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Simplicial Quantum Gravity*

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ABSTRACT

Quantum gravity on a lattice in a formulation due to Regge is reviewed in view of possible applications to renormalizable asymptotically free higher derivative theories of gravity.

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Contents

1. Higher derivative gravity

1.1 Introduction

It has been known for some time that if one attempts to quantize the Einstein theory of gravity one encounters two major difficulties. The field equations for the metric are derived from an action that is unbounded from below, and the path integral is therefore mathematically ill-defined. Furthermore the coupling constant in Einstein gravity (Newton's constant) has dimension of inverse mass squared (in units $\hbar = c = 1$), and this leads to a non-renormalizable quantum theory, as can be verified by doing explicit Feynman diagram perturbation theory[1-3].

One possible attitude is to hope that these problems will be cured in the context of a grand unified theory like supergravity. Alternatively, one might argue that the above problems hint to a fundamental incompatibility between gravity and quantum mechanics, and any modification of the Einstein action will in general lead to new undetermined parameters.

Of course the argument about naturalness and simplicity of the Einstein theory can be turned around, in the sense that a quantum theory of gravity should just provide the answer for why, starting with the most general microscopic theory consistent with general invariance principles, some terms appear in the low energy effective Lagrangian and others do not. The question then becomes of course a dynamical one. In other field theoretic contexts this phenomenon is connected with the flow of the coupling constants as the length scale at which the theory is probed is changed, and the concept of scaling dimensions and operator relevance[4].

Within the framework of continuum local field theories, the alternative possibility is thus to include in the action those terms that are generated by renormalization, and see whether the resulting action leads to a tractable and perhaps meaningful quantum theory[5]. It turns out that only two additional terms, involving fourth derivatives of the metric, need to be added to the Einstein action in order to obtain a perturbatively renormalizable theory, and cure at the same time the unboundedness problem[6]. More remarkably, the resulting theory is asymptotically free in the new coupling constants (if they are chosen with the right sign) and is thus ultraviolet stable[7-9]. This suggests that some class of higher derivative gravity theories with a cutoff can be defined, such that a truly cutoff independent continuum limit exists, and can be constructed using the renormalization group.

Unfortunately, as will be further discussed below, it appears that the theory has some potential problems with unitarity. If one looks at the tree level graviton propagator one finds that it exhibits a massive ghost pole. It has been argued that radiative corrections restore the unitarity of the theory (by decoupling the ghosts), but it seems unlikely that these and other questions (like the recovery of a Newtonian limit) can be answered within the context of weak coupling perturbation theory. In fact the complexity of the theory resembles Quantum Chromo-Dynamics, for which other tools (like the lattice regularization and nonperturbative methods) are needed to control and understand the low energy, large distance properties.

It is in this spirit that one has decided to turn to a discrete formulation of quantum gravity. The lattice is introduced as an aid to formulating and calculating the theory in the same way that one uses finite differences both to define derivatives and to obtain approximate numerical solutions of differential equations. For recent reviews of Yang-Mills theories defined on a lattice see references (11). Gravity on a lattice was in fact formulated some time ago by Regge[(12)]. In his work he showed that lattice gravity can be described by a simplicial net in which the elementary variables are the edge lengths connecting the points in the net[(13-16)]. The idea that continuum Riemannian geometry should be defined by some limiting procedure on piecewise flat manifolds goes back to Riemann himself. Here Regge's formulation and its extension to higher derivative gravity will be presented and discussed in some detail.

It is clear that if some class of higher derivative gravity theories is well defined and has a sensible low energy limit, it should agree with the experimental evidence. This suggests that the effects of the higher derivative terms should vanish in the low energy limit, and that the observed smallness of the cosmological constant be explained in a natural way as a consequence of renormalization effects. These are questions that presumably can best be answered by studying the dynamics of the theory on a space-time lattice.

1.2 Unboundedness of the Einstein action

Consider the euclidean Einstein action without a cosmological constant term

$$I_E = -\frac{1}{16\pi G} \int d^4x \sqrt{g} R. \tag{1.1}$$

where G is Newton's constant, $\sqrt{g}$ is the determinant of the metric $g_{\mu\nu}$ and R is the scalar curvature. Here boundary terms have been dropped, and couplings to matter fields are not considered. Variation with respect to the metric leads to the classical equations of motion for the gravitational field in a vacuum

$$\delta I_E = 0 \quad => \quad R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = 0 \tag{1.2}$$

If one attempts to write down a path integral of the form

$$Z = \int_{geometries} e^{-I_E} \tag{1.3}$$

(which will in general depend on a specified initial and final three-geometry) one soon realizes that it appears ill defined due to the fact that the scalar curvature can become arbitrarily positive (or negative). The gravitational action is unbounded from below and the functional integral strongly depends on how the unboundedness is cut off[(1)].

A second serious problem is connected to the fact that the coupling constant G^{-1} has dimension of mass to the power $(d-2)$ and suggests that the theory is not perturbatively renormalizable above two dimensions. It has been shown that close to four dimensions in order to renormalize the theory at one loop, one needs to introduce higher derivative terms, which are needed to cancel the divergences proportional (in dimensional regularization) to[2]

$$\Delta I = \frac{1}{16\pi^2(d-4)} \int d^4x\sqrt{g}\, (\frac{7}{40} R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{23}{120} R^2) \tag{1.4}$$

A possible solution to the unboundedness problem has been described by Hawking [1], who suggests performing the integration in a conformal gauge in which the Einstein action is bounded from below, and then integrating over all conformal factors by distorting the integration contour in the complex plane.

A second possibility is to add to the Einstein action extra terms, including higher derivative ones like R^2, in a carefully chosen combination which makes the total action bounded from below. It turns out that only up to fourth derivative terms need to be considered in order to cure the renormalizabilty problem. Thus one is led to consider the extended gravitational action[7]

$$I = \int d^4x\sqrt{g}\, [\lambda - kR + \frac{1}{4} a R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \frac{1}{3}(b - \frac{a}{4})R^2] \tag{1.5}$$

with a cosmological constant term (proportional to λ), the Einstein term ($k = 1/16\pi G$ is proportional to the inverse of the bare Newton constant), and two higher derivative terms with additional *dimensionless* coupling constants a^{-1} and b^{-1}. An action for pure gravity of this form was first considered by Pauli[17].

It is shown in the appendix that, even though there are four possible higher derivative terms in four dimensions which do not give rise to topological invariants (proportional to integrals of $C^2_{\mu\nu\rho\sigma}, R^2_{\mu\nu\rho\sigma}, R^2_{\mu\nu}$ and R^2), only two are found to be independent if one uses some identities for the Riemann tensor and the integral expression for the Euler charachteristic[18].

1.3 Renormalizable Asymptotically Free Theories of Gravity

The higher derivative action of eq. (1.5) was shown to be power counting renormalizable [5] and, later, renormalizable to all orders in perturbation theory[6]. Perturbation theory is usually performed around flat space, which requires $\lambda = 0$. The theory is asymptotically free in the couplings a and b and the action is bounded from below for $a > 0$, $b > 0$. To one loop order (small a^{-1}, b^{-1}) the renormalization group equations for the two higher derivative couplings are[7]

$$\frac{\partial a}{\partial \ln L} = \beta_a = \frac{1}{16\pi^2}\frac{133}{10} + \cdots \tag{1.6}$$

and

$$a\frac{\partial (b/a)}{\partial \ln L} = \beta_b - \frac{b}{a}\beta_a = \frac{1}{16\pi^2}[\frac{10b^2}{3a^2} - \frac{183b}{10a} - \frac{1}{12}] + \cdots \tag{1.7}$$

where L is the cutoff in momentum space. The first equation gives an ultraviolet fixed point at $a^{-1} = 0$, and the second one shows that there is an ultraviolet fixed point at $b/a = (549 - \sqrt{302401})/200 \approx -0.0046$. (remarkably, this number is rather small, and it is not clear whether its finiteness could be an artifact of the one-loop computation. Note that while a^{-1} and b^{-1} are assumed to be small to start with, b/a is neither small nor large). Therefore also the coupling b is asymptoticaly free, under the assumption $b < 0$, which is also necessary for the correspondence with the Einstein theory in the spin-zero sector (see below). The theory with the Weyl C^2 term alone ($b = 0$) is conformally invariant, but not perturbatively renormalizable because of the conformal anomaly[7].

In order to see the problems with unitarity consider the graviton propagator for higher derivative gravity in the weak field limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $\eta_{\mu\nu} = diag(1,1,1,1)$ and $h_{\mu\nu}$ small. Then the free propagator for $\lambda = 0$ in momentum space can be written as

$$\frac{1}{\mu^2} < h_{\mu\nu} h_{\rho\sigma} >_k = \frac{2P^A_{\mu\nu\rho\sigma}(k)}{k^2} - \frac{2P^B_{\mu\nu\rho\sigma}(k)}{k^2 + m_2^2} + \frac{2P^C_{\mu\nu\rho\sigma}(k)}{k^2 + m_0^2} + gauge\ terms \tag{1.8}$$

where the spin projection operators $P^{A,B,C}_{\mu\nu\rho\sigma}$ are functions of $\eta_{\mu\nu} - k_\mu k_\nu / k^2$ and $k_\mu k_\nu / k^2$. Their explicit form can be found in ref. (6). The higher derivative terms improve the ultraviolet behavior of the theory since the propagator now falls of as $1/k^4$ for large k^2. The P^A contribution is the same as in ordinary Einstein gravity, whereas the P^B (massive spin-two ghost) and the P^C (massive spin 0 particle) contributions are new. The mass of the spin-two ghost is given by $m_2 = \mu/\sqrt{a}$ and the mass of the spin two particle by $m_0 = \mu/\sqrt{-2b}$ where $\mu = \sqrt{k}$ is the Planck mass.

The presence of massive states in the tree level graviton propagator indicates also short distance deviations from the static Newtonian potential, which in higher derivative gravity (in the weak field limit) has the form

$$h_{00} \sim \frac{1}{\mu}\left[\frac{1}{r} - \frac{4}{3}\frac{e^{-m_2 r}}{r} + \frac{1}{3}\frac{e^{-m_0 r}}{r}\right] \tag{1.9}$$

In the absence of the Einstein term ($k = 0$) the potential is linear in r and the theory is strongly infrared divergent, and it is not completely clear whether weak coupling perturbation theory around the tree level solution is trustworthy. The masses that give the potentially dangerous exponential corrections to the $1/r$ behavior are of course *bare* masses, and the *full renormalized gauge invariant part* of the potential should be computed nonperturbatively (as in QCD) before any meaningful comparison with experiment is attempted. In fact the one loop result for the running coupling a (β_a is the one loop coefficient of the beta function for the coupling a)

$$a^{-1}(r) = \frac{a^{-1}(r_0)}{1 + a^{-1}(r_0)\beta_a \ln(r_0/r)} \tag{1.10}$$

suggests strongly that the asymptotic freedom (running) coupling constants grow indefinitely in the infrared regime, and the massive ghost becomes increasingly heavy as r_0/r goes to zero (low energy limit), and possibly decouples completely

$$\lim_{r/r_0 \to \infty} m_2(r) = \frac{\mu}{\sqrt{a(r_0)}}(1 + a^{-1}(r_0)\beta_a \ln \frac{r}{r_0})^{\frac{1}{2}} = \infty \tag{1.11}$$

Here r_0 is the cutoff scale, not to be confused with the inverse Planck mass. The same reasoning can be applied to the massive spin-zero state.

These considerations are in fact far from being rigorous, but one should keep in mind that the one-loop result is *qualitatively* correct in $\lambda\phi^4$ and Yang-mills theories in four dimensions. Also one should notice the fact that the graviton propagator as defined above is not gauge invariant. This is analogous to the gluon propagator in QCD : the perturbative massless gluons are in fact confined in massive glueballs and there is no gluon state in the physical gauge invariant spectrum. (The situation in gravity has of course to be different to some extent, since massless gravitons presumably do exist). Therefore the real question to answer is whether the massive additional states contribute to gauge invariant correlation functions in the low energy, large distance limit, in a way which is consistent with present experimental evidence.

Another potential problem is connected with the cosmological constant λ, whose value is observed to be of the order of 10^{-122} or less, in units of the Planck mass. (Experimentally one has $1/\sqrt{G} = 1.2 \times 10^{19} GeV$ and $\lambda < (0.003eV)^4$). In higher derivative gravity one would expect on dimensional grounds a quartic divergence

$$\lambda_R = \lambda_0 + c_4 L^4 + c_2 L^2 + O(\ln L) \tag{1.12}$$

($L \sim r_0^{-1}$ is the ultraviolet cutoff), which has then to be canceled by fine-tuning the bare cosmological constant to one part in 10^{122}, a rather unnatural procedure.

The procedure of setting all quartic and quadratic divergences equal to zero, as in dimensional regularization(2), seems somewhat formal and ad hoc, and clearly does not provide a physical explanation. On the other hand it has been argued that the quartic divergence for λ is absent ($c_4 = 0$) for an appropriate choice of measure for the $g_{\mu\nu}$ fields(7). The local gauge invariant measure is

$$d\mu(g) = \prod_x g^{-\frac{d+1}{2}} \prod_{\mu \geq \nu} dg_{\mu\nu} \tag{1.13}$$

which is also scale invariant(19,20). It is known that the measure can play a delicate role in canceling some spurious divergences in loop diagrams that arise when a continuous symmetry is explicitly broken.

For completeness we list here the quadratic one-loop divergences(7)

$$\begin{aligned} \lambda_R =& \lambda_0 + \frac{k_0}{2a}(5 - \frac{a}{2b})\frac{L^2}{16\pi^2} + O(\ln L) \\ k_R =& k_0 + \frac{1}{2}(\frac{10b}{3a} - 5)\frac{L^2}{16\pi^2} + O(\ln L) \end{aligned} \tag{1.14}$$

where the subscript R denotes renormalized quantities and 0 bare ones.

In conclusion it appears clear that further study of higher derivative gravity requires non-perturbative methods. If the theory is correct, it should allow one to explain the smallness of the cosmological constant and should give definite predictions (in units of the renormalized, effective low energy Planck mass) for its value and the value of the masses of gauge invariant states that set the scale for possible deviations from the predictions of Einstein gravity. It is in this spirit that Regge's lattice formulation of gravity is now described.

2. Gravity on Piecewise Linear Spaces

2.1 Triangulations of Smooth Manifolds

The following sections are based on the description of gravity known as Regge calculus [12–16], in which the Einstein theory is expressed in terms of simplicial decompositions of space-time manifolds. Its use in quantum gravity is prompted by the desire to make use of techniques developed in lattice gauge theories, but with a lattice which reflects the structure of space-time rather than just providing a flat passive background. It also allows one to use powerful nonperturbative analytical techniques of statistical mechanics and numerical methods. A regularized lattice version of the continuum field theory is also a necessary prerequisite for any rigorous study of the latter.

Most lattice formulations of gravity so far have been based on flat hypercubical lattices[21]. For such lattices, the formulation of R^2-type terms in four dimensions involves constraints between the connections and the tetrads, which are difficult to handle. Also there is no simple way of writing down topological invariants, which are either related to the Einstein action (in two dimensions), or are candidates for extra terms to be included in the action[1]. A flat hypercubic lattice action has been written with higher derivative terms which appears to be reflection positive but has a very cumbersome form. We shall see how these difficulties need not be present on a simplicial lattice (except that it is not known how to write the Hirzebruch signature in lattice terms[22]).

In Regge gravity the infinite number of degrees of freedom in the continuum is restricted by considering Riemannian spaces that are described by only a finite number of variables, the geodesic distances between neighboring points. Such spaces are taken to be flat almost everywhere and are called *piecewise linear* [23–26]. The elementary building blocks for d-dimensional space-time are *simplices* σ^d of dimension d. A 0-simplex is a point, a 1-simplex is an edge, a 2-simplex is a triangle, a 3-simplex is a tetrahedron. A d-simplex is a d-dimensional object with $d+1$ vertices and $d(d+1)/2$ edges connecting them. It has the important property that the values of its edge lengths specify the shape (and therefore the relative angles) uniquely.

A simplicial complex can be viewed as a set of simplices glued together to each other in such a way that either two simplices are disjoint or they touch at a common face. The relative position of points on the lattice is thus completely specified by the *incidence matrix* (it tells which point is next to which) and the *edge lengths*, and this in turn induces a metric structure on the piecewise linear space. Finally the polyhedron constituting the union of all the simplices of dimension d is called a geometrical complex or *skeleton*. The transition from a smooth triangulation of a sphere to the corresponding secant approximation is illustrated in fig. 1.

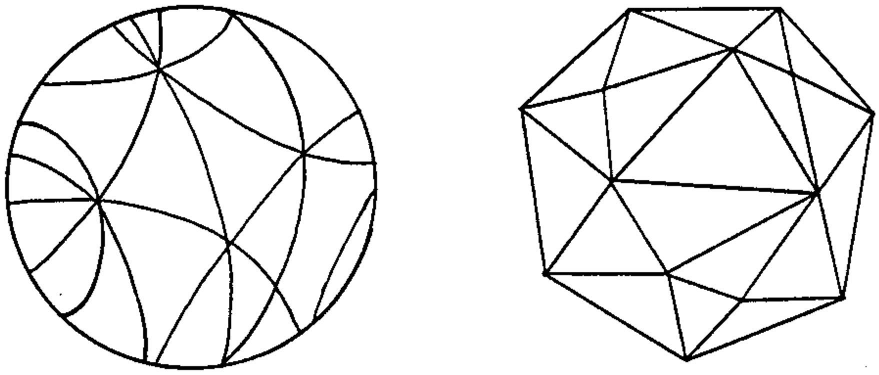

Figure 1

A manifold can then be defined by its relationship to a piecewise linear space: a topological space is called a closed d-dimensional manifold if it is homeomorphic to a connected polyhedron, and furthermore, if its points possess neighborhoods which are homeomorphic to the interior of the d-dimensional sphere[23].

2.2 Description of Local Curvature in Terms of Deficit Angles

The curvature on a two-dimensional surface is defined locally by the ratio

$$(curvature) = \frac{rotation\ of\ vector}{area\ circumnavigated} \tag{2.1}$$

and the area centered on the point one is considering is supposed to be small. A similar definition applies in higher dimension, but with the difference that in d dimension there are $d(d-1)/2$ independent surfaces to be considered and d components for the rotated vector. This leads in the continuum to a description of local curvature in terms of the $d^2(d^2-1)/12$ components of the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$[27]. The Riemann tensor can be completely reconstructed from a set of independent sectional (gaussian) curvatures[28–30].

Curvature on the piecewise linear space is described in terms of *deficit angles* $\delta_h(\sigma^{d-2})$ assigned to each simplex ('hinge' h) σ^{d-2} of dimension $d-2$. In two dimension one considers a number of triangles that meet on a common vertex. A *dihedral angle* θ_d is associated with each triangle at that vertex. The deficit angle δ_h is then defined as

$$\delta_h = 2\pi - \sum_{\substack{\text{triangles}\\ \text{meeting on h}}} \theta_d \tag{2.2}$$

Figure 2

The two-dimensional case is illustrated in fig. 2.

In three dimensions several tetrahedra meet on a common edge, and with each tetrahedron a dihedral angle can be associated at the given edge. The three-dimensional case is illustrated in fig. 3.

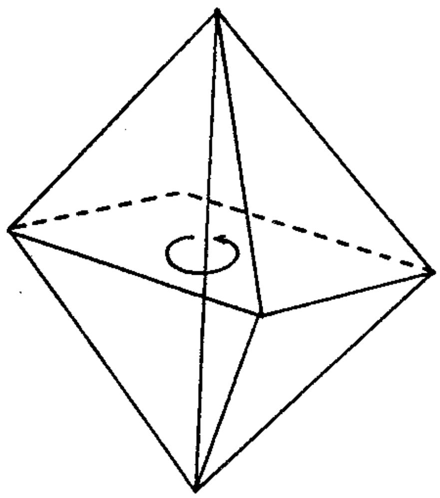

Figure 3

In four dimensions several four-simplices meet on a common triangle, and with each four-simplex a dihedral angle can be associated at the given triangle. This is illustrated in fig. 4.

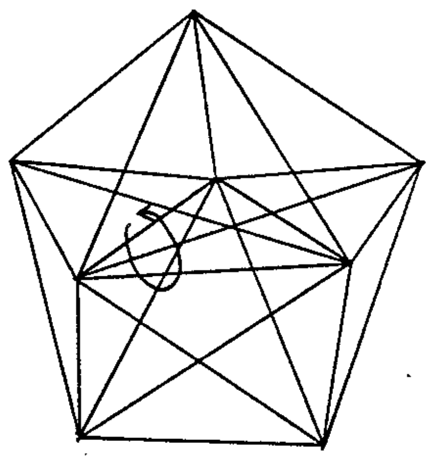

Figure 4

Thus in d dimensions several d-simplices meet on a $(d-2)$-dimensional hinge, and the deficit angle is defined by

$$\delta_h = 2\pi - \sum_{\substack{\text{d-simplices} \\ \text{meeting on h}}} \theta_d \tag{2.3}$$

The sine of the dihedral angle is given by the formula[27]

$$\sin\theta_d = \frac{d}{d-1}\frac{V_d V_{d-2}}{V_{d-1}V'_{d-1}} \tag{2.4}$$

where V_{d-2} is the volume of the hinge, V_d is the volume of the d-simplex, and V_{d-1}, V'_{d-1} the volumes of the two $(d-1)$-dimensional faces that meet on the hinge. But since the sine does not determine the angle, another formula is needed which uniquely determines the angle. This formula is given in the appendix.

In piecewise linear spaces curvature is detected by going around elementary loops which are dual to a $(d-2)$-dimensional subspace. Still the area of the loop is not well defined, since any loop inside the d-dimensional simplices bordering the hinge will give the same result for the deficit angle. On the other hand the hinge has a content (the length of the edge in $d=3$ and the area of the triangle in $d=4$), and there is a natural volume associated with each hinge, defined by dividing the volume of each simplex touching the hinge into a contribution belonging to that hinge, and other contributions belonging to the other hinges on that simplex[31]. The contribution belonging to that simplex will be called *dihedral volume* V_d. The volume V_h associated with the hinge h is then naturally the sum of the dihedral volumes V_d belonging to each simplex

$$V_h = \sum_{\substack{d-\text{simplices} \\ \text{meeting on } h}} V_d \tag{2.5}$$

The dihedral volume associated with each hinge in a simplex can be defined using dual volumes, a baricentric subdivision[23] or some other natural way of dividing the volume of a d-simplex in $d(d+1)/2$ parts. If the theory has some reasonable continuum limit, then the final result should not depend on the detailed choice of volume type. This assumption is the analog of the statement that the value of the Riemann integral is independent of the particular choice of the subdivision of the interval of integration occurring in the Riemann sum.

There is a well-established procedure for constructing a dual lattice for any given lattice[32]. This involves constructing polyhedral cells, known in the literature as Voronoi polyhedra, around each vertex, in such a way that the cell around each particular vertex contains all points which are nearer to that vertex than to any other vertex. Thus the cell is made up from $(d-1)$-dimensional subspaces which are the perpendicular bisectors of the edges in the original lattice, $(d-2)$-dimensional subspaces which are orthogonal to the 2-dimensional subspaces of the original lattice, and so on. Formulas for dual volumes are given in Appendix B. In the case of the baricentric subdivision, the dihedral volume is just $2/d(d+1)$ times the volume of the simplex.

This leads one to conclude that there is a natural area A_{Γ_h} associated with each hinge

$$Area\ of\ loop = A_{\Gamma_h} \propto \frac{V_h}{A_h^{d-2}} \tag{2.6}$$

obtained by dividing the volume per hinge (which is d-dimensional) by the volume of the hinge (which is $(d-2)$-dimensional).

2.3 Regge Action and Equations of Motion

Before discussing the construction of higher derivative terms, the lattice analogue of the Einstein action will next be introduced. In a d-dimensional piecewise linear space-time, the expression analogous to the Einstein action was given by Regge[12] as

$$I_R = \sum_{hinges\ h} A_h^{d-2} \delta_h \tag{2.7}$$

where A_h^{d-2} is the volume of the hinge and δ_h is the deficit angle there. The action is the equivalent for a simplicial decomposition of the continuum expression

$$I_E = \frac{1}{2} \int d^d x \sqrt{g} R \tag{2.8}$$

and indeed it has been shown[25,26,32–35] that I_R tends to the continuum expression as the Regge block size (or the average edge length) tends to zero. In two dimension the discrete analogue of the Gauss-Bonnet theorem holds

$$I_R = \sum_h \delta_h = 2\pi\chi \tag{2.9}$$

where χ is the Euler characteristic (two minus twice the number of handles of the surface). This remarkable identity ensures that two-dimensional lattice R gravity is as trivial as in the continuum, since the variation of the local action density under a small variation of an edge length l_p is still zero (more precisely, it can be written as a boundary term)

$$\sum_{h \subset l_p} \delta(\delta_h) = 0 \tag{2.10}$$

Here the sum is over hinges that are affected by the change of the edge l_p. In higher dimensions variation of I_R with respect to the edge lengths gives the simplicial analogue of Einstein's equations, whose derivation is particularly simplified by the fact that the variation of the deficit angle is zero in any dimensions

$$\delta I_R = \sum_h \delta(A_h^{d-2})\delta_h \tag{2.11}$$

(In the continuum one also finds that to first order the variation of the curvature gives a total derivative).

This then implies the equations of motion $\delta_h = 0 \ \forall h$ in three dimensions. In four dimensions variation with respect to l_p yields[12]

$$\frac{1}{2}\, l_p \sum_{h \supset l_p} \delta_h \cot\theta_{ph} = 0 \tag{2.12}$$

where the sum is over hinges (triangles) labeled by h meeting on the common edge p, and θ_{ph} is the angle in the hinge h opposite to the edge p. This is illustrated in fig. 5.

A solution to the skeleton equation can then be found by adjusting the edge lengths. Since the equations are non-linear in the edge length variables, the existence of multiple solutions cannot in general be ruled out. Several authors have discussed the applications of the Regge equations to problems in classical general relativity such as the Schwarzshild and Reissner-Nordstrom geometries, the Friedman and Tolman universes [36,37], and the problem of radial motion and circular (actually polygonal) orbits[38].

2.4 Local Gauge Invariance and Bianchi Identities

Consider the two-dimensional flat skeleton shown in fig. 6. It is clear that one can move around a point on the surface, keeping all the neighbors fixed, without violating the triangle inequalities and leave all curvature invariants unchanged.

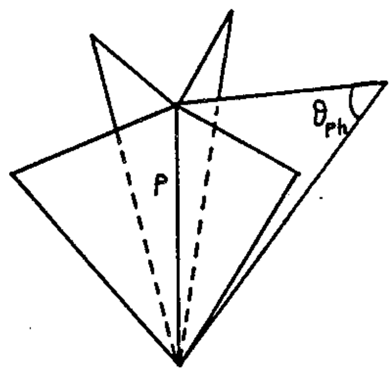

Figure 5

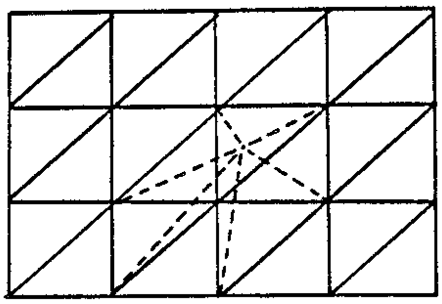

Figure 6

In d dimensions this transformation has d parameters and is an exact invariance of the action. When space is slightly curved, the invariance is in general only an approximate one, even though for piecewise linear spaces piecewise diffeomorphisms can still be defined as the set of local motions of points that leave the action, the measure and the lattice curvature invariants unchanged. In the limit when the number of edges becomes very large, the continuum diffeomorphism group should be recovered.

Before discussing the Bianchi identities, it is useful to interpret some of the above definitions and results in terms of the parallel transport of a test vector around a small loop. Consider a closed path Γ encircling a hinge h and passing through each of the simplices that meet at that hinge. In particular one may take Γ to be the boundary of the polyhedral dual area surrounding the hinge.

For each neighboring pair of simplices $j, j+1$, we may write down a Lorentz transformation $L_\mu^{\ \nu}$, which describes how a given vector ϕ_μ transforms between the local coordinate systems in these two simplices

$$\phi'_\mu = \left[L(j,j+1)\right]_\mu^{\ \nu}\phi_\nu \tag{2.13}$$

(Note that it is possible to choose coordinates so that $L_\mu^{\ \nu}$ is the unit matrix for one pair of simplices, but it will not then be unity for other pairs). The Lorentz transformation is related to the path-ordered (P) exponential of the integral of the connection $(\Gamma_\lambda)_\mu^\nu = \Gamma^\nu_{\mu\lambda}$ by

$$L_\mu^{\ \nu} = [Pe^{\int_{\text{path between simplices}} \Gamma_\lambda dx^\lambda}]_\mu^\nu \tag{2.14}$$

The connection here has support only on the common interface between the two simplices. The product of these Lorentz transformations around a closed elementary loop Γ is then given, for smooth enough manifolds, by

$$[\prod_{\substack{\text{pairs of}\\ \text{simplices on }\Gamma}} L(j,j+1)]_\mu^\nu \approx [e^{R^{\cdot}_{\cdot\rho\sigma}\Sigma^{\rho\sigma}}]_\mu^\nu \tag{2.15}$$

where $(R^{\cdot}_{\cdot\rho\sigma})_\mu^\nu = R^\nu_{\ \mu\rho\sigma}$ is the curvature tensor and $\Sigma^{\rho\sigma}$ is a bivector in the plane of Γ, with magnitude equal to $1/\sqrt{2}$ times the area of the loop Γ. (For a parallelogram with edges a^ρ and b^ρ, $\Sigma^{\rho\sigma} = \frac{1}{2}(a^\sigma b^\rho - a^\rho b^\sigma)$.

The total change in a vector ϕ_μ which undergoes parallel transport around Γ is then given by

$$\phi'_\mu = \phi_\mu + \delta\phi_\mu = [\prod_{\substack{\text{pairs of}\\ \text{simplices on }\Gamma}} L(j,j+1)]_\mu^\nu \phi_\nu \tag{2.16}$$

which reproduces to lowest order the usual parallel transport formula

$$\Delta\phi_\mu = R^\nu_{\ \mu\rho\sigma}\Sigma^{\rho\sigma}\phi_\nu \tag{2.17}$$

On the Regge skeleton the effect of parallel transport around Γ is described by

$$[\prod_j L(j,j+1)]_{\mu\nu} = [e^{\delta_h U^{(h)}_{\cdot\cdot}}]_{\mu\nu} \tag{2.18}$$

where $U^{(h)}_{\mu\nu}$ is a bivector orthogonal to the hinge h, defined in 4 dimensions by

$$U^{(h)}_{\mu\nu} = \frac{1}{2A_h}\epsilon_{\mu\nu\rho\sigma} l^\rho_{(a)} l^\sigma_{(b)} \tag{2.19}$$

and $l^\rho_{(a)}$ and $l^\rho_{(b)}$ are the vectors forming two sides of the hinge h. Note that the validity of the simplicial parallel transport formula given above is not restricted to small deficit angles.

Comparison of equations (2.15) and (2.18) means that we may make the identification

$$R_{\mu\nu\rho\sigma}\Sigma^{\rho\sigma} \quad \rightarrow \quad \delta_h U^{(h)}_{\mu\nu} \tag{2.20}$$

It is important to notice here that this relation does not give complete information about the Riemann tensor, but only about its projection in the plane of the loop Γ orthogonal to the hinge. In fact the deficit angle divided by the area of the loop can be taken as a definition of the *local sectional curvature* K_h[28]

$$\frac{\delta_h}{A_{\Gamma_h}} = K_h = \frac{R_{\mu\nu\rho\sigma}e^\mu_a e^\nu_b e^\rho_a e^\sigma_b}{(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})e^\mu_a e^\nu_b e^\rho_a e^\sigma_b} \tag{2.21}$$

which represents the projection of the Riemann curvature in the direction of the bivector $\mathbf{e_a} \wedge \mathbf{e_b}$.

For a continuous space of constant curvature the K' are independent of the direction of the bivector and one has

$$R_{\mu\nu\rho\sigma} = K(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma}) \tag{2.22}$$

This is analogous to the expression suggested in ref. (31) for the Riemann tensor at a hinge,

$$R^{(h)}_{\mu\nu\rho\sigma} = \frac{\delta_h}{A_{\Gamma_h}} U^{(h)}_{\mu\nu} U^{(h)}_{\rho\sigma} \tag{2.23}$$

and one is possibly neglecting terms in $R_{\mu\nu\rho\sigma}$ which vanish when projected in the plane of the loop Γ. This is a reason why one encounters problems if one tries to use (2.23) as a formula giving full information about $R_{\mu\nu\rho\sigma}$ in Regge calculus. (This point is further discussed in section 3.3). Note however that (2.23) does have all the correct symmetries of the Riemann tensor.

The parallel transporters around closed elementary loops satisfy the lattice analogues of the Bianchi identities. In the continuum the Bianchi identities read

$$\frac{\partial R_{\mu\nu\rho\sigma}}{\partial x^\lambda} + \frac{\partial R_{\mu\nu\lambda\rho}}{\partial x^\sigma} + \frac{\partial R_{\mu\nu\sigma\lambda}}{\partial x^\rho} = 0 \tag{2.24}$$

On the simplicial lattice the Bianchi identities are derived by considering closed paths in four dimensions that can be shrunk to a point without entangling any hinge. The product of rotation matrices associated with the path then has to give the identity matrix[12,16]. Thus, for example, the ordered product of rotation matrices associated with the triangles meeting on a given edge has to give one, since a path can be constructed which sequentially encircles all the triangles and is topologically trivial

$$\prod_{\substack{\text{hinges h} \\ \text{meeting on edge p}}} [e^{\delta_h U^{(h)}_{\cdot\cdot}}]_{\mu\nu} = 1 \tag{2.25}$$

Other identities might be derived by considering paths that encircle hinges meeting on one point. Regge has shown that the above lattice relations correspond to the continuum Bianchi identities[12]. These equations also parallel the lattice Bianchi identities in lattice gauge theories.

3. Formulation of Higher Derivative Terms

3.1 Construction of $\mathbf{R}^2$-type Terms

Next generalizations of the Regge calculus equivalent of the Einstein action will be constructed. First consider a cosmological constant term, which in the continuum theory takes the form $\lambda \int d^d x \sqrt{g}$. The expression for the cosmological constant term on the lattice involves the total volume of the simplicial complex. This may be written as

$$V_{total} = \sum_{d-simplices} V_{simplices} \tag{3.1}$$

or equivalently as

$$V_{total} = \sum_{hinges\ h} V_h \tag{3.2}$$

where V_h is the volume associated with each hinge, as described above. Thus one may regard the invariant volume element $\sqrt{g} d^d x$ as being represented by either V_h or $V_{simplex}$.

Secondly, one wishes to find a term equivalent to the continuum expression $\frac{1}{4} \int d^d x \sqrt{g} R^2$, and the remainder of this section will be concerned with this problem[31]. It may be objected that since in Regge calculus where the curvature is restricted to the hinges which are subspaces of dimension 2 less than that of the space considered, then the curvature tensor involves δ-functions with support on the hinges, and so higher powers of the curvature tensor are not defined[33,35]. (This argument clearly does not apply to the Euler characteristic

$$\chi = \frac{1}{128\pi^2} \int d^4 x \sqrt{g} R_{\mu\nu\rho\sigma} R^{\kappa\lambda\omega\tau} \epsilon^{\mu\nu\kappa\lambda} \epsilon^{\rho\sigma\omega\tau} \tag{3.3}$$

and the Hirzebruch signature

$$\tau = \frac{1}{96\pi^2} \int d^4 x \sqrt{g} R_{\mu\nu\rho\sigma} R^{\mu\nu}_{\kappa\lambda} \epsilon^{\rho\sigma\kappa\lambda} \tag{3.4}$$

which are both integrals of 4-forms). However it is a common procedure in lattice field theory to take powers of fields defined at the same point, and there is no reason why one should not consider similar terms in lattice gravity. Of course one would like the expressions to correspond to the continuum ones as the edge lengths of the simplicial lattice become smaller and smaller.

Since the curvature is restricted to the hinges, it is natural that expressions for curvature integrals should involve sums over hinges as in (2.7). The curvature tensor, which involves second derivatives of the metric, is of dimension L^{-2}. Therefore $\frac{1}{4}\int d^dx\sqrt{g}R^n$ is of dimension L^{d-2n}. Thus if one postulates that an R^2 term will involve the square of $A_h\delta_h$, which is of dimension $L^{2(d-2)}$, then one will need to divide by some d-dimensional volume to obtain the correct dimension for the extra term in the action. Now any hinge is surrounded by a number of d-dimensional simplices, so the procedure of dividing by a d-dimensional volume seems ambiguous. The crucial step is to realize that there *is* a unique d-dimensional volume associated with each hinge, as described above.

If one regards the invariant volume element $\sqrt{g}d^dx$ as being represented by V_h when one performs the sum over hinges as in equation (2.7), then this means that one may regard the scalar curvature R as being represented at each hinge by $2A_h\delta_h/V_h$

$$\frac{1}{2}\int d^dx\sqrt{g}R \quad\rightarrow\quad \sum_{hinges\ h} V_h\frac{A_h\delta_h}{V_h} \equiv \sum_{hinges\ h} A_h\delta_h \tag{3.5}$$

It is then straightforward to see that

$$\frac{1}{4}\int d^dx\sqrt{g}R^2 \quad\rightarrow\quad \sum_{hinges\ h} V_h\Big(\frac{A_h\delta_h}{V_h}\Big)^2 \equiv \sum_{hinges\ h} V_h\Big(\frac{\delta_h}{A_{\Gamma_h}}\Big)^2 \tag{3.6}$$

If one takes equation (2.23) for the Riemann tensor on a hinge and contracts one obtains

$$R^{(h)} = 2\frac{\delta_h}{A_{\Gamma_h}} \tag{3.7}$$

which agree with the form we have used for R, and shows that the numerical factors have been chosen correctly in (2.23).

3.2 Convergence to the Continuum; the Regular Tessellations of S^n

It is of interest to see how the formulae (3.2), (3.5) and (3.6) for an R^2-type term compare with the continuum values for the regular tessellations of the two-sphere, the three-sphere and the four-sphere. (These correspond to the regular polyhedra in three, four and five dimensions[39]). For regular tessellations, the volumes V_h take a very simple form since each d-dimensional simplex has its volume divided into $\mathbf{p}$ equal parts, where $\mathbf{p}$ is the number of hinges per simplex. If $\mathbf{q}$ d-simplices meet at each hinge, then V_h is just the sum of $\mathbf{q}$ of these contributions

$$V_h = \frac{q}{p}V \tag{3.8}$$

where V is the volume of the d-simplex. Then in 2 dimensions, $\{\mathbf{p},\mathbf{q}\}$, with $\mathbf{p}$ and $\mathbf{q}$ as defined here, is just the Schläfli symbol. In 3 dimensions, the Schläfli symbol is $\{\mathbf{a},\mathbf{b},\mathbf{q}\}$, where $\{\mathbf{a},\mathbf{b}\}$, is the Schläfli symbol of the 2-dimensional simplex used to build the 3-dimensional ones. (Thus $\{\mathbf{a},\mathbf{b}\}$, determines the value of $\mathbf{p}$ as defined above).

For regular tessellations, the dihedral angles θ_d also take particularly simple forms. For example, the dihedral angle θ_d at the $(d-2)$-dimensional hinge in a d-dimensional simplex satisfies

$$\cos\theta_d = \frac{1}{d} \tag{3.9}$$

The results for the regular tessellations of $\mathbf{S^2}$, $\mathbf{S^3}$ and $\mathbf{S^4}$ are listed below. The scale for each tessellation is set by requiring the edge lengths l to give the same total volume as a sphere in that dimension, of radius r. The scalar curvature for $\mathbf{S^n}$ is $n(n-1)/r^2$ [41]. The full analytic expressions and more details can be found in ref. (31,40).

Table I Regular Tessellations of $\mathbf{S^2}$

Tessellation	N_0	Volume	$\sum_h \delta_h$ $(\equiv \frac{1}{2}\int d^2x\sqrt{g}R)$	$\sum_h \frac{\delta_h^2}{V_h}$ $(\equiv \frac{1}{4}\int d^2x\sqrt{g}R^2)$
Tetrahedron α_3	4	$\sqrt{3}l^2$	4π	$\frac{16\pi^2}{\sqrt{3}l^2} \equiv \frac{4\pi}{r^2}$
Octahedron β_3	6	$2\sqrt{3}l^2$	4π	$\frac{8\pi^2}{\sqrt{3}l^2} \equiv \frac{4\pi}{r^2}$
Cube γ_3	8	$6l^2$	4π	$\frac{8\pi^2}{3l^2} \equiv \frac{4\pi}{r^2}$
Icosahedron	12	$5\sqrt{3}l^2$	4π	$\frac{16\pi^2}{5\sqrt{3}l^2} \equiv \frac{4\pi}{r^2}$
Dodecahedron	20	$\frac{3\,5^{3/4}}{2^{3/2}}(1+\sqrt{5})^{3/2}l^2$	4π	$\frac{32\sqrt{2}\pi^2}{3\,5^{3/4}(1+\sqrt{5})^{3/2}l^2} \equiv \frac{4\pi}{r^2}$
Continuum	-	$4\pi r^2$	4π	$\frac{4\pi}{r^2}$

Table I shows the simplicial lattice predictions for the various tessellations of $\mathbf{S^2}$. In two dimensions the number of hinges is equal to the number of sites N_0. The second expression in the last column is the form taken by the first expression there when the length scale is set in the way described above. For $\mathbf{S^2}$, the Regge calculus equivalent of the Einstein action is exact, as indeed it must be by the Gauss-Bonnet theorem. Note that the Regge calculus expression for $\frac{1}{4}\int d^dx\sqrt{g}R^2$ is also exact in this case!

In table II the results for the regular tessellations of $\mathbf{S^3}$ are listed. As the number of vertices increases, the values of $\sum_h l_h\delta_h$ and $\sum_h l_h^2\delta_h^2/V_h$ tend towards the continuum expression $\frac{1}{2}\int d^3x\sqrt{g}R$ and $\frac{1}{4}\int d^3x\sqrt{g}R^2$.

For both operators the approach to the continuum is very close to an $N_0^{-2/3}$ (or $N_1^{-2/3}$) behavior. These results are further evidence that the formula (2.18) is indeed the appropriate representation for an R^2-type term in the lattice action.

Table II Regular Tessellations of S^3

Tessellation	sites N_0	hinges N_1	$\sum_h l_h \delta_h$ $(\equiv \frac{1}{2}\int d^3x\sqrt{g}R)$	$\sum_h \frac{l_h^2 \delta_h^2}{V_h}$ $(\equiv \frac{1}{4}\int d^3x\sqrt{g}R^2)$
5-cell	5	10	$8.461\pi^2 r$	$35.789\pi^2 r^{-1}$
16-cell	8	24	$7.231\pi^2 r$	$26.144\pi^2 r^{-1}$
Tessaract	16	32	$6.880\pi^2 r$	$23.681\pi^2 r^{-1}$
24-cell	24	96	$6.455\pi^2 r$	$20.836\pi^2 r^{-1}$
600-cell	120	720	$6.121\pi^2 r$	$18.735\pi^2 r^{-1}$
120-cell	600	1200	$6.077\pi^2 r$	$18.467\pi^2 r^{-1}$
Continuum	-	-	$6.000\pi^2 r$	$18.000\pi^2 r^{-1}$

The convergence of the numerical results for S^4 shown in table III is not as impressive as for S^3. The problem lies in the fact that there are no other regular tessellations of S^4, and ones with 6, 10 or even 32 vertices are certainly very crude approximations to the continuum. Hence one cannot expect in this case strong evidence from the regular tessellations on the convergence to the continuum of the simplicial lattice expressions.

Table III Regular Tessellations of S^4

Tessellation	sites N_0	hinges N_2	$\sum_h A_h \delta_h$ $(\equiv \frac{1}{2}\int d^4x\sqrt{g}R)$	$\sum_h \frac{A_h^2 \delta_h^2}{V_h}$ $(\equiv \frac{1}{4}\int d^4x\sqrt{g}R^2)$
α_5	6	20	$28.04\pi^2 r^2$	$294.9\pi^2$
β_5	10	80	$21.08\pi^2 r^2$	$166.6\pi^2$
γ_5	32	80	$20.70\pi^2 r^2$	$160.0\pi^2$
Continuum	-	-	$16.00\pi^2 r^2$	$96.00\pi^2$

3.3 Other Higher Derivative Terms

So far the inclusion of only one of many possible higher derivative terms in the lattice action was discussed. As shown in appendix A, there are six fourth derivative terms in four dimensions, two of which are topological invariants, the Euler characteristic and the Hirzebruch signature. Let us consider these last two quantities first.

The Euler characteristic χ for a simplicial decomposition may be obtained from a particular case of the general formula for the analogue of the Lipschitz-Killing curvatures of smooth Riemannian manifolds for piecewise flat spaces(26). In two dimensions, the formula of Cheeger, Müller and Schrader reduces of course to

$$\chi = \frac{1}{2\pi} \sum_h \delta_h \tag{3.10}$$

which is the exact equivalent of the Gauss-Bonnet theorem

$$\chi = \frac{1}{4\pi} \int d^2x \sqrt{g} R \tag{3.11}$$

In four dimensions the formula becomes

$$\chi = \sum_{\sigma^0} [1 - \sum_{\sigma^2 \supset \sigma^0} (0,2) - \sum_{\sigma^4 \supset \sigma^0} (0,4) + \sum_{\sigma^4 \supset \sigma^2 \supset \sigma^0} (0,2)(2,4)] \tag{3.12}$$

where σ^i denotes an i-dimensional simplex and (i,j) denotes the (internal) dihedral angle at an i-dimensional face of a j-dimensional simplex. Thus, for example, $(0,2)$ is the angle at the vertex of a triangle and $(2,4)$ is the dihedral angle at a triangle in a 4-simplex (The normalization of the angles is such that the volume of a sphere in any dimension is one; thus planar angles are divided by 2π, 3-dimensional solid angles by 4π and so on).

Of course there is a much simpler formula for the Euler characteristic of a simplicial complex

$$\chi = \sum_{i=0}^{d} (-1)^i N_i \tag{3.13}$$

where N_i is the number of simplices of dimension i. However, it may turn out to be useful in quantum gravity calculations to have a formula for χ in terms of the angles, and hence of the edge lengths, of the simplicial decomposition. In practice, an obstacle to the use of (4.3) is that there is no simple formula for $(0,4)$, the solid angle at the vertex of a general 4-simplex. This is equivalent to the long-standing problem of the volume of a spherical tetrahedron(42). For a regular 4-simplex, it can be shown that $(0,4) = -\frac{1}{5} + \frac{1}{2\pi} \cos^{-1} \frac{1}{4}$. Furthermore, there seems to be no equivalent formula for the Hirzebruch signature for a simplicial decomposition.

Formula (3.12) does not appear to be bilinear in the deficit angles, as one would have expected from our general arguments about R^2 type terms. However this may be due to the fact that the Euler characteristic is a total divergence, and so this formula is probably equivalent in some sense to evaluating the surface integral of the curvature two-form times the connection one-forms.

Let us now look in more detail at the other possible higher derivative terms mentioned at the beginning of this section. In fact in two dimensions, as shown in the appendix A, there is only *one* independent higher derivative term, so the R^2 term, which we have already written down, is the only possible term of dimension four. In three dimensions one needs to find also an expression for $\int d^3x\sqrt{g}R_{\mu\nu}R^{\mu\nu}$. In 4 dimensions there are two independent higher derivative terms, which can be taken to be $\int d^4x\sqrt{g}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ and $\int d^4x\sqrt{g}R^2$.

If one considers the expression for the Riemann tensor on a given hinge

$$R^{(h)}{}_{\mu\nu\rho\sigma} = \frac{\delta_h}{A_{\Gamma_h}} U^{(h)}_{\mu\nu} U^{(h)}_{\rho\sigma} \tag{3.14}$$

one finds that the higher derivative terms are all proportional to each other

$$\frac{1}{4} R^{(h)}{}_{\mu\nu\rho\sigma} R^{(h)\mu\nu\rho\sigma} = \frac{1}{2} R^{(h)}{}_{\mu\nu} R^{(h)\mu\nu} = \frac{1}{2} R^{(h)2} = \frac{\delta_h}{A_{\Gamma_h}} \tag{3.15}$$

Furthermore if one uses the above expression for the Riemann tensor to evaluate the contribution to the Euler characteristic on each hinge one obtains zero, and it is therefore clear that one needs cross terms involving contributions from different hinges. Even then it seems unlikely that one would obtain the correct integer value for a particular simplicial decomposition by this method, and formula (3.12) or (3.13) has to be used.

Thus one is faced with the puzzling situation that only *one* higher derivative term can be constructed at a given hinge, while in the continuum there appear to be *two* terms in four dimensions. The regular tessellations of $\mathbf{S^n}$ are not able to distinguish between different terms since one has in this case

$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \propto R_{\mu\nu} R^{\mu\nu} \propto R^2 \propto \frac{1}{r^2} \tag{3.16}$$

for a sphere of radius r.

The next step is then to construct the full Riemann tensor by considering more than one hinge. The simplest possibilities would be to consider all the hinges that are the faces of a simplex or, alternatively, all the hinges that have one point in common. On the other hand one notes that since the value of (3.14) depends on the coordinate system used, one should consider, in this formulation, terms only from those hinges which can be covered by the same coordinate system. Define the Riemann tensor for a simplex as a weighted sum of hinge contributions

$$\left[R_{\mu\nu\rho\sigma}\right]_{\sigma^4} = \sum_{\sigma^2 \subset \sigma^4} \omega_{\sigma^2} \left[\frac{\delta}{A_\Gamma} U_{\mu\nu} U_{\rho\sigma}\right]_{\sigma^2} \tag{3.17}$$

where the ω_{σ^2} are dimensionless weights, to be determined later. After squaring one obtains

$$\left[R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\right]_{\sigma^4} = \sum_{\sigma^2 \subset \sigma^4} \sum_{\sigma^{2\prime} \subset \sigma^4} \omega_{\sigma^2} \omega_{\sigma^{2\prime}} \left[\frac{\delta}{A_\Gamma} U_{\mu\nu} U_{\rho\sigma}\right]_{\sigma^2} \left[\frac{\delta}{A_\Gamma} U^{\mu\nu} U^{\rho\sigma}\right]_{\sigma^{2\prime}} \tag{3.18}$$

Consider two hinges labeled by i and j. By using formula (2.19) for the bivectors $U_{\mu\nu}$ the product of the last two square brackets can be worked out[43].

$$
\begin{aligned}
R^{(i)}{}_{\mu\nu\rho\sigma}R^{(j)\mu\nu\rho\sigma} &\equiv \Big[\frac{\delta}{A_\Gamma}U_{\mu\nu}U_{\rho\sigma}\Big]_{(i)}\Big[\frac{\delta}{A_\Gamma}U^{\mu\nu}U^{\rho\sigma}\Big]_{(j)} \\
&= \frac{\delta_i\delta_j}{A_{\Gamma_i}A_{\Gamma_j}}\frac{1}{4{A_i}^2{A_j}^2}\Big[(a\cdot c)(b\cdot d)-(a\cdot d)(b\cdot c)\Big]^2
\end{aligned}
\tag{3.19}
$$

where a and b are two edges in hinge i, and c and d are two edges in hinge j. For the square of the Ricci tensor one needs the expressions

$$
\begin{aligned}
R^{(i)}{}_{\mu\nu}R^{(j)\mu\nu} \equiv&\Big[\frac{\delta}{A_\Gamma}U^\rho_\mu U_{\rho\nu}\Big]_{(i)}\Big[\frac{\delta}{A_\Gamma}U^{\sigma\mu}U^\nu_\sigma\Big]_{(j)} \\
=&\frac{\delta_i\delta_j}{A_{\Gamma_i}A_{\Gamma_j}}\frac{1}{16A_i^2A_j^2} \\
&\times\Big[a^2c^2(b\cdot d)^2+a^2d^2(b\cdot c)^2+b^2c^2(a\cdot d)^2+b^2d^2(a\cdot c)^2 \\
&-2[a^2(b\cdot c)(c\cdot d)(d\cdot b)+b^2(a\cdot d)(c\cdot d)(d\cdot a)+c^2(a\cdot b)(b\cdot d)(d\cdot a) \\
&+d^2(a\cdot b)(b\cdot c)(c\cdot a)]+2[(a\cdot b)(c\cdot d)[(a\cdot c)(b\cdot d)+((a\cdot d)(b\cdot c)]]\Big]
\end{aligned}
\tag{3.20}
$$

and for the scalar curvature squared

$$
\begin{aligned}
R^{(i)}R^{(j)} &\equiv \Big[\frac{\delta}{A_\Gamma}U^{\mu\nu}U_{\mu\nu}\Big]_{(i)}\Big[\frac{\delta}{A_\Gamma}U^{\rho\sigma}U_{\rho\sigma}\Big]_{(j)} \\
&= 4\frac{\delta_i\delta_j}{A_{\Gamma_i}A_{\Gamma_j}}
\end{aligned}
\tag{3.21}
$$

and the Euler characteristic

$$
\begin{aligned}
R^{(i)}{}_{\mu\nu\rho\sigma}R^{(j)\kappa\lambda\omega\tau}\epsilon^{\mu\nu\kappa\lambda}\epsilon^{\rho\sigma\omega\tau} &\equiv \Big[\frac{\delta}{A_\Gamma}U_{\mu\nu}U_{\rho\sigma}\Big]_{(i)}\Big[\frac{\delta}{A_\Gamma}U_{\kappa\lambda}U_{\omega\tau}\Big]_{(j)}\epsilon^{\mu\nu\kappa\lambda}\epsilon^{\rho\sigma\omega\tau} \\
&= \frac{\delta_i\delta_j}{A_{\Gamma_i}A_{\Gamma_j}}\frac{1}{A_i^2A_j^2}\Big[\epsilon_{\mu\nu\rho\sigma}a^\mu b^\nu c^\rho d^\sigma\Big]^2
\end{aligned}
\tag{3.22}
$$

and finally for the Hirzebruch signature

$$
\begin{aligned}
R^{(i)}{}_{\mu\nu\rho\sigma}R^{(j)\mu\nu}{}_{\kappa\lambda}\epsilon^{\rho\sigma\kappa\lambda} &\equiv \Big[\frac{\delta}{A_\Gamma}U_{\mu\nu}U_{\rho\sigma}\Big]_{(i)}\Big[\frac{\delta}{A_\Gamma}U^{\mu\nu}U_{\kappa\lambda}\Big]_{(j)}\epsilon^{\rho\sigma\kappa\lambda} \\
&= \frac{\delta_i\delta_j}{A_{\Gamma_i}A_{\Gamma_j}}\frac{1}{2A_i^2A_j^2}\epsilon_{\mu\nu\rho\sigma}a^\mu b^\nu c^\rho d^\sigma\Big[(a\cdot c)(b\cdot d)-(a\cdot d)(b\cdot c)\Big]
\end{aligned}
\tag{3.23}
$$

In the above formulae A_i and A_j are the areas of the triangles i and j, respectively.

The question of the weights ω_{σ^2} introduced in equation (3.17) will now be addressed. Consider the expression for the scalar curvature of a simplex defined as

$$
\Big[R\Big]_{\sigma^4} = \sum_{\sigma^2\subset\sigma^4}\omega_{\sigma^2}\Big[2\frac{\delta}{A_\Gamma}\Big]_{\sigma^2}
\tag{3.24}
$$

It is clear from the formulae given above for the lattice curvature invariants (constructed in a simplex by summing over hinge contributions) that there is again a natural volume associated with them : the sum of the volumes of the hinges that form the faces of the simplex

$$V_{\sigma^4} = \sum_{\sigma^2 \subset \sigma^4} V_{\sigma^2} \tag{3.25}$$

where $V_{\sigma^2} \equiv V_h$ is the volume defined in equation (2.5). Summing the scalar curvature over all simplices, one should recover Regge's expression

$$\sum_{\sigma^4} V_{\sigma^4} [R]_{\sigma^4} = \sum_{\sigma^4} \sum_{\sigma^2 \subset \sigma^4} \omega_{\sigma^2} [2 \frac{\delta}{A_\Gamma}]_{\sigma^2} = \sum_{\sigma^2} \delta_{\sigma^2} A_{\sigma^2} \tag{3.26}$$

which implies

$$N_{2,4} V_{\sigma^4} \omega_{\sigma^2} \frac{\delta_{\sigma^2}}{A_{\Gamma_{\sigma^2}}} \equiv N_{2,4} V_{\sigma^4} \omega_{\sigma^2} \frac{\delta_{\sigma^2} A_{\sigma^2}}{V_{\sigma^2}} = \delta_{\sigma^2} A_{\sigma^2} \tag{3.27}$$

where $N_{2,4}$ is the number of simplices meeting on that hinge. Therefore the natural choice for the weights is

$$\omega_\sigma^2 = \frac{V_{\sigma^2}}{N_{2,4} V_{\sigma^4}} = \frac{V_{\sigma^2}}{N_{2,4} \sum_{\sigma^2 \subset \sigma^4} V_{\sigma^2}} \tag{3.28}$$

Thus the weighting factors that reproduce Regge's formula for the Einstein action are just the volume fractions occupied by the various hinges in a simplex, which is not surprising. Of course the above formulae are not unique, since one might have done the above construction of higher derivative terms by considering a point σ^0 instead of a 4-simplex σ^4. The Riemann tensor is then constructed by averaging with the appropriate weights the contributions of different hinges meeting at one point, and the volume V_{σ^4} becomes the sum of all hinge volume contributions coming from the hinges touching the point.

The above formulae for higher derivative terms are still rather involved. Of course in dealing with the quantum theory one could consider the two simpler expressions which contain some of the structure of the previous terms

$$\int \sqrt{g} R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} \sim 2 \sum_{\sigma^2} V_{\sigma^2} \left([\frac{\delta}{A_\Gamma}]_{\sigma^2} \right)^2 \tag{3.29}$$

which vanishes if and only if the Riemann tensor projected on all the hinges vanishes (it is in fact a rewriting of the expression (3.6) for the 'naive' higher derivative term constructed in the previous section), and

$$\int \sqrt{g} (R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} - \frac{1}{3} R^2) \sim \frac{1}{6} \sum_{\sigma^4} V_{\sigma^4} \sum_{\sigma^2, \sigma^{2\prime} \subset \sigma^4} \left(\omega_{\sigma^2} [\frac{\delta}{A_\Gamma}]_{\sigma^2} - \omega_{\sigma^{2\prime}} [\frac{\delta}{A_\Gamma}]_{\sigma^{2\prime}} \right)^2 \tag{3.30}$$

which introduces a short range coupling between deficit angles. Note that this interaction term has the remarkable property that it requires neighboring deficit angles to have similar values, but it does *not* require them to be small.

Alternatively, of course the summation can be done over the σ^0's, instead of the σ^4's, as in the expression (3.12) for the Euler characteristic. The second choice (sum over sites) appears in fact more natural when one considers the coupling of gravity to matter fields, which will be represented here for simplicity by a scalar field $\phi(x)$. In the continuum an invariant action, up to terms quadratic in ϕ, is[3]

$$I_{matter} = \int d^4x\sqrt{g}\,[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}m^2\phi^2 + g_1 R\phi^2 + g_2 R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \cdots] \tag{3.31}$$

On the skeleton define the fields ϕ_n living on the sites. If R is defined on the sites, then the third interaction term (proportional to g_1 is just a point coupling term. As far as the first term is concerned, introduce lattice (forward) derivatives

$$\Delta_i\phi_n = (\phi_{n+i} - \phi_n)/l_{n,n+i} \tag{3.32}$$

where i labels the possible directions in which one can move from a point in a given simplex (there are d of them in d dimensions), and $l_{n,n+i}$ is the length of the edge connecting the two points. The metric g_{ij} at point n in a simplex is

$$g^{ij}(n) = \frac{1}{2}(l^2_{n,n+i} + l^2_{n,n+j} - l^2_{n+i,n+j}) \tag{3.33}$$

and $\sqrt{g}$ is proportional to the volume of the simplex. The lattice analogue of the first term in the action I_{matter} of eq. (3.31) is then, in four dimensions,

$$\sum_{\sigma^0}\sum_{\sigma^4\supset\sigma^0} V_{\sigma^4} g^{ij}(\sigma_0,\sigma^4)\Delta_i\phi_{\sigma^0}\Delta_j\phi_{\sigma^0} \tag{3.34}$$

and double-counting can be avoided by summing only over simplices with sides pointing in the positive lattice direction. In two dimensions the corresponding expression is

$$\sum_{\sigma^0}\sum_{\sigma^2\supset\sigma^0} V_{\sigma^2} g^{ij}(\sigma_0,\sigma^2)\Delta_i\phi_{\sigma^0}\Delta_j\phi_{\sigma^0} \tag{3.35}$$

On a regular triangular lattice as the one in fig. 6 one can associate with each point n two adjacent triangles in the positive (1,2) direction, which can be labeled by γ = 1,2. Then the action is simply

$$\sum_n\sum_{\gamma=1}^{2} V_n^\gamma\left[(\Delta_1\phi_n)^2 + (\Delta_2\phi_n)^2 + 2\cos\alpha^\gamma_{12,n}\Delta_1\phi_n\Delta_2\phi_n\right] \tag{3.36}$$

where $\alpha^\gamma_{12,n}$ is the (dihedral) angle between the two edges l^1_n and l^2_n coming out of the point n in the triangle labeled by γ.

3.4 Expansion Around Flat Space

One of the simplest possible problems that can be attacked in quantum Regge calculus is the analysis of small fluctuations about a fixed flat simplicial geometry[34]. The Regge action for R-gravity will be considered for simplicity. It was shown above that the first variation of the Regge involves only the variation of the area of the hinge. The second variation is thus

$$\delta^2 I_R = \sum_{\sigma^2} \Big(\sum_{\sigma^0} \frac{\partial A_h}{\partial l} \delta l\Big) \Big(\sum_{\sigma^0} \frac{\partial \delta_h}{\partial l} \delta l\Big) \tag{3.37}$$

where δ_h is the deficit angle, not to be confused with the variation symbol.

Next a flat lattice is chosen as background geometry. A natural choice is to choose a flat hypercubic lattice which is made rigid by introducing face diagonals, body diagonals and hyperbody diagonals. This implies that there are $2^d - 1 = 15$ fields per point corresponding to the edge lengths emanating from one vertex.

The edge lengths are then allowed to fluctuate around the equilibrium point l_i^0

$$l_i' = l_i^0 (1 + \epsilon_i) \tag{3.38}$$

The second variation of the action then becomes a quadratic form in the small fluctuation vector ϵ_i with 15 components per site

$$\delta^2 I_R = \sum_{ij} \epsilon_i^\dagger M_{ij} \epsilon_j \tag{3.39}$$

and M is the small fluctuation matrix whose inverse determines the graviton propagator. The matrix M can more easily be computed by going to momentum space. One finds that the matrix M has four zero modes corresponding to periodic translations of the lattice, and a fifth zero mode corresponding to periodic fluctuations in the hyperbody diagonal. After block-diagonalization it is found that 4 modes completely decouple and are constrained to vanish, and thus the remaining degrees of freedom are 10, as in the continuum, where the metric has 10 independent components. After gauge fixing and the introduction of the appropriate ghost terms the remaining 10-dimensional matrix can be reduced to the form[34]

$$(matrix\ of\ numbers) \times [2 \sum_{\mu} (1 - \cos k_\mu)] + gauge\ terms \tag{3.40}$$

which shows the correct $1/k^2$ behavior for the lattice graviton propagator at small k^2.

4. Quantum Gravity Beyond Perturbation Theory

4.1 Choice of Underlying Lattice Structure

In principle a natural setting for lattice quantum gravity calculations would be a random lattice, in which the coordination number at each site is itself a random variable. Unfortunately such a lattice is rather difficult to deal with, both analytically and numerically[44].

Another possibility is to use the regular tessellations of the n-sphere[45]. Since the maximum number of edges allowed in such tesselations is not very largè, a refinement of the same could be achieved by considering further regular subdivisions, such as the barycentric one. Thus the degree of irregularity is kept at a minimum.

A third possibility is to start with a hypercubical lattice, which can be made topologically equivalent to that of a hypertorus, by identifying opposite faces. Finite volume effects are minimized for this lattice, since the boundary is formed by a replica of the same lattice. The advantage of the hypercubic lattice lies in the fact that the number of edges can be increased arbitrarily, keeping the local incidence matrix unchanged. If the theory has some reasonable continuum limit, then this limit should not depend on the detailed lattice structure at short distances. Of course different topologies can be obtained by changing the boundary conditions.

The lattice actions for gravity written in the previous sections do not contain terms which allow tunnelling from one topology to another. Thus initially one would like to keep the topology fixed, and vary the metric within the given sector. Eventually it will be important to verify that the results obtained do not depend on the particular topology chosen[45]. This is likely to happen for correlations of local operators over distances that are much smaller then the size of the system. In fact the renormalization properties of the lattice operators can be extracted by looking at the dependence of the low energy effective hamiltonian (and its correlation functions) on the ultraviolet cutoff. Here by low energy one means energies that are still above the infrared cutoff set by the finite box (universe) size.

It is not clear at the present moment how the integration over topologies should be performed, and how the weighting should be assigned. For a discussion of this point see reference (45). Arguments have been given for suggesting that a sum over topologies in Regge calculus cannot give a finite functional integral, because the number of manifolds with a given topology increases too rapidly as a function of the number of simplices[46]. In fact the problem of enumerating all possible manifolds constructed out of a given finite number of edges is likely to be NP-complete.

The asymptotic freedom of higher derivative gravity further restricts the short distance fluctuations in the metric, implying that the field configurations become smooth at the scale of the ultraviolet cutoff[7]. Furthermore it seems unlikely that a unitary theory can be defined by summing over topologies. The time-slice factorization property of the functional integral needed to construct a time evolution operator no longer holds if this summation is performed, even if the action is reflection positive[47]. This is connected with the fact that the weighting factors for individual topologies, being necessarily topological invariants, are only globally defined.

In the following only results obtained with the hypercubic lattice will be discussed. The two-dimensional square lattice with diagonals was shown in fig. 6. In fig. 9 a single hypercube is drawn, with the relevant body principals, face diagonals, body diagonals and hyperbody diagonals. The diagonals have to be introduced to make the lattice rigid. Otherwise the values of the edge lengths do not determine the angles, and therefore the geometry, uniquely. The hypercube is then replicated in four directions to construct the full skeleton.

4.2 Functional Integral and Definition of the Measure

After having chosen an appropriate lattice, the next step is to define and evaluate the functional integral for simplicial quantum gravity restricted to a manifold of fixed topology, say a hypertorus.

$$Z = \int d\mu[l] e^{-I[l]} \tag{4.1}$$

In the following the scale invariant form of the measure

$$\int d\mu_\epsilon[l] = \prod_i \int_0^\infty \frac{dl_i^2}{l_i^2} F_\epsilon[l] \tag{4.2}$$

will be used, where $F_\epsilon[l]$ is a rather complicated function of the edge lengths which is non-vanishing when the triangle inequalities for the simplicial complex are satisfied, and zero otherwise. These inequalities ensure that the edge lengths, triangle areas, tetrahedron and four-simplex volumes are positive. The positive real parameter ϵ is introduced as an ultraviolet cutoff at small edge lengths : the function $F_\epsilon[l]$ is zero if any of the edges is equal or less than ϵ.

Of course the measure suggested above is not unique, but is certainly the most attractive one since it is local and scale invariant as the continuum measure(19). Other measures one might consider would involve an integration over edge lengths divided by some volume to the appropriate power, such that the total measure is scale invariant. However there are several volumes that are touching a given edge, and the measure then becomes rather complicated, involving some odd powers of volumes in the denominator.

A possible approach to evaluate the functional integral is by using numerical Monte Carlo methods that do not rely on an expansion in a small parameter. Then the edges of the skeleton are varied individually (or in small groups) by a small amount, and the difference in action is compared. If the action is lowered, the new edge value is accepted, if it is raised then the new edge length is only accepted with a probability given by the exponential of the action difference. The same procedure is then applied to another edge, and so on. After many edges have been changed, the probability distribution for the edges approaches the equilibrium one. For a more detailed discussion of the procedure see ref. (11).

An alternative procedure to evaluate the functional integral is to introduce in four dimensions a fictitious fifth time t and solve the stochastic differential equation (for pure R-gravity without a cosmological constant term)

$$\frac{1}{l_p(t)} \frac{dl_p(t)}{dt} = \frac{1}{2}\, k\, l_p^2(t) \sum_{h \supset l_p} \delta_h(t) \cot \theta_p h(t) + \sqrt{2}\eta_p(t) \tag{4.3}$$

where the sum is over hinges (triangles) labeled by h meeting on the common edge p, and θ_{ph} is the angle in the hinge h opposite to the edge p. The field $\eta_p(t)$ is a gaussian white noise with zero mean and unit variance. The constraint that the triangle inequalities be satisfied implies that the force term is infinite when they are violated. Averaging over the noise $\eta_p(t)$ reproduces then the averages computed by the functional integral method, in the limit of large times.

4.3 Numerical Results : Two Dimensions

In two dimensions the action of pure higher derivative gravity is

$$I[l] = \sum_h [\lambda A_h + k\delta_h + a\frac{\delta_h^2}{A_h}] \tag{4.4}$$

We do not consider here the case of long-range interactions of the type

$$I_{lr}[l] = \frac{1}{2}\sum_{h,h'} \delta_h [\frac{1}{-\Delta + m^2}]_{h,h'} \delta_{h'} \tag{4.5}$$

where Δ is the nearest-neighbor covariant lattice Laplacian, and m^2 is a mass regulator. This interaction term corresponds to the continuum contribution

$$\frac{1}{2}\int d^2x d^2y R\sqrt{g}(x) < x|\frac{1}{-\partial^2 + m^2}|y > R\sqrt{g}(y) \tag{4.6}$$

considered (for $m^2 = 0$) by Polyakov[48] in the context of the problem of random surfaces embedded in higher dimensional space[49].

The coupling k is irrelevant as long as the topology is fixed. In the following only the torus will be discussed, and $k = 0$ will be set. The triangular lattice (as in fig. 6) is of size $N \times N$ with $3N^2$ edges (later only $N = 32$ will be considered). Integration over the edge lengths is cut off at small edge lengths

$$l_i \geq \epsilon \tag{4.7}$$

and the two couplings have dimensions $[\lambda] \sim \epsilon^{-2}$ and $[a] \sim \epsilon^2$. The cosmological constant in two dimensions has at most a quadratic divergence

$$\lambda_R = \lambda_0 + c_2 L^2 + c_0 \ln L + \cdots \tag{4.8}$$

with $L \sim \epsilon^{-1}$ the ultraviolet cutoff, while Newton's constant does not get renormalized.

Consider now the path integral

$$Z[\lambda, a, \epsilon] = \int d\mu_\epsilon[l] e^{-I[l]} \tag{4.9}$$

Because of the scale invariance of the measure, the lengths can be rescaled $l_i \to (\frac{a}{\lambda})^{1/4} l_i$ and one obtains

$$Z[\lambda, a, \epsilon] = Z[\sqrt{a\lambda}, \sqrt{a\lambda}, (\frac{\lambda}{a})^{1/4}\epsilon] \tag{4.10}$$

If ϵ can be sent to zero, then Z depends only on $\sqrt{a\lambda}$, once all lengths are expressed in units of the length scale $l_0 \equiv \left(\frac{a}{\lambda}\right)^{1/4}$.

The following expectation values are of interest

$$< A >= \frac{1}{N_h} < \sum_h A_h >$$
$$< R^2 >= \frac{4}{N_h} < \sum_h \frac{\delta_h^2}{A_h} > \tag{4.11}$$

where $N_h = N^2$ is the number of hinges. Dual volumes are used in the following. A numerical evaluation of the path integral by Monte Carlo methods leads to the following results. For $a = 0$ the results are trivial

$$< A >= \begin{cases} 0, & \text{if } \lambda > 0; \\ \infty; & \text{if } \lambda \leq 0. \end{cases} \tag{4.12}$$

and $< R^2 >= \infty$ for both signs of λ. These results are valid as the cutoff ϵ is sent to zero.

For strictly positive a the path integral exists for positive λ, while for negative λ one has results similar to $a = 0$ (no nontrivial equilibrium distribution of edge lengths). A typical equilibrium distribution of edge lengths is shown in fig. 7 for $a = 1$ and $\lambda = 0.2$. It is insensitive to a small edge length cutoff ϵ, unless λ is very large.

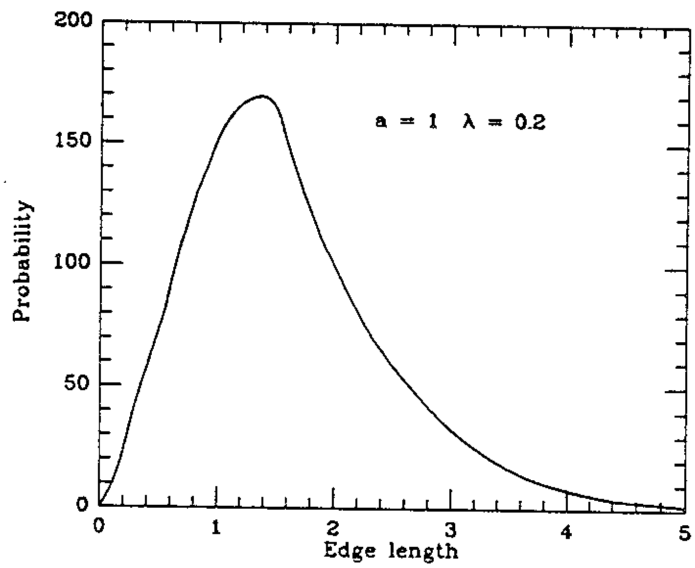

Figure 7

From now on a will be kept fixed and ϵ sent to zero. As λ goes to zero for fixed a the distribution flattens out. The expectation value of the total volume of space-time

$< A >$ diverges as λ approaches zero from the positive side, while the average curvature squared $< R^2 >$ goes to zero. For the case $a = 1$ the results are shown in table IV. There $\sqrt{< l^2 >}$ denotes the square root of the average edge length squared, and $< A >$ and $< R^2 >$ were defined before. $\sqrt{< l^2 >}$ here plays the role of the lattice spacing, the fundamental unit of length. μ denotes the 'mass' of the ghost.

Table IV Results for the 2-dimensional Model

λ	$\sqrt{< l^2 >}$	$< A >$	$\frac{<A>}{2<l^2>}$	$\frac{1}{4} < R^2 >$	$\frac{<R^2>}{4<A>}$	μ
0.4	1.28(3)	0.96(5)	0.292(5)	0.408(3)	0.42(3)	0.99(8)
0.3	1.52(3)	1.34(5)	0.289(5)	0.396(3)	0.30(1)	0.64(8)
0.2	1.78(3)	1.82(5)	0.287(5)	0.371(3)	0.20(1)	0.62(8)
0.1	2.34(3)	3.12(5)	0.285(5)	0.324(3)	0.10(1)	0.69(8)
0.0	∞	∞	0.288	0	0	0

Thus the phase boundary appears to be at $\lambda = 0$ *for any a*, and the cosmological constant has no quadratic divergence (the renormalized cosmological constant is zero if the bare one is zero). This behavior is connected to the strong infrared divergence of the 'graviton' propagator in two dimensions. (Higher derivative gravity is super-renormalizable in two dimensions). For small k^2 the inverse of the quadratic fluctuation matrix around flat space in momentum space is proportional to

$$\frac{1}{4\mu^4 + k^4}, \qquad \mu = \left(\frac{\lambda}{a}\right)^{1/4} \tag{4.13}$$

and is thus strongly convergent for large k^2. (Of course tha flat space expansion is not really justified in the presence of a cosmological constant term).

For $\epsilon = 0$ one has from scale invariance the exact identity

$$\frac{< R^2 >}{4 < A >} = \frac{\lambda}{a} \tag{4.14}$$

which is well satisfied, as can be seen from the table. Also, the average area of a triangle divided by the average edge length squared is independent of λ. One finds

$$\frac{< A >}{2 < l^2 >} = 0.288 \approx \frac{\sqrt{3}}{6} \tag{4.15}$$

which shows that the triangles are not equilateral. (For equilateral triangles the ratio is $\sqrt{3}/4 = 0.433$).

In the region of λ and a considered the space-time volume and the integrated curvature squared can be reasonably well fitted by simple functions of the form

$$\lambda < A > = \frac{2}{B + C \ln \frac{1}{\lambda a}}$$
$$\frac{a}{4} < R^2 > = \frac{2}{B + C \ln \frac{1}{\lambda a}} \tag{4.16}$$

with $B = 4.0$ and $C = 1.0$, but other fits are equally possible at this point.

The 'mass' of the ghost can be extracted by looking at the large distance decay of correlation function. A natural choice is the volume-volume correlation at geodesic separation $d(h, h')$

$$G_V(d) = \sum_{h,h'} < V_h V_h' \, \delta(d(h,h') - d) > \tag{4.17}$$

As can be seen from fig. 8 (there $\lambda = 0.3$ and $a = 1$) $< V_h V_h' >$ itself is not positive as a function of the flat lattice distance $|h - h'|$ (the distance on the original flat space with edges of length one, which does not account for the fact that the lattice spacing is not equal to one and space is not flat), indicating a violation of unitarity already at this level.

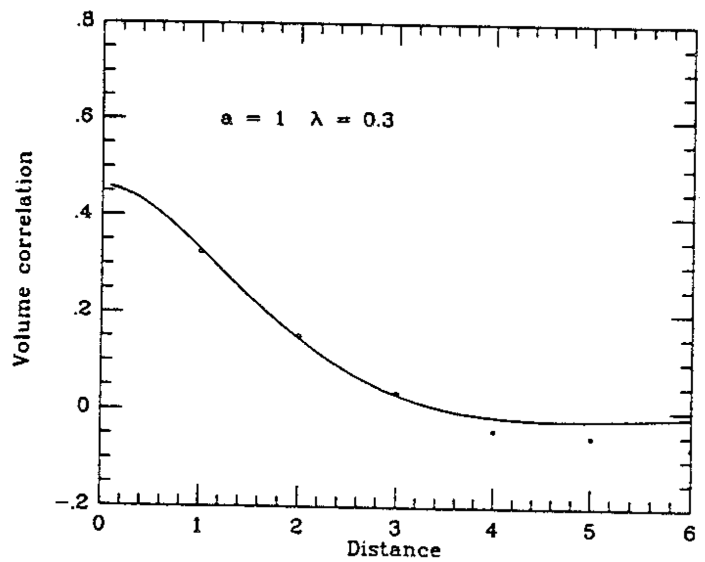

Figure 8

In this particular case the correlation function is fitted rather well by a pure k^4 propagator with finite 'mass' $\mu = 0.64$. A more careful computation of the correlation function would involve the evaluation of the correlation function (4.17). Assuming that the fits in this case are truly reliable, one obtains

$$\mu = C\left(\frac{\lambda}{a}\right)^{\nu} F(\sqrt{\lambda a}) \tag{4.18}$$

with $C = 1.0(15)$ and $\nu = 0.25(5)$ and $F = 1$, independent of λ, within errors. An estimate of the physical 'mass' of the ghost can be obtained by multiplying the 'lattice mass' by the unit of length $\sqrt{< l^2 >}$

$$\mu_{physical} = \sqrt{< l^2 >}\mu \tag{4.19}$$

As a is sent to zero for fixed λ the ghost becomes infinitely massive.

In conclusion it appears that the phase diagram of a model of higher derivative gravity can be worked out with relative ease, even though the detailed dependence of physical observables on the bare parameters is more difficult to determine and would require further work. The lack of unitarity of the theory in two dimensions (where no R term is generated by radiative corrections) can be seen by analyzing the decay of physical correlation functions in real space.

4.3 Numerical Results : Four Dimensions

In four dimensions the action for pure higher derivative gravity on a simplicial lattice was described in section (3.3). The full action is

$$\begin{aligned} I[l] = \sum_{\sigma^2} [\lambda V_{\sigma^2} - 2k A_{\sigma^2}\delta_{\sigma^2} + 2b\frac{A^2_{\sigma^2}\delta^2_{\sigma^2}}{V_{\sigma^2}}] \\ + 6(\frac{a}{4} - b)\sum_{\sigma^0} V_{\sigma^0} \sum_{\sigma^2,\sigma^{2\prime}\supset\sigma^0} (\frac{\delta_{\sigma^2}A_{\sigma^2}}{V_{\sigma^0}} - \frac{\delta_{\sigma^{2\prime}}A_{\sigma^{2\prime}}}{V_{\sigma^0}})^2 \end{aligned} \tag{4.20}$$

with $V_{\sigma^0} = \sum_{\sigma^2\supset\sigma^0} V_{\sigma^2}$. The complexity of the interactions is clearly an order of magnitude greater than in the two-dimensional case.

As in the two-dimensional case, the lattice is chosen to be regular and built out of rigid hypercubes, one of which is illustrated in fig. 9.

The lattice is of size $N \times N \times N \times N$ with $15N^4$ edges, and up to now only $N = 4$ (3840 edges) has been considered. Periodic boundary conditions are used, and the topology is therefore restricted to a hypertorus. Again integration over the edge lengths is cut off at small edge lengths $l_i \geq \epsilon$, and baricentric volumes are used.

For $a = b = 0$ the action is unbounded from below, and the path intergal does not converge.

This is shown, for $2k = 1$, in fig. 10, where the evolution of the total space-time volume (circles) and of the lattice analogue of the integral of R (the term multiplying $2k$

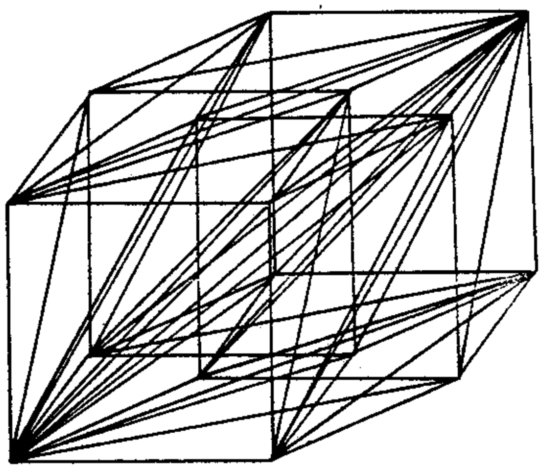

Figure 9

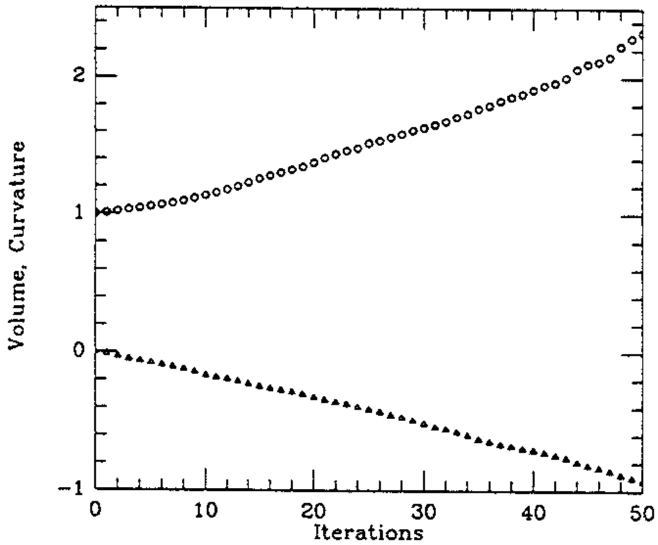

Figure 10

in the action) (triangles) is shown as a function of the iteration number, starting from a flat space configuration with body principals equal to one[(43)]. (The initial volume per site has been normalized to one at the beginning) The cutoff ϵ in this particular case was set equal to zero. A detailed analysis of the phase diagram with couplings λ, k, a, b different from zero has not been done yet, but appears to be well within reach of present computers capabilities.

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Appendix A

Some formulae in Riemannian geometry

The square of the line element is defined by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{A.1}$$

where the metric $g_{\mu\nu}$ is a symmetric positive definite tensor field. General coordinate transformations act on the metric as

$$g'_{\mu\nu} = \frac{\partial x^\lambda}{\partial x'^\mu} g_{\lambda\sigma} \frac{\partial x^\sigma}{\partial x'^\nu} \tag{A.2}$$

The connection is given by

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}\, g^{\sigma\lambda} \left[\frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right] \tag{A.3}$$

Under an infinitesimal general coordinate transformation $\delta x_\mu = \eta_\mu(x)$ the metric and the connection change as[19]

$$\begin{aligned} \delta g^{\mu\nu} &= -\,\eta^\lambda \frac{\partial g^{\mu\nu}}{\partial x^\lambda} + g^{\mu\lambda} \frac{\partial \eta^\nu}{\partial x^\lambda} + g^{\nu\lambda} \frac{\partial \eta^\mu}{\partial x^\lambda} \\ \delta \Gamma^\rho_{\mu\nu} &= -\,\eta^\lambda \frac{\partial \Gamma^\rho_{\mu\nu}}{\partial x^\lambda} - \Gamma^\rho_{\mu\lambda} \frac{\partial \eta^\lambda}{\partial x^\lambda} - \Gamma^\rho_{\nu\lambda} \frac{\partial \eta^\lambda}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu} \frac{\partial \eta^\rho}{\partial x^\lambda} - \frac{\partial^2 \eta^\rho}{\partial x^\mu \partial x^\nu} \end{aligned} \tag{A.4}$$

The covariant derivative of an arbitrary vector V^μ is then

$$D_\lambda V^\mu = \frac{\partial V^\mu}{\partial x^\lambda} + \Gamma^\mu_{\lambda\sigma} V^\sigma \tag{A.5}$$

and the parallel transport formula for a displacement dx^ν is

$$\delta V^\mu = -\Gamma^\mu_{\lambda\nu} dx^\nu V^\lambda \tag{A.6}$$

Parallel transport around a closed loop rotates a vector by

$$\Delta V_\mu = -R^\sigma{}_{\mu\nu\lambda} d\sigma^{\nu\lambda} V_\sigma \tag{A.7}$$

where $R^\sigma{}_{\mu\nu\lambda}$ is the Riemann tensor, and $d\sigma^{\nu\lambda} = \frac{1}{2}\int x^\nu dx^\lambda$ is the area of the loop. In terms of the connection

$$R^\lambda{}_{\mu\nu\sigma} = \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\sigma} - \frac{\partial \Gamma^\lambda_{\mu\sigma}}{\partial x^\nu} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\sigma\eta} - \Gamma^\eta_{\mu\sigma}\Gamma^\lambda_{\nu\eta} \tag{A.8}$$

Covariant differentiation is symmetric

$$[D_\mu, D_\nu]_- V^\rho = R^\rho{}_{\sigma\nu\mu} V^\sigma \tag{A.9}$$

The Riemann tensor satisfies the symmetry relations

$$\begin{gathered} R_{\mu\nu\lambda\sigma} = -R_{\nu\mu\lambda\sigma} = -R_{\mu\nu\sigma\lambda} = R_{\nu\mu\sigma\lambda} \\ R_{\mu\nu\lambda\sigma} = R_{\lambda\sigma\mu\nu} \\ R_{\mu\nu\lambda\sigma} + R_{\mu\lambda\sigma\nu} + R_{\mu\sigma\nu\lambda} = 0 \end{gathered} \tag{A.10}$$

With one contraction one obtains the Ricci tensor $R_{\mu\nu}$, and with two the scalar curvature R

$$R_{\mu\nu} = g^{\lambda\sigma} R_{\lambda\mu\sigma\nu} \qquad R = g^{\mu\nu} g^{\lambda\sigma} R_{\mu\lambda\nu\sigma} \tag{A.11}$$

The conformal curvature or Weyl tensor is constructed from the Riemann tensor as

$$C_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma} - R_{\lambda[\mu} g_{\nu\sigma]} + R_{\sigma[\mu} g_{\nu\lambda]} + \frac{1}{3} R g_{\lambda[\mu} g_{\nu\sigma]} \tag{A.12}$$

The square brackets denote antisymmetrization. The Weyl tensor is traceless

$$g^{\lambda\sigma} C_{\lambda\mu\sigma\nu} = g^{\mu\nu} g^{\lambda\sigma} C_{\mu\lambda\nu\sigma} = 0 \tag{A.13}$$

The scalar curvature R and the volume element $d^4x\sqrt{g}$, where g is the determinant of the metric, are invariant under general coordinate transformations, and so are the action contributions

$$\begin{gathered} \int d^4x\sqrt{g} \\ \int d^4x\sqrt{g}R \end{gathered} \tag{A.14}$$

Possible terms quadratic in the curvature are[18]

$$\begin{aligned} &\int d^4x\sqrt{g}\; R^2 \\ &\int d^4x\sqrt{g}\; R_{\mu\nu}R^{\mu\nu} \\ &\int d^4x\sqrt{g}\; R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} \\ &\int d^4x\sqrt{g}\; C_{\mu\nu\lambda\sigma}C^{\mu\nu\lambda\sigma} \\ &\int d^4x\sqrt{g}\; R_{\mu\nu\rho\sigma}R^{\kappa\lambda\omega\tau}\epsilon^{\mu\nu\kappa\lambda}\epsilon^{\rho\sigma\omega\tau} = 128\pi^2\chi \\ &\int d^4x\sqrt{g}\; R_{\mu\nu\rho\sigma}R^{\mu\nu}_{\kappa\lambda}\epsilon^{\rho\sigma\kappa\lambda} = 96\pi^2\tau \end{aligned} \tag{A.15}$$

where χ is the Euler characteristic and τ the Hirzebruch signature. Not all these quantities are independent. In two dimensions one has the identity

$$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} = \frac{1}{2}R_{\mu\nu}R^{\mu\nu} = R^2 \qquad C_{\mu\nu\lambda\sigma}C^{\mu\nu\lambda\sigma} = 0 \tag{A.16}$$

and in three dimensions

$$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} = 4R_{\mu\nu}R^{\mu\nu} - R^2 \qquad C_{\mu\nu\lambda\sigma}C^{\mu\nu\lambda\sigma} = 0 \tag{A.17}$$

and finally in four dimensions

$$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} = C_{\mu\nu\lambda\sigma}C^{\mu\nu\lambda\sigma} + 2R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2 \tag{A.18}$$

Furthermore the expression for the Euler characteristic can be rewritten as

$$\chi = \frac{1}{32\pi^2}\int d^4x\sqrt{g}\,[R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2] \tag{A.19}$$

and thus only *two* curvature squared terms for the action are independent in four dimensions.

To get more a feeling for the difference between different higher derivative terms consider the following example. A particular case of Riemannian manifold is one of dimension d which is locally isometrically embedded in $(d+1)$-dimensional euclidean space with the canonical euclidean metric. If the manifold is locally convex, the principal curvatures k_λ are of the same sign everywhere. Then the manifold is the called a locally convex hypersurface[(50)]. Every point of the manifold admits a neighborhood in which the vectors tangent to the lines of curvature form an orthonormal frame such that

$$R_{\mu\nu\rho\sigma} = k_\mu k_\nu(\delta_{\nu\rho}\delta_{\mu\sigma} - \delta_{\nu\sigma}\delta_{\mu\rho}) \tag{A.20}$$

Define $K_{\mu\nu} = k_\mu k_\nu$. Then one derives in four dimensions

$$\begin{aligned} R &= 2(K_{12}+K_{13}+K_{14}+K_{23}+K_{24}+K_{34}) \\ R^2 &= 4(K_{12}+K_{13}+K_{14}+K_{23}+K_{24}+K_{34})^2 \\ R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} &= 2(K_{12}^2+K_{13}^2+K_{14}^2+K_{23}^2+K_{24}^2+K_{34}^2) \end{aligned} \tag{A.21}$$

and therefore

$$\begin{aligned} C_{\mu\nu\lambda\sigma}C^{\mu\nu\lambda\sigma} &\sim \frac{1}{2}(R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - \frac{1}{3}R^2) \\ = \frac{1}{6}\big[&(K_{12}-K_{13})^2 + (K_{12}-K_{14})^2 + (K_{12}-K_{23})^2 + (K_{12}-K_{24})^2 + (K_{12}-K_{34})^2 \\ &+ \mathit{other}\ 10\ \mathit{terms}\big] \end{aligned} \tag{A.22}$$

This simple example shows some of the difference between R^2, $R^2_{\mu\nu\rho\sigma}$ and $C^2_{\mu\nu\rho\sigma}$. While $R^2_{\mu\nu\rho\sigma}$ tries to make all $K_{\mu\nu}$'s small when inserted in the functional integral, $C^2_{\mu\nu\rho\sigma}$ tries to make all $K_{\mu\nu}$'s equal to each other.

Appendix B

Some useful formulae in Regge calculus

Consider an n-dimensional simplex with vertices 1, 2, 3, ... n+1 and square edge lengths $l^2_{12} = l^2_{21}$, ... Its vertices are specified by a set of vectors $v_0 = 0$, v_1, ... v_n. The matrix[14,26]

$$g_{ij} = < v_i | v_j > \tag{B.1}$$

with $1 \leq i, j \leq n$ is positive definite, and in terms of the edge lengths $l_{ij} = |v_i - v_j|$ it is given by

$$g_{ij} = \frac{1}{2}[l^2_{0i} + l^2_{0j} - l^2_{ij}] \tag{B.2}$$

The volume of an n-simplex is then given by the generalization of Tartaglia's formula for a tetrahedron

$$V_n = \frac{1}{n!}\sqrt{\det g_{ij}} \tag{B.3}$$

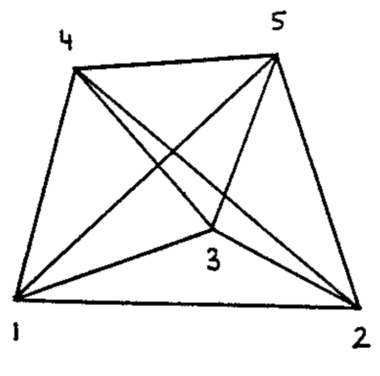

Figure 11

An equivalent form is

$$V_n = \frac{(-1)^{\frac{n+1}{2}}}{n!2^{n/2}} \begin{vmatrix} 0 & 1 & 1 & \cdots \\ 1 & 0 & l_{12}^2 & \cdots \\ 1 & l_{21}^2 & 0 & \cdots \\ 1 & l_{31}^2 & l_{32}^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1 & l_{n1}^2 & l_{n2}^2 & \cdots \\ 1 & l_{n+1,1}^2 & l_{n+1,2}^2 & \cdots \end{vmatrix}^2 \qquad (B.4)$$

In the particular case of a triangle with sides of length l_1, l_2, l_3 the formula becomes

$$A_{triangle} = \frac{1}{4}\left[2(l_1^2 l_2^2 + l_2^2 l_3^2 + l_3^2 l_1^2) - l_1^4 - l_2^4 - l_3^4\right]^{\frac{1}{2}} \qquad (B.5)$$

For the tetrahedron with edge lengths $l_1 \cdots l_6$ one has

$$\begin{aligned} V_{tetrahedron} = \frac{1}{12}\big[&4l_1^2 l_3^2 l_4^2 - l_1^2(l_3^2 - l_6^2 + l_4^2)^2 - l_3^2(l_1^2 + l_4^2 - l_5^2)^2 \\ &+ l_4^2(l_1^2 - l_2^2 + l_3^2)^2 + (l_3^2 - l_6^2 + l_4^2)(l_1^2 + l_4^2 - l_5^2)(l_1^2 - l_2^2 + l_3^2)\big]^{\frac{1}{2}} \end{aligned} \qquad (B.6)$$

For a 4-simplex with edge lengths $l_1 \cdots l_{10}$ (see fig. 11) define the quantity[43]

$$\begin{aligned} K(A, V_1, V_2) = \frac{1}{288 V_1 V_2}\Big[&16A^2(l_4^2 - l_{10}^2 + l_7^2) - 2l_1^2(l_3^2 - l_6^2 + l_4^2)(l_3^2 - l_9^2 + l_7^2) \\ &- 2l_3^2(l_1^2 - l_5^2 + l_4^2)(l_1^2 - l_8^2 + l_7^2) + (l_1^2 - l_2^2 + l_3^2) \\ &\times \left[(l_3^2 - l_6^2 + l_4^2)(l_1^2 - l_8^2 + l_7^2) + (l_1^2 - l_5^2 + l_4^2)(l_3^2 - l_9^2 + l_7^2)\right]\Big]^{\frac{1}{2}} \end{aligned} \qquad (B.7)$$

where A is the area of the triangle with edge lengths l_1, l_2, l_3, and V_1 and V_2 the volumes of the tetrahedra with edge lengths $l_1, l_2, l_3, l_4, l_5, l_6$ and $l_1, l_2, l_3, l_7, l_8, l_9$ respectively. l_{10} corresponds then to the edge in the simplex which is opposite to the triangle. Then the dihedral angle at the triangle with edge lengths l_1, l_2, l_3 is given by

$$\cos\theta_d = K(A, V_1, V_2) \qquad (B.8)$$

and the volume of the simplex is

$$V_{simplex} = \frac{3V_1 V_2 \sqrt{1 - K^2}}{4A} \qquad (B.9)$$

In two dimensions the formula for the dihedral angle is of course

$$\cos\theta_d = \frac{l_1^2 + l_2^2 - l_3^2}{2 l_1 l_2} \qquad (B.10)$$

where l_3 is the length of the edge opposite to the vertex considered. For the same vertex the dihedral dual volume contribution is

$$A_d = \frac{1}{32A}\left[l_2^2(l_1^2 + l_2^2) - (l_1^2 - l_2^2)^2\right] \qquad (B.11)$$

The baricentric dihedral volume is simply $A_d = A/3$. In four dimensions first introduce the quantity[31]

$$\Sigma(l_1,l_2,l_3,l_4,l_5,l_6) = -2l_1^2l_2^2l_3^2+l_1^2l_6^2(l_2^2-l_1^2+l_3^2)+l_2^2l_4^2(l_1^2-l_2^2+l_3^2)+l_3^2l_5^2(l_1^2+l_2^2-l_3^2) \quad (B.12)$$

which is characteristic of the tetrahedron with edges (123)456 containing the hinge (triangle) with edges 123. Then the dihedral dual volume attached to the triangle with edge lengths l_1, l_2, l_3 is given by

$$V_d = \frac{1}{(384)^2A^2V_1V_2V_{simplex}}\left[2\Sigma_1\Sigma_2V_1V_2 - K(\Sigma_1^2V_2^2 + \Sigma_2^2V_1^2)\right] \qquad (B.13)$$

where Σ_1 and Σ_2 relate to tetrahedron (123)456 and (123)789 respectively. The dihedral baricentric volume is $V_d = V_{simplex}/10$.

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