

NONLOCAL EFFECTIVE FIELD EQUATIONS FOR QUANTUM COSMOLOGY

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The possibility that the strength of gravitational interactions might slowly increase with distance, is explored by formulating a set of effective field equations, which incorporate the gravitational, vacuum-polarization induced, running of Newton's constant G . The resulting long distance (or large time) behavior depends on only one adjustable parameter ξ , and the implications for the Robertson–Walker universe are calculated, predicting an accelerated power-law expansion at later times $t \sim \xi \sim 1/H$.

In the Standard Model of particle interactions, all gauge couplings are known to run with energy. Recent nonperturbative studies of quantum gravity have suggested that the gravitational coupling may also depend on a scale related to curvature, and therefore macroscopic in size. Here, we investigate the effects of a running gravitational constant G at large distances. This scale dependence is assumed to be driven by gravitational vacuum polarization effects, which produce an anti-screening effect some distance away from the primary source, and therefore tend to increase the strength of the gravitational coupling. A power law running of G will be implemented via manifestly covariant nonlocal terms in the effective gravitational action and field equations.

We start by assuming^{1–7} that for Newton's constant one has

$$G(r) = G(0)[1 + c_\xi(r/\xi)^{1/\nu} + O((r/\xi)^{2/\nu})], \quad (1)$$

where the exponent ν is generally related to the derivative of the beta function for pure gravity evaluated at the nontrivial ultraviolet fixed point. Recent studies have ν^{-1} varying between 3.0 and 1.7.^{4,5,8–18} These estimates rely on three different,

and unrelated, nonperturbative approaches to quantum gravity, based on the lattice path integral formulation, the two plus epsilon expansions of continuum gravity, and a momentum slicing scheme combined with renormalization group methods in the vicinity of flat space, respectively. In all three approaches a nonvanishing, positive bare cosmological constant is required for the consistency of the renormalization group procedure. The mass scale $m = \xi^{-1}$ in Eq. (1) is supposed to determine the magnitude of quantum deviations from the classical theory. It seems natural to identify $1/\xi^2$ with either some very large average spatial curvature scale, or perhaps more appropriately with the Hubble constant (as measured today) determining the macroscopic expansion rate of the universe, via the correspondence

$$\xi = \frac{1}{H}, \quad (2)$$

in a system of units for which the speed of light equals one. A possible concrete scenario is one in which $\xi^{-1} = H_\infty = \lim_{t \rightarrow \infty} H(t) = \sqrt{\Omega_\Lambda} H_0$ with $H_\infty^2 = \frac{\Lambda}{3}$, where Λ is the observed cosmological constant, and for which the horizon radius is H_∞^{-1} .

As it stands, the formula for the running of G is coordinate dependent, and we therefore replace it with a manifestly covariant expression involving the covariant d'Alembertian operator

$$\square = g^{\mu\nu} \nabla_\mu \nabla_\nu, \quad (3)$$

whose Green's function in d spatial dimensions is known to behave as

$$\langle x | \frac{1}{\square} | y \rangle \delta(r - d(x, y|g)) \sim \frac{1}{r^{D-2}}, \quad (4)$$

where d is the minimum distance between points x and y in a background with metric $g_{\mu\nu}$. We therefore write, in four dimensions,

$$G \rightarrow G(\square) = G(0) \left[1 + c_\square \left(\frac{1}{\xi^2 \square} \right)^{1/2\nu} + O((\xi^2 \square)^{-1/\nu}) \right]. \quad (5)$$

One way of incorporating this is to replace the gravitational action

$$I = \frac{1}{16\pi G} \int dx \sqrt{g} R \quad (6)$$

by

$$I = \frac{1}{16\pi G} \int dx \sqrt{g} \left(1 - c_\square \left(\frac{1}{\xi^2 \square} \right)^{1/2\nu} + O((\xi^2 \square)^{-1/\nu}) \right) R. \quad (7)$$

The above prescription has in fact been used successfully to systematically incorporate the effects of radiative corrections in an effective action formalism.^{19–25} It should be noted that the coefficient c_ξ in Eq. (1) is expected to be a calculable number of order one, not necessarily the same as the coefficient c_\square , as r and $1/\sqrt{\square}$ are clearly rather different entities to begin with. One should recall here that in

general the form of the covariant d'Alembertian operator \square depends on the specific tensor nature of the object it is acting on.

The details of the incorporation of this modified G in the gravitational side of the Einstein equations have been given elsewhere.¹ Here we shall describe instead its incorporation on the matter side of Einstein's equations, giving the effective field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G(1 + A(\square))T_{\mu\nu}, \quad (8)$$

where we have replaced $G(r)$ by $G(0)(1 + A(\square))$. These can be written in the form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G\tilde{T}_{\mu\nu}, \quad (9)$$

with $\tilde{T}_{\mu\nu} = (1 + A(\square))T_{\mu\nu}$ defined as an effective, or gravitationally dressed, energy-momentum tensor. Just like the ordinary Einstein gravity case, in general $\tilde{T}_{\mu\nu}$ might not be covariantly conserved *a priori*, $\nabla^\mu \tilde{T}_{\mu\nu} \neq 0$, but ultimately the consistency of the effective field equations demands that it be exactly conserved in consideration of the Bianchi identity satisfied by the Riemann tensor.

The ensuing new covariant conservation law

$$\nabla^\mu \tilde{T}_{\mu\nu} \equiv \nabla^\mu [(1 + A(\square))T_{\mu\nu}] = 0 \quad (10)$$

can then be viewed as a constraint on $\tilde{T}_{\mu\nu}$ (or $T_{\mu\nu}$) which, for example, in the specific case of a perfect fluid, will imply again a definite relationship between the density $\rho(t)$, the pressure $p(t)$ and the Robertson-Walker scale factor $R(t)$, just as it does in the standard case.

For simplicity we set the cosmological constant Λ to zero from now on and consider first the trace of the effective field equations

$$R = 8\pi G(1 + A(\square))T_\mu{}^\mu. \quad (11)$$

The advantage of this is that, initially, we need to consider the action of \square only on a scalar function, $S(x)$ say, which is given by

$$\frac{1}{\sqrt{g}}\partial_\mu g^{\mu\nu}\sqrt{g}\partial_\nu S(x). \quad (12)$$

For the Robertson-Walker metric,

$$ds^2 = -dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right\}, \quad (13)$$

\square acting on a scalar function of t only, is

$$-\frac{1}{R^3(t)}\frac{\partial}{\partial t}\left[R^3(t)\frac{\partial}{\partial t}\right]. \quad (14)$$

The energy-momentum tensor for a perfect fluid is given by

$$T_{\mu\nu} = (p(t) + \rho(t))u_\mu u_\nu + g_{\mu\nu}p(t). \quad (15)$$

We consider a pressureless fluid $p(t) = 0$ and assume the density and scale factor are given by powers of t , as in the classical solution for the RW metric: $\rho(t) = \rho_0 t^\beta$, $R(t) = R_0 t^\alpha$. Then

$$\square^n(T_\mu{}^\mu) = \square^n(-\rho(t)) \rightarrow 4^n(-1)^{n+1} \frac{\Gamma\left(\frac{\beta}{2} + 1\right) \Gamma\left(\frac{\beta+3\alpha+1}{2}\right)}{\Gamma\left(\frac{\beta}{2} + 1 - n\right) \Gamma\left(\frac{\beta+3\alpha+1}{2} - n\right)} \rho_0 t^{\beta-2n}. \quad (16)$$

We may analytically continue the exponent to negative fractional n ,^{26–28} and obtain with $n = -1/(2\nu)$, an expression for $(1 + A(\square))$ acting on the trace of $T_{\mu\nu}$, given by

$$-\left(1 + c_\xi \left(\frac{t}{\xi}\right)^{1/\nu}\right) \rho_0 t^\beta, \quad (17)$$

with

$$c_\nu = 4^{-1/2\nu} (-1)^{1-1/2\nu} \frac{\Gamma\left(\frac{\beta}{2} + 1\right) \Gamma\left(\frac{\beta+3\alpha+1}{2}\right)}{\Gamma\left(\frac{\beta}{2} + 1 + \frac{1}{2\nu}\right) \Gamma\left(\frac{\beta+3\alpha+1}{2} + \frac{1}{2\nu}\right)}. \quad (18)$$

Using the value of the scalar curvature for the Robertson–Walker metric in the $k = 0$ case,

$$R = 6(\dot{R}^2(t) + R(t)\ddot{R}(t))/R^2(t), \quad (19)$$

gives

$$\frac{6\alpha(2\alpha - 1)}{t^2} = -\left(1 + c_\xi \left(\frac{t}{\xi}\right)^{1/\nu}\right) \rho_0 t^\beta. \quad (20)$$

For large t , when the correction term starts to take over, we see from the powers of t that

$$\beta = -2 - 1/\nu. \quad (21)$$

Next we will examine the full effective field equations (as opposed to just their trace part) of Eq. (8) with $\Lambda = 0$,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G(1 + A(\square))T_{\mu\nu}. \quad (22)$$

Here the d'Alembertian operator

$$\square = g^{\mu\nu}\nabla_\mu\nabla_\nu \quad (23)$$

acts on a second rank tensor,

$$\begin{aligned} \nabla_\nu T_{\alpha\beta} &= \partial_\nu T_{\alpha\beta} - \Gamma_{\alpha\nu}^\lambda T_{\lambda\beta} - \Gamma_{\beta\nu}^\lambda T_{\alpha\lambda} \equiv I_{\nu\alpha\beta}, \\ \nabla_\mu(\nabla_\nu T_{\alpha\beta}) &= \partial_\mu I_{\nu\alpha\beta} - \Gamma_{\nu\mu}^\lambda I_{\lambda\alpha\beta} - \Gamma_{\alpha\mu}^\lambda I_{\nu\lambda\beta} - \Gamma_{\beta\mu}^\lambda I_{\nu\alpha\lambda}, \end{aligned} \quad (24)$$

and would thus seem to require the calculation of 1920 terms, of which fortunately many vanish by symmetry. Assuming that $T_{\mu\nu}$ describes a pressureless perfect fluid, we obtain

$$\begin{aligned}
 (\square T_{\mu\nu})_{tt} &= 6\rho(t) \left(\frac{\dot{R}(t)}{R(t)} \right)^2 - 3\dot{\rho}(t) \frac{\dot{R}(t)}{R(t)} - \ddot{\rho}(t), \\
 (\square T_{\mu\nu})_{rr} &= \frac{1}{1-kr^2} (2\rho(t) \dot{R}(t)^2), \\
 (\square T_{\mu\nu})_{\theta\theta} &= r^2(1-kr^2) (\square T_{\mu\nu})_{rr}, \\
 (\square T_{\mu\nu})_{\varphi\varphi} &= r^2(1-kr^2) \sin^2 \theta (\square T_{\mu\nu})_{rr},
 \end{aligned} \tag{25}$$

with the remaining components equal to zero. Note that a nonvanishing pressure contribution is generated in the effective field equations, even if one assumes initially a pressureless fluid. As before, repeated applications of the d'Alembertian \square to the above expressions leads to rapidly escalating complexity (for example, eighteen distinct terms are generated by \square^2 for each of the above contributions), which can only be tamed by introducing some further simplifying assumptions. In the following we will therefore assume as before that $k = 0$, $\rho(t) = \rho_0 t^\beta$ and $R(t) = R_0 t^\alpha$. We obtain

$$\begin{aligned}
 (\square T_{\mu\nu})_{tt} &= (6\alpha^2 - \beta^2 - 3\alpha\beta + \beta) \rho_0 t^{\beta-2}, \\
 (\square T_{\mu\nu})_{rr} &= 2R_0^2 t^{2\alpha} \alpha^2 \rho_0 t^{\beta-2},
 \end{aligned} \tag{26}$$

which again shows that the tt and rr components get mixed by the action of the \square operator, and that a nonvanishing rr component gets generated, even though it was not originally present.

The geometric side of the gravitational field equations, the Einstein tensor, has the following components for the RW metric:

$$\begin{aligned}
 G_{tt} &= 3\dot{R}^2(t)/R^2(t), \\
 G_{rr} &= \frac{-1}{1-kr^2} (\dot{R}^2(t) + 2R(t)\ddot{R}(t)), \\
 G_{\theta\theta} &= r^2(1-kr^2)G_{rr}, \\
 G_{\varphi\varphi} &= \sin^2 \theta G_{\theta\theta}.
 \end{aligned} \tag{27}$$

Then with $k = 0$ and $R(t) = R_0 t^\alpha$, these will all behave like t^{-2} so in fact a solution can only be achieved at order \square^n provided the exponent β satisfies $\beta = -2 + 2n$, or since $n = -1/(2\nu)$,

$$\beta = -2 - 1/\nu, \tag{28}$$

as was found previously from the trace equation, Eqs. (11) and (21).

We must now determine α . By repeated application of \square , for general n one can then write

$$(\square^n T_{\mu\nu})_{tt} \rightarrow c_{tt}(\alpha, \nu) \rho_0 t^{-2} \quad (29)$$

and similarly for the rr component

$$(\square^n T_{\mu\nu})_{rr} \rightarrow c_{rr}(\alpha, \nu) R_0^2 t^{2\alpha} \rho_0 t^{-2}. \quad (30)$$

But remarkably (see also Eq. (25)) one finds for the two coefficients the simple identity

$$c_{rr}(\alpha, \nu) = \frac{1}{3} c_{tt}(\alpha, \nu) \quad (31)$$

as well as $c_{\theta\theta} = r^2 c_{rr}$ and $c_{\varphi\varphi} = r^2 \sin^2 \theta c_{rr}$. Then for large times, when the quantum correction starts to become important, the tt and rr field equations reduce to

$$3\alpha^2 t^{-2} = 8\pi G c_{tt}(\alpha, \nu) \rho_0 t^{-2} \quad (32)$$

and

$$-\alpha(3\alpha - 2) R_0^2 t^{2\alpha-2} = 8\pi G c_{rr}(\alpha, \nu) R_0^2 t^{2\alpha} \rho_0 t^{-2} \quad (33)$$

respectively. But the identity $c_{rr} = \frac{1}{3} c_{tt}$ implies, simply from the consistency of the tt and rr effective field equations at large times,

$$\frac{c_{rr}(\alpha, \nu)}{c_{tt}(\alpha, \nu)} \equiv \frac{1}{3} = -\frac{3\alpha - 2}{3\alpha}, \quad (34)$$

whose only possible solution finally gives the second sought-for result, namely

$$\alpha = \frac{1}{2}. \quad (35)$$

We still need to check whether the above solution is consistent with covariant energy conservation. With the assumed form for $T_{\mu\nu}$ it is easy to check that energy conservation yields for the t component

$$(\nabla^\mu (\square^n T_{\mu\nu}))_t \rightarrow -((3\alpha + \beta + 1/\nu) c_{tt} + 3\alpha c_{rr}) \rho_0 t^{\beta+1/\nu-1} = 0 \quad (36)$$

when evaluated for $n = -1/2\nu$, and zero for the remaining three spatial components. But from the solution for the matter density $\rho(t)$ at large times one has $\beta = -2 - 1/\nu$, so the above zero condition again gives $c_{rr}/c_{tt} = -(3\alpha - 2)/3\alpha$, exactly the same relationship previously implied by the consistency of the tt and rr field equations.

Let us emphasize that the values for $\alpha = 1/2$ of Eq. (35) and $\beta = -2 - 1/\nu$ of Eq. (28), determined from the effective field equations at large times, are found to be consistent with *both* the field equations *and* covariant energy conservation. More importantly, the above solution is also consistent with what was found previously by looking at the trace of the effective field equations, Eq. (11), which also implied the result $\beta = -2 - 1/\nu$, Eq. (21).

The classical unmodified matter-dominated RW equations have solutions $\alpha = 2/3$, $\beta = -2$, which mean that the scale factor behaves as

$$R(t) \sim t^\alpha \sim t^{2/3} \quad (37)$$

and the density as

$$\rho(t) \sim t^\beta \sim t^{-2} \sim (R(t))^{-3}. \quad (38)$$

This will also be the behavior for our model at early times, but at later times, when the effect of the gravitational vacuum-polarization modification dominates, the behavior is rather different: for the scale factor, we have

$$R(t) \sim t^\alpha \sim t^{1/2} \quad (39)$$

and for the density

$$\rho(t) \sim t^\beta \sim t^{-2-1/\nu} \sim (R(t))^{-2(2+1/\nu)}. \quad (40)$$

Thus the density decreases significantly faster in time than the classical value ($\rho(t) \sim t^{-2}$), a signature of an accelerating expansion at later times.

Within the Friedmann–Robertson–Walker (FRW) framework the gravitational vacuum polarization term behaves in many ways like a positive pressure term. The value $\alpha = 1/2$ corresponds to $\omega = 1/3$ in

$$\alpha = \frac{2}{3(1 + \omega)}, \quad (41)$$

(this follows from the consistency of the rr and tt equations in the general case) where we have taken the pressure and density to be related by $p(t) = \omega\rho(t)$, which is therefore characteristic of radiation. One can therefore visualize the gravitational vacuum polarization contribution as behaving like ordinary radiation, in the form of a dilute virtual graviton gas. It should be emphasized though that the relationship between density $\rho(t)$ and scale factor $R(t)$ is very different from the classical case.

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