Ultraviolet Divergences and Scale-Dependent Gravitational Couplings

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I review the field-theoretic renomalization group approach to quantum gravity, built around the existence of a non-trivial ultraviolet fixed point in four dimensions. I discuss the implications of such a fixed point, found in three largely unrelated non-perturbative approaches, and how it relates to the vacuum state of quantum gravity, and specifically to the running of G. One distinctive feature of the new fixed point is the emergence of a second genuinely non-perturbative scale, analogous to the scaling violation parameter in non-abelian gauge theories. I argue that it is natural to identify such a scale with the small observed cosmological constant, which in quantum gravity can arise as a non-perturbative vacuum condensate. I then show how the lattice cutoff theory of gravity can in principle provide quantitative predictions on the running of G, which can then be used to construct manifestly covariant effective field equations, and from there estimate the size of non-local quantum corrections to the standard GR framework.

Keywords: quantum gravitation, path integrals, renormalization group

1. Perturbative Non-renormalizability and Feynman Path Integral

In a quantum theory of gravity the coupling constant is dimensionful, $G \sim \mu^{2-d}$, and within the standard perturbative treatment of radiative corrections one expects trouble in four dimensions, based on purely on dimensional grounds. The divergent one loop corrections are proportional to $G\Lambda^{d-2}$ where Λ is the ultraviolet cutoff, which then leads to a bad high momentum behavior, with an effective running Newton's constant

$$G(k^2) \, / \, G \; \sim \; 1 + c_1(d) \; G \, k^{d-2} + \, O(G^2) \; . \tag{1}$$

A more general argument for perturbative non-renormalizability starts by considering the gravitational action with scalar curvature term R, which involves two derivatives of the metric. Then the graviton propagator in momentum space goes like $1/k^2$, and the vertex functions like k^2. In d dimensions each loop integral with involve a momentum integration d^dk, so that the superficial degree of divergence $\mathcal{D}$ of a Feynman diagram with L loops is given by

$$\mathcal{D} = 2 + (d-2) \, L \; , \tag{2}$$

independent of the number of external lines. One therefore concludes that for $d > 2$ the degree of divergence for Einstein gravity increases rapidly with loop order L, and that the theory cannot be renormalized in naive perturbation theory. A consequence

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of the lack of perturbative renormalizability is the fact that new higher derivative counterterms arise to one-loop order [1]

$$\Delta \mathcal{L}_g = \frac{\sqrt{g}}{8\pi^2 (d-4)} \left(\frac{1}{120} R^2 + \frac{7}{20} R_{\mu\nu} R^{\mu\nu} \right) , \tag{3}$$

with even higher derivatives appearing at the next order. One concludes that the standard approach based on a perturbative expansion of the pure Einstein theory in four dimensions is not convergent; in fact it is badly divergent.

A number of possible options have been proposed, the simplest of which is to just add the above higher derivative terms to the original action. The resulting extended theory is perturbatively renormalizabe to all orders, but suffers potentially from unitarity problems. But these cannot be satisfactorily addressed in perturbation theory, as the theory is now asymptotically free in the higher derivative couplings and presumably exhibits a non-trivial vacuum. Alternatively, the gravity divergences can be cancelled by adding new unobserved massless particles and invoking supersymmetry; in fact it has been claimed recently that $N = 8$ supergravity might not be just renormalizable, but indeed finite to some relatively high loop order. The downside of this somewhat contrived approach is the artificial introduction of a plethora of unobserved massless particles of spin $0, 1/2, 1, 3/2$, added to the original action in order to cancel the gravitational ultraviolet divergences. Finally, string theory solves the problem of ultraviolet divergences by postulating the existence of fundamental extended objects, thus in part bypassing the requirement of supersymmetry and providing a natural cutoff for gravity, related to a fundamental string scale [2].

Nevertheless one important point that cannot be overlooked is the fact that in other field theories, which to some extent share with gravity the same set of ultraviolet problems (the non-linear sigma model is the most notable one, and the best studied case), the analogous result of Eq. (1) is in fact known to be *incorrect*. It appears as an artifact of naive perturbation theory, which in four dimensions does not converge, and seems to lead therefore to fundamentally incorrect answers. The correct answer in these models is found instead either by expanding around the dimension in which the theory *is* perturbatively renormalizable, or by solving it exactly in the large N limit and then computing $1/N$ corrections, or by solving it numerically on a lattice. The generic new feature in these models is the existence of a non-trivial fixed point of the renormalization group [3-7], which is inaccessible by perturbation theory in four dimensions, and radically alters the ultraviolet behavior of the theory. [a] The key question for gravity is therefore: are the ultraviolet problems

[a] After QED, the second most accurate prediction of quantum field theory to date is for a perturbatively non-renormalizable theory, the $O(N)$ non-linear σ-model in three dimensions, whose field theoretic treatment based on a non-trivial fixed point of the renormalization group, either on the lattice or in the continuum, eventually provides detailed predictions for scaling behavior and anomalous dimensions in the vicinity of the fixed point [5,8]. These have recently been verified experimentally to high accuracy in a sophisticated space shuttle experiment [9] for critical superfuid Helium, whose order parameter corresponds to $N = 2$ in the non-linear σ-model.

just an artifact of a naive application of perturbation theory in four dimensions, as clearly happens in other perturbatively non-renormalizable theories that also contain dimensionful couplings in four dimensions?

In the following I will limit my discussion to the approach based on traditional quantum field theory methods and the renormalization group, applied to the Einstein action with a cosmological term, an avenue which in the end is intimately tied with the existence of a non-trivial ultraviolet fixed point in G in four dimensions. The nature of such a fixed point was first discussed in detail by K. Wilson for scalar and fermionic theories [3], and the methods later applied to gravity in [6], where they were referred to as asymptotic safety. As discussed above, it is fair to say that so far this is the only approach known to work in other not perturbatively renormalizable theories.

If non-perturbative effects play an important role in quantum gravity, then one would expect the need for an improved formulation of the quantum theory, which does not rely exclusively on the framework of perturbation theory. After all, the fluctuating quantum metric field $g_{\mu\nu}$ is dimensionless, and carries therefore no natural scale. For the somewhat simpler cases of a scalar field and non-Abelian gauge theories a consistent non-perturbative formulation based on the Feynman path integral has been used for some time, and is by now well developed. In a nutshell, the Feynman path integral formulation for quantum gravitation can be expressed by the functional integral formula

$$Z = \int_{\text{geometries}} e^{\frac{i}{\hbar} I_{\text{geometry}}} \quad . \tag{4}$$

Furthermore a bit of thought reveals that for gravity, to all orders in the weak field expansion, there is really no difference of substance between the Lorentzian (or pseudo-Riemannian) and the Euclidean (or Riemannian) formulation, which can be mapped into each other by analytic continuation. In the following therefore the Euclidean formulation will be assumed, unless stated otherwise.

In function space one needs a metric before one can define a volume element. Therefore, following DeWitt, one first defines an invariant norm for metric deformations

$$\| \delta g \|^2 \; = \; \int d^d x \; \delta g_{\mu\nu}(x) \, G^{\mu\nu,\alpha\beta} \big[g(x) \big] \, \delta g_{\alpha\beta}(x) \quad , \tag{5}$$

with the supermetric G given by the ultra-local expression

$$G^{\mu\nu,\alpha\beta} \big[g(x) \big] \; = \; \tfrac{1}{2} \sqrt{g(x)} \; \big[g^{\mu\alpha}(x) \, g^{\nu\beta}(x) + g^{\mu\beta}(x) \, g^{\nu\alpha}(x) - \lambda \, g^{\mu\nu}(x) \, g^{\alpha\beta}(x) \big] \tag{6}$$

with λ a real parameter, $\lambda \neq 2/d$. The DeWitt supermetric then defines a suitable functional volume element $\sqrt{G}$ in four dimensions,

$$\int [d \, g_{\mu\nu}] \; = \; \int \prod_x \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \; . \tag{7}$$

The Euclidean Feynman path integral for pure Einstein gravity with a cosmological constant term is then written as

$$Z_{cont} \;=\; \int [d\, g_{\mu\nu}] \;\exp \left\{ -\lambda_0 \int dx \sqrt{g} \;+\; \frac{1}{16\pi G} \int dx \sqrt{g}\, R \right\} \;. \tag{8}$$

An important aspect of this path integral is connected with the global scaling properties of the action and the measure [10]. First one notices that in pure Einstein gravity with a bare cosmological constant term

$$\mathcal{L} = - \frac{1}{16\pi G_0} \sqrt{g}\, R \;+\; \lambda_0 \sqrt{g} \tag{9}$$

one can rescale the metric by $g_{\mu\nu} = \omega\, g'_{\mu\nu}$ with ω a constant, giving

$$\mathcal{L} = - \frac{1}{16\pi G_0} \,\omega^{d/2-1} \sqrt{g'}\, R' \;+\; \lambda_0\, \omega^{d/2} \sqrt{g'} \;. \tag{10}$$

This can then be interpreted as a rescaling of the two bare couplings $G_0 \rightarrow \omega^{-d/2+1} G_0$, $\lambda_0 \rightarrow \lambda_0\, \omega^{d/2}$, leaving the dimensionless combination $G_0^d \lambda_0^{d-2}$ unchanged. Therefore only the latter combination has physical meaning in pure gravity. In particular, one can always choose the scale $\omega = \lambda_0^{-2/d}$, so as to adjust the volume term to have a unit coefficient. The implication of this last result is that pure gravity only contains one bare coupling G_0, besides the ultraviolet cutoff Λ needed to regulate the quantum theory.

2. Gravity in $2 + \epsilon$ Dimensions and Non-Trivial UV Fixed Point

In two dimensions the gravitational coupling becomes dimensionless, $G \sim \Lambda^{2-d}$, and the theory appears perturbatively renormalizable. In spite of the fact that the gravitational action reduces to a topological invariant, it is meaningful to attempt to construct, in analogy to what was suggested originally by Wilson for scalar field theories, the theory perturbatively as a double series in $\epsilon = d - 2$ and G. The $2 + \epsilon$ expansion for pure gravity then proceeds as follows [6,11]. First the gravitational part of the action

$$\mathcal{L} = - \frac{\mu^{\epsilon}}{16\pi G} \sqrt{g}\, R \;+\; \lambda_0 \sqrt{g} \;, \tag{11}$$

with G now dimensionless and μ an arbitrary momentum scale, is expanded by setting

$$g_{\mu\nu} \;\rightarrow\; \bar{g}_{\mu\nu} = g_{\mu\nu} \;+\; h_{\mu\nu} \;, \tag{12}$$

where $g_{\mu\nu}$ is the classical background field, and $h_{\mu\nu}$ a small quantum fluctuation. The quantity $\mathcal{L}$ in Eq. (11) is naturally identified with the bare Lagrangian, and the scale μ with a microscopic ultraviolet cutoff Λ, corresponding to the inverse lattice spacing in the lattice formulation. After the quantum fluctuations in $h_{\mu\nu}$ are integrated out and the cosmological constant term gets rescaled, one obtains the following result for the renormalization group beta function for G: with N_S scalar fields and N_F Majorana fermion fields the result to two loops reads [12]

$$\mu \frac{\partial}{\partial \mu} G = \beta(G) = \epsilon \, G \; - \; \beta_0 \, G^2 \; - \; \beta_1 \, G^3 \; + \; O(G^4, G^3 \epsilon, G^2 \epsilon^2) \; , \tag{13}$$

with $\beta_0 = \frac{2}{3} \, (25 - c)$ and $\beta_1 = \frac{20}{3} \, (25 - c)$, and $c \equiv N_S + N_F/2$. The physics of this result is contained in the fact that the gravitational β-function determines the scale dependence of Newton's constant G, and has the shape shown in Fig. 1.

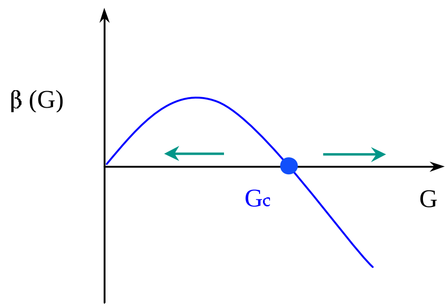

Figure 1. Renormalization group β-function for gravity close to two dimensions. The arrows indicate the coupling constant flow towards increasingly larger distance scales.

A closer examination of the results shows that depending on whether one is on the right ($G > G_c$) or on the left ($G < G_c$) of the non-trivial ultraviolet fixed point at

$$G_c = \frac{d-2}{\beta_0} + O((d-2)^2) \tag{14}$$

(with G_c positive, provided one has $c < 25$) the coupling will either flow to increasingly larger values of G, or flow towards the Gaussian fixed point at $G = 0$, respectively. Furthermore the running of G as a function of the sliding momentum scale $\mu = k$ can be obtained by integrating Eq. (13), and one has to lowest order

$$G(k^2) \; \simeq \; G_c \left[1 \, \pm \, c_0 \left(\frac{m^2}{k^2} \right)^{(d-2)/2} + \, \dots \right] \; , \tag{15}$$

with c_0 a positive constant, and m a new nonperturbative scale. As in non-abelian gauge theories and QCD, this last quantity arises naturally as an integration constant of the renormalization group equations. The choice of + or − sign is then determined from whether one is to the left (+), or to right (-) of G_c, in which case the effective $G(k^2)$ decreases or, respectively, increases as one flows away from the ultraviolet fixed point towards lower momenta, or larger distances. Physically therefore the two solutions represent a gravitational screening ($G < G_c$), and a gravitational anti-screening ($G > G_c$) situation [10]. Finally, at energies sufficiently high to become comparable to the ultraviolet cutoff, the gravitational coupling G eventually flows towards the ultraviolet fixed point $G(k^2) \sim_{k^2 \to \Lambda^2} G(\Lambda)$, where $G(\Lambda)$ is the coupling at the cutoff scale Λ, to be identified with the bare or lattice coupling.

One message therefore is that the quantum corrections involves a new physical, renormalization group invariant scale $\xi = 1/m$, which cannot be fixed perturbatively and whose size determines the scale for the new quantum effects. In terms of the bare coupling $G(\Lambda)$, it is given by

$$\xi^{-1} = m = A_m \cdot \Lambda \, \exp\left(- \int^{G(\Lambda)} \frac{dG'}{\beta(G')} \right) . \tag{16}$$

The constant A_m on the r.h.s. of Eq. (16) cannot be determined by perturbation theory; it needs to be computed by non-perturbative lattice methods.

At the fixed point $G = G_c$ the theory is scale invariant by definition; in statistical field theory language the fixed point corresponds to a phase transition. In the vicinity of the fixed point one can write

$$\beta(G) \underset{G \to G_c}{\sim} \beta'(G_c)\,(G - G_c) \, + \, O((G - G_c)^2) \, . \tag{17}$$

If one defines the exponent ν by $\beta'(G_c) = -1/\nu$, then from Eq. (16) one has by integration

$$m \underset{G \to G_c}{\sim} \Lambda \cdot A_m \, | \, G(\Lambda) - G_c |^{\nu} \;\; , \tag{18}$$

with ν the correlation length exponent. To two loops the results of [12] imply

$$\nu^{-1} = \epsilon + \frac{15}{25 - c} \, \epsilon^2 \, + \, \dots \tag{19}$$

which gives, for pure gravity without matter ($c = 0$) in four dimensions, to lowest order the scaling exponent $\nu^{-1} = 2$, and $\nu^{-1} \approx 4.4$ at the next order. The key question raised by these $2 + \epsilon$ perturbative calculations is therefore: what remains of the above phase transition in four dimensions, how are the two phases of gravity characterized non-perturbatively, and what is the value of the exponent ν determining the running of G in the vicinity of the fixed point in four dimensions ? To answer this question in a controlled way would seem to require the introduction of a non-perturbative regulator, based on the lattice formulation (since no other reliable non-perturbative regulator for field theories is known to date).

3. Lattice Regularized Quantum Gravity

On the lattice the infinite number of degrees of freedom in the continuum is restricted, by considering Riemannian spaces described by only a finite number of variables, to the geodesic distances between neighboring points. Such spaces are taken to be flat almost everywhere, and referred to as piecewise linear. The elementary building blocks for d-dimensional space-time are then simplices of dimension d.

A 0-simplex is a point, a 1-simplex is an edge, a 2-simplex is a triangle, a 3-simplex is a tetrahedron. A d-simplex is a d-dimensional object with $d+1$ vertices and $d(d+1)/2$ edges connecting them [13].

The geometry of the interior of a d-simplex is assumed to be flat, and is therefore completely specified by the lengths of its $d(d+1)/2$ edges. When focusing on one such n-simplex, it is convenient to label the vertices by $0, 1, 2, 3, \ldots, n$ and denote the square edge lengths by $l_{01}^2 = l_{10}^2$, ... , l_{0n}^2. The simplex can then be spanned by the set of n vectors e_1, ... e_n connecting the vertex 0 to the other vertices. To the remaining edges within the simplex one then assigns vectors $e_{ij} = e_i - e_j$ with $1 \leq i < j \leq n$. Within each n-simplex one can define a metric $g_{ij}(s) = e_i \cdot e_j$, and then in terms of the edge lengths $l_{ij} = |e_i - e_j|$ the metric is given by

$$g_{ij}(s) = \tfrac{1}{2} \left(l_{0i}^2 + l_{0j}^2 - l_{ij}^2 \right) . \tag{20}$$

The volume of a general n-simplex can be found by the n-dimensional generalization of the well-known formula for a tetrahedron, namely

$$V_n(s) = \frac{1}{n!} \sqrt{\det g_{ij}(s)} . \tag{21}$$

In a piecewise linear space curvature is detected by going around elementary loops which are dual to a $(d-2)$-dimensional subspace. From the dihedral angles $\theta(s, h)$ associated with the faces of the simplices meeting at a given hinge h one computes the deficit angle $\delta(h)$, defined as [13]

$$\delta(h) = 2\pi - \sum_{s \supset h} \theta(s, h) , \tag{22}$$

where the sum extends over all simplices s meeting on h. It then follows that the deficit angle δ is a measure of the local curvature at h. By considering rotation matrices around a hinge one can obtain an expression for the Riemann tensor at the hinge h

$$R_{\mu\nu\lambda\sigma}(h) = \frac{\delta(h)}{A_C(h)} U_{\mu\nu}(h) \, U_{\lambda\sigma}(h) , \tag{23}$$

which is expected to be valid in the limit of small curvatures, with $A_C(h)$ the area of the loop entangling the hinge h, and U a bivector describing the hinge's orientation. From the expression for the Riemann tensor at a hinge given in Eq. (23) one obtains by contraction the Ricci scalar $R(h) = 2\,\delta(h)/A_C(h)$, and the continuum expression $\sqrt{g}\,R$ is then obtained by multiplication with the volume element $V(h)$ associated with a hinge. The curvature and cosmological constant terms then lead to the combined Regge lattice action

$$I_{\rm latt}(l^2) = \lambda_0 \sum_{\rm simplices\ s} V_s^{(d)} - k \sum_{\rm hinges\ h} \delta_h \, V_h^{(d-2)} . \tag{24}$$

One key aspect of this formulation is the local gauge invariance of the lattice action, in analogy to the local gauge invariance of the Wilson action for gauge theories. Already on a flat 2-d lattice it is clear that one can move around a point on a surface, keeping all the neighbors fixed, without violating the triangle inequalities, and leaving local curvature invariants unchanged. In d dimensions this transformation has d parameters and is an exact invariance of the action. When space is slightly curved, piecewise linear diffeomorphisms can still be defined as the set of local motions of points that leave the local contribution to the action, the measure and the lattice analogs of continuum curvature invariants unchanged. In the limit when the number of edges becomes very large one expects the full continuum diffeomorphism group to be recovered [14].

In order to write down a lattice path integral, one needs, besides the action, a functional measure. As the edge lengths l_{ij} play the role of the continuum metric $g_{\mu\nu}(x)$, one expects the discrete measure to involve an integration over the squared edge lengths. After choosing coordinates along the edges emanating from a vertex, the relation between metric perturbations and squared edge length variations for a given simplex based at 0 in d dimensions is from Eq. (20)

$$\delta g_{ij}(l^2) \; = \; \tfrac{1}{2} \left(\delta l_{0i}^2 + \delta l_{0j}^2 - \delta l_{ij}^2 \right) . \tag{25}$$

For one d-dimensional simplex labeled by s the integration over the metric is thus equivalent to an integration over the edge lengths, and one has the identity

$$\left(\frac{1}{d!} \sqrt{\det g_{ij}(s)} \right) \prod_{i \geq j} dg_{ij}(s) = \left(-\tfrac{1}{2} \right)^{\frac{d(d-1)}{2}} \left[V_d(l^2) \right] \prod_{k=1}^{d(d+1)/2} dl_k^2 \, . \tag{26}$$

Indeed there are $d(d+1)/2$ edges for each simplex, just as there are $d(d+1)/2$ independent components for the metric tensor in d dimensions. In addition, a certain set of simplicial inequalities need to be imposed on the edge lengths. These represent conditions conditions on the edge lengths l_{ij} such that the sites i can be considered as vertices of a d-simplex embedded in flat d-dimensional Euclidean space. After summing over all simplices one derives what is regarded as the lattice functional measure representing the continuum DeWitt measure in four dimensions

$$\int [dl^2] \; = \; \int_0^\infty \prod_{ij} dl_{ij}^2 \; \Theta[l_{ij}^2] \quad . \tag{27}$$

Here $\Theta[l_{ij}^2]$ is a (step) function of the edge lengths, with the property that it is equal to one whenever the triangle inequalities and their higher dimensional analogs are satisfied and zero otherwise. The lattice action of Eq. (24) for pure four-dimensional Euclidean gravity then leads to the regularized lattice functional integral [7]

$$Z_{latt} \; = \; \int [d\,l^2] \; \exp \left\{ -\lambda_0 \sum_h V_h \; + \; k \sum_h \delta_h A_h \right\} \; , \tag{28}$$

where, as customary, the lattice ultraviolet cutoff is set equal to one (i.e. all length scales are measured in units of the lattice cutoff). Furthermore, λ_0 sets the overall scale, and can therefore be set equal to one without any loss of generality, according to the scaling arguments presented before. The lattice partition function Z_{latt} should then be compared to the continuum Euclidean Feynman path integral for pure gravity of Eq. (8).

In closing we note that what makes the Regge theory stand out compared to other possible discretization of gravity is the fact that it is the *only* lattice theory known to have the correct spectrum of continuous excitations in the weak field limit, i.e. transverse traceless modes, or, equivalently, helicity-two massless gravitons. Indeed one of the simplest possible problems that can be treated in lattice quantum gravity is the analysis of small fluctuations about a fixed flat simplicial background. In this case one finds that the lattice graviton propagator in a De Donder-like gauge is identical to the continuum expression [7].

4. Strongly Coupled Gravity and Gravitational Wilson Loop

As in non-abelian gauge theories, important information about the non-perturbative ground state of the theory can be gained by considering the strong coupling limit. In lattice gravity an expansion can be performed for large G or small $k = 1/8\pi G$, and the resulting series is in general expected to be useful up to some $k = k_c$, where k_c is the lattice critical point, at which point the partition function Z eventually develops a singularity. One starts from the lattice regularized path integral with action Eq. (24) and measure Eq. (27). The four-dimensional Euclidean lattice action usually contains a cosmological constant and scalar curvature term as in Eq. (24),

$$I_{latt} \;=\; \lambda \sum_h V_h(l^2) \;-\; k \sum_h \delta_h(l^2)\, A_h(l^2) \; . \tag{29}$$

The action only couples edges which belong either to the same simplex or to a set of neighboring simplices, and can therefore be considered *local* just like the continuum action; it leads to the lattice partition function defined in Eq. (28). When doing an expansion in the kinetic term proportional to k, it is convenient to include the λ-term in the measure, and Z_{latt} can then be expanded in powers of k,

$$Z_{latt}(k) \;=\; \int d\mu(l^2)\; e^{k \sum_h \delta_h A_h} \;=\; \sum_{n=0}^{\infty} \frac{1}{n!}\, k^n \int d\mu(l^2) \left(\sum_h \delta_h\, A_h \right)^n \quad . \tag{30}$$

$Z(k) \;=\; \sum_{n=0}^{\infty} a_n\, k^n$ is analytic at $k = 0$, so this expansion is well defined up to the nearest singularity in the complex k plane.

In the gravity case the analogs of the gauge variables of Yang-Mills theories are given by the connections, so it is natural when computing the gravitational Wilson loop [15] to look for a first order formulation of Regge gravity [16]. For each neighboring pair of simplices $s, s+1$ one can associate a Lorentz transformation $R^{\mu}{}_{\nu}(s, s+1)$, and one then might want to consider a near-planar closed loop C, such as the one shown schematically in Fig.2. Along a closed loop the overall rotation matrix is given by

$$R^{\mu}{}_{\nu}(C) \;=\; \Big[\prod_{s \subset C} R_{s,s+1} \Big]^{\mu}{}_{\nu} \; . \tag{31}$$

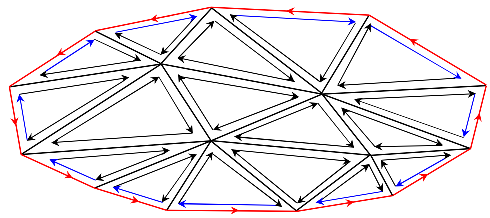

Figure 2. Gravitational analog of the Wilson loop. A vector is parallel-transported along the larger outer loop. The enclosed minimal surface is tiled with parallel transport polygons, here chosen to be triangles for illustrative purposes. For each link of the dual lattice, the elementary parallel transport matrices $\mathbf{R}(s, s')$ are represented by arrows.

In the quantum theory one is interested in averages of the above product of rotations along a given path. If the curvature of the manifold is small, then classically the expression for the rotation matrix $\mathbf{R}(C)$ associated with a near-planar loop can be re-written in terms of a surface integral of the large-scale Riemann tensor, projected along the surface area element bivector $A^{\alpha\beta}(C)$ associated with the orientation of the loop,

$$R^{\mu}{}_{\nu}(C) \approx \left[e^{\frac{1}{2} \int_S R^{\cdot}{}_{\cdot\,\alpha\beta} \, A^{\alpha\beta}(C)} \right]^{\mu}{}_{\nu} . \tag{32}$$

Thus a direct calculation of the quantum Wilson loop could in principle provide a way of determining the *effective* curvature on very large distance scales, even in the case where short distance fluctuations in the metric may be significant.

A detailed lattice calculation in the strong coupling limit then gives the following result. First one defines the lattice Wilson loop as

$$W(C) = < Tr[(U_C + \epsilon \, I_4) \, R_1 \, R_2 \dots \dots R_n] > . \tag{33}$$

where the R_i's are the rotation matrices along the path and the factor $(U_C + \epsilon I_4)$ contains some average direction bivector U_C for the loop, which is assumed to be close to planar. Then for sufficiently strong coupling one can show that one naturally obtains an area law, which here we express as

$$W(C) \simeq \exp(- A_C/\xi^2) , \tag{34}$$

where ξ is the gravitational correlation length. The appearance of ξ follows from dimensional arguments, given that the correlation length is the only relevant length scale in the vicinity of the fixed point; the results can thus be considered analogous to the well-known behavior for the Wilson loop in non-abelian gauge theories [17]. In the actual calculation the rapid decay of the quantum gravitational Wilson loop as a function of the area can be seen as a general consequence of the assumed disorder in the uncorrelated fluctuations of the parallel transport matrices $\mathbf{R}(s, s')$ at large G. A careful identification of (a suitable trace of) Eq. (32) with the expression in

Eq. (34), and in particular the comparison of the two area-dependent terms, then yields the following estimate for the macroscopic, large scale, average curvature in the large G limit

$$\bar{R} \sim 1/\xi^2 \,, \tag{35}$$

where ξ is the quantity in Eq. (16). An equivalent way of phrasing the last result is the suggestion that $1/\xi^2$, where ξ is the renormalization group invariant gravitational correlation length of Eq. (16), should be identified, up to a constant of proportionality of order one, with the observationally determined, large scale cosmological constant λ.

5. Nonperturbative Gravity

The exact evaluation of the lattice functional integral for quantum gravity by numerical methods allows one, in principle, to investigate a regime which is generally inaccessible by perturbation theory: where the coupling G is strong and quantum fluctuations in the metric are large. The hope is, in the end, to make contact with the analytic results obtained in the $2+\epsilon$ expansion, and determine which scenarios are physically realized in the lattice regularized model. The main question one would therefore like to answer is whether there is any indication that the non-trivial ultraviolet fixed point scenario is realized in the lattice theory, in four dimensions. This would imply, as in the non-linear sigma model and similar models, the existence of at least two physically distinct phases, and associated non-trivial scaling dimensions. A clear physical characterization of the two gravitational phases would also allow one, at least in principle, to decide which phase, if any, could be realized in nature. As discussed below, the lattice continuum limit is taken in the vicinity of the fixed point, so close to it is perhaps the physically most relevant regime.

At the next level one would hope to be able to establish a quantitative connection with the continuum perturbative results, such as the $2+\epsilon$ expansion discussed earlier. Since the lattice cutoff and the method of dimensional regularization cut the theory off in the ultraviolet in rather different ways, one needs to compare universal quantities which are *cutoff-independent.* An example is the critical exponent ν, as well as any other non-trivial scaling dimension that might arise. One should note that within the $2+\epsilon$ expansion only *one* such exponent appears, to *all* orders in the loop expansion, as $\nu^{-1} = -\beta'(G_c)$. Therefore one central issue in the four-dimensional lattice regularized theory is the value of the universal scaling exponent ν [10,18] [in Eqs. (15), (18) and (19)] .

The starting point is again the lattice regularized path integral with action as in Eq. (24) and measure as in Eq. (27). Among the simplest quantum mechanical averages that one can compute is one associated with the local curvature,

$$\mathcal{R}(k) \;\sim\; \frac{< \int dx \sqrt{g}\, R(x) >}{< \int dx \sqrt{g} >} \;\; . \tag{36}$$

But the curvature associated with this quantity is one that would be detected when parallel-transporting vectors around very small infinitesimal loops. Furthermore when computing correlations in quantum gravity new subtle issues arise, due to the fact that the physical distance between any two points x and y

$$d(x, y \,|\, g) \;=\; \min_{\xi} \int_{\tau(x)}^{\tau(y)} d\tau \, \sqrt{g_{\mu\nu}(\xi) \frac{d\xi^{\mu}}{d\tau} \frac{d\xi^{\nu}}{d\tau}} \tag{37}$$

is a fluctuating function of the background metric $g_{\mu\nu}(x)$. Consequently physical correlations have to be defined at fixed geodesic distance d, as in the following connected correlation between observables O

$$< \int dx \int dy \, \sqrt{g} \, O(x) \, \sqrt{g} \, O(y) \, \delta(|x-y|-d) >_c \; . \tag{38}$$

Based on general arguments one expects such correlations to either follow a power law decay at short distances, or an exponential decay characterized by a correlation length ξ at larger distances

$$< \sqrt{g} \, O(x) \, \sqrt{g} \, O(y) \, \delta(|x-y|-d) >_c \; \underset{d \gg \xi}{\sim} \; e^{-d/\xi} \quad . \tag{39}$$

In practice such correlations at fixed geodesic distance are difficult to compute numerically, and therefore not the best route to study the critical properties of the theory. But scaling arguments allow one to determine the scaling behavior of correlation functions from critical exponents characterizing the singular behavior of the free energy $F(k) = -(1/V) \ln Z$ and of various local averages in the vicinity of the critical point. In general a divergence of the correlation length ξ

$$\xi(k) \; \underset{k \to k_c}{\sim} \; A_{\xi} \, |k_c - k|^{-\nu} \tag{40}$$

signals the presence of a phase transition, and leads to the appearance of a non-analyticty in the free energy $F(k)$. One way to determine ν is from the curvature fluctuation, for which one can show

$$\chi_{\mathcal{R}}(k) \; \underset{k \to k_c}{\sim} \; A_{\chi_{\mathcal{R}}} \, |k_c - k|^{-(2-d\nu)} \quad . \tag{41}$$

From such averages and fluctutations one can therefore, in principle, extract the correlation length exponent ν of Eq. (40), without having to compute an invariant correlation function at fixed geodesic distance.

In general for the measure in Eq. (27) one finds a well behaved ground state only for $k < k_c$ [10]. The system then resides in the 'smooth' phase, with an effective dimensionality close to four. On the other hand, for $k > k_c$ the curvature becomes very large and the lattice collapses locally into degenerate configurations with very long, elongated simplices. This last phenomenon is usually interpreted as a lattice remnant of the conformal mode instability of Euclidean gravity.

There are a number of ways by which the critical exponents can be determined to some accuracy from numerical simulations,and it is beyond the scope of this short review to go into more details; one obtains eventually in $d = 4$ $k_c \simeq 0.0636$ and $\nu \simeq 0.335$, which suggests

$$\nu = 1/3 \tag{42}$$

for pure quantum gravity in four dimensions [18]. Note that at the critical point the gravitational coupling is not weak, since $G_c \approx 0.626$ in units of the ultraviolet cutoff. Fig. 3 shows a comparison of the critical exponent ν obtained by three independent methods, namely the original lattice result in $d = 2, 3, 4$ [18], the recent $2 + \epsilon$ expansion (to one and two loops) [12], and the even more recent renormalization group truncation method [19-21].

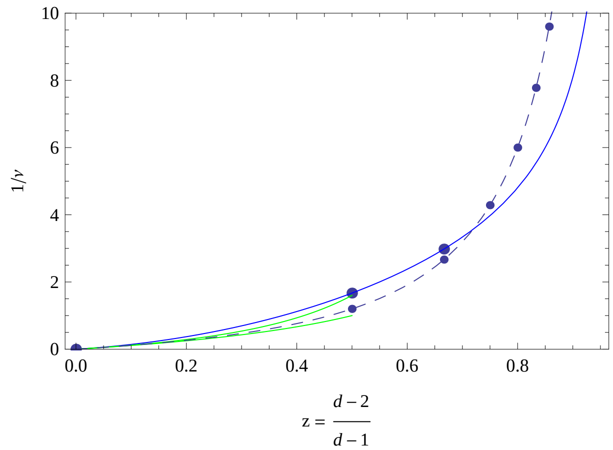

Figure 3. Universal renormalization group scaling exponent $1/\nu$ of Eq. (17), computed in the lattice theory of gravity (large dots for $d = 2$, $d = 3$ and $d = 4$, and continous interpolating line) [18]. For comparison the $2 + \epsilon$ result is shown (two lower curves, to one (lower) and two (upper) loops) [12], as well as the recent Einstein-Hilbert truncation result (smaller dots and connecting dashed line) [21]. The abscissa is, for convenience, a variable related to the space-time dimension d through $z = (d - 2)/(d - 1)$, which maps $d = \infty$ (where it is known that $\nu = 0$) to $z = 1$. Note that for a scalar theory one has $1/\nu = 2$ for $d \geq 4$ [3].

6. Renormalization Group, Lattice Continuum Limit and the Running of G

The lattice theory points to the existence of a phase transition in pure quantum gravity, with a divergent correlation length in the vicinity of the critical point, as in Eq. (40), which can be re-written in terms of the inverse correlation length $m \equiv 1/\xi$

$$\xi^{-1} = m = \Lambda \, A_m \, | \, k_c - k \, |^{\nu} \ . \tag{43}$$

In the above expression the correct dimensions have been restored, by inserting explicitly on the r.h.s. the ultraviolet cutoff Λ. Here k and k_c are dimensionless quantities, corresponding to bare microscopic couplings at the cutoff scale, $k \equiv k(\Lambda) \equiv 1/8\pi G(\Lambda)$. A_m is related to A_ξ in Eq. (40) by $A_m = A_\xi^{-1}$. It is worth pointing out that the above expression for $m(k)$ is identical in structure to the $2+\epsilon$ result for continuum gravity, Eq. (18).

Then the lattice continuum limit corresponds to a large cutoff limit, taken at *fixed* m or ξ,

$$\Lambda \to \infty \; , \quad k \to k_c \; , \quad m \text{ fixed} \; , \tag{44}$$

which shows that the continuum limit is in fact reached in the vicinity of the ultraviolet fixed point, $k \to k_c$. In practice, since the cutoff ultimately determines the physical value of Newton's constant G, the cutoff Λ cannot be taken to ∞, and it persists as a fundamental scale in the theory. A very large value will suffice, $\Lambda^{-1} \sim 10^{-33} cm$, for which it will still be true that $\xi \gg \Lambda^{-1}$, which is all that is required for the continuum limit.

In order to discuss the renormalization group behavior of the coupling in the lattice theory it is convenient to re-write the result of Eq. (43) directly in terms of Newton's constant G as

$$m \; = \; \Lambda \left(\frac{1}{c_0} \right)^{\nu} \left[\frac{G(\Lambda)}{G_c} - 1 \right]^{\nu} \; , \tag{45}$$

with the dimensionless constant c_0 related to A_m by $A_m = 1/(c_0 k_c)^\nu$. The above expression only involves the dimensionless ratio $G(\Lambda)/G_c$, which is the only relevant quantity here. From the knowledge of the dimensionless constant A_m in Eq. (43) one can estimate from first principles the value of c_0 in Eq. (45) and later in Eq. (49). Lattice results for the correlation functions at fixed geodesic distance give a value for $A_m \approx 0.72$ with a significant uncertainty, which, when combined with the values $k_c \simeq 0.0636$ and $\nu \simeq 0.335$ given above, gives $c_0 = 1/(k_c \, A_m^{1/\nu}) \simeq 42$. Then the renormalization group invariance of $m = \xi^{-1}$ requires that the running gravitational coupling $G(\mu)$ varies in the vicinity of the fixed point in accordance with the above equation, with $\Lambda \to \mu$, where μ is an arbitrary momentum scale,

$$m \; = \; \mu \left(\frac{1}{c_0} \right)^{\nu} \left[\frac{G(\mu)}{G_c} - 1 \right]^{\nu} \; . \tag{46}$$

The latter is equivalent to the renormalization group invariance requirement

$$\mu \, \frac{d}{d\,\mu} \, m(\mu, G(\mu)) \; = \; 0 \; , \tag{47}$$

provided $G(\mu)$ is varied in a specific way. Thus Eq. (47) can be used to obtain a Callan-Symanzik β-function for the coupling $G(\mu)$ in units of the ultraviolet cutoff,

$$\mu \, \frac{\partial}{\partial \, \mu} \, G(\mu) \;\; = \;\; \beta(G(\mu)) \;\; , \tag{48}$$

with $\beta(G)$ given in the vicinity of the non-trivial fixed point, from Eq. (46), by $\beta(G) \sim_{G \to G_c} - \frac{1}{\nu} \, (G - G_c)$. Or one can obtain the scale dependence of the gravitational coupling directly from Eq. (46), which then gives

$$G(\mu) \;=\; G_c \left[1 \,+\, c_0 (m^2/\mu^2)^{1/2\nu} \,+\, O(\,(m^2/k^2)^{\frac{1}{\nu}}\,) \right] \tag{49}$$

in the physical anti-screening phase. Again, this last expression can be compared directly to the lowest order $2+\epsilon$ result of Eq. (15). The physical dimensions of G can be restored, by multiplying the above expression on both sides by the ultraviolet cutoff Λ, if one so desires. Physically the above lattice result implies anti-screening: the gravitational coupling G increases slowly with distance.

7. Curvature Scales and Gravitational Condensate

The renormalization group running of $G(\mu)$ in Eq. (49) involves an invariant scale $\xi = 1/m$. At first it would seem that such a scale could take any value, including a very small one based on the naive estimate $\xi \sim l_P$ - which would then preclude any observable quantum effects in the foreseeable future. But the results from the gravitational Wilson loop at strong coupling would suggest otherwise, namely that the non-perturbative scale ξ is in fact related to macroscopic *curvature*. From astrophysical observation the average curvature is very small [22], so one would conclude that ξ has to be very large and possibly macroscopic,

$$\lambda_{obs} \;\simeq\; \frac{1}{\xi^2} \tag{50}$$

with λ_{obs} the observed small, but non-vanishing, scaled cosmological constant. A further indication that the identification of the observed cosmological constant with a mass-like - and therefore renormalization group invariant - term might make sense beyond the weak field limit can be seen, for example, by comparing the structure of the three classical field equations

$$\begin{aligned} R_{\mu\nu} \,-\, \tfrac{1}{2}\, g_{\mu\nu}\, R \,+\, \lambda\, g_{\mu\nu} \;&=\; 8\pi G\, T_{\mu\nu} \\ \partial^\mu F_{\mu\nu} \,+\, \mu^2\, A_\nu \;&=\; 4\pi e\, j_\nu \\ \partial^\mu \partial_\mu\, \phi \,+\, m^2\, \phi \;&=\; \frac{g}{3!}\, \phi^3 \end{aligned} \tag{51}$$

for gravity, QED (made massive via the Higgs mechanism) and a self-interacting scalar field, respectively. Nevertheless it seems so far that the strongest argument suggesting the identification of the scale ξ with λ is derived from the calculation of the gravitational Wilson loop at strong [16].

This relationship, taken at face value, implies a very large, cosmological value for $\xi \sim 10^{28} cm$, given the present bounds on λ_{phys}. Thus a set of modified Einstein equations, incorporating the quantum running of G, would read

$$R_{\mu\nu} \,-\, \tfrac{1}{2}\, g_{\mu\nu}\, R \,+\, \lambda\, g_{\mu\nu} \;=\; 8\pi\, G(\mu)\, T_{\mu\nu} \;, \tag{52}$$

with $\lambda \simeq 1/\xi^2$, and $G(\mu)$ on the r.h.s. scale-dependent, in accordance with Eq. (49). The precise meaning of $G(\mu)$ in a covariant framework is given below.

8. Effective Covariant Field Equations

The result of Eq. (49) implies a running gravitational coupling in the vicinity of the ultraviolet fixed point, with $m = 1/\xi$, $c_0 > 0$ and $\nu \simeq 1/3$. Since ξ is expected to be very large, the quantity G_c in the above expression should now be identified with the laboratory scale value $\sqrt{G_c} \sim 1.6 \times 10^{-33} cm$. The effective interaction in real space is then obtained by Fourier transform, but since the above expression is singular as $k^2 \to 0$, the infrared divergence needs to be regulated, which can be achieved by utilizing as the lower limit of momentum integration $m = 1/\xi$. A properly infrared regulated version of the above would read

$$G(k^2) \simeq G_c \left[1 + c_0 \left(\frac{m^2}{k^2 + m^2} \right)^{\frac{1}{2\nu}} + \dots \right] . \tag{53}$$

Then at very large distances $r \gg \xi$ the gravitational coupling is expected to approach the finite value $G_\infty = (1 + c_0 + \dots) \, G_c$.

The first step in analyzing the consequences of a running of G is to re-write the expression for $G(k^2)$ in a coordinate-independent way, for example by the use of a non-local Vilkovisky-type effective action. Since in going from momentum to position space one usually employs $k^2 \to -\Box$, to obtain a quantum-mechanical running of the gravitational coupling one has to make the replacement $G \to G(\Box)$. Therefore from Eq. (49) one obtains

$$G(\Box) = G_c \left[1 + c_0 \left(\frac{1}{\xi^2 \Box} \right)^{1/2\nu} + \dots \right] , \tag{54}$$

and the running of G is expected to lead to a non-local gravitational action, for example of the form

$$I_{eff} = \frac{1}{16 \pi G} \int dx \sqrt{g} \left[1 - c_0 \left(\frac{1}{\xi^2 \Box} \right)^{1/2\nu} + \dots \right] R \ . \tag{55}$$

Due to the appearance of a fractional exponent, the covariant operator appearing in the above expression has to be suitably defined by analytic continuation. The latter can be done, for example, by computing $\Box^n$ for positive integer n and then analytically continuing to $n \to -1/2\nu$.

Alternatively, had one not considered the action of Eq. (55) as a starting point for constructing the effective theory, one would naturally be led (as suggested by Eq. (54)) to consider instead the following effective field equation

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G(\Box) \, T_{\mu\nu} \ , \tag{56}$$

the argument again being the replacement $G \to G(\Box)$ in the classical Einstein field equations. Being manifestly covariant, these expressions at least satisfy some of the requirements for a set of consistent field equations incorporating the running of G.

The effective field equations of Eq. (56) can in fact be re-cast in a form very similar to the classical field equations but with a $\tilde{T}_{\mu\nu} = [G(\Box)/G] \; T_{\mu\nu}$ defined as an effective, or gravitationally dressed, energy-momentum tensor. Just like the ordinary Einstein gravity case, in general $\tilde{T}_{\mu\nu}$ might not be covariantly conserved a priori, $\nabla^\mu \tilde{T}_{\mu\nu} \neq 0$, but ultimately the consistency of the effective field equations demands that it be exactly conserved, in consideration of the Bianchi identity satisfied by the Riemann tensor. In this picture therefore the running of G can be viewed as contributing to a sort of "vacuum fluid", introduced in order to to account for the gravitational vacuum polarization contribution.

9. Static Isotropic Solutions

One can show that the quantum correction due to the running of G can be described - at least in the non-relativistic limit of Eq. (53) when applied to Poisson's equation - in terms of a vacuum energy density $\rho_m(r)$, distributed around the static source of strength M in accordance with

$$\rho_m(r) \; = \; \frac{1}{8\pi} \, c_\nu \, c_0 \, M \, m^3 \, (m \, r)^{-\frac{1}{2}(3-\frac{1}{\nu})} \, K_{\frac{1}{2}(3-\frac{1}{\nu})}(m \, r) \; , \tag{57}$$

and with $c_\nu \; \equiv \; 2^{\frac{1}{2}(5-\frac{1}{\nu})} / \sqrt{\pi} \, \Gamma(\frac{1}{2\nu})$, and

$$4\,\pi \int_0^\infty r^2 \, dr \, \rho_m(r) \; = \; c_0 \, M \quad . \tag{58}$$

More generally in the fully relativistic case, after solving the covariant effective field equations with $G(\Box)$ for $\nu = 1/3$ one finds in Schwarzschild coordinates, and in the limit $r \gg 2MG$,

$$A^{-1}(r) \; == \; B(r) \; = \; 1 - \frac{2\,M\,G}{r} \; + \; \frac{4\,c_0\,M\,G\,m^3}{3\,\pi} \, r^2 \, \ln{(m\,r)} \; + \; \dots \; . \tag{59}$$

The last expressions for $A(r)$ and $B(r)$ are therefore consistent with a gradual slow increase in G with distance, in accordance with the formula

$$G \; \rightarrow \; G(r) \; = \; G \left(1 \, + \, \frac{c_0}{3\,\pi} \, m^3 \, r^3 \, \ln \, \frac{1}{m^2 \, r^2} \; + \; \dots \right) \tag{60}$$

in the regime $r \gg 2\,M\,G$. The last result is in some ways reminiscent of the QED small-r result

$$Q \; \rightarrow \; Q(r) \; = \; Q \left(1 \, + \, \frac{\alpha}{3\,\pi} \, \ln \frac{1}{m^2 \, r^2} \; + \; \dots \right) \; . \tag{61}$$

In the gravity case, the correction vanishes as r goes to zero: in this limit one is probing the bare mass, unobstructed by its virtual graviton cloud. In some ways the running G term acts as a local cosmological constant term, for which the r dependence of the vacuum solution for small r is fixed by the nature of the Schwarzschild solution with a cosmological constant term. One could in fact wonder what these solutions might look like in d dimensions, and after some straightforward calculations one finds that in $d \geq 4$ space-time dimensions a solution to the effective field equations can only be found if in Eq. (54) $\nu = 1/(d-1)$ exactly [23].

10. Cosmological Solutions

A scale dependent Newton's constant is expected to lead to small modifications of the standard cosmological solutions to the Einstein field equations. Here I will summarize what modifications are expected from the effective field equations on the basis of $G(\Box)$, as given in Eq. (54), which itself originates in Eqs. (53) and (49). The starting point are the quantum effective field equations of Eq. (56), with $G(\Box)$ defined in Eq. (54). In the Friedmann-Robertson-Walker (FRW) framework these are applied to the standard homogeneous isotropic metric

$$d\tau^2 \;=\; dt^2 - a^2(t) \left\{ \frac{dr^2}{1 - k\, r^2} + r^2 \, \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \right\} \; . \qquad (62)$$

It should be noted that there are in fact *two* related quantum contributions to the effective covariant field equations. The first one arises because of the presence of a non-vanishing cosmological constant $\lambda \simeq 1/\xi^2$, caused by the non-perturbative vacuum condensate of Eq. (50). As in the case of standard FRW cosmology, this is expected to be the dominant contributions at large times t, and gives an exponential (for $\lambda > 0$), or cyclic (for $\lambda < 0$) expansion of the scale factor. The second contribution arises because of the explicit running of $G(\Box)$ in the effective field equations. The next step is therefore a systematic examination of the nature of the solutions to the full effective field equations, with $G(\Box)$ involving the relevant covariant d'Alembertian operator

$$\Box \;=\; g^{\mu\nu} \nabla_\mu \nabla_\nu \qquad (63)$$

acting on second rank tensors as in the case of $T_{\mu\nu}$,

$$\nabla_\nu T_{\alpha\beta} \;=\; \partial_\nu T_{\alpha\beta} - \Gamma_{\alpha\nu}^{\lambda} T_{\lambda\beta} - \Gamma_{\beta\nu}^{\lambda} T_{\alpha\lambda} \;\equiv\; I_{\nu\alpha\beta}$$

$$\nabla_\mu \left(\nabla_\nu T_{\alpha\beta} \right) = \partial_\mu I_{\nu\alpha\beta} - \Gamma_{\nu\mu}^{\lambda} I_{\lambda\alpha\beta} - \Gamma_{\alpha\mu}^{\lambda} I_{\nu\lambda\beta} - \Gamma_{\beta\mu}^{\lambda} I_{\nu\alpha\lambda} \; . \qquad (64)$$

To start the process, one assumes for example that $T_{\mu\nu}$ has a perfect fluid form, for which one obtains the action of $\Box^n$ on $T_{\mu\nu}$, and then analytically continues to negative fractional values of $n = -1/2\nu$. Even in the simplest case, with $G(\Box)$ acting on a *scalar* such as the trace of the energy-momentum tensor $T_\lambda^{\ \lambda}$, one finds for the choice $\rho(t) = \rho_0 \, t^\beta$ and $a(t) = r_0 \, t^\alpha$ the rather unwieldy expression

$$\Box^n \left[-\rho(t) \right] \rightarrow 4^n \; (-1)^{n+1} \, \frac{\Gamma\left(\frac{\beta}{2}+1\right) \Gamma\left(\frac{\beta + 3\,\alpha + 1}{2}\right)}{\Gamma\left(\frac{\beta}{2}+1-n\right) \Gamma\left(\frac{\beta + 3\,\alpha + 1}{2} - n\right)} \, \rho_0 \, t^{\beta - 2n} \; , \qquad (65)$$

with integer n then analytically continued to $n \rightarrow -\frac{1}{2\,\nu}$, with $\nu = 1/3$.

A more general calculation shows that a non-vanishing pressure contribution is generated in the effective field equations, even if one initially assumes a pressureless fluid, $p(t) = 0$. After a somewhat lengthy computation one obtains for a universe filled with non-relativistic matter (p=0) the following set of effective Friedmann equations, namely

$$\begin{aligned} \frac{k}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)} &= \frac{8\pi G(t)}{3} \rho(t) + \frac{1}{3\,\xi^2} \\ &= \frac{8\pi G}{3} \left[1 + c_t \, (t/\xi)^{1/\nu} + \dots \right] \rho(t) + \tfrac{1}{3} \lambda \end{aligned} \tag{66}$$

for the tt field equation, and

$$\frac{k}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)} + \frac{2\,\ddot{a}(t)}{a(t)} = -\frac{8\pi G}{3} \left[c_t \, (t/\xi)^{1/\nu} + \dots \right] \rho(t) + \lambda \tag{67}$$

for the rr field equation. In the above equations the running of G appropriate for the Robertson-Walker metric is given by

$$G(t) = G \left[1 + c_t \left(\frac{t}{\xi} \right)^{1/\nu} + \dots \right] , \tag{68}$$

with c_t of the same order as c_0 of Eq. (49). Note that it is the running of G that induces an effective pressure term in the second (rr) equation, corresponding to the presence of a relativistic fluid arising from the vacuum polarization contribution. The second important feature of the new equations is an additional power-law acceleration contribution, in addition to the standard one due to λ.

11. Quantum Gravity and Cosmological Density Perturbations

Besides the cosmic scale factor evolution and the static isotropic solutions just discussed, the running of $G(\Box)$ also affects the nature of matter density perturbations on very large scales. In discussing these effects, it is customary to introduce a perturbed metric of the form

$$d\tau^2 = dt^2 - a^2 \left(\delta_{ij} + h_{ij} \right) dx^i dx^j \ , \tag{69}$$

with $a(t)$ the unperturbed scale factor and $h_{ij}(\vec{x}, t)$ a small metric perturbation. The next step is to determine the effects of the running of G on the relevant matter and metric perturbations, again by the use of the modified field equations. For sufficiently small perturbations, one can expand $G(\Box)$ appearing in the effective covariant field equations in powers of the metric perturbation h_{ij} as

$$G(\Box) = G_0 \left[1 + \frac{c_0}{\xi^{1/\nu}} \left(\frac{1}{\Box^{(0)}} \right)^{1/2\nu} \left(1 - \frac{1}{2\nu} \frac{1}{\Box^{(0)}} \Box^{(1)}(h) + \dots \right) \right] . \tag{70}$$

It is also customary at this point to expand the density, pressure and metric trace perturbation modes in spatial Fourier components

$$\delta\rho(\vec{x}, t) = \delta\rho(t)\, e^{i\,\vec{k}\cdot\vec{x}} \quad \delta p(\vec{x}, t) = \delta p(t)\, e^{i\,\vec{k}\cdot\vec{x}} \quad h(\vec{x}, t) = h(t)\, e^{i\,\vec{k}\cdot\vec{x}} \ . \tag{71}$$

Normally the Einstein field equations $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda \, g_{\mu\nu} = 8\pi G \, T_{\mu\nu}$ are given to first order in the small perturbations by

$$\frac{\dot{a}(t)}{a(t)} \, \dot{h}(t) = 8\pi G \rho(t) \, \delta(t)$$
$$\ddot{h}(t) + 3 \, \frac{\dot{a}(t)}{a(t)} \, \dot{h}(t) = -24\pi \, G \, w \, \rho(t) \delta(t) \tag{72}$$

with $\delta(t) = \delta\rho(t)/\rho(t)$ and $w = 0$ for non-relativistic matter, yielding then a single equation for the trace of the metric perturbation $h(t)$,

$$\ddot{h}(t) + 2 \, \frac{\dot{a}(t)}{a(t)} \dot{h}(t) \;=\; -8\pi G (1 + 3 \, w) \rho(t) \delta(t) \; . \tag{73}$$

Combined with the first order energy conservation $-\frac{1}{2} \, (1 + w) \, h(t) \;=\; \delta(t)$, this then gives a single equation for the density contrast $\delta(t)$,

$$\ddot{\delta}(t) + 2 \, \frac{\dot{a}}{a} \, \dot{\delta}(t) - 4\pi \, G \, \rho(t) \, \delta(t) = 0 \; . \tag{74}$$

In the case of a running $G(\Box)$ these equations need to be re-derived from the effective covariant field equations, and lead to several additional terms not present at the classical level [23]. In other words, the correct field equations for a running G are not given simply by a naive replacement $G \to G(t)$, which would lead to incorrect results, and violate general covariance.

It is common practice at this point to write an equation for the density contrast $\delta(a)$ as a function not of t, but of the scale factor $a(t)$, by utilizing the identities

$$\dot{f}(t) = a \, H \, \frac{\partial f(a)}{\partial a} \tag{75}$$

$$\ddot{f}(t) = a^2 \, H^2 \left(\frac{\partial \ln H}{\partial a} + \frac{1}{a} \right) \frac{\partial f(a)}{\partial a} + a^2 \, H^2 \, \frac{\partial^2 f(a)}{\partial a^2} \tag{76}$$

where f is any function of t, and $H \equiv \dot{a}(t)/a(t)$ is the Hubble constant. This last quantity can then be obtained from the zero-th order tt field equation

$$3 \left(\frac{\dot{a}}{a} \right)^2 = 8 \, \pi \, G_0 \, \rho + \lambda \tag{77}$$

re-written in terms of $H(a)$ as

$$H^2(a) \equiv \left(\frac{\dot{a}}{a} \right)^2 = \left(\frac{\dot{z}}{1+z} \right)^2 = H_0^2 \left[\Omega \, (1+z)^3 + \Omega_R \, (1+z)^2 + \Omega_\lambda \right] \tag{78}$$

with $a = \frac{1}{1+z}$ where z is a red shift, H_0 the Hubble constant evaluated today, and Ω the matter (baryonic and dark) density, Ω_R the space curvature contribution corresponding to a curvature k term, and $\Omega_\lambda$the dark energy part,

$$\Omega_\lambda \equiv \frac{\lambda}{3 \, H^2} \qquad \Omega \equiv \frac{8 \, \pi \, G_0 \, \rho}{3 \, H^2} \qquad \text{with} \quad \Omega + \Omega_\lambda = 1 \; . \tag{79}$$

After introducing the parameter θ as the cosmological constant fraction

$$\theta \equiv \Omega_\lambda \left(\frac{8\,\pi\,G\,\rho}{3\,H^2} \right)^{-1} \;=\; \frac{\Omega_\lambda}{\Omega} \;=\; \frac{1-\Omega}{\Omega} \tag{80}$$

one then obtains an equation for the density contrast $\delta(a)$ in the normal (i.e. non-running G) case

$$\frac{\partial^2 \delta(a)}{\partial a^2} + \left[\frac{\partial \ln H(a)}{\partial a} + \frac{3}{a} \right] \frac{\partial \delta(a)}{\partial a} - 4\,\pi\,G_0\,\frac{1}{a^2 H(a)^2}\,\rho(a)\,\delta(a) = 0 \tag{81}$$

with growing solution

$$\delta_0(a) \;\sim\; a \cdot {}_2F_1 \left(\frac{1}{3}, 1; \frac{11}{6}; -\,a^3\,\theta \right) \;, \tag{82}$$

where ${}_2F_1$ is a hypergeometric function.

To determine the quantum correction to $\delta(a)$ originating from $G(\Box)$ in Eq. (54), one sets

$$\delta(a) \;=\; \delta_0(a)\,[1 + c_0\,f(a)] \;, \tag{83}$$

and then uses this linear Ansatz to find the form of $f(a)$ to lowest order in c_0. The correction is in principle unambiguously determined from the field equations with a running Netwon constant $G(\Box)$, but here the running of G (due to the choice of variables) naturally takes on the form

$$G(a) = G_0 \left[1 + c_a \left(\frac{a}{a_0} \right)^\gamma + \ldots \right] \tag{84}$$

with a scale factor $a \approx a_0$ corresponding to "today". Furthermore one has $\gamma = 3/2\,\nu$ with $\nu = 1/3$, as determined from lattice gravity in four dimensions.

What then remains to be done is to compute the growth index $f \equiv \partial \ln \delta / \partial \ln a$, and from it the growth index exponent γ defined through $f = \Omega^\gamma$ [24]. Ultimately one is interested in the value for this quantities in the vicinity of a current matter fraction $\Omega \approx 0.25$. For a constant (i.e. not scale dependent) Newton's constant one has the well know result $f = 0.6028$ and exponent $\gamma = 0.5562$. An explicit calculation in the presence of a running G and for a matter fraction $\Omega \approx 0.25$ gives, to lowest linear order in the small quantum correction c_0 [25],

$$\gamma \;=\; 0.5562 - c_q\,c_a \;. \tag{85}$$

Here c_a is the coefficient of the quantum correction in the expression for $G(a)$, and therefore fixed by the underlying lattice gravity calculations, and c_q an explicitly calculable numerical constant that comes out of the solution of the full effective covariant field equations for $\delta(a)$.

The perturbed RW metric is well suited for discussing matter perturbations, but occasionally one finds it more convenient to use a different metric parametrization, such as the one derived from the conformal Newtonian (cN) gauge line element

$$d\tau^2 \;=\; a^2(t) \left\{ (1 + 2\,\psi)\,dt^2 \;-\; (1 - 2\,\phi)\,\delta_{ij}\,dx^i dx^j \right\} \tag{86}$$

with Conformal Newtonian potentials $\psi(\vec{x},t)$ and $\phi(\vec{x},t)$. In this gauge, and in the absence of a $G(\Box)$, the unperturbed equations are

$$\begin{aligned} \left(\frac{\dot a}{a}\right)^2 &= \frac{8\pi}{3}\, G\, a^2\, \bar\rho \\ \frac{d}{dt}\left(\frac{\dot a}{a}\right) &= -\frac{4\pi}{3}\, G\, a^2\, (\bar\rho + 3\bar p)\ , \end{aligned} \tag{87}$$

in the absence of spatial curvature ($k = 0$). In the presence of a running G these again need to be modified, in accordance with Eqs. (67), (66) and (68). A cosmological constant can be conveniently included in the $\bar\rho$ and $\bar p$, with $\bar\rho_\lambda = \lambda/8\pi G = -\bar p_\lambda$. In this gauge scalar perturbations are characterized by Fourier modes $\psi(\vec{k},t)$ and $\phi(\vec{k},t)$, and the first order Einstein field equations in the absence of $G(\Box)$ read [26]

$$\begin{aligned} k^2\,\phi + 3\,\frac{\dot a}{a}\left(\dot\phi + \frac{\dot a}{a}\,\psi\right) &= 4\pi\, G\, a^2\, \delta T^0{}_0 \\ k^2\left(\dot\phi + \frac{\dot a}{a}\,\psi\right) &= 4\pi\, G\, a^2\, (\bar\rho + \bar p)\,\theta \\ \ddot\phi + \frac{\dot a}{a}\left(2\dot\phi + \dot\psi\right) + \left(2\,\frac{\ddot a}{a} - \frac{\dot a^2}{a^2}\right)\psi + \frac{k^2}{3}\,(\phi - \psi) &= \frac{4\pi}{3}\, G\, a^2\, \delta T^i{}_i \\ k^2\,(\phi - \psi) &= 12\pi\, G\, a^2\, (\bar\rho + \bar p)\,\sigma \end{aligned} \tag{88}$$

where the perfect fluid energy-momentum tensor is given to linear order in the perturbations $\delta\rho = \rho - \bar\rho$ and $\delta p = p - \bar p$ by

$$\begin{aligned} T^0{}_0 &= -(\bar\rho + \delta\rho) \\ T^0{}_i &= (\bar\rho + \bar p)\, v_i \;=\; -T^i{}_0 \\ T^i{}_j &= (\bar p + \delta p)\,\delta^i{}_j + \Sigma^i{}_j \quad \Sigma^i{}_i = 0 \end{aligned} \tag{89}$$

and one has allowed for an anisotropic shear perturbation $\Sigma^i{}_j$ to the perfect fluid form $T^i{}_j$. The two quantities θ and σ are commonly defined by

$$(\bar\rho + \bar p)\,\theta \;\equiv\; i\, k^j\, \delta T^0_j \quad (\bar\rho + \bar p)\,\sigma \;\equiv\; -(\hat k_i \hat k_j - \frac{1}{3}\delta_{ij})\Sigma^i{}_j \tag{90}$$

with $\Sigma^i{}_j \equiv T^i_j - \delta^i{}_j T^k_k/3$ the traceless component of $T^i{}_j$. For a perfect fluid θ is the divergence of the fluid velocity, $\theta = i k^j v_j$, with $v^j = dx^j/dt$ the small velocity of the fluid. The field equations imply, by consistency, the covariant energy momentum conservation law

$$\begin{aligned} \dot\delta &= -(1+w)\,(\theta - 3\dot\phi) - 3\,\frac{\dot a}{a}\left(\frac{\delta p}{\delta\rho} - w\right)\delta \\ \dot\theta &= -\frac{\dot a}{a}\,(1-3w)\,\theta - \frac{\dot w}{1+w}\,\theta + \frac{1}{1+w}\frac{\delta p}{\delta\rho}\,k^2\delta - k^2\sigma + k^2\psi \end{aligned} \tag{91}$$

and relate the matter fields δ, σ and θ to the metric perturbations ϕ and ψ. where δ is the matter density contrast $\delta = \delta\rho/\rho$, and w is the equation of state parameter $w = p/\rho$.

In the presence of a $G(\Box)$ the above equations need to be re-derived and amended [25], starting from the covariant field equations of Eq. (56) in the cN gauge of Eq. (86), with zero-th order modified field equations as in Eqs. (66) and (67), using the expansion for $G(\Box)$ given in Eq. (70), but now in terms of the new cN gauge potentials ϕ and ψ. One key question is then the nature of the vacuum-polarization induced anisotropic shear perturbation correction $\Sigma^i{}_j$ appearing in the covariant effective field equations analogous to Eqs. (88), but derived with a $G(\Box)$. In particular one would expect the quantum correction to the energy momentum tensor appearing on the r.h.s. of Eq. (56) to contribute new terms to the last of Eqs. (88), which could then account for a non-zero stress σ, and thus for a small deviation from the classical result for a perfect fluid, $\phi = \psi$.

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