. If the triangle inequalities or their higher-dimensional analogues are violated, the new edge length is rejected. In order to compute the variation in the action under the change of one edge length, a large number of adjoining triangles and their deficit angles has to be considered.

We have considered mostly lattices of size up to 8^4 (with 61 440 edges); we have also done some short runs on 16^4 lattices (with 983 040 edges), but the results will not be discussed here in detail since the statistics at this point are still rather low. The lengths of our runs typically vary between 30 000 Metropolis, Monte Carlo iterations on the 2^4 lattice, 10 000 iterations on the 4^4 lattice, and 1000–2000 iterations on the 8^4 lattice. On the 16^4 lattice the results are from a few hundred sweeps. As starting configurations on the larger lattices duplicated copies of the smaller lattice are used for each k, allowing for additional equilibration sweeps after duplicating the lattice. One should emphasize that at this point the nature of the results is still rather preliminary. A more detailed description of the results will be given in a separate publication [12].

Quantities of physical interest which can be computed include the average curvature $\mathcal{R}$,

$$\mathcal{R} = \langle l^2 \rangle \frac{\left\langle 2\sum_h \delta_h A_h \right\rangle}{\left\langle \sum_h V_h \right\rangle} \sim \frac{\left\langle \int \sqrt{g} R \right\rangle}{\left\langle \int \sqrt{g} \right\rangle} , \tag{3.1}$$

and the average curvature squared $\mathcal{R}^2$,

$$\mathcal{R}^2 = \langle l^2 \rangle^2 \frac{\left\langle 4\sum_h \delta_h^2 A_h^2 / V_h \right\rangle}{\left\langle \sum_h V_h \right\rangle} \sim \frac{\left\langle \int \sqrt{g} R_{\mu\nu\rho\sigma}^2 \right\rangle}{\left\langle \int \sqrt{g} \right\rangle} . \tag{3.2}$$

Previous studies indicated the presence of a transition between a "smooth" (small curvature) and a "rough" (very large curvature) phase of spacetime. Besides $\mathcal{R}$ and $\mathcal{R}^2$, one can also estimate the lattice analogues of the fluctuations in the local curvatures

$$\chi_{\mathcal{R}} = \frac{1}{\left\langle \sum_h V_h \right\rangle} \left[\left\langle \left[2\sum_h \delta_h A_h \right]^2 \right\rangle - \left\langle 2\sum_h \delta_h A_h \right\rangle^2 \right] \tag{3.3}$$

and of the fluctuations in the local volumes

$$\chi_V = \frac{1}{\left\langle \sum_h V_h \right\rangle} \left[\left\langle \left[\sum_h V_h \right]^2 \right\rangle - \left\langle \sum_h V_h \right\rangle^2 \right] . \tag{3.4}$$

A divergence in the fluctuation is then indicative of long-range correlations (a massless particle) in the relevant channel. The results obtained for the average curvature $\mathcal{R}$ are shown in Fig. 1 for $a=0.005$, $\lambda=1$, and for different lattice sizes. The statistical errors in $\mathcal{R}$ are estimated by the usual binning procedure, and represent one standard deviation (one finds that as long as one does not move too close to k_c, the autocorrelations are contained). One notices that as k is varied, the curvature appears to go to zero at some finite value k_c. For k close to, but less than, k_c (and $\lambda=1$) one can write

$$\begin{aligned} \mathcal{R}(k) &\underset{k \to k_c}{\sim} A_{\mathcal{R}} (k_c - k)^{\delta} , \\ \chi_{\mathcal{R}}(k) &\underset{k \to k_c}{\sim} A_{\chi} (k_c - k)^{\delta - 1} , \end{aligned} \tag{3.5}$$

where δ is a universal exponent characteristic of the transition. Performing a simultaneous fit to $\mathcal{R}$ in $A_{\mathcal{R}}$, k_c, and the exponent δ, one finds the results that are summarized in Table I. In general the quality of the fits appears to be quite good. Altogether four values of a were considered, with five to six values of k for each a. The most accurate results with the smallest errors are for $a=0.005$ and 0.02, since for $a=0.1$ the curvature is quite small and more difficult to measure accurately, while for $a=0$ the fluctuations are significant due to the fact that one is quite close to the singularity at $k \approx 0, a \approx -0.0011$, where the curvature becomes infinite (see below). A weighted average of all the 4^4 lattice results gives $\delta = 0.70 \pm 0.13$, while a weighted average of all the 8^4 lattice results gives $\delta = 0.59 \pm 0.09$. If one combines the results for the curvature from both lattice sizes before attempting the fits, one obtains $\delta = 0.60 \pm 0.06$, and presumably also independent of a as expected from universality. If a possible logarithmic correction is taken into account in the fit

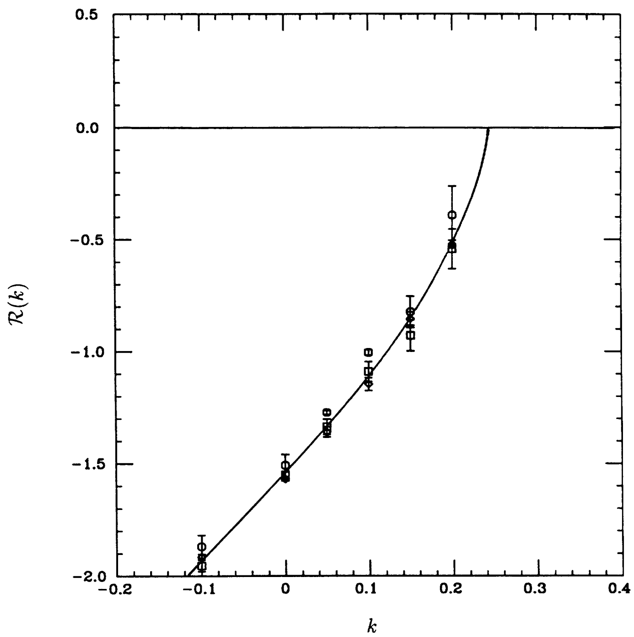

FIG. 1. Average curvature $\mathcal{R}$ as a function of k, for $\lambda=1$ and $a=0.005$ (dl^2 measure). The circles refer to $L=2$, the squares to $L=4$, and the diamonds to $L=8$.

$$\mathcal{R}(k) \underset{k \to k_c}{\sim} A_{\mathcal{R}} (k_c - k)^{\delta} |\ln(k_c - k)|^{\omega} , \tag{3.6}$$

one finds that the correction must be small ($\omega < 0.09 \pm 0.12$) and that the exponent δ appears to stay within the quoted errors. For different values of a the curvature vanishes along a line in the (k,a) plane which resembles quite closely a parabola,

$$a(k_c) = a_0 + a_1 k_c^2 . \tag{3.7}$$

Assuming this form, one finds $a_0 = -0.0011(9)$ and $a_1 = 0.083(12)$. Alternatively, one can try to determine a_0 by assuming that the average curvature amplitude (close to k_c) diverges for small a like

$$A_{\mathcal{R}}(a) \sim A_0 (a - a_0)^{-\sigma} . \tag{3.8}$$

One finds from the results in Table I $a_0 = -0.0011(4)$ and $\sigma = 1.38(5)$, in good agreement with the previous estimate for the critical value a_0.

The function $k_c(a)$ from Eq. (3.7) appears to have two branches, and along the negative k branch one finds that the curvature diverges; this then would seem to suggest that the curvature has a sharp discontinuity at $a=a_0$ and $k=0$, where it jumps from zero to infinity. In the continuum the presence of such a phase transition line is inferred from the domain of boundedness of the Euclidean action. For comparison, in the continuum one has from the classical action $a_0=0$ and $a_1=3/4$, whereas, for example, from the regular tessellation of the sphere α_5 one gets to lowest order $a_0=0$ and $a_1=0.4710$. Here instead a_0 is nonvanishing and negative, as a result of higher-order classical as well as radiative corrections, which seem to induce an effective positive R^2-type term and therefore lead to a stabilization of the pure Regge action

TABLE I. Estimates, for different values of a, of the critical amplitude $A_{\mathcal{R}}$, the critical point k_c and the critical exponent δ. Different lattice sizes (4^4 and 8^4) are considered, and estimates are presented also from the curvature averaged over both lattice sizes in order to reduce the statistical noise, and finally from a determination which always assumes $\delta \approx 0.60$.

a	L^4	$A_{\mathcal{R}}$	k_c	δ	$\chi^2/N_{\rm DF}$
0	4^4	−49.5(74)	0.101(23)	0.70(11)	2.21
	8^4	−35.4(32)	0.073(17)	0.49(9)	3.42
	4^4+8^4	−40.9(53)	0.083(9)	0.57(8)	2.92
	$4^4,8^4$	−43.3(16)	0.088(4)	[0.60]	
0.005	4^4	−3.91(8)	0.251(6)	0.66(4)	0.23
	8^4	−3.69(7)	0.239(6)	0.60(3)	0.22
	4^4+8^4	−3.76(8)	0.242(4)	0.62(2)	0.09
	$4^4,8^4$	−3.73(6)	0.240(6)	[0.60]	
0.02	4^4	−0.712(11)	0.412(11)	0.58(5)	0.78
	8^4	−0.751(5)	0.455(9)	0.71(4)	0.27
	4^4+8^4	−0.725(11)	0.422(7)	0.61(2)	0.09
	$4^4,8^4$	−0.721(5)	0.420(5)	[0.60]	
0.1	4^4	−0.063(5)	1.363(18)	1.01(22)	0.67
	8^4	−0.078(6)	0.978(21)	0.44(16)	0.85
	4^4+8^4	−0.077(5)	1.084(20)	0.60(21)	0.55
	$4^4,8^4$	−0.077(3)	1.092(21)	[0.60]	

($a=0$). This latter action is known to be equivalent to the Euclidean Einstein action only up to higher-order lattice corrections, which in principle can be computed within the framework of the lattice weak-field expansion. Furthermore, one can argue that since the lattice higher-derivative theories we have considered here ($a>0$) and the reflection positive (and therefore unitary) [13] pure Regge theory ($a=0$) appear to belong to the same phase, at least for large enough G, they should possess the same unitary quantum continuum limit.

From the analysis of the curvature fluctuation $\chi_{\mathcal{R}}$ one obtains similar and consistent values for δ and k_c, but as expected with larger errors, since the fluctuation is more difficult to compute accurately. In Fig. 2 the curvature susceptibility is shown, again for $a=0.005$; other values of a show a rather similar behavior. The results are consistent with the picture of a vanishing curvature and a divergent curvature fluctuation, at the same value of k_c. At this point one cannot entirely exclude a discontinuous (first-order) transition at k_c, with a rather small discontinuity. But from our results there is clearly no evidence for such a discontinuous transition. The results presented above correctly give a negative sign for the average curvature, which is needed for $k<k_c$ in order to have a positive fluctuation $\chi_{\mathcal{R}}$. Furthermore, the average curvature becomes complex for $k>k_c$, a reflection of the fact that the theory becomes unstable in that regime. The curvature is really infinite (or very large on a large lattice) in this phase, and the simplices collapse into degenerate configurations with very small volumes ($\langle V_h \rangle / \langle l^2 \rangle^2 \sim 0$). This is the region of the usual weak-field expansion ($G \rightarrow 0$), and it is therefore not surprising that such an expansion has difficulties in extending to the region where a sensible path integral for pure gravity can be defined. A qualitative picture of the phase diagram for pure gravity with the above lattice action is sketched in Fig. 3. If on the other hand one computes the volume susceptibility χ_V, one finds that it approaches a finite value at k_c, suggesting the absence of critical volume fluctuations. These appear to be desirable properties in a quantum theory of gravity, where the excitations in the continuum are expected to be massless gravitons, without massless scalar

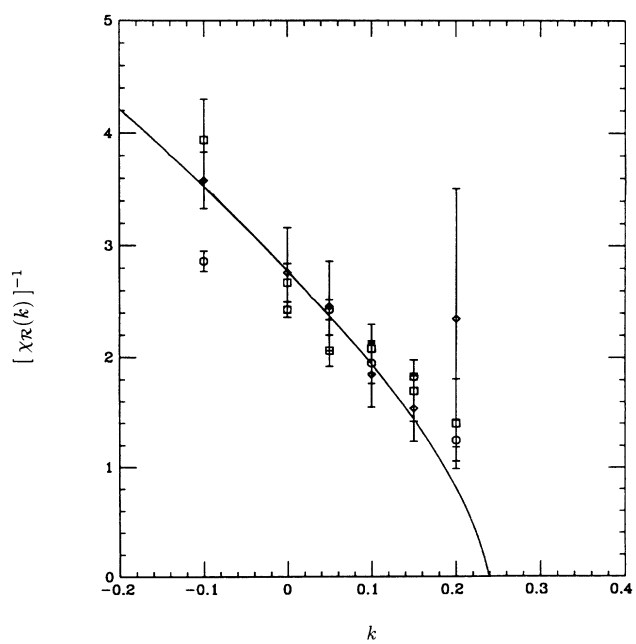

FIG. 2. Inverse of the curvature fluctuation $\chi_{\mathcal{R}}$ as a function of k, for the same parameters as in Fig. 1.

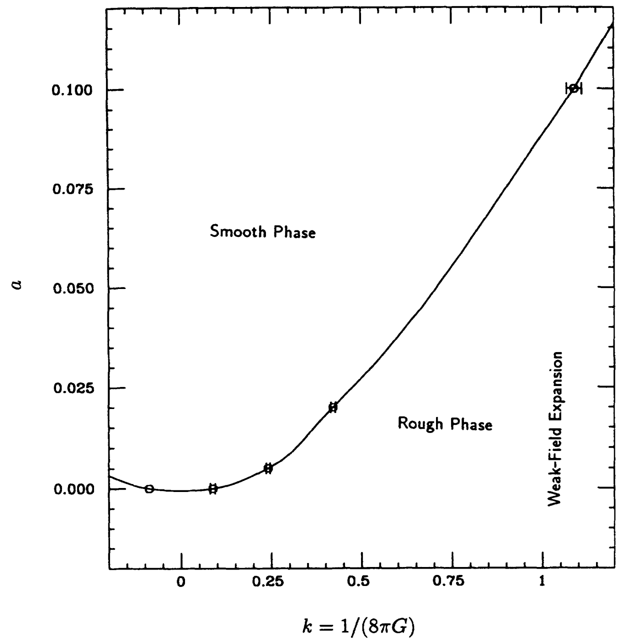

FIG. 3. Phase diagram for pure four-dimensional simplicial gravity with a higher-derivative coupling a (for fixed $\lambda=1$).

states and without massless volume density fluctuations. This situation should be contrasted to the two-dimensional case, where the volume fluctuations (the Liouville mode) are found to be massless, as expected from continuum arguments [10].

The results for the average curvature $\mathcal{R}$ are not inconsistent with known results within the weak-field expansion in the continuum (for small a). Substituting $k^{-1}=8\pi G$, and setting $k_c=c\Lambda^2$, where c is a constant independent of k, and Λ is the ultraviolet cutoff (here of the order of the average inverse lattice spacing $\sim\langle l^2\rangle^{-1/2}$), one obtains from Eq. (3.5)

$$\begin{aligned}\mathcal{R} &\sim A_{\mathcal{R}}\left[\frac{-1}{8\pi G}\right]^{\delta}(1-c\Lambda^2 8\pi G)^{\delta}\\ &\sim A_{\mathcal{R}}\left[\frac{-1}{8\pi G}\right]^{\delta}\left[1+\delta c\Lambda^2(-8\pi G)+\frac{\delta(\delta-1)}{2}\right.\\ &\qquad\left.\times(c\Lambda^2)^2(-8\pi G)^2+\cdots\right]. \end{aligned} \tag{3.9}$$

One can see that $\mathcal{R}$ is possibly not analytic at $G=0$, and an expansion in powers of G involves increasingly higher powers of the ultraviolet cutoff Λ, as expected from a theory which is not perturbatively renormalizable in G.

If one assumes that the curvature $\mathcal{R}$ has scaling dimension d_R, $\mathcal{R}\sim m^{d_R}$, then one obtains from Eq. (3.5) and from the definition of the curvature fluctuation in Eq. (3.3).

$$\begin{aligned}\mathcal{R}(k) &\underset{k\to k_c}{\sim} (\Delta k)^{\delta}\sim m^{d_R},\\ \chi_{\mathcal{R}}(k) &\underset{k\to k_c}{\sim} (\Delta k)^{\delta-1}\sim m^{2d_R-d}\end{aligned} \tag{3.10}$$

with $\Delta k=k_c-k$. Classically one would expect $d_R=2$; here instead $d_R=d\delta/(1+\delta)$ and $m\sim(\Delta k)^{\nu}$ with $\nu=(\delta+1)/d=0.40\pm0.02$. The same result is found alternatively by setting the action fluctuation, or specific heat, exponent $\alpha\equiv2-d\nu=\delta-1$ (since the average curvature represents one of the contributions to the average action). Therefore in this model the average curvature appears to be related to a dynamical "graviton mass" via

$$\mathcal{R}(k)\sim m^{d\delta/(1+\delta)}. \tag{3.11}$$

Only at k_c do both the curvature and the "graviton mass" vanish. Away from k_c, the size of this mass is related to the average curvature of the Universe via the above equation.

It would appear that close to the transition one is dealing with two rather different length scales. One can define first a length scale R_0 associated exclusively with the spacetime average of the curvature, and therefore related to some average curvature radius,

$$\frac{\left\langle 2\sum_h \delta_h A_h\right\rangle}{\left\langle \sum_h V_h\right\rangle}\equiv\pm\frac{1}{R_0^2}\sim\left[\frac{4\lambda}{k}\right]_{\mathrm{eff}} \tag{3.12}$$

and the choice of sign depends on whether the average curvature is positive or negative. As one approaches the fixed point at k_c this length scale becomes very large. Naturally another length scale M_0^{-1} can then be associated with the average volume (per hinge or per site)

$$\frac{N}{\left\langle \sum_h V_h\right\rangle}\equiv\frac{M_0}{R_0^3}\sim\lambda_{\mathrm{eff}}. \tag{3.13}$$

Close to and below the critical point this particular combination approaches some constant value. The results suggest that one is dealing simultaneously with a very large (R_0) and a very small (M_0^{-1}) length scale, associated with the curvature and density of the Universe, respectively. The renormalized cosmological constant can be made very small, and in the model this is simply a consequence of the fact that the curvature is small in units of the volume, as long as one does not cross over into the "rough" phase of gravity ($k>k_c$). There no sensible ground state seems to exist, at least within the context of this model.

Still it would be more desirable if one could compute the effective, renormalized Newton's constant directly. In order to achieve this goal some correlations at fixed geodesic distance have to be computed, a notoriously difficult task [4]. As a step in this direction we have computed both the edge-edge and curvature-curvature connected correlation functions at fixed geodesic distance d and close to k_c, where we expect the power-law decay

$$\begin{aligned}G^{(l)}_{\alpha\beta}(d)&=\langle l_\alpha^2(d)\,l_\beta^2(0)\rangle_c \underset{d\to\infty}{\sim}\langle l^2\rangle^2 T^{(l)}_{\alpha\beta}\frac{C_l}{4\pi^2 d^2},\\ G^{(R)}_{\alpha\beta}(d)&=\langle \delta_\alpha A_\alpha(d)\,\delta_\beta A_\beta(0)\rangle_c \underset{d\to\infty}{\sim}\langle l^2\rangle^2 T^{(R)}_{\alpha\beta}\frac{C_R}{4\pi^2 d^8}.\end{aligned} \tag{3.14}$$

Here the indices α,β label the edges and hinges (triangles) within a hypercube respectively. On the specific simplicial lattice we are considering $G^{(l)}_{\alpha\beta}(d)$ is a 15×15 matrix for each d, while $G^{(R)}_{\alpha\beta}(d)$ is a 50×50 matrix.

In general the above correlations will contain particles of different spin $(0,1,2,\ldots)$, but only the lightest (massless) state with spin two should dominate at large distances. Therefore one expects that the largest eigenvalue $\lambda^{(l)}_{\max}(d)$ of the edge-edge correlation matrix $G^{(l)}_{\alpha\beta}(d)$ will decay like $1/d^2$ for large geodesic distance d. The quantity $C_l=4\pi^2 d^2\lambda^{(l)}_{\max}(d)$ should approach a constant, which can be taken as a possible definition of the effective Newton's constant in units of the ultraviolet cutoff, $1/k_{\text{eff}}\equiv 8\pi G_{\text{eff}}=C_l$. It would seem from our results that this quantity tends to a finite value as k tends to k_c. For example, from the edge-edge correlations up to about geodesic distance 8 on the 16^4 lattice, and for $a=0.005$ and $k=k_c=0.239$, one finds $1/k_{\text{eff}}\sim C_l\approx 3.85$, which is quite close to the bare value $1/k\approx 4.18$. More generally it would seem that the average curvature in units of the ultraviolet cutoff tends to zero, in the way described before, as one approaches the fixed point, while the effective Newton's constant approaches some finite value which is of the same order as the cutoff. Alternatively one could determine $1/k_{\text{eff}}$ by computing the analogues of Wilson loops, which involve the determination of the dependence of the deficit angle associated with a large loop on its physical perimeter length.

It is of interest to explore other correlations that are of a purely geometric nature. We have found that some of the geometric properties of the discrete simplicial manifold are close to being Euclidean (for $k\lesssim k_c$). As an example we have considered how the number of points within geodesic distance d and $d+\Delta d$ scales with the geodesic distance itself. This quantity is equivalent, up to a constant which depends on the average lattice spacing, to the physical three-volume within geodesics distance d and $d+\Delta d$. In practice the geodesic distance between two arbitrary points on the simplicial lattice can be determined from a fixed configuration of edge lengths by selecting, among all the possible random walks between the two points that are less than some cutoff length ($\approx 2L$), the one with the shortest length. One finds for the small distances considered in this work ($d\leq 8\sqrt{\langle l^2\rangle}\approx 16$) and which correspond therefore to only a few lattice spacings,

$$N(d)\underset{d\to\infty}{\sim} d^{d_v} \tag{3.15}$$

with $d_v\approx 3.4$, which is roughly consistent with the flat-space value of three.

Many questions have remained open. It would be of interest to investigate how our results depend on a for even large values, and complete the picture for the phase diagram for pure gravity. The sensitivity of the results on the measure should be explored further. It would also be of interest to investigate these questions in the presence of matter fields, as well as for surfaces with boundaries. Finally it would be useful to reproduce some of the above results in the context of a model for lattice gravity based not on a regular lattice with fixed coordination number as we have done here, but instead on a random simplicial lattice. This would provide further evidence that one is dealing with the correct continuum theory.

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