PHYSICAL REVIEW D **82,** 043518 (2010)

Cosmological density perturbations with a scale-dependent Newton's constant G

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(Received 24 June 2010; published 16 August 2010)

We explore possible cosmological consequences of a running Newton's constant $G(\Box)$, as suggested by the nontrivial ultraviolet fixed point scenario in the quantum field-theoretic treatment of Einstein gravity with a cosmological constant term. In particular, we focus here on what possible effects the scale-dependent coupling might have on large scale cosmological density perturbations. Starting from a set of manifestly covariant effective field equations derived earlier, we systematically develop the linear theory of density perturbations for a nonrelativistic, pressureless fluid. The result is a modified equation for the matter density contrast, which can be solved and thus provides an estimate for the growth index parameter γ in the presence of a running G. We complete our analysis by comparing the fully relativistic treatment with the corresponding results for the nonrelativistic (Newtonian) case, the latter also with a weakly scale-dependent G.

DOI: 10.1103/PhysRevD.82.043518 PACS numbers: 98.80.Qc, 04.60.−m, 04.60.Gw

I. INTRODUCTION

Recent years have seen the development of a bewildering variety of alternative theories of gravity, in addition to the more traditional alternate theories, which used to include Brans-Dicke, tensor-scalar, tensor-vector-scalar, higher derivative, effective quantum gravity, and supergravity theories. Some of the new additions to the already rather long list include dilaton gravity, $f(R)$ and $f(G)$ gravity, Chern-Simons gravity, conformal gravity, torsion gravity, loop quantum gravity, holographic modified gravity, modified gravity (MoG), asymmetric brane gravity, massive gravity, and minimally modified self-dual gravity, just to cite a few representative examples. All of these theories eventually predict some level of deviation from classical gravity, which is often parametrized either by a suitable set of post-Newtonian parameters, or more recently by the introduction of a slip function [1,2]. The latter has been quite useful in describing deviations from classical general relativity (GR), and specifically from the standard ΛCMD model, when analyzing the latest cosmological cosmic microwave background (CMB), weak lensing, supernovae, and galaxy clustering data.

In this paper, we will focus on the analysis of departures from GR in the growth history of matter perturbations, within the narrow context of the nontrivial ultraviolet fixed point scenario for Einstein gravity with a cosmological term. Thus, instead of looking at deviations from GR at very short distances, due to new interactions such as the ones suggested by string theories [3], we will be considering here infrared effects, which could therefore become manifest at very large distances. The classical theory of small density perturbations is by now well developed in standard textbooks, and the resulting theoretical predictions for the growth exponents are simple to state, and well understood. Except possibly on the very largest scales, where the data so far is still rather limited, the predictions agree quite well with current astrophysical observations. Here we will be interested in computing and predicting possible small deviations in the growth history of matter perturbations, and specifically in the values of the growth exponents, arising from a very specific scenario, namely, a weak scale-dependent gravitational coupling, whose value very gradually increases with distance.

The specific nature of the scenario we will be investigating here is motivated by the treatment of field-theoretic models of quantum gravity, based on the Einstein action with a bare cosmological term. Its long distance scaling properties are derived from the existence of a nontrivial ultraviolet fixed point of the renormalization group in Newton's constant G. The latter is inaccessible by direct perturbation theory in four dimensions, and can be shown to radically alter the short and long distance behavior of the theory when compared to more naive expectations. The renormalization group origin of such fixed points was first discussed in detail by Wilson for scalar and self-coupled fermion theories [4]. The general field-theoretic methods were later extended and applied to gravity, where they are now referred to as the nontrivial fixed point scenario or asymptotic safety [5]. It is fair to say that so far this is the only field-theoretic approach known to work consistently

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in other not perturbatively renormalizable theories, such as the nonlinear sigma model. While perhaps still a bit mundane in the context of gravity, such nontrivial fixed points are well studied and well understood in statistical field theory, where they generally describe phase transitions between ordered and disordered ground states, or between weakly coupled and condensed states.

The paper is organized as follows. First we recall the effective covariant field equations describing the running of G, and describe the nature of the objects and parameters entering the quantum nonlocal corrections. We then discuss the zeroth order (in the fluctuations) field equations and energy-momentum conservation equations for the standard homogeneous isotropic metric, with a running G. Later we extend the formalism to deal with small metric and matter perturbations, and derive the relevant field and energy conservation equations to first order in the perturbations. After showing the overall consistency of the derived equations, we proceed to derive the modified differential equation for the density contrast $\delta(t)$. Later this is rewritten, following customary procedures, as a function of the scale factor as $\delta(a)$. The resulting differential equation for the density contrast is then solved and the results for the growth exponents compared to the standard classical result. The conclusions provide an interpretation of the theoretical results and their associated uncertainties *vis-à-vis* present and future high precision galaxy clustering measurements.

II. RUNNING NEWTON'S CONSTANT $G(\Box)$

Originally the running of G was computed either on the lattice directly in four dimensions [6–8], or in the continuum within the framework of the background field expansion applied to $2+\epsilon$ spacetime dimensions [5,9] and later using truncation methods applied in 4D [10]. In either case, one obtains a momentum dependent $G(k^2)$, which needs to be eventually reexpressed in a coordinate-independent way, so that it can be usefully applied to more general problems involving arbitrary background geometries.

The first step in analyzing the consequences of a running of G is therefore to rewrite the expression for $G(k^2)$ in a coordinate-independent way, either by the use of a nonlocal Vilkovisky-type effective gravity action [11,12], or by the use of a set of consistent effective field equations. In going from momentum to position space, one usually employs $k^2 \to -\Box$, which then gives for the quantum-mechanical running of the gravitational coupling the replacement $G \to G(\Box)$. One then finds that the running of G is given in the vicinity of the UV fixed point by

$$G(\Box) = G_0\left[1 + c_0\left(\frac{1}{\xi^2\Box}\right)^{1/2\nu} + \ldots\right], \tag{2.1}$$

where $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the covariant d'Alembertian, and the dots represent higher order terms in an expansion in $1/(\xi^2\Box)$. Current evidence from Euclidean lattice quantum gravity points toward $c_0 > 0$ (implying infrared growth) and $\nu \simeq \frac{1}{3}$ [8]. Within the quantum-field-theoretic renormalization group treatment, the quantity ξ arises as an integration constant of the Callan-Symanzik renormalization group equations. One challenging issue therefore, and of great relevance to the physical interpretation of the results, is a correct identification of the renormalization group invariant scale ξ. A number of arguments can be given (see below) in support of the suggestion that the infrared scale ξ (very much analogous to the $\Lambda_{\overline{MS}}$ of QCD) can in fact be very large, even cosmological, in the gravity case. From these arguments, one would then infer that the constant G_0 can, to a very close approximation, be identified with the laboratory value of Newton's constant, $\sqrt{G_0} \sim 1.6 \times 10^{-33}$ cm.

The appearance of the d'Alembertian $\Box$ in the running of G naturally leads to both a nonlocal effective gravitational action, and a corresponding set of nonlocal modified field equations. Instead of the ordinary Einstein field equations with constant G

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.2}$$

one is now lead to consider the modified effective field equations

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = 8\pi G(\Box) T_{\mu\nu} \tag{2.3}$$

with a new nonlocal term due to the $G(\Box)$. By being manifestly covariant, they still satisfy some of the basic requirements for a set of consistent field equations incorporating the running of G. Not unexpectedly though, the new nonlocal equations are much harder to solve than the original classical field equations for constant G.

It is instructive to note, as already pointed out in [13], that the effective nonlocal field equations of Eq. (2.3) can be recast in a form very similar to the classical field equations, but with a new source term $\tilde{T}_{\mu\nu} = [G(\Box)/G_0]T_{\mu\nu}$ defined as the effective, or gravitationally dressed, energy-momentum tensor. Ultimately, the consistency of the effective field equations demands that it be exactly conserved, in consideration of the contracted Bianchi identity satisfied by the Ricci tensor. In this picture, therefore, the running of G can be viewed as contributing to a sort of a vacuum fluid, introduced in order to account for the new gravitational vacuum-polarization contribution.

More on the technical side, and mainly due the appearance of a negative fractional exponent in Eq. (2.1), the covariant operator appearing in the expression for $G(\Box)$ has to be suitably defined by analytic continuation. This can be done, for example, by computing $\Box^n$ for positive integer n, and then analytically continuing to $n \to -1/2\nu$ [13]. Equivalently, $G(\Box)$ can be defined via a suitable regulated parametric integral representation [14], such as

$$\left(\frac{1}{-\Box(g)+m^2}\right)^{1/2\nu} = \frac{1}{\Gamma(\frac{1}{2\nu})}\int_0^\infty d\alpha\,\alpha^{1/2\nu-1}e^{-\alpha(-\Box(g)+m^2)}. \tag{2.4}$$

As far as the calculations in this paper are concerned, it will not be necessary to commit oneself to an unduly specific form for the running of $G(\Box)$. Thus, for example, although the lattice gravity results only allow for a nondegenerate phase for the case $c_0 > 0$, it will nevertheless be possible later to have either sign for the correction in Eq. (2.1), in the sense that the very existence of a nontrivial ultraviolet fixed point implies in principle the appearance of two physically distinct phases, each of which might or might not be physically realized due to issues of nonperturbative stability. Observation could then be used, in principle, to constrain one or the other choice. Furthermore, the value of the exponent ν needs not to be specified until the very end of the calculation, so that most of the results can be kept general.[1]

The situation regarding the running of G is perhaps most easily illustrated close and above two dimensions, where the gravitational coupling becomes dimensionless, $G \sim \Lambda^{2-d}$ with Λ the ultraviolet cutoff required to regularize the theory (a similar and completely parallel line of arguments and results can in fact be presented for the 4D lattice theory as well, but a discussion of renormalization on the lattice ends up being inevitably quite a bit less transparent [6,8]). There the theory appears perturbatively renormalizable, so that the full machinery of covariant renormalization and of the renormalization group can in principle be applied, following Wilson's dimensional expansion method, now formulated as a double expansion in G and $\epsilon = d - 2$ [5,9]. Both here and on the lattice, a renormalization of the bare cosmological constant, besides being gauge dependent, is also physically meaningless, as it can be reabsorbed by a trivial rescaling of the metric; the latter is needed in order to recover the proper normalization of the volume term in the path integral, thus avoiding spurious renormalization effects, as discussed in [6,8,9].

In momentum space, the result corresponding to Eq. (2.1), and allowing now possibly for either sign in front of the correction, is

$$G(k^2) \simeq G_0\left[1 \pm c_0\left(\frac{1}{\xi^2k^2}\right)^{1/2\nu} + \ldots\right], \tag{2.5}$$

with c_0 a positive constant, and ξ the new, genuinely nonperturbative, gravity scale.[2] Consequently, the above expression for $G(k^2)$ can be used whenever the full generality of the manifestly covariant expression in Eq. (2.1) is not really needed, for example, when dealing with the Newtonian (nonrelativistic) limit.

The choice of the + or − sign is ultimately determined from whether one is initially to the left (+), or to the right (−) of the fixed point G_0, in which case the effective $G(k^2)$ decreases or, respectively, increases as one flows away from the ultraviolet fixed point towards lower momenta, or larger distances. Physically the two solutions represent of course gravitational screening ($G < G_0$) or antiscreening ($G > G_0$).

It is crucial that the quantum correction involves a new physical, renormalization group invariant, scale ξ, whose value cannot be fixed by a perturbative calculation, and whose absolute size determines the comparison scale for the new nonlocal quantum effects. It should therefore be rightfully considered as the gravity analog of the celebrated gauge theory scaling violation parameter $\Lambda_{\overline{MS}}$. In terms of the bare gravitational coupling $G(\Lambda)$ it is given by

$$\xi^{-1} = A_\xi \cdot \Lambda \exp\left(-\int^{G(\Lambda)}\frac{dG'}{\beta(G')}\right), \tag{2.7}$$

where $\beta(G)$ is the Callan-Symanzik beta function for G (which can be given explicitly, for example, in the $2+\epsilon$ expansion to a given loop order, or can be computed on the lattice). It is then more or less a direct consequence of the renormalization group that the value of the constant A_ξ determines the coefficient c_0 in Eq. (2.1), $c_0 = 1/(A_\xi^{1/\nu}G_0)$. The nonperturbative lattice formulation of quantum gravity then allows an explicit and direct computation of A_ξ, and therefore of the coefficient c_0 in $G(\Box)$ [6,8].

Physically it would seem at first, based on renormalization group considerations alone, that the nonperturbative (renormalization group integration constant) scale ξ could in principle take any value, including a very small one—based on the naive estimate $\xi \sim l_P$—which would then of course preclude any observable quantum effects in the foreseeable future. But a number of recent results for the gravitational Wilson loop on the Euclidean lattice at strong

[1] A running cosmological constant $\lambda(k) \to \lambda(\Box)$ causes a number of mathematical inconsistencies [13] within the manifestly covariant framework, described here by the effective field equations of Eq. (2.3). Indeed if one assumes for the running part of $\lambda(\Box) \sim (\xi^2\Box)^{-\sigma}$, where σ is a (positive or negative) power, then the infrared regulated expression in Eq. (2.4) gives no running of λ, after using $\nabla_\lambda g_{\mu\nu} = 0$. This last conclusion is in agreement with the field-theoretic results of the nontrivial renormalization group fixed point scenario, thereby providing perhaps an independent consistency check. Note that this general argument also applies to possible additional contributions from nonzero vacuum expectation values of matter fields.

[2] A properly infrared regulated version of the above expression, here with the choice of the + sign, would read

$$G(k^2) \simeq G_0\left[1 + c_0\left(\frac{\xi^{-2}}{k^2+\xi^{-2}}\right)^{1/2\nu} + \ldots\right]. \tag{2.6}$$

Then for large distances $r \gg \xi$ the gravitational coupling no longer exhibits the spurious infrared divergence, but instead approaches the finite value $G_\infty \simeq (1 + c_0 + \ldots)G_0$.

coupling, giving an area law, and their subsequent interpretation in light of the observed large scale semiclassical curvature [15], would suggest otherwise, namely, that the nonperturbative scale ξ appears in fact to be related to *macroscopic curvature*. From astrophysical observation, the average curvature on very large scales, or, stated in somewhat better terms, the measured physical cosmological constant λ, is very small. This would then suggest that the new scale ξ can be very large, even cosmological,

$$\frac{1}{\xi^2} \simeq \frac{\lambda}{3}, \tag{2.8}$$

which would then give a more concrete quantitative estimate for the scale in the $G(\Box)$ of Eq. (2.1), namely, $\xi \sim 1/\sqrt{\lambda/3} \sim 1.51 \times 10^{28}$ cm. Indeed for quantum gravity, no other suitable infrared cutoff presents itself, so that λ can almost be considered as the only ''natural'' candidate to take on the role of a (generally covariant) infrared regulator or graviton masslike parameter.

Finally, let us mention here briefly and for completeness that for a limited number of metrics it has been possible, after some considerable work, to find exact solutions, in some regime, to the above effective nonlocal field equations. One such case is the static isotropic metric, where in the limit $r \gg 2MG$ one can obtain an explicit solution for the metric coefficients $A(r) = 1/B(r)$, leading eventually to the rather simple result [16]

$$G \to G(r) = G_0\left(1 + \frac{c_0}{3\pi} m^3 r^3 \ln\frac{1}{m^2 r^2} + \ldots\right) \tag{2.9}$$

with $m \equiv \xi^{-1}$, consistent with a gradual slow increase of $G(r)$ with distance.[3] One amusing aspect of the exact solution in the static isotropic case is that no consistent solution can be found unless $\nu = 1/3$ exactly in four dimensions, and similarly $\nu = 1/(d-1)$ in dimensions $d \geq 4$ [16], lending further support, and independently of the lattice theory results, to this particular value for ν in four dimensions.

A. (Zeroth order) effective field equations with $G(\Box)$

A scale-dependent Newton's constant is expected to lead to small modifications of the standard cosmological solutions to the Einstein field equations. Here we will summarize what modifications are expected from the effective field equations on the basis of $G(\Box)$, as given in Eq. (2.1), which itself originates in Eq. (2.5). The starting point is the quantum effective field equations of Eq. (2.3), with $G(\Box)$ defined in Eq. (2.1). In the Friedmann-Lemaître-Robertson-Walker (FLRW) framework, these are applied to the standard homogeneous isotropic metric

$$d\tau^2 = dt^2 - a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right\}$$
$$k = 0, \pm 1. \tag{2.11}$$

In the following, we will mainly consider the case $k = 0$ (spatially flat universe). It should be noted that there are in fact *two* related quantum contributions to the effective covariant field equations. The first one arises because of the presence of a nonvanishing cosmological constant $\lambda \simeq 3/\xi^2$, caused by the nonperturbative quantum vacuum condensate $\langle R \rangle \neq 0$ [15]. As in the case of standard FLRW cosmology, this is expected to be the dominant contributions at large times t, and gives an exponential (for $\lambda > 0$), or cyclic (for $\lambda < 0$) expansion of the scale factor. The second contribution arises because of the explicit running of $G(\Box)$ in the effective field equations. The next step therefore is a systematic examination of the nature of the solutions to the full effective field equations, with $G(\Box)$ involving the relevant covariant d'Alembertian operator

$$\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu \tag{2.12}$$

acting on second rank tensors as in the case of $T_{\mu\nu}$,

$$\nabla_\nu T_{\alpha\beta} = \partial_\nu T_{\alpha\beta} - \Gamma^\lambda_{\alpha\nu} T_{\lambda\beta} - \Gamma^\lambda_{\beta\nu} T_{\alpha\lambda} \equiv I_{\nu\alpha\beta}$$
$$\nabla_\mu(\nabla_\nu T_{\alpha\beta}) = \partial_\mu I_{\nu\alpha\beta} - \Gamma^\lambda_{\nu\mu} I_{\lambda\alpha\beta} - \Gamma^\lambda_{\alpha\mu} I_{\nu\lambda\beta} - \Gamma^\lambda_{\beta\mu} I_{\nu\alpha\lambda}, \tag{2.13}$$

and in general it requires the calculation of 1920 terms, of which fortunately many vanish by symmetry due to specific choice of metric.

To start the process, one assumes, for example, that $T_{\mu\nu}$ has a perfect fluid form,

$$T_{\mu\nu} = [p(t) + \rho(t)]u_\mu u_\nu + g_{\mu\nu} p(t) \tag{2.14}$$

for which one needs to compute the action of $\Box^n$ on $T_{\mu\nu}$, and then analytically continues the answer to negative fractional values of $n = -1/2\nu$. Even in the simplest case, with $G(\Box)$ acting on a *scalar* such as the trace of the energy-momentum tensor $T_\lambda{}^\lambda$, one finds for the choice $\rho(t) = \rho_0 t^\beta$ and $a(t) = a_0 t^\alpha$ the rather unwieldy expression

[3] We have pointed out before that the result for $G(r)$ is in a number of ways reminiscent of the analogous QED result (known as the Uehling correction to the Coulomb potential in atoms)

$$Q \to Q(r) = Q\left(1 + \frac{\alpha}{3\pi}\ln\frac{1}{m^2 r^2} + \ldots\right). \tag{2.10}$$

In the gravity case, the correction is not a log but a power, which is what one would naively expect from a perturbatively nonrenormalizable theory. In gravity, the infrared cutoff due in QED to the finite physical electron mass is naturally replaced by the physical cosmological constant; the magnitude of neither one of these two quantities can be predicted by the fundamental theory.

$$\Box^n[-\rho(t)] \to 4^n(-1)^{n+1} \times \frac{\Gamma(\frac{\beta}{2}+1)\Gamma(\frac{\beta+3\alpha+1}{2})}{\Gamma(\frac{\beta}{2}+1-n)\Gamma(\frac{\beta+3\alpha+1}{2}-n)}\rho_0 t^{\beta-2n}, \tag{2.15}$$

with an integer n later analytically continued to $n \to -\frac{1}{2\nu}$, with $\nu = \frac{1}{3}$.

A more general calculation shows that a nonvanishing pressure contribution is generated in the effective field equations, even if one initially assumes a pressureless fluid, $p(t) = 0$. After a somewhat lengthy derivation, one obtains for a universe filled with nonrelativistic matter ($p = 0$) the following set of effective Friedmann equations:

$$\frac{k}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G(t)}{3}\rho(t) + \frac{\lambda}{3} = \frac{8\pi G_0}{3}[1 + c_t(t/\xi)^{1/\nu} + \ldots]\rho(t) + \frac{\lambda}{3} \tag{2.16}$$

for the tt field equation, and

$$\frac{k}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)} + \frac{2\ddot{a}(t)}{a(t)} = -\frac{8\pi G_0}{3}[c_t(t/t_0)^{1/\nu} + \ldots]\rho(t) + \lambda \tag{2.17}$$

for the rr field equation. In the above expressions, the running of G appropriate for the Robertson-Walker (RW) metric is

$$G(t) \equiv G_0\left(1 + \frac{\delta G(t)}{G_0}\right) = G_0\left[1 + c_t\left(\frac{t}{t_0}\right)^{1/\nu} + \ldots\right] \tag{2.18}$$

with c_t of the same order as c_0 in Eq. (2.5), and $t_0 = \xi$ [13]; in the quoted reference it was estimated $c_t = 0.450c_0$ for the tensor box operator. Note that it is the running of G that induces an effective pressure term in the second (rr) equation, corresponding to the presence of a relativistic fluid due to the vacuum-polarization contribution. One important feature of the new equations is an additional power-law acceleration contribution, on top of the standard one due to λ.

B. Introduction of the w_{vac} parameter

It was noted in [13] that the field equations with a running G, Eqs. (2.16) and (2.17), can be recast in an equivalent, but slightly more appealing, form by defining a vacuum-polarization pressure p_{vac} and density ρ_{vac}, such that for the FLRW background one has

$$\rho_{\text{vac}}(t) = \frac{\delta G(t)}{G_0}\rho(t) \qquad p_{\text{vac}}(t) = \frac{1}{3}\frac{\delta G(t)}{G_0}\rho(t). \tag{2.19}$$

Consequently, the source term in the tt field equation can be regarded as a combination of two density terms

$$\left(1 + \frac{\delta G(t)}{G_0}\right)\rho(t) \equiv \rho(t) + \rho_{\text{vac}}(t), \tag{2.20}$$

while the rr equation involves the new vacuum-polarization pressure term

$$\frac{1}{3}\frac{\delta G(t)}{G_0}\rho(t) \equiv p_{\text{vac}}(t). \tag{2.21}$$

Form this viewpoint, the inclusion of a vacuum-polarization contribution in the FLRW framework seems to amount to a replacement

$$\rho(t) \to \rho(t) + \rho_{\text{vac}}(t) \qquad p(t) \to p(t) + p_{\text{vac}}(t) \tag{2.22}$$

in the original field equations. Just as one introduces the parameter w, describing the matter equation of state,

$$p(t) = w\rho(t) \tag{2.23}$$

with $w = 0$ for nonrelativistic matter, one can do the same for the remaining contribution by setting

$$p_{\text{vac}}(t) = w_{\text{vac}}\rho_{\text{vac}}(t). \tag{2.24}$$

Then in terms of the two w parameters

$$\left(w + w_{\text{vac}}\frac{\delta G(t)}{G_0}\right)\rho(t) = p(t) + p_{\text{vac}}(t) \tag{2.25}$$

with, according to Eqs. (2.16) and (2.17) and following the results of [13], $w_{\text{vac}} = \frac{1}{3}$ in a FLRW background. We should remark here that the calculations of [13] also indicate that $w_{\text{vac}} = \frac{1}{3}$ is obtained *generally* for the given class of $G(\Box)$ considered, and is not tied therefore to a specific choice of ν, such as $\nu = \frac{1}{3}$.

The previous, slightly more compact, notation allows one to rewrite the field equations for the FLRW background in an equivalent form, which we will describe below. First, we note though that in the following we will restrict our attention mainly to a spatially flat geometry, $k = 0$. Furthermore, when dealing with density perturbations we will have to distinguish between the background, which will involve a background pressure ($\bar{p}$) and background density ($\bar{\rho}$), from the corresponding perturbations which will be denoted here by δp and $\delta\rho$. Then with this notation and for constant G_0, the FLRW field equations for the background are written as

$$3\frac{\dot{a}^2(t)}{a^2(t)} = 8\pi G_0\bar{\rho}(t) + \lambda$$
$$\frac{\dot{a}^2(t)}{a^2(t)} + 2\frac{\ddot{a}(t)}{a(t)} = -8\pi G_0\bar{p}(t) + \lambda. \tag{2.26}$$

Now in the presence of a running $G(\Box)$, and in accordance with the results of Eqs. (2.16) and (2.17), the modified FLRW equations for the background read

$$3\frac{\dot{a}^2(t)}{a^2(t)} = 8\pi G_0\left(1 + \frac{\delta G(t)}{G_0}\right)\bar{\rho}(t) + \lambda$$
$$\frac{\dot{a}^2(t)}{a^2(t)} + 2\frac{\ddot{a}(t)}{a(t)} = -8\pi G_0\left(w + w_{\rm vac}\frac{\delta G(t)}{G_0}\right)\bar{\rho}(t) + \lambda, \tag{2.27}$$

using the definitions in Eqs. (2.23) and (2.24), here with $\bar{p}_{\rm vac}(t) = w_{\rm vac}\bar{\rho}_{\rm vac}(t)$.

We note here that the procedure of defining a $\rho_{\rm vac}$ and a $p_{\rm vac}$ contribution, arising entirely from quantum vacuum-polarization effects, is not necessarily restricted to the FLRW background metric case [13]. In general one can decompose the full source term in the effective nonlocal field equations of Eq. (2.3), making use of

$$G(\Box) = G_0\left(1 + \frac{\delta G(\Box)}{G_0}\right) \quad \text{with}$$
$$\frac{\delta G(\Box)}{G_0} \equiv c_0\left(\frac{1}{\xi^2\Box}\right)^{1/2\nu}, \tag{2.28}$$

as two contributions,

$$\frac{1}{G_0}G(\Box)T_{\mu\nu} = \left(1 + \frac{\delta G(\Box)}{G_0}\right)T_{\mu\nu} = T_{\mu\nu} + T_{\mu\nu}^{\rm vac}. \tag{2.29}$$

The latter involves the nonlocal part[4]

$$T_{\mu\nu}^{\rm vac} \equiv \frac{\delta G(\Box)}{G_0}T_{\mu\nu}. \tag{2.30}$$

In addition, consistency of the full nonlocal field equations requires that the sum be conserved,

$$\nabla^\mu(T_{\mu\nu} + T_{\mu\nu}^{\rm vac}) = 0. \tag{2.31}$$

It is important to note at this stage that the nature of the covariant d'Alembertian $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is such that the result depends on the type of the object it acts on. Here $T_{\mu\nu}$ is a second rank tensor [as in Eq. (2.13)], which causes a reshuffling of components in $T_{\mu\nu}$ due to the matrix nature of both tensor $\Box$ and tensor $G(\Box)$, and eventually accounts for the generation of a nonvanishing induced pressure term. This is clearly seen in the effective field equations of Eqs. (2.16) and (2.17), and in the ensuing definitions of Eq. (2.19).

In general though, one cannot expect that the contribution $T_{\mu\nu}^{\rm vac}$ will always be expressible in the perfect fluid form of Eq. (2.14), even if the original $T_{\mu\nu}$ for matter (or radiation) has such a form. The former will in general contain, for example, nonvanishing shear stress contributions, even if they were originally absent in the matter part.

[4]One normally does not include the left-hand side field equation contribution $+\lambda g_{\mu\nu}$ as part of the right-hand side matter part $T_{\mu\nu}^{\rm vac}$, although it might be sensible to do so, given its large radiative (quantum) content [12]. We note here that the former is expected to contain the fundamental length scale ξ as well, in the form $\simeq +(3/\xi^2)g_{\mu\nu}$.

Nevertheless, the interesting question arises of whether, for example, $w_{\rm vac} = \frac{1}{3}$ continues to hold beyond the FLRW case treated above. In part this question will be answered affirmatively below, in the case of matter density perturbations.

III. RELATIVISTIC TREATMENT OF MATTER DENSITY PERTURBATIONS

Besides the modified cosmic scale factor evolution just discussed, the running of $G(\Box)$ given in Eq. (2.28) also affects the nature of matter density perturbations on very large scales. In computing these effects, it is customary to introduce a perturbed metric of the form

$$d\tau^2 = dt^2 - a^2(\delta_{ij} + h_{ij})dx^i dx^j, \tag{3.1}$$

with $a(t)$ the unperturbed scale factor and $h_{ij}(\mathbf{x}, t)$ a small metric perturbation, and $h_{00} = h_{i0} = 0$ by choice of coordinates. As will become clear later, we will mostly be concerned here with the trace mode $h_{ii} \equiv h$, which determines the nature of matter density perturbations. After decomposing the matter fields into background and fluctuation contribution, $\rho = \bar{\rho} + \delta\rho$, $p = \bar{p} + \delta p$, and $\mathbf{v} = \bar{\mathbf{v}} + \delta\mathbf{v}$, it is customary in these treatments to expand the density, pressure, and metric trace perturbation modes in spatial Fourier modes,

$$\delta\rho(\mathbf{x}, t) = \delta\rho_{\mathbf{q}}(t)e^{i\mathbf{q}\cdot\mathbf{x}} \qquad \delta p(\mathbf{x}, t) = \delta p_{\mathbf{q}}(t)e^{i\mathbf{q}\cdot\mathbf{x}}$$
$$\delta\mathbf{v}(\mathbf{x}, t) = \delta\mathbf{v}_{\mathbf{q}}(t)e^{i\mathbf{q}\cdot\mathbf{x}} \qquad h_{ij}(\mathbf{x}, t) = h_{\mathbf{q}ij}(t)e^{i\mathbf{q}\cdot\mathbf{x}} \tag{3.2}$$

with $\mathbf{q}$ the comoving wave number. Once the Fourier coefficients have been determined, the original perturbations can later be obtained from

$$\delta\rho(\mathbf{x}, t) = \int\frac{d^3\mathbf{x}}{(2\pi)^{3/2}}e^{-i\mathbf{q}\cdot\mathbf{x}}\delta\rho_{\mathbf{q}}(t) \tag{3.3}$$

and similarly for the other fluctuation components. Then the field equations with a constant G_0 [Eq. (2.2)] are given to zeroth order in the perturbations by Eq. (2.26), which fixes the three background fields $a(t)$, $\bar{\rho}(t)$, and $\bar{p}(t)$. Note that in a comoving frame the four velocity appearing in Eq. (2.14) has components $u^i = 1$, $u^0 = 0$. To first order in the perturbations and in the limit $\mathbf{q} \to 0$ the field equations give

$$\frac{\dot{a}(t)}{a(t)}\dot{h}(t) = 8\pi G_0\bar{\rho}(t)\delta(t)$$
$$\ddot{h}(t) + 3\frac{\dot{a}(t)}{a(t)}\dot{h}(t) = -24\pi G_0 w\bar{\rho}(t)\delta(t) \tag{3.4}$$

with the matter density contrast defined as $\delta(t) \equiv \delta\rho(t)/\bar{\rho}(t)$, $h(t) \equiv h_{ii}(t)$ the trace part of h_{ij}, and $w = 0$ for nonrelativistic matter. When combined together, these last two equations then yield a single equation for the trace of the metric perturbation,

$$\ddot{h}(t) + 2\frac{\dot{a}(t)}{a(t)}\dot{h}(t) = -8\pi G_0(1+3w)\bar{\rho}(t)\delta(t). \tag{3.5}$$

From first order energy conservation, one has $-\frac{1}{2}\times(1+w)h(t) = \delta(t)$, which then allows one to eliminate $h(t)$ in favor of $\delta(t)$. This finally gives a single second order equation for the density contrast $\delta(t)$,

$$\ddot{\delta}(t) + 2\frac{\dot{a}}{a}\dot{\delta}(t) - 4\pi G\bar{\rho}(t)\delta(t) = 0. \tag{3.6}$$

In the case of a running $G(\square)$, these equations need to be rederived from the effective covariant field equations of Eq. (2.3), and lead to several additional terms not present at the classical level. Not surprisingly, as we shall see below, the correct field equations with a running G are not given simply by a naive replacement $G \to G(t)$, which would lead to incorrect results, and violate general covariance.

A. Zeroth order energy-momentum conservation

As a first step in computing the effects of density matter perturbations, one needs to examine the consequences of energy and momentum conservation, to zeroth and first order in the relevant perturbations. If one takes the covariant divergence of the field equations in Eq. (2.3), the left-hand side has to vanish identically because of the Bianchi identity. The right-hand side then gives $\nabla^\mu(T_{\mu\nu} + T^{\mathrm{vac}}_{\mu\nu}) = 0$, where the fields in $T^{\mathrm{vac}}_{\mu\nu}$ can be expressed, at least to lowest order, in terms of the p_{vac} and ρ_{vac} fields defined in Eqs. (2.19) and (2.24). The first equation one obtains is the zeroth (in the fluctuations) order energy conservation in the presence of $G(\square)$, which reads

$$3\frac{\dot{a}(t)}{a(t)}\left[(1+w) + (1+w_{\mathrm{vac}})\frac{\delta G(t)}{G_0}\right]\bar{\rho}(t) + \frac{\delta\dot{G}(t)}{G_0}\bar{\rho}(t) + \left(1+\frac{\delta G(t)}{G_0}\right)\dot{\bar{\rho}}(t) = 0. \tag{3.7}$$

For $w = 0$ and $w_{\mathrm{vac}} = \frac{1}{3}$ this reduces to

$$\left[3\frac{\dot{a}(t)}{a(t)} + 4\frac{\dot{a}(t)}{a(t)}\frac{\delta G(t)}{G_0} + \frac{\delta\dot{G}(t)}{G_0}\right]\bar{\rho}(t) + \left(1+\frac{\delta G(t)}{G_0}\right)\dot{\bar{\rho}}(t) = 0, \tag{3.8}$$

or equivalently in terms of the variable $a(t)$ only

$$\left[\frac{3}{a} + \frac{4}{a}\frac{\delta G(a)}{G_0} + \frac{\delta G'(a)}{G_0}\right]\bar{\rho}(a) + \left(1+\frac{\delta G(a)}{G_0}\right)\bar{\rho}'(a) = 0. \tag{3.9}$$

In the absence of a running G these equations reduce to the ordinary mass conservation equation for $w = 0$,

$$\dot{\bar{\rho}}(t) = -3\frac{\dot{a}(t)}{a(t)}\bar{\rho}(t). \tag{3.10}$$

It will be convenient in the following to solve the energy conservation equation not for $\bar{\rho}(t)$, but instead for $\bar{\rho}(a)$. This requires that, instead of using the expression for $G(t)$ in Eq. (2.18), one uses the equivalent expression for $G(a)$

$$G(a) = G_0\left(1+\frac{\delta G(a)}{G_0}\right), \quad \text{with} \quad \frac{\delta G(a)}{G_0} \equiv c_a\left(\frac{a}{a_0}\right)^{\gamma_\nu} + \ldots \tag{3.11}$$

In this last expression, the power is $\gamma_\nu = 3/2\nu$, since from Eq. (2.18) one has for nonrelativistic matter $a(t)/a_0 \approx (t/t_0)^{2/3}$ in the absence of a running G. In the following, we will almost exclusively consider the case $\nu = \frac{1}{3}$ [8] for which therefore $\gamma_\nu = 9/2$.[5] Then in the above expression $c_a \approx c_t$ if a_0 is identified with a scale factor appropriate for a universe of size ξ; to a good approximation this should correspond to the Universe "today," with the relative scale factor customarily normalized at such a time to $a/a_0 = 1$. Consequently, and with the above proviso, the constant c_a in Eq. (3.11) can safely be taken to be of the same order as the constant c_0 appearing in the original expressions for $G(\square)$ in Eq. (2.28).

Then the solution to Eq. (3.8) can be written as

$$\bar{\rho}(a) = \mathrm{const}\exp\left\{-\int\frac{da}{a}\left(3+\frac{\delta G(a)}{G_0} + a\frac{\delta G'(a)}{G_0}\right)\right\}, \tag{3.12}$$

or, more explicitly, as

$$\bar{\rho}(a) = \bar{\rho}_0\left(\frac{a_0}{a}\right)^3\left(\frac{1+c_a}{1+c_a(\frac{a}{a_0})^{\gamma_\nu}}\right)^{(1+\gamma_\nu)/\gamma_\nu} \simeq \bar{\rho}_0\left(\frac{a_0}{a}\right)^3\frac{1+(1+\gamma_\nu^{-1})c_a}{1+(1+\gamma_\nu^{-1})c_a(\frac{a}{a_0})^{\gamma_\nu}} \tag{3.13}$$

with $\bar{\rho}(a)$ normalized so that $\bar{\rho}(a=a_0) = \bar{\rho}_0$. For $c_a = 0$, the above expression reduces of course to the usual result for nonrelativistic matter,

$$\bar{\rho}(t) = \bar{\rho}_0\left(\frac{a_0}{a}\right)^3. \tag{3.14}$$

Furthermore, here one also finds that the zeroth order momentum conservation equation is identically satisfied, just as in the case of constant G.

B. Zeroth order field equations with running $G(\square)$

The zeroth order field equations with the running of G included were already given in Eq. (2.27). One can subtract the two equations from each other to get an equation that does not contain λ,

[5] This implicitly assumes that the cosmic evolution is largely matter dominated, if $p = w\rho$ then $a(t)/a_0 = (t/t_0)^{2/3(1+w)}$. In the opposite regime where a cosmological constant can eventually prevail one has instead $a(t)/a_0 = \exp\sqrt{\lambda/3}(t-t_0)$. Then $\frac{t}{t_0} = 1 + \frac{1}{t_0}\sqrt{\frac{3}{\lambda}}\log\frac{a}{a_0}$ and for $t_0 \simeq \xi$ and $\sqrt{\frac{3}{\lambda}} \simeq \xi$ one has simply $\frac{t}{t_0} = 1 + \log\frac{a}{a_0}$.

$$\frac{\dot{a}^2(t)}{a^2(t)} - \frac{\ddot{a}(t)}{a(t)} = 4\pi G_0\left[(1+w) + (1+w_{\rm vac})\frac{\delta G(t)}{G_0}\right]\bar{\rho}(t). \tag{3.15}$$

Alternatively, from Eqs. (2.27) one can obtain a single equation that only involves the acceleration term with $\ddot{a}(t)$,

$$3\frac{\ddot{a}(t)}{a(t)} = -4\pi G_0\left[(1+3w) + (1+w_{\rm vac})\frac{\delta G(t)}{G_0}\right]\bar{\rho}(t) + \lambda. \tag{3.16}$$

It is also rather easy to check the overall consistency of the energy conservation equation, Eq. (3.8), and of the two field equations in Eq. (2.27). This is done by (i) taking the time derivative of the first tt equation in Eq. (2.27), (ii) replacing terms involving $\dot{\bar{\rho}}$ by $\bar{\rho}$ using the energy conservation equation, Eq. (3.8), and (iii) finally by substituting again the result of the first (tt) equation into Eq. (2.27) to obtain the second (rr) equation in Eq. (2.27).

C. Effective energy-momentum tensor $\rho_{\rm vac}$, $p_{\rm vac}$

The next step consists in obtaining the equations which govern the effects of small field perturbations. These equations will involve, apart from the metric perturbation h_{ij}, the matter and vacuum-polarization contributions. The latter arise from [see Eq. (2.29)]

$$\left(1 + \frac{\delta G(\Box)}{G_0}\right)T_{\mu\nu} = T_{\mu\nu} + T_{\mu\nu}^{\rm vac} \tag{3.17}$$

with a nonlocal $T_{\mu\nu}^{\rm vac} \equiv (\delta G(\Box)/G_0)T_{\mu\nu}$. Fortunately to zeroth order in the fluctuations the results of Ref. [13] indicated that the modifications from the nonlocal vacuum-polarization term could simply be accounted for by the substitution

$$\bar{\rho}(t) \to \bar{\rho}(t) + \bar{\rho}_{\rm vac}(t) \qquad \bar{p}(t) \to \bar{p}(t) + \bar{p}_{\rm vac}(t). \tag{3.18}$$

Here we will apply this last result to the small field fluctuations as well, and set

$$\begin{aligned} \delta\rho_{\mathbf{q}}(t) &\to \delta\rho_{\mathbf{q}}(t) + \delta\rho_{\mathbf{q}\,{\rm vac}}(t) \\ \delta p_{\mathbf{q}}(t) &\to \delta p_{\mathbf{q}}(t) + \delta p_{\mathbf{q}\,{\rm vac}}(t). \end{aligned} \tag{3.19}$$

The underlying assumptions is of course that the equation of state for the vacuum fluid still remains roughly correct when a small perturbation is added. Furthermore, just like we had $\bar{p}(t) = w\bar{\rho}(t)$ [Eq. (2.23)] and $\bar{p}_{\rm vac}(t) = w_{\rm vac}\bar{\rho}_{\rm vac}(t)$ [Eq. (2.24)] with $w_{\rm vac} = \frac{1}{3}$, we now write for the fluctuations

$$\delta p_{\mathbf{q}}(t) = w\delta\rho_{\mathbf{q}}(t) \qquad \delta p_{\mathbf{q}\,{\rm vac}}(t) = w_{\rm vac}\delta\rho_{\mathbf{q}\,{\rm vac}}(t), \tag{3.20}$$

at least to leading order in the long wavelength limit, $\mathbf{q} \to 0$. In this limit we then have simply

$$\begin{aligned} \delta p(t) &= w\delta\rho(t) \\ \delta p_{\rm vac}(t) &= w_{\rm vac}\delta\rho_{\rm vac}(t) \equiv w_{\rm vac}\frac{\delta G(t)}{G_0}\delta\rho(t), \end{aligned} \tag{3.21}$$

with $G(t)$ given in Eq. (2.18), and we have used Eq. (2.19), now applied to the fluctuation $\delta\rho_{\rm vac}(t)$,

$$\delta\rho_{\rm vac}(t) = \frac{\delta G(t)}{G_0}\delta\rho(t) + \ldots \tag{3.22}$$

where the dots indicate possible additional $O(h)$ contributions.

A bit of thought reveals that the above treatment is incomplete, since $G(\Box)$ in the effective field equation of Eq. (2.3) contains, for the perturbed RW metric of Eq. (3.1), terms of order h_{ij}, which need to be accounted for in the effective $T_{\rm vac}^{\mu\nu}$. Consequently, the covariant d'Alembertian has to be Taylor expanded in the small field perturbation h_{ij},

$$\Box(g) = \Box^{(0)} + \Box^{(1)}(h) + O(h^2), \tag{3.23}$$

and similarly for $G(\Box)$

$$G(\Box) = G_0\left[1 + \frac{c_0}{\xi^{1/\nu}}\left(\frac{1}{\Box^{(0)} + \Box^{(1)}(h) + O(h^2)}\right)^{1/2\nu} + \ldots\right], \tag{3.24}$$

which requires the use of the binomial expansion for the operator $(A+B)^{-1} = A^{-1} - A^{-1}BA^{-1} + \ldots$. Thus for sufficiently small perturbations it should be adequate to expand $G(\Box)$ entering the effective field equations in powers of the metric perturbation h_{ij}. Since a number of subtleties arise in this expansion, we shall first consider the simpler case of a scalar box, where some of the issues we think can be clearly identified and addressed. After that, we will consider the more complicated case of the tensor box. This will be followed by a determination of the effects of the running of G on the relevant matter and metric perturbations, by the use of the modified field equations, now expanded to first order in the perturbations.

D. $O(h)$ correction using scalar box

In this section the term $O(h)$ in $\delta\rho_{\rm vac}$ of Eq. (3.19) will be determined, using a set of formal manipulations involving the covariant scalar box operator. Instead of considering the full field equations with a running $G(\Box)$, as given in Eq. (2.3),

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = 8\pi G_0\left(1 + \frac{\delta G(\Box)}{G_0}\right)T_{\mu\nu} \tag{3.25}$$

we will consider here instead the action of a scalar $G(\Box)$ on the trace of the field equations for $\lambda = 0$,

$$R = -8\pi G_0\left(1 + \frac{\delta G(\Box)}{G_0}\right)T_\lambda{}^\lambda, \tag{3.26}$$

or equivalently, by having the operator $G(\Box)$ act on the left-hand side,

$$\left(1 - \frac{\delta G(\Box)}{G_0} + \ldots\right)R = -8\pi G_0 T_\lambda{}^\lambda. \tag{3.27}$$

For a perfect fluid, one has simply $T_\lambda{}^\lambda = -\rho$, which then gives [13]

$$G_0\left(1 + \frac{\delta G(\Box)}{G_0}\right)T_\lambda{}^\lambda \to G_0\left[1 + c_t\left(\frac{t}{t_0}\right)^{1/\nu} + \ldots\right]T_\lambda{}^\lambda \equiv G(t)T_\lambda{}^\lambda, \tag{3.28}$$

or equivalently,

$$G_0\left[1 + c_t\left(\frac{t}{t_0}\right)^{1/\nu} + \ldots\right]\bar\rho(t) \equiv G(t)\bar\rho(t), \tag{3.29}$$

with $c_t \simeq 0.785c_0$, and $t_0 = \xi$ [13] (in the tensor box case a slightly smaller value was found, $c_t \simeq 0.450c_0$). The two terms in Eq. (3.29) are of course recognized, up to a factor of G_0, as the combination

$$\bar\rho(t) + \bar\rho_{\rm vac}(t) \tag{3.30}$$

of Eq. (3.18), with $\bar\rho_{\rm vac}(t) \equiv \delta G(t)/G_0 \cdot \bar\rho(t)$. Thus the zeroth order result obtained by the use of the scalar d'Alembertian acting on the trace of the field equations is consistent with what has been used so far for $G(t)$.

To compute the higher order terms in the h_{ij}'s appearing in the metric of Eq. (3.1) one needs to expand $G(\Box)$ according to Eq. (3.24) giving

$$G(\Box) = G_0\left[1 + \frac{c_0}{\xi^{1/\nu}}\left(\left(\frac{1}{\Box^{(0)}}\right)^{1/2\nu} - \frac{1}{2\nu}\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h)\cdot\left(\frac{1}{\Box^{(0)}}\right)^{1/2\nu} + \ldots\right)\right]. \tag{3.31}$$

Here we are interested in the correction of order h_{ij}, when the above operator acts on the scalar $T_\lambda{}^\lambda = -\bar\rho$. This would then give the correction $O(h)$ to $\delta\rho_{\rm vac}$, namely, the second term in

$$\delta\rho_{\rm vac}(t) = \frac{\delta G(\Box^{(0)})}{G_0}\delta\rho(t) + \frac{\delta G(\Box)(h)}{G_0}\bar\rho(t), \tag{3.32}$$

with the first term being simply given in the FLRW background by $\delta G(t)/G_0 \cdot \delta\rho(t)$. Here the $O(h)$ correction is given explicitly by the expression

$$\frac{\delta G(\Box)(h)}{G_0}\bar\rho = -\frac{1}{2\nu}\frac{c_0}{\xi^{1/\nu}}\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h)\cdot\left(\frac{1}{\Box^{(0)}}\right)^{1/2\nu}\cdot\bar\rho. \tag{3.33}$$

The effect of the $(\Box^{(0)})^{-1/2\nu}$ term is essentially to make the coupling time dependent, i.e. to correctly reproduce the required overall time-dependent factor $\delta G(t)/G_0$.

Now the scalar d'Alembertian $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ acting on scalar functions $S(x)$ has the form

$$\Box S(x) \equiv \frac{1}{\sqrt{g}}\partial_\mu g^{\mu\nu}\sqrt{g}\partial_\nu S(x). \tag{3.34}$$

In the absence of h_{ij} fluctuations, this gives for the metric in Eq. (3.1)

$$\Box^{(0)}S(x) = \frac{1}{a^2}\nabla^2 S - 3\frac{\dot a}{a}\dot S - \ddot S \to \left(-\partial_t^2 - 3\frac{\dot a}{a}\partial_t\right)S(t), \tag{3.35}$$

where in the second expression we have used the properties of the RW background metric: we only need to consider functions that are time dependent, so that $S(\mathbf{x}, t) \to S(t)$. To first order in the field fluctuation h_{ij} of Eq. (3.1) one computes

$$\Box^{(1)}(h)S(x) = -\frac{1}{2}\dot h\dot S - \frac{1}{a^2}h_{xx}\partial_x^2 S + \frac{1}{a^2}(-\partial_x h_{xx})\cdot\partial_x S + \frac{1}{2a^2}\partial_x h\cdot\partial_x S + \ldots \tag{3.36}$$

with the trace $h(t) = h_{xx}(t) + h_{yy}(t) + h_{zz}(t)$. But for a function of the time only, one obtains

$$\Box^{(1)}(h)\rho(t) = -\tfrac{1}{2}\dot h(t)\dot S(t). \tag{3.37}$$

Thus to first order in the fluctuations one obtains the expression

$$\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h)\cdot(\delta G\bar\rho) = \frac{1}{-\partial_t^2 - 3\frac{\dot a}{a}\partial_t}\cdot\frac{1}{2}\dot h\left(3\frac{\dot a}{a}\delta G - \dot{\delta G}\right)\bar\rho \tag{3.38}$$

where use has been made of the zeroth order mass conservation equation in Eq. (3.10). Note that this result also correctly incorporates the effect of $G(\Box^{(0)})$ on functions of t, as given, for example, in Eq. (3.28), which ensures the proper running of $\delta G(t)$.

Now in our treatment we are generally interested in mass density and metric perturbations around a near-static background described by $\dot a/a = H(t)$, and $\bar\rho(t)$. For these we expect the relevant time variations in $\delta\rho$ and h to be somewhat larger than for the background itself. Thus for sufficiently slowly varying background fields we retain only $h(t)$ and its derivatives, and for a sufficiently slowly varying $h(t)$ only $h(t)$ and the lowest derivatives. Then the factors of $\dot a/a$ are seen to cancel out at leading order between the numerator and denominator in Eq. (3.38), and one is left simply with

$$\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h)\cdot\delta G(t)\bar\rho(t) = -\frac{1}{2}\delta G(t)h(t)\bar\rho(t) + \ldots \tag{3.39}$$

Putting everything together, one finds for the $O(h)$ correction

$$\frac{\delta G(\Box)(h)}{G_0}\bar{\rho}(t) \simeq +\frac{1}{4\nu}\frac{\delta G(t)}{G_0}h(t)\bar{\rho}(t). \tag{3.40}$$

The scalar box calculation just described allows one to compute the correction $O(h)$ to $\delta\rho_{\rm vac}(t)$ in Eq. (3.32), and leads to the following $O(h)$ modification of Eq. (3.22)

$$\delta\rho_{\rm vac}(t) = \frac{\delta G(t)}{G_0}\delta\rho(t) + \frac{1}{2\nu}c_h\frac{\delta G(t)}{G_0}h(t)\bar{\rho}(t) \tag{3.41}$$

and similarly from $\delta p_{\rm vac}(t) = w_{\rm vac}\delta\rho_{\rm vac}(t)$,

$$\delta p_{\rm vac}(t) = w_{\rm vac}\left(\frac{\delta G(t)}{G_0}\delta\rho(t) + \frac{1}{2\nu}c_h\frac{\delta G(t)}{G_0}h(t)\bar{\rho}(t)\right) \tag{3.42}$$

with $w_{\rm vac} = \frac{1}{3}$. The second $O(h)$ terms in both expressions account for the feedback of the metric fluctuations h on the vacuum density $\delta\rho_{\rm vac}$ and pressure $\delta p_{\rm vac}$ fluctuations.

The potential flaw with the preceding argument is that it assumes that certain very specific functions of the background stay constant, or at least very slowly varying. In the case at hand, this was $\dot{a}/a \equiv H(a) \approx$ const and $\rho \approx$ const, which in principle is not the only possibility, and would seem therefore a bit restrictive. A slightly more general approach, and a check, to the evaluation of the expression in Eq. (3.38) goes as follows. One assumes instead a harmonic time dependence for the metric fluctuation $h(t) = h_0 e^{i\omega t}$, and similarly for $a(t) = a_0 e^{i\Gamma t}$, $\bar{\rho}(t) = \bar{\rho}_0 e^{i\Gamma t}$, and $\delta G(t) = \delta G_0 e^{i\Gamma t}$; different frequencies for a and $\bar{\rho}$ could be considered as well, but here we will just stick with the simplest possibility. Then from the last expression in Eq. (3.38) one has

$$\begin{aligned}\frac{1}{-\partial_t^2 - 3\frac{\dot{a}}{a}\partial_t}\cdot\frac{1}{2}\dot{h}\left(3\frac{\dot{a}}{a}\delta G - \dot{\delta G}\right)\bar{\rho} \\ = \frac{1}{\omega^2 + 7\Gamma\omega + 10\Gamma^2}\cdot(-\Gamma\omega\delta G h\bar{\rho}).\end{aligned} \tag{3.43}$$

In the limit $\omega \gg \Gamma$, corresponding to $\dot{h}/h \gg \dot{a}/a$, one obtains for the above expression

$$-\frac{\Gamma}{\omega}\delta G(t)h(t)\bar{\rho}(t) \simeq -\left(\frac{\dot{a}}{a}\frac{h}{\dot{h}}\right)\delta G(t)h(t)\bar{\rho}(t), \tag{3.44}$$

after substituting back $\dot{h}/h = i\omega$ and $\dot{a}/a = i\Gamma$ in the last expression. Then $\delta\rho_{\rm vac}(t)$ in Eq. (3.41) now involves the quantity c_h

$$c_h = \frac{\dot{a}}{a}\frac{h}{\dot{h}}. \tag{3.45}$$

At first this last factor (a function and not a constant) would seem rather hard to evaluate, and perhaps not even close to constant in time. But a bit of thought reveals that, to the order we are working, one can write

$$\frac{\dot{h}}{h}\frac{a}{\dot{a}} = \frac{\partial\log h(a)}{\partial\log a} = \frac{\partial\log\delta(a)}{\partial\log a} \equiv f(a), \tag{3.46}$$

where $\delta(a)$ is the matter density contrast, and $f(a)$ the known density growth index [17]. In the absence of a running G (which is all that is needed, to the order one is working here) an explicit form for $f(a)$ is known in terms of derivatives of a Gauss hypergeometric function, which will be given below. One can then either include the explicit form for $f(a)$ in the above formula for $\delta\rho_{\rm vac}(t)$, or use the fact that for a scale factor referring to today $a/a_0 \approx 1$, and for a matter fraction $\Omega \approx 0.25$, one knows that $f(a = a_0) \simeq 0.4625$, and thus in Eq. (3.41) one obtains the improved result $c_h \simeq 2.1621$. This can then be compared to the earlier result, which gave $c_h \simeq 1/2$.

A similar analysis can now be done in the opposite, but in our opinion less physical, $\omega \ll \Gamma$ limit, for which one now obtains for the expression in Eq. (3.43)

$$-\frac{1}{10}\left(\frac{a}{\dot{a}}\frac{\dot{h}}{h}\right)\delta G(t)h(t)\bar{\rho}(t). \tag{3.47}$$

This new limit is less physical because of the fact that now the background is assumed to be varying more rapidly in time than the metric perturbation itself, $\dot{a}/a \gg \dot{h}/h$. For $\delta\rho_{\rm vac}(t)$ one then obtains a similar expression to the one in Eq. (3.41), with a different coefficient

$$c_h = \frac{1}{10}\frac{a}{\dot{a}}\frac{\dot{h}}{h} \tag{3.48}$$

still involving the quantity $(a/\dot{a})(\dot{h}/h) \equiv f(a)$. By the same chain of arguments used in the previous paragraph, one can now either include the explicit form for $f(a)$ in the formula for $\delta\rho_{\rm vac}(t)$, or use the fact that for a scale factor referring to today $a/a_0 \approx 1$ and a matter fraction $\Omega \approx 0.25$ one knows that $f(a = a_0) \simeq 0.4625$. In this case one then has in Eq. (3.41) $c_h \simeq (1/10)\times 0.4625 = 0.0463$. One disturbing, but not entirely surprising, general aspect of the whole calculation in this second $\omega \ll \Gamma$ limit (as opposed to the previous treatment in the opposite limit) is its rather significant sensitivity, in the final result, to the set of assumptions initially made about the time development of the background as specified by the functions $a(t)$ and $\bar{\rho}(t)$. Therefore in the following we shall not consider this limit further.

To summarize, the results for a scalar box and a slowly varying background, $\dot{h}/h \gg \dot{a}/a$, give the $O(h)$ corrected expression for $\delta\rho_{\rm vac}(t)$ in Eq. (3.41) and $\delta p_{\rm vac}(t) = w_{\rm vac}\delta\rho_{\rm vac}(t)$, with $c_h \simeq +2.1621$.

E. $O(h)$ correction using tensor box

The results of Eqs. (3.32) and (3.41) for the vacuum-polarization contribution,

$$\delta\rho_{\rm vac}(t) = \frac{\delta G(t)}{G_0}\delta\rho(t) + \frac{1}{2\nu}c_h\frac{\delta G(t)}{G_0}h(t)\bar{\rho}(t) \tag{3.49}$$

and similarly for $\delta p_{\rm vac}(t) = w_{\rm vac} \delta\rho_{\rm vac}(t)$ with $w_{\rm vac} = \frac{1}{3}$, were obtained using a scalar d'Alembertian to implement $G(\Box)$ by considering the trace of the field equation, Eq. (3.26). In this section, we will discuss instead the result for the full tensor d'Alembertian, as it appears originally in the effective field equations of Eqs. (2.3) and (3.25).

Now the d'Alembertian operator $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ acts on the second rank tensor $T_{\mu\nu}$ as in Eq. (2.13), and should therefore be regarded as a four by four matrix, transforming $T_{\mu\nu}$ into $[\Box T]_{\mu\nu}$. Indeed it is precisely this matrix nature of $\Box$, and therefore of $G(\Box)$, that accounts for the fact that a vacuum pressure is induced in the first place, leading to a $w_{\rm vac} \neq 0$.

To compute the correction of $O(h)$ to $\delta\rho_{\rm vac}(t)$ one needs to consider the relevant term in the expansion of $(1 + \delta G(\Box)/G_0)T_{\mu\nu}$, which we write as

$$-\frac{1}{2\nu}\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h)\cdot\frac{\delta G(\Box^{(0)})}{G_0}\cdot T_{\mu\nu} = -\frac{1}{2\nu}\frac{c_0}{\xi^{1/\nu}}\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h)\cdot\left(\frac{1}{\Box^{(0)}}\right)^{1/2\nu}\cdot T_{\mu\nu}. \quad (3.50)$$

This last form allows us to use the results obtained previously for the FLRW case in [13], namely,

$$\frac{\delta G(\Box^{(0)})}{G_0}T_{\mu\nu} = T_{\mu\nu}^{\rm vac} \quad (3.51)$$

with here

$$T_{\mu\nu}^{\rm vac} = [p_{\rm vac}(t) + \rho_{\rm vac}(t)]u_\mu u_\nu + g_{\mu\nu}p_{\rm vac}(t) \quad (3.52)$$

and [see Eqs. (2.19) and (2.30)] to zeroth order in h,

$$\rho_{\rm vac}(t) = \frac{\delta G(t)}{G_0}\bar\rho(t) \qquad p_{\rm vac}(t) = w_{\rm vac}\frac{\delta G(t)}{G_0}\bar\rho(t), \quad (3.53)$$

with $w_{\rm vac} = \frac{1}{3}$. Therefore, in light of the results of Ref. [13], the problem has been dramatically reduced to just computing the much more tractable expression

$$-\frac{1}{2\nu}\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h)\cdot T_{\mu\nu}^{\rm vac}, \quad (3.54)$$

and in fact the only ordering for which the expression $(\delta G(\Box)/G_0)T_{\mu\nu}$ is calculable within reasonable effort. Still, in general the resulting expression for $\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h)$ is rather complicated if evaluated for arbitrary functions, although it does have a structure similar to the one found for the scalar box in Eq. (3.38).

Here we will resort, for lack of better insights, to a treatment that parallels what was done before for the scalar box, where one assumed a harmonic time dependence for the metric trace fluctuation $h(t) = h_0 e^{i\omega t}$, and similarly for $a(t) = a_0 e^{i\Gamma t}$ and $\rho(t) = \rho_0 e^{i\Gamma t}$. In the limit $\omega \gg \Gamma$, corresponding to $\dot h/h \gg \dot a/a$, one finds for the fluctuation $\delta\rho_{\rm vac}(t)$ in Eq. (3.41)

$$\delta\rho_{\rm vac}(t) = \frac{\delta G(t)}{G_0}\delta\rho(t) + \frac{1}{2\nu}c_h\frac{\delta G(t)}{G_0}h(t)\bar\rho(t). \quad (3.55)$$

The $O(h)$ correction factor c_h for the tensor box is now given by

$$c_h = \frac{11}{3}\frac{\dot a}{a}\frac{h}{\dot h}, \quad (3.56)$$

with all other off-diagonal matrix elements vanishing. Furthermore, one finds to this order, but only for the specific choice $w_{\rm vac} = \frac{1}{3}$ in the zeroth order $T_{\mu\nu}^{\rm vac}$,

$$\delta p_{\rm vac}(t) = \tfrac{1}{3}\delta\rho_{\rm vac}(t), \quad (3.57)$$

i.e., the $O(h)$ correction preserves the original result $w_{\rm vac} = \frac{1}{3}$. In other words, the first order result $O(h)$ just obtained for the tensor box would have been somewhat inconsistent with the zeroth order result, unless one had $w_{\rm vac} = \frac{1}{3}$ to start with. Now, one would not necessarily expect that the first order correction could be still be cast in the form of the same equation of state $p_{\rm vac} \simeq \frac{1}{3}\rho_{\rm vac}$ as the zeroth order result, but it would nevertheless seem attractive that such a simple relationship can be preserved beyond the lowest order.

As far as the magnitude of the correction c_h in Eq. (3.56), one can argue again, as was done in the scalar box case, that from Eq. (3.46) one can relate the combination $(\dot h/h)(a/\dot a)$ to the growth index $f(a)$. Then, in the absence of a running G (which is all that is needed here, to the order one is working), an explicit form for $f(a)$ is known in terms of suitable derivatives of a Gauss hypergeometric function. These can then be inserted into Eq. (3.56). Alternatively, one can make use again of the fact that for a scale factor referring to today $a/a_0 \approx 1$, and for a matter fraction $\Omega \approx 0.25$, one knows that $f(a = a_0) \simeq 0.4625$, and thus in Eq. (3.41) $c_h \simeq (11/3) \times 2.1621 = +7.927$. This last result can then be compared to the earlier scalar result which gave $c_h \simeq +2.162$ using the same set of approximations (slowly varying background fields). It is encouraging that the new correction is a bit larger but not too different from what was found before in the scalar box case. Note that so far the sign of the $O(h)$ correction is the same in all physically relevant cases examined.

Next, as in the scalar box case, one can do the same analysis in the opposite, but less physical, limit $\omega \ll \Gamma$ or $\dot h/h \ll \dot a/a$. One now obtains from the tt matrix element the $O(h)$ correction in the expression for $\delta\rho_{\rm vac}$ given in Eq. (3.41), namely

$$\frac{1}{2\nu}c_h\frac{\delta G(t)}{G_0}h(t)\bar\rho(t), \quad (3.58)$$

with a coefficient

$$c_h = -\frac{121}{60}\frac{\omega^2}{\Gamma^2} =\simeq -\frac{121}{60}\left(\frac{a}{\dot a}\right)^2\frac{\ddot h}{h} = O(\ddot h/h). \quad (3.59)$$

Similarly for the ii matrix element of the $O(h)$ correction, one finds

$$\frac{1}{2\nu} a^2(t) c_h' \frac{\delta G(t)}{G_0} h(t) \bar{\rho}(t), \qquad (3.60)$$

with

$$c_h' = -\tfrac{5}{18} \qquad (3.61)$$

giving now the $\delta p_{\rm vac}(h)$ correction. Again, all off-diagonal matrix elements are equal to zero. It seems therefore that this limit, $\omega \ll \Gamma$ or $\dot{h}/h \ll \dot{a}/a$, leads to rather different results compared to what had been obtained before: the only surviving contribution to $O(h)$ is a rather large pressure contribution, with a sign that is opposite to all other cases encountered previously. Furthermore, here the relationship $w_{\rm vac} = \frac{1}{3}$ is no longer preserved to $O(h)$. But, as emphasized in the previous discussion of the scalar box case, this second limit is in our opinion less physical, because of the fact that now the background is assumed to be varying more rapidly in time than the metric perturbation itself, $\dot{a}/a \gg \dot{h}/h$. Furthermore, as in the scalar box calculation, one disturbing but not entirely surprising general aspect of the whole calculation in this second $\omega \ll \Gamma$ limit, is its extreme sensitivity as far as magnitudes and signs of the results are concerned, to the set of assumptions initially made about the time development of the background. As a final sample calculation, let us mention here the case, similar to what was done originally for the scalar box, where one assumes instead $\dot{a}/a \equiv H(a) \approx$ const and $\bar{\rho} \approx$ const, which, as we mentioned previously, seems now a bit restrictive. Nevertheless, we find it instructive to show how sensitive the calculations are to the nature of the background, and, in particular, its assumed time dependence. In the notation of Eqs. (3.58), (3.59), and (3.61) one finds in this case

$$c_h = +\frac{625}{192}\frac{\omega^2}{H^2} = -\frac{625}{192}\frac{1}{H^2}\frac{\ddot{h}}{h} \qquad c_h' = -\frac{4}{9}. \qquad (3.62)$$

Again, here the pressure contribution $\delta p_{\rm vac}(h)$ is the dominant contribution, the $\delta \rho_{\rm vac}(h)$ part being negligible, $O(\ddot{h})$. For the reasons mentioned, in the following we will no longer consider this limit of rapid background fluctuations any further.

To summarize, the results for a scalar box and for a very slowly varying background, $\dot{h}/h \gg \dot{a}/a$, give the $O(h)$ corrected expression for $\delta \rho_{\rm vac}(t)$ in Eq. (3.41) and $\delta p_{\rm vac}(t) = w_{\rm vac} \delta \rho_{\rm vac}(t)$ with $c_h \simeq +2.162$, while the tensor box calculation, under essentially the same assumptions, gives the somewhat larger result $c_h \simeq +7.927$. From now on, these will be the only two choices we shall consider here.

F. First order energy-momentum conservation

The next step in the analysis involves the derivation of the energy-momentum conservation to first order in the fluctuations, and a derivation of the relevant field equations to the same order. After that, energy conservation will be used to eliminate the h field entirely, and thus obtain a single equation for the matter density fluctuation δ.

The results so far can be summarized as follows. For the metric in Eq. (3.1), and in the limit $\mathbf{q} \rightarrow 0$, the field equations in Eq. (2.3) can now be written as

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G_0 (T_{\mu\nu} + T_{\mu\nu}^{\rm vac}), \qquad (3.63)$$

with $T_{\mu\nu}^{\rm vac} \equiv (\delta G(\Box)/G_0) T_{\mu\nu}$. Here $T_{\mu\nu}$ describes the ordinary matter contribution, in the form of a perfect fluid as given in Eq. (2.14), here with $p = w\rho$ and $w \simeq 0$, while $T_{\mu\nu}^{\rm vac}$ describes the additional vacuum-polarization contribution

$$T_{\mu\nu}^{\rm vac} = [p_{\rm vac}(t) + \rho_{\rm vac}(t)] u_\mu u_\nu + g_{\mu\nu} p_{\rm vac}(t) \qquad (3.64)$$

with $p_{\rm vac} = w_{\rm vac} \rho_{\rm vac}$ and $w_{\rm vac} = \frac{1}{3}$, as in Eq. (2.24). Furthermore, each field now contains both a background and a perturbation contribution,

$$\rho(t) = \bar{\rho}(t) + \delta\rho(t) \qquad p(t) = w\rho(t), \qquad (3.65)$$

and similarly,

$$\rho_{\rm vac}(t) = \bar{\rho}_{\rm vac}(t) + \delta\rho_{\rm vac}(t) \qquad p_{\rm vac}(t) = w_{\rm vac}\rho_{\rm vac}(t). \qquad (3.66)$$

From Eq. (2.19) one has

$$\bar{\rho}_{\rm vac}(t) = \frac{\delta G(t)}{G_0} \rho(t), \qquad (3.67)$$

while from Eq. (3.41) on has

$$\delta\rho_{\rm vac}(t) = \frac{\delta G(t)}{G_0} \delta\rho(t) + \frac{1}{2\nu} c_h \frac{\delta G(t)}{G_0} h(t) \bar{\rho}(t), \qquad (3.68)$$

and similarly, $\delta p_{\rm vac}(t) = w_{\rm vac} \delta\rho_{\rm vac}(t)$. The second $O(h)$ terms in both expressions physically account for the feedback of the metric fluctuations h on the vacuum density $\delta\rho_{\rm vac}$ and pressure $\delta p_{\rm vac}$ fluctuations. In light of the discussion of the previous section, we will limit our derivations below to the case of constant c_h; the case of a nonconstant c_h as in Eq. (3.46) can be dealt with as well, but the resulting equations are found to be quite a bit more complicated to write down.

Consequently, all quantities in the effective field equations of Eq. (3.63) have been specified to the required order in the field perturbation expansion. First we will look here at the implications of energy-momentum conservation, $\nabla^\mu (T_{\mu\nu} + T_{\mu\nu}^{\rm vac}) = 0$, to first order in the fluctuations. The zeroth order energy conservation equation was already obtained in Eq. (3.7), and its explicit solution for $\bar{\rho}(a)$ given in Eq. (3.13). After defining the matter density contrast $\delta(t)$ as the ratio $\delta(t) \equiv \delta\rho(t)/\bar{\rho}(t)$, the energy conservation equation to first order in the perturbations is found to be

$$\left[-\frac{1}{2}\left((1+w)+(1+w_{\rm vac})\frac{\delta G(t)}{G_0}\right)-\frac{1}{2\nu}c_h\frac{\delta G(t)}{G_0}\right]\dot{h}(t) + \left[\frac{1}{2\nu}c_h\left(3(w-w_{\rm vac})\frac{\dot{a}(t)}{a(t)}\frac{\delta G(t)}{G_0}-\frac{\delta\dot{G}(t)}{G_0}\right)\right]h(t) = \left[1+\frac{\delta G(t)}{G_0}\right]\dot{\delta}(t). \tag{3.69}$$

In the absence of a running G ($\delta G(t)=0$), this reduces simply to $-\frac{1}{2}(1+w)\dot{h}(t)=\dot{\delta}(t)$, and thus to the standard result for the metric trace perturbation in terms of the density contrast

$$-\tfrac{1}{2}(1+w)h(t)=\delta(t). \tag{3.70}$$

This last result then allows us to solve explicitly, at the given order, i.e., to first order in the fluctuations and to first order in δG, for the metric perturbation $\dot{h}(t)$ in terms of the matter density fluctuation $\delta(t)$ and $\dot{\delta}(t)$,

$$\dot{h}(t) = -\frac{2}{1+w}\left[1+\frac{1}{1+w}\left((w-w_{\rm vac})-2c_h\frac{1}{2\nu}\right)\frac{\delta G(t)}{G_0}\right]\dot{\delta}(t)-\frac{1}{2\nu}\frac{4c_h}{(1+w)^2}\left[3(w-w_{\rm vac})\frac{\dot{a}(t)}{a(t)}\frac{\delta G(t)}{G_0}-\frac{\delta\dot{G}(t)}{G_0}\right]\delta(t). \tag{3.71}$$

Similarly, by differentiating the above relationship, an expression for $\ddot{h}(t)$ in terms of δ and its derivatives can be obtained as well.

G. First order field equations

To first order in the perturbations, the tt and ii effective field equations become, respectively,

$$\frac{\dot{a}(t)}{a(t)}\dot{h}(t)-8\pi G_0\frac{1}{2\nu}c_h\frac{\delta G(t)}{G_0}\bar{\rho}(t)h(t) = 8\pi G_0\left(1+\frac{\delta G(t)}{G_0}\right)\bar{\rho}(t)\delta(t) \tag{3.72}$$

and

$$\ddot{h}(t)+3\frac{\dot{a}(t)}{a(t)}\dot{h}(t)+24\pi G_0\frac{1}{2\nu}c_h w_{\rm vac}\frac{\delta G(t)}{G_0}\bar{\rho}(t)h(t) = -24\pi G_0\left(w+w_{\rm vac}\frac{\delta G(t)}{G_0}\right)\bar{\rho}(t)\delta(t). \tag{3.73}$$

In the second ii equation, the zeroth order ii field equation of Eq. (2.27) has been used to achieve some simplification.

As a final exercise, it is easy to check the overall consistency of the first order energy conservation equation of Eq. (3.69), and of the two field equations given in Eqs. (3.72) and (3.73). To do so, one needs to (i) take the time derivative of the tt equation in Eq. (3.72); (ii) get rid of $\dot{\bar{\rho}}$ consistently by using energy conservation to zeroth order in δG and in the fluctuations from Eq. (3.69) for terms of order δG times a fluctuation, combined with the use of energy conservation to first order in δG, but without fluctuations as in Eq. (3.8) for the terms that are already of first order in the fluctuations; (iii) eliminate the $\dot{\delta}$ terms using the energy conservation equation to first order in δG without field fluctuations [Eq. (3.8)] for terms proportional to δG times a fluctuation, and using the energy conservation equation to first order in δG and in the fluctuation [again Eq. (3.69)] for terms of zeroth order in the fluctuations; (iv) use the combination of Eqs. (2.27) that does not contain λ, Eq. (3.15), to get rid of $\ddot{a}/a$ terms; (v) finally, use the tt equation for the fluctuation, Eq. (3.72), to eliminate some terms proportional to $\bar{\rho}$ times a fluctuation so as to finally obtain the second ii field equation Eq. (3.73).

H. Matter density contrast equation in t

To obtain an equation for the matter density contrast $\delta(t)=\delta\rho(t)/\bar{\rho}(t)$, one needs to eliminate the metric trace field $h(t)$ from the field equations. This is first done by taking a suitable linear combination of the two field equations in Eqs. (3.72) and (3.73) to get the equivalent equation

$$\ddot{h}(t)+2\frac{\dot{a}(t)}{a(t)}\dot{h}(t)+8\pi G_0\frac{1}{2\nu}c_h(1+3w_{\rm vac})\frac{\delta G(t)}{G_0}\bar{\rho}(t)h(t) = -8\pi G_0\left[(1+3w)+(1+3w_{\rm vac})\frac{\delta G(t)}{G_0}\right]\bar{\rho}(t)\delta(t). \tag{3.74}$$

Then the first order energy conservation equations to zeroth [Eq. (3.70)] and first [Eq. (3.71)] order in δG allows one to completely eliminate the h, $\dot{h}$, and $\ddot{h}$ fields in terms of the matter density perturbation $\delta(t)$ and its derivatives. The resulting equation reads, for $w=0$ and $w_{\rm vac}=\frac{1}{3}$,

$$\ddot{\delta}(t)+\left[\left(2\frac{\dot{a}(t)}{a(t)}-\frac{1}{3}\frac{\dot{\delta G}(t)}{G_0}\right)-\frac{1}{2\nu}\cdot 2c_h\cdot\left(\frac{\dot{a}(t)}{a(t)}\frac{\delta G(t)}{G_0}+2\frac{\dot{\delta G}(t)}{G_0}\right)\right]\dot{\delta}(t)$$
$$+\left[-4\pi G_0\left(1+\frac{7}{3}\frac{\delta G(t)}{G_0}-\frac{1}{2\nu}\cdot 2c_h\cdot\frac{\delta G(t)}{G_0}\right)\bar{\rho}(t)-\frac{1}{2\nu}\cdot 2c_h\right.$$
$$\left.\cdot\left(\frac{\dot{a}^2(t)}{a^2(t)}\frac{\delta G(t)}{G_0}+3\frac{\dot{a}(t)}{a(t)}\frac{\dot{\delta G}(t)}{G_0}+\frac{\ddot{a}(t)}{a(t)}\frac{\delta G(t)}{G_0}+\frac{\ddot{\delta G}(t)}{G_0}\right)\right]\delta(t)=0. \quad (3.75)$$

This last equation then describes matter density perturbations to linear order, taking into account the running of $G(\Box)$, and is therefore the main result of this paper. The terms proportional to c_h, which can be clearly identified in the above equation, describe the feedback of the metric fluctuations h on the vacuum density $\delta\rho_{\rm vac}$ and pressure $\delta p_{\rm vac}$ fluctuations. The equation given above can now be compared with the corresponding, much simpler, equation obtained for constant G, i.e., for $G\rightarrow G_0$ and still $w=0$ (see for example [17,18])

$$\ddot{\delta}(t)+2\frac{\dot{a}}{a}\dot{\delta}(t)-4\pi G_0\bar{\rho}(t)\delta(t)=0 \quad (3.76)$$

from which one obtains for the growing mode

$$\delta_{\mathbf{q}}(t)=\delta_{\mathbf{q}}(t_0)\left(\frac{t}{t_0}\right)^{2/3}, \quad (3.77)$$

which is the standard result in the matter-dominated era.

I. Matter density contrast equation in $a(t)$

It is common practice at this point to write an equation for the density contrast $\delta(a)$ as a function not of t, but of the scale factor $a(t)$. This is done by utilizing the following simple derivative identities

$$\dot{f}(t)=aH(a)\frac{\partial f(a)}{\partial a} \quad (3.78)$$

$$\ddot{f}(t)=a^2H^2(a)\left(\frac{\partial \ln H(a)}{\partial a}+\frac{1}{a}\right)\frac{\partial f(a)}{\partial a}+a^2H^2(a)\frac{\partial^2 f(a)}{\partial a^2}, \quad (3.79)$$

where f is any function of t, and $H\equiv\dot{a}(t)/a(t)$ the Hubble constant. This last quantity can be obtained from the zeroth order tt field equation

$$H^2(a)\equiv\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G_0}{3}\bar{\rho}+\frac{\lambda}{3}. \quad (3.80)$$

Often this last equation is written in terms of current density fractions,

$$H^2(a)\equiv\left(\frac{\dot{a}}{a}\right)^2=\left(\frac{\dot{z}}{1+z}\right)^2$$
$$=H_0^2[\Omega(1+z)^3+\Omega_R(1+z)^2+\Omega_\lambda] \quad (3.81)$$

with $a/a_0=1/(1+z)$, where z is the red shift, and a_0 the scale factor today. Then H_0 is the Hubble constant evaluated today, Ω the (baryonic and dark) matter density, Ω_R the space curvature contribution corresponding to a curvature k term, and Ω_λ the dark energy or cosmological constant part, all again measured *today*. In the absence of spatial curvature $k=0$ one has today

$$\Omega_\lambda\equiv\frac{\lambda}{3H_0^2}\qquad \Omega\equiv\frac{8\pi G_0\bar{\rho}_0}{3H_0^2}\qquad \Omega+\Omega_\lambda=1. \quad (3.82)$$

In terms of the scale factor $a(t)$, the equation for matter density perturbations for constant $G=G_0$, Eq. (3.76), becomes

$$\frac{\partial^2\delta(a)}{\partial a^2}+\left[\frac{\partial \ln H(a)}{\partial a}+\frac{3}{a}\right]\frac{\partial\delta(a)}{\partial a}-4\pi G_0\frac{1}{a^2H(a)^2}\bar{\rho}(a)\delta(a)=0. \quad (3.83)$$

The quantity $H(a)$ is most simply obtained from the FLRW field equations

$$H(a)=\sqrt{\frac{8\pi}{3}G_0\bar{\rho}(a)+\frac{\lambda}{3}}, \quad (3.84)$$

with the matter density given in Eq. (3.14), which can in principle be solved for $a(t)$,

$$t-t_0=\int\frac{da}{a\sqrt{\frac{8\pi}{3}G_0\bar{\rho}_0(\frac{a_0}{a})^3+\frac{\lambda}{3}}}. \quad (3.85)$$

It is convenient at this stage to introduce a parameter θ describing the cosmological constant fraction as measured today,

$$\theta\equiv\frac{\lambda}{8\pi G_0\bar{\rho}_0}=\frac{\Omega_\lambda}{\Omega}=\frac{1-\Omega}{\Omega}. \quad (3.86)$$

While the following discussion will continue with some level of generality, in practice one is mostly interested in the observationally favored case of a current matter fraction $\Omega\approx 0.25$, for which $\theta\approx 3$. In terms of the parameter θ the equation for the density contrast $\delta(a)$ for constant G can then be recast in the slightly simpler form

$$\frac{\partial^2\delta(a)}{\partial a^2}+\frac{3(1+2a^3\theta)}{2a(1+a^3\theta)}\frac{\partial\delta(a)}{\partial a}-\frac{3}{2a^2(1+a^3\theta)}\delta(a)=0. \quad (3.87)$$

A general solution of the above equation is given by a linear combination of the two solutions

$$\delta_0(a) = c_1 \cdot \sqrt{1 + a^3\theta}\, a^{-3/2} + c_2 \cdot a \cdot {}_2F_1\left(\tfrac{1}{3}, 1; \tfrac{11}{6}; -a^3\theta\right), \tag{3.88}$$

where c_1 and c_2 are arbitrary constants, and ${}_2F_1$ is the Gauss hypergeometric function. The subscript 0 in $\delta_0(a)$ is to remind us that this solution is appropriate for the case of constant $G = G_0$. Since one is only interested in the growing solution, the constant $c_1 = 0$.

To evaluate the correction to $\delta_0(a)$ coming from the terms proportional to c_a one sets

$$\delta(a) \propto \delta_0(a)[1 + c_a \mathcal{F}(a)], \tag{3.89}$$

and inserts the resulting expression in Eq. (3.75), written now as a differential equation in $a(t)$, after using Eqs. (3.78) and (3.79) to replace

$$\dot{a}(t) = aH \qquad \ddot{a}(t) = a^2H^2\left(\frac{\partial \ln H}{\partial a} + \frac{1}{a}\right). \tag{3.90}$$

One only needs to determine the differential equations for density perturbations δ up to first order in the fluctuations, so it will be sufficient to obtain an expression for Hubble constant H from the tt component of the effective field equation to zeroth order in the fluctuations, namely, the first of Eqs. (2.27). One has

$$H(a) = \sqrt{\frac{8\pi}{3} G_0\left(1 + \frac{\delta G(a)}{G_0}\right)\bar{\rho}(a) + \frac{\lambda}{3}} \tag{3.91}$$

with $G(a)$ given in Eq. (3.11) and $\bar{\rho}(a)$ given in Eq. (3.12).[6]

In this last expression, the exponent is $\gamma_\nu = 3/2\nu \simeq 9/2$ for a matter-dominated background universe, although more general choices, such as $\gamma_\nu = 3(1 + w)/2\nu$ or even the use of Eq. (3.85), are possible and should be explored (see discussion later). Also, $c_a \approx c_t$ if a_0 is identified with a scale factor corresponding to a universe of size ξ; to a good approximation this corresponds to the universe today, with the relative scale factor customarily normalized at that time to $a/a_0 = 1$. In [13], it was found that in Eq. (2.18) $c_t \simeq 0.785c_0$ in the scalar box case, and $c_t \simeq 0.450c_0$ in the tensor box case; in the following we will use the average of the two values.

After the various substitutions and insertions have been performed, one obtains, after expanding to linear order in a_0, a second order linear differential equation for the correction $\mathcal{F}(a)$ to $\delta(a)$, as defined in Eq. (3.89). Since this equation looks rather complicated for general $\delta G(a)$ it will not be recorded here, but it is easily obtained from Eq. (3.75) by a sequence of straightforward substitutions and expansions. The resulting equation can then be solved for $\mathcal{F}(a)$, giving the desired density contrast $\delta(a)$ as a function of the parameter Ω.

Nevertheless, with the specific choice for $G(a)$ given in Eq. (3.11) an explicit form for the equation for $\delta(a)$ reads

$$\frac{\partial^2 \delta(a)}{\partial a^2} + A(a)\frac{\partial \delta(a)}{\partial a} + B(a)\delta(a) = 0, \tag{3.92}$$

with the two coefficients given by

$$A(a) = \frac{3(1 + 2a^3\theta)}{2a(1 + a^3\theta)} - \frac{c_a(9a^3(1 + \gamma_\nu)\theta\nu + a^{\gamma_\nu}(6c_h\gamma_\nu(1 + 2\gamma_\nu)(1 + a^3\theta)^2 + (-9a^3\theta + \gamma_\nu(1 + a^3\theta)(3 + 2\gamma_\nu(1 + a^3\theta)))\nu))}{6a\nu\gamma_\nu(1 + a^3\theta)^2} \tag{3.93}$$

and

$$B(a) = -\frac{3}{2a^2(1 + a^3\theta)} - \frac{c_a(3a^3(1 + \gamma_\nu)\theta\nu + a^{\gamma_\nu}(c_h\gamma_\nu(2 + \gamma_\nu)(1 + a^3\theta)(-1 + 2\gamma_\nu + 2a^3(1 + \gamma_\nu)\theta) + (4\gamma_\nu + a^3(-3 + 4\gamma_\nu)\theta)\nu))}{2\nu\gamma_\nu a^2(1 + a^3\theta)^2} \tag{3.94}$$

and the variable a considered just as a stand-in for what should really be the variable a/a_0. To obtain an explicit solution to the above equation, one needs to know the coefficient c_a and the exponent γ_ν in Eq. (3.11), whose likely values are discussed above and right after the quoted expression for $G(a)$. For the exponent ν one has $\nu \simeq \frac{1}{3}$, whereas for the value for c_h one finds, according to the discussion in the previous sections, $c_h \simeq 7.927$ for the tensor box case. Furthermore, one needs at some point to insert a value for the matter density fraction parameter θ as given in Eq. (3.86), which based on current observation is close to $\theta = (1 - \Omega)/\Omega \simeq 3$.

[6]We have noted before that Eq. (3.91) is suggestive of a small additional matter contribution, $\Omega_{\mathrm{vac}} \simeq (8\pi/3)\delta G(a)\bar{\rho}_0/H_0^2$, to the overall balance in Eq. (3.82).

IV. RELATIVISTIC GROWTH INDEX WITH $G(\Box)$

The solution of the above differential equation for the matter density contrast in the presence of a running Newton's constant $G(\Box)$ leads to an explicit form for the function $\delta(a) = \delta_0(a)[1 + c_a \mathcal{F}(a)]$. From it, an estimate of the size of the corrections coming from the new terms due to the running of G can be obtained. It is clear from the previous discussion, and the form of $G(\Box)$, that such corrections will become increasingly important in the present era $t \approx t_0$ or $a \approx a_0$. When discussing the growth of density perturbations in classical general relativity it is customary at this point to introduce a scale-factor-dependent *growth index* $f(a)$ defined as

$$f(a) \equiv \frac{\partial \ln\delta(a)}{\partial \ln a}, \tag{4.1}$$

which is in principle obtained from the differential equation for any scale factor $a(t)$. Nevertheless, here one is mainly interested in the neighborhood of the present era, $a(t) \approx a_0$. One therefore introduces today's *growth index parameter* γ via

$$f(a = a_0) \equiv \frac{\partial \ln\delta(a)}{\partial \ln a}\bigg|_{a=a_0} \equiv \Omega^\gamma, \tag{4.2}$$

so that the exponent γ itself is obtained via

$$\gamma \equiv \frac{\ln f}{\ln \Omega}\bigg|_{a=a_0}. \tag{4.3}$$

The solution of the above differential equation for $\delta(a)$ then determines an explicit value for the growth index γ parameter, for any value of the current matter fraction Ω. In the end, because of observational constraints, one is mostly interested in the range $\Omega \approx 0.25$, so the following discussion will be limited to this case only, although from the original differential equation for $\delta(a)$ one can in principle obtain a solution for any sensible Ω. Numerically, the differential equation for $\delta(a)$ can in principle be solved for any value of the parameters. In practice we have found it convenient, and adequate, to obtain the solution as a power series in either Ω or $1 - \Omega$. In the first case, the resulting series is asymptotic and only slowly convergent around $\Omega \approx 0.25$, while in the latter case the convergence is much more rapid. In this last case, we have carried therefore the expansion up to eighth order, which gives the answers given below (see, also, Figs. 1–4) to an accuracy of several decimals.

It is known that in the absence of a running Newton's constant G ($G \to G_0$, thus $c_a = 0$) one has $f(a = a_0) = 0.4625$ and $\gamma = 0.5562$ for the standard ΛCDM scenario with $\Omega = 0.25$ [17]. On the other hand, when the running of $G(\Box)$ is taken into account, one finds from the solution to Eq. (3.75) for the growth index parameter γ at $\Omega = 0.25$ the following set of results.

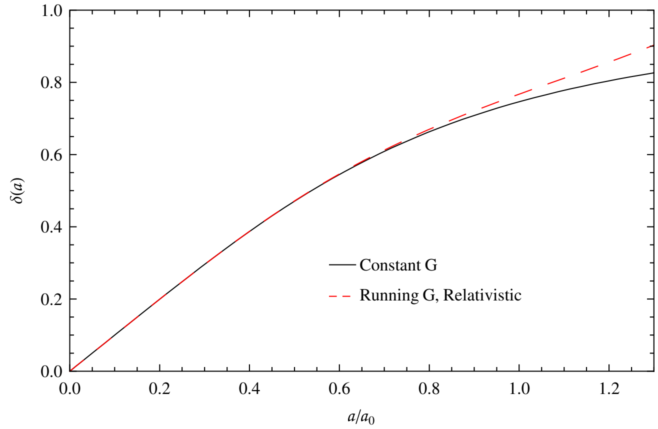

FIG. 1 (color online). Illustration of the matter density contrast $\delta(a)$ as a function of the scale factor $a(t)$, in the fully relativistic treatment (tensor box) and for a given matter fraction $\Omega = 0.25$, obtained from the solution of the density contrast equation of Eq. (3.76), with $G(a)$ given in Eq. (3.11) with $\gamma_\nu = 9/2$ and for $c_a = 0.001$. In the case of a running $G(\Box)$, one generally observes a slightly faster growth rate for later times, as compared to the solution for the case of constant G and with the same choice of Ω, described by Eq. (3.87).

For the tensor box case discussed in Sec. III E, one has the value $c_h = (11/3) \times 2.1621 = 7.927$ in Eqs. (3.41) and (3.68), which gives

$$\gamma = 0.5562 - 199.2 c_a + O(c_a^2). \tag{4.4}$$

For the scalar box case discussed in Sec. III D, one has instead $c_h = 2.1621$ and in this case one finds

$$\gamma = 0.5562 - 54.8 c_a + O(c_a^2). \tag{4.5}$$

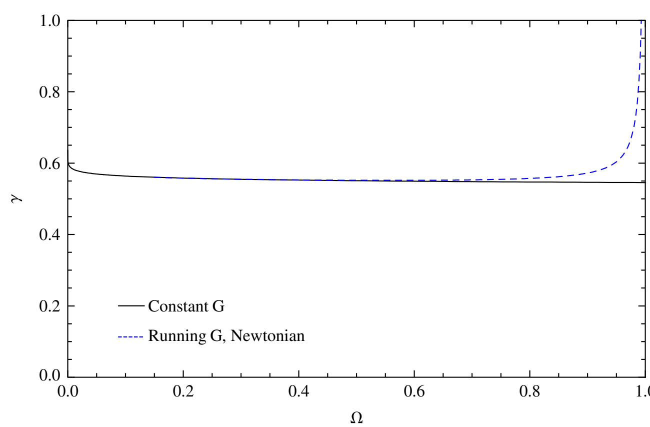

FIG. 2 (color online). Illustration of the growth index parameter γ of Eq. (4.3) as a function of the matter density fraction Ω, computed in the Newtonian (nonrelativistic) theory with a running $G(a)$ given in Eq. (3.11), and obtained by solving Eq. (A36), here with $\gamma_\nu = 9/2$ and $c_a = 0.01$. For the specific choice of matter fraction $\Omega = 0.25$, suggested by ΛCDM models, one then obtains the estimates for the growth index parameter given in Eq. (4.6).

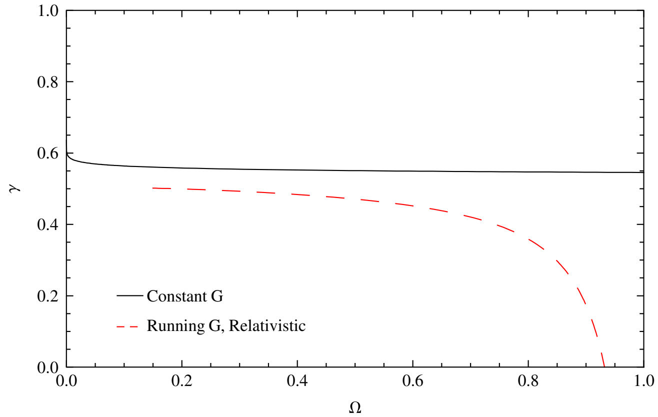

FIG. 3 (color online). Illustration of the growth index parameter γ of Eq. (4.3) as a function of the matter density fraction Ω, computed in the fully relativistic (tensor box) theory with a running $G(a)$ as given in Eq. (3.11), and obtained by solving Eq. (3.75) with $\gamma_\nu = 9/2$ and $c_a = 0.0003$. For the specific choice of matter fraction $\Omega = 0.25$ one then obtains the estimates given for the tensor box in Eq. (4.4). Not surprisingly, the deviations from the standard result for γ become more visible for larger values of Ω.

As a comparison, we have also computed the exponent γ for the case $c_h = 0$ in Eqs. (3.41) and (3.68). This corresponds to a case where the $O(h)$ correction to $\delta\rho_{\rm vac}$ is entirely neglected, and one obtains $\gamma = 0.5562 - 0.703c_a + O(c_a^2)$. Finally, for the Newtonian (nonrelativistic) treatment, described in Appendix A, one finds the much smaller correction

$$\gamma = 0.5562 - 0.0142c_a + O(c_a^2). \qquad (4.6)$$

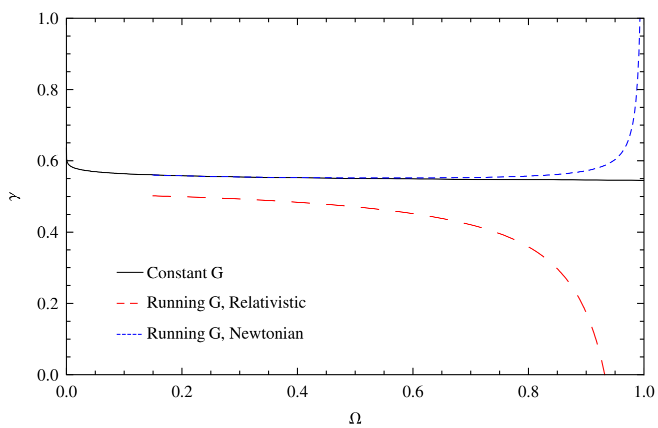

FIG. 4 (color online). Qualitative comparison of the growth index parameters γ of Eq. (4.3) as a function of the matter density fraction Ω, computed first in the relativistic (tensor box) theory with a running $G(a)$ and $c_a = 0.0003$, then in the Newtonian (nonrelativistic) treatment also with a running $G(a)$ and $c_a = 0.01$, both with $\gamma_\nu = 9/2$, and finally compared to the usual treatment with constant G. In both cases, the deviations from the standard result for γ are most visible for larger values of Ω, corresponding to a greater matter fraction.

Among these last expressions, the tensor box case is supposed to give ultimately the correct answer; the scalar box case only serves as a qualitative comparison, and the $c_h = 0$ case is done to estimate independently the size of the correction coming from the ubiquitous $O(h)$ or $\frac{1}{2\nu}c_h$ terms [see, for example, the differential equation for the density perturbations $\delta(t)$ in Eq. (3.75)]. Note that the $c_h = 0$, scalar and tensor box results can be summarized into the slightly more general formula

$$\gamma = 0.5562 - (0.703 + 25.04c_h)c_a + O(c_a^2), \qquad (4.7)$$

showing again the overall importance of the c_h contribution to $\delta\rho_{\rm vac}$ in Eq. (3.41). This last term is responsible for the feedback of the metric fluctuations h on the vacuum density $\delta\rho_{\rm vac}$ and pressure $\delta p_{\rm vac}$ fluctuations.

It should be emphasized here once again that all of the above results have been obtained by solving the differential equation for $\delta(a)$, Eq. (3.92), with $G(a)$ given in Eq. (3.11), and exponent $\gamma_\nu = 3/2\nu \simeq 9/2$ relevant for a matter-dominated background universe. It is this last choice that needs to be critically analyzed, as it might give rise to a definite bias. Our value for γ_ν so far reflects our choice of a matter-dominated background. More general choices, such as an "effective" $\gamma_\nu = 3(1 + w)/2\nu$ with an effective w, or even the use of Eq. (3.85), are in principle possible. Then, although Eq. (3.75) for $\delta(t)$ remains unchanged, Eq. (3.92) for $\delta(a)$ would have to be solved with new parameters. In the next section we will discuss a number of options which should allow one to increase on the accuracy of the above result, and, in particular, correct the possible shortcomings coming so far from the specific choice of the exponent γ_ν.

A. Possible physical interpretation of the results

Looking at these last results (see, also, Figs. 1–4), they seem to indicate that (a) the correction due to the h (or $1/2\nu$) terms in Eq. (3.41) and in the differential equation, Eq. (3.75), for $\delta(a)$ is rather large, and that (b) it is more than twice as large in the tensor box case than it is in the scalar box case. Furthermore, they seem to suggest that (c) the Newtonian (nonrelativistic) result, which does not contain a $\rho_{\rm vac}$ contribution, substantially underestimates the size of the quantum correction. To quantitatively estimate the actual size of the correction in the above expressions for the growth index parameter γ, and make some preliminary comparison to astrophysical observations, some additional information is needed.

The first item is the coefficient $c_0 \approx 33.3$ in Eq. (2.28) as obtained from lattice gravity calculations of invariant correlation functions at fixed geodesic distance [19]. We have reanalyzed the results of [19], which involve rather large uncertainties for this particular quantity, nevertheless it would seem difficult to accommodate values for c_0 that are more than an order of magnitude smaller than the quoted value. A renewed more accurate lattice calculation of c_0, obtained from the computation of invariant curvature correlation functions at fixed geodesic distance, would seem rather desirable at this point.

The next item that is needed here is a quantitative estimate for the magnitude of the coefficient c_a in Eq. (3.11) in terms of c_t in Eq. (2.18), and therefore in terms of c_0 in the original Eq. (2.28). First of all, one has $c_a \approx c_t$, if a_0 is identified with a scale factor corresponding to a universe of size ξ; to a good approximation this corresponds to the Universe today, with the relative scale factor customarily normalized at that time to $a/a_0 = 1$, although some large conversion factor might be hidden in this perhaps naive identification (see below).

Regarding the numerical value of the coefficient c_t itself, it was found in [13] that in Eq. (2.18) $c_t \simeq 0.785 c_0$ in the scalar box case, and $c_t \simeq 0.450 c_0$ in the tensor box case. In both cases, these estimates refer to values obtained from the zeroth order covariant effective field equations. In the following we will take for concreteness the average of the two values, thus $c_t \approx 0.618 c_0$. Then for all three covariant calculations recorded above $c_a \approx 0.618 \times 33.3 \approx 20.6$, a rather large coefficient.

From all of these considerations one would tend to get estimates for the growth parameter γ with rather large corrections. For example, in the tensor box case the corrections would add up to $-199.0 c_a = -199.0 \times 0.618 \times 33.3 = -4095.0$. Even in the Newtonian (nonrelativistic) case, where the correction is found to be the smallest, the corresponding result appears to be quite large. In this last case $c_a \approx c_t \approx 2.7 c_0$ (see Appendix A), so the correction to the index γ becomes $-0.0142 \times 2.7 \times 33.3 = -1.28$.

It would seem though that one should account somewhere for the fact that the largest galaxy clusters and superclusters studied today up to redshifts $z \simeq 1$ extend for only about, at the very most, $1/20$ the overall size of the visible universe. This would suggest then that the corresponding scale for the running coupling $G(t)$ or $G(a)$ in Eqs. (2.18) and (3.11), respectively, should be reduced by a suitable ratio of the two relevant length scales, one for the largest observed galaxy clusters or superclusters, and the second for the very large, cosmological scale $\xi \sim 1/\sqrt{\lambda/3} \sim 1.51 \times 10^{28}$ cm entering the expression for $\delta G(\Box)$ in Eqs. (2.3) and (2.28). This would dramatically reduce the magnitude of the quantum correction by as much as a factor of the order of $(1/20)^{\gamma_\nu} = (1/20)^{4.5} \approx 1.398 \times 10^{-6}$. When this correction factor is roughly taken into account, one obtains the more reasonable (and perhaps observationally more compatible) estimates for the tensor box case

$$\gamma = 0.5562 - 0.0057 c_a + O(c_a^2), \tag{4.8}$$

and for the scalar box case

$$\gamma = 0.5562 - 0.0016 c_a + O(c_a^2), \tag{4.9}$$

while in the nonrelativistic (Newtonian) case one finds $\gamma \approx 0.5562 - 4.08 \times 10^{-7}$. In the tensor box case, this would then amount to a slightly reduced value for the growth index γ *at these scales* as compared to the constant G case, by as much as a few percent, which could perhaps be observable in the not too distant future. Of course, on larger scales the effects would be more significant, and somewhat bigger for larger values of Ω.

A second possibility we will pursue here briefly is to consider a shortcoming, mentioned previously, in the use of $a(t) \sim a_0 (t/t_0)^{2/3}$ in relating $G(a)$ in Eq. (3.11) to $G(t)$ in Eq. (2.18). In general, if w is not small, one should use instead Eq. (3.85) to relate the variable t to $a(t)$. The problem here is that, loosely speaking, for $w \neq 0$ at least two w's are involved, $w = 0$ (matter) and $w = -1$ (λ term). Unfortunately, this issue complicates considerably the problem of relating $\delta G(t)$ to $\delta G(a)$, and therefore the solution to the resulting differential equation for $\delta(a)$. As a tractable approximation though, one should set instead $a(t) \sim a_0 (t/t_0)^{2/3(1+w)}$, and then use an effective value of $w \approx -7/9$, which would seem more appropriate for the final target value of $\Omega \approx 0.25$. For this choice, one then obtains a significantly reduced power in Eq. (3.11), namely, $\gamma_\nu = 3(1+w)/2\nu = 1$. Furthermore, the resulting differential equation for $\delta(a)$, Eq. (3.92), is still relatively easy to solve, by the same methods used in the previous section. One now finds

$$\gamma = 0.5562 - (0.92 + 7.70 c_h) c_a + O(c_a^2), \tag{4.10}$$

which should be compared to the previous result of Eq. (4.7). In particular, for the tensor box case one has again $c_h = 7.927$, which can the be used to compare to the previous result of Eq. (4.4). Thus by reducing the value of γ_ν by about a factor of 4, the c_a coefficient in the above expression has been reduced by about a factor of 3, a significant change.

After using this improved value for the power γ_ν, the problem of correcting for relative scales needs to be addressed again, in light of the corrected estimate for the growth exponent parameter of Eq. (4.10). Given this new choice for $\gamma_\nu = 1$, on can now consider, for example, the types of galaxy clusters studied recently in [20–22], which typically involve comoving radii of $\sim$8.5 Mpc and viral radii of $\sim$1.4 Mpc. For these one would obtain an approximate overall scale reduction factor of $(1.4/4890)^1 \approx 2.9 \times 10^{-4}$. Note that in these units (Mpcs) the reference scale appearing in $G(\Box)$ is of the order of $\xi \simeq 4890$ Mpc. This would give for the tensor box ($c_h = 7.927$) correction to the growth index γ in Eq. (4.10) the more reasonable order of magnitude estimate $-62. \times 20.6 \times 2.9 \times 10^{-4} \approx -0.37$, and for γ itself the reduced value would end up at ≈ 0.19. Clearly, at this point these should only be considered as rough order of magnitude estimates.[7]

Nevertheless, this last case is suggestive of a trend, quite independently of the specific value of c_h and therefore of the overall numerical coefficient of the correction in Eq. (4.10): namely, that the correction to the growth index parameter will increase close to linearly (for γ_ν close to 1, as we have argued) in the size of the cluster. Consequently, one expects that the deviations will increase tenfold in going from a cluster size of 1 Mpc to one of 10 Mpc, and a hundredfold in going from 1 to 100 Mpc.

Finally, another possible, and ultimately much more conservative, approach would be to take—at least for the time being—with some caution the rather large value for c_0 obtained from nonperturbative lattice quantum gravity calculations. One could then use instead the observational bounds on x-ray studies of large galactic clusters at distance scales of up to about 1.4 to 8.5 Mpc [21], namely $\gamma = 0.50 \pm 0.08$, to constrain the value of the constant c_a *at that scale*, giving, for example, from Eq. (4.10) the bound $c_a \lesssim 8 \times 10^{-4}$ in the case of tensor box, and the much less stringent bound $c_a \lesssim O(1)$ for the Newtonian (nonrelativistic) case of Eq. (4.6).

B. Density perturbations in the conformal Newtonian gauge with $G(\Box)$

In this section, we will outline briefly what other avenues can be pursued to determine quantitatively and systematically the cosmological effects of a running $G(\Box)$. The perturbed RW metric is well suited for discussing matter perturbations, but occasionally one finds it more convenient to use a different metric parametrization, such as the one derived from the conformal Newtonian (cN) gauge line element (see, for example, [23,24], and references therein)

$$d\tau^2 = a^2(t)\{(1+2\psi)dt^2 - (1-2\phi)\delta_{ij}dx^i dx^j\} \quad (4.11)$$

with conformal Newtonian potentials $\psi(\mathbf{x}, t)$ and $\phi(\mathbf{x}, t)$. In the simplest framework, the two potentials ψ and ϕ give rise separately to Newton's equation for a point particle, and Poisson's equation, respectively,

$$\ddot{\mathbf{x}} = -\nabla\psi \qquad \nabla^2\phi = 4\pi G a^2 \delta\rho. \quad (4.12)$$

In this gauge, and in the absence of a $G(\Box)$, the unperturbed equations are

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3} G a^2 \bar\rho \qquad \frac{d}{dt}\left(\frac{\dot a}{a}\right) = -\frac{4\pi}{3} G a^2(\bar\rho + 3\bar p), \quad (4.13)$$

in the absence of spatial curvature ($k = 0$). In the presence of a running G these again need to be modified, in accordance with Eqs. (2.17), (2.16), and (2.18). A cosmological constant can be conveniently included in the $\bar\rho$ and $\bar p$, with $\bar\rho_\lambda = \lambda/8\pi G = -\bar p_\lambda$. In this gauge scalar perturbations are characterized by Fourier modes $\psi(\mathbf{q}, t)$ and $\phi(\mathbf{q}, t)$, and the first order Einstein field equations in the absence of $G(\Box)$ read [23]

$$\begin{aligned}
&k^2\phi + 3\frac{\dot a}{a}\left(\dot\phi + \frac{\dot a}{a}\psi\right) = 4\pi G a^2 \delta T^0{}_0 \\
&k^2\left(\dot\phi + \frac{\dot a}{a}\psi\right) = 4\pi G a^2(\bar\rho + \bar p)\theta \\
&\ddot\phi + \frac{\dot a}{a}(2\dot\phi + \dot\psi) + \left(2\frac{\ddot a}{a} - \frac{\dot a^2}{a^2}\right)\psi + \frac{k^2}{3}(\phi - \psi) \\
&= \frac{4\pi}{3} G a^2 \delta T^i{}_i \\
&k^2(\phi - \psi) = 12\pi G a^2(\bar\rho + \bar p)\sigma,
\end{aligned} \quad (4.14)$$

where the perfect fluid energy-momentum tensor is given to linear order in the perturbations $\delta\rho = \rho - \bar\rho$ and $\delta p = p - \bar p$ by

$$\begin{aligned}
T^0{}_0 = -(\bar\rho + \delta\rho) \qquad T^0{}_i = (\bar\rho + \bar p)v_i = -T^i{}_0 \\
T^i{}_j = (\bar p + \delta p)\delta^i{}_j + \Sigma^i{}_j \Sigma^i{}_i = 0
\end{aligned} \quad (4.15)$$

and one has allowed for an anisotropic shear perturbation $\Sigma^i{}_j$ to the perfect fluid form $T^i{}_j$. The two quantities θ and σ are commonly defined by

$$(\bar\rho + \bar p)\theta \equiv i k^j \delta T^0{}_j \qquad (\bar\rho + \bar p)\sigma \equiv -(\hat k_i \hat k_j - \frac{1}{3}\delta_{ij})\Sigma^i{}_j \quad (4.16)$$

with $\Sigma^i{}_j \equiv T^i{}_j - \delta^i{}_j T^k{}_k/3$ the traceless component of $T^i{}_j$. For a perfect fluid, θ is the divergence of the fluid velocity, $\theta = i k^j v_j$, with $v^j = dx^j/dt$ the small velocity of the fluid. The field equations imply, by consistency, the covariant energy-momentum conservation law

$$\begin{aligned}
\dot\delta &= -(1+w)(\theta - 3\dot\phi) - 3\frac{\dot a}{a}\left(\frac{\delta p}{\delta\rho} - w\right)\delta \\
\dot\theta &= -\frac{\dot a}{a}(1 - 3w)\theta - \frac{\dot w}{1+w}\theta + \frac{1}{1+w}\frac{\delta p}{\delta\rho}k^2\delta \\
&\quad - k^2\sigma + k^2\psi
\end{aligned} \quad (4.17)$$

and relate the matter fields δ, σ, and θ to the metric perturbations ϕ and ψ, where δ is the matter density contrast $\delta = \delta\rho/\rho$, and w is the equation of state parameter $w = p/\rho$. In general relativity $\phi = \psi$ as long as there is no anisotropic stress, but in extended theories of gravity,

[7]One might perhaps think that the running of G envisioned here might lead to small observable consequences on much shorter, galactic length scales. That this is not the case can be seen, for example, from the following argument. For a typical galaxy, one has a size $\sim$30 kpc, giving for the quantum correction the estimate, from Eq. (2.9) for the correction to the static potential, $(30\ \mathrm{kpc}/4890 \times 10^3\ \mathrm{kpc})^3 \sim 2.31 \times 10^{-16}$, which is tiny given the large size of ξ. It is therefore unlikely that such a correction will be detectable at these length scales, or that it could account for large anomalies in the galactic rotation curves.

such as the one described here, the relation between ϕ and ψ can become scale dependent.

In the presence of a $G(\Box)$, the above equations need to be rederived and amended, starting from the covariant field equations of Eq. (2.3) in the cN gauge of Eq. (4.11), with zeroth order modified field equations as in Eqs. (2.16) and (2.17), using the expansion for $G(\Box)$ given in Eq. (3.24), but now in terms of the new cN gauge potentials ϕ and ψ. One key question is then the nature of the vacuum-polarization induced anisotropic shear perturbation correction $\Sigma^i{}_j$ appearing in the covariant effective field equations analogous to Eqs. (4.14), but derived with a $G(\Box)$. In particular, one would expect the quantum correction to the energy-momentum tensor appearing on the right-hand side of Eq. (2.3) to contribute new terms to the last of Eqs. (4.14), which could then account for a nonzero stress σ, and thus for a small deviation from the classical GR result for a perfect fluid, $\phi = \psi$. Naively, one would expect $\psi/\phi = 1 + O(\delta G/G_0)$. An explicit calculation with $G(\Box)$ [25] gives

$$\begin{aligned} \frac{\psi}{\phi} &= 1 + \left(1 - \frac{1}{2\nu(1+w)}\right) 3 w_{\mathrm{vac}} \frac{\delta G(t)}{G_0} \\ &= 1 + \left(1 - \frac{1}{2\nu}\right) \frac{\delta G(t)}{G_0} \end{aligned} \tag{4.18}$$

for $w = 0$ and $w_{\mathrm{vac}} = \frac{1}{3}$. It is often customary (see, e.g., [23,24,26,27]) to parametrize deviations from general relativity in terms of a slip function Σ and of the growth rate parameter γ introduced previously. These two quantities are defined by

$$\nabla^2(\phi + \psi) = 3\Sigma\Omega H^2 \delta \qquad \gamma = \frac{\log f}{\log \Omega} \tag{4.19}$$

with δ the density contrast and f the density contrast exponent. Occasionally, the parameter $\eta = \psi/\phi - 1$ is introduced as well. In classical general relativity $\psi/\phi = 1$, $\eta = 0$, $\Sigma = 1$, and then the growth exponent $\gamma \approx 0.55$ for $\Omega \approx 0.25$. The calculations presented in the previous sections have already suggested to some extent what changes to expect for the exponent γ, which then leaves the problem of determining the structure of the Σ correction. In addition, the Newtonian (nonrelativistic) calculation of Appendix A has determined, from the form of the modified Poisson equation, one of the relevant equations, namely, the one for the potential ϕ. We plan to discuss these interesting questions in a future publication [25].

V. CONCLUSIONS

In this paper, we have attempted to systematically analyze the effects on matter density perturbations of a running $G(\Box)$ appearing in the original effective, nonlocal covariant field equations of Eq. (2.3). The specific form of $G(\Box)$ in Eq. (2.1) is inspired by the nonperturbative treatment of covariant path integral quantum gravity, and follows from the existence of a nontrivial fixed point in G of the renormalization group in four dimensions. The resulting effective field equations are manifestly covariant, and in principle besides the genuinely nonperturbative scale ξ there are no adjustable parameters, since the coefficients (c_0) and scaling dimensions (ν) entering $G(\Box)$ are, again in principle, calculable by systematic field theory and lattice methods (see [6] and references therein).

The present work can be viewed in broad terms as consisting of two parts. In the first part, we have systematically developed the general formalism necessary to deal with small matter density fluctuations in the presence of a running gravitational coupling $G(\Box)$. Most, if not all, of the results in the first part have been formulated in a way that assumes as little as possible about specific aspects related to how exactly G does run with scale. Indeed, many of the equations we have obtained are not restricted to $\nu = \frac{1}{3}$, and are found to be valid for a wide range of powers ν and coefficients c_0 appearing, for example, in the original expression for $G(\Box)$ as given in Eq. (2.28). Furthermore, the zeroth order (in the fluctuations) results of [13], on which the present work builds up, do not rely on any specific value for these parameters either, since the expressions obtained there follow from general properties of the covariant d'Alembertian and its powers, as they appear in $G(\Box)$. In particular the flow in the vicinity of the ultraviolet fixed point could in principle allow for c_0 being either negative (gravitational screening) or positive (gravitational antiscreening), and both cases could in principle be described by the results obtained above, for example, for the growth index f and the growth index parameter γ. It is only the latter option though that is favored by studies of nonperturbative Euclidean lattice gravity (the weak coupling phase is unstable and found to describe a collapsed degenerate two-dimensional spacetime), hence the choice here to discuss primarily this last case. But in principle the fact remains that the sign of c_0 will ultimately determine the direction of the corrections given above, which could eventually become constrained by observation. In the end, the only result that is extensively used in the first part is the result of [13] that $w_{\mathrm{vac}} = \frac{1}{3}$, apart from the fact that we choose to restrict our attention from the very beginning primarily to the nonrelativistic matter case $w = 0$, and to the long wavelength limit $\mathbf{q} \to 0$. Later on it was found that for sufficiently slowly varying backgrounds the result $w_{\mathrm{vac}} = \frac{1}{3}$ is preserved also to first order in the perturbations, which seems to suggest some level of consistency in the treatment of the field perturbations.

In spite of the nonlocality of the original effective field equations in Eq. (2.3), one finds quite in general that small perturbations can be treated, in first approximations, in terms of local terms, described by quantities ρ_{vac} and p_{vac} as they appear in the effective description of $T_{\mu\nu}^{\mathrm{vac}}$ in terms of a perfect fluid. The latter should then be regarded as the leading term in a derivative expansion of the nonlocal contribution to the effective field equations, as they apply here to the rather specific case of the FLRW background. Under the physically motivated assumption of a comparatively slowly varying (both in space and time) background, it is then possible to obtain a complete and consistent set of effective field equations, describing small perturbations for the metric trace and matter modes [Eqs. (3.69), (3.72), (3.73), and (3.74)]. From these, a single equation for the matter density contrast is eventually obtained, Eq. (3.75), which is the main result of this work. The only input needed in this last equation is $\delta G(t)$, the zeroth order (in the fluctuations) running of G as written in Eq. (2.18), with given more or less known parameters ν and c_t. The corresponding result in the Newtonian (nonrelativistic) treatment is obtained in Appendix A, leading to Eq. (A30).

The next step was a translation of the equation for the density contrast $\delta(t)$ into the corresponding equation for $\delta(a)$, involving a related running coupling $G(a)$, instead of the original $G(t)$. Since in general the transformation from one variable to the other is not entirely trivial, some simplification had to be assumed, i.e., that the quantum correction in $G(a)$ can be written as a power, with an exponent γ_ν, a choice that could in the future be relaxed as part of a broader more systematic investigation. Subsequently, a solution for the differential equation for $\delta(a)$ was obtained, leading to expressions for the growth index $f(a)$ and for the growth index parameter γ. A number of general features can be observed, the first one being the fact that generally the correction to the growth index parameter γ is found to be negative, indicating a less steep rise of f with Ω.

The second part of the paper describes a number of attempts to provide a semiquantitative estimate for the corrections obtained, in order to see whether these corrections could be related in some way to current astrophysical observations. In order to do so, one needs to adapt the theoretical calculation for the growth index parameter γ to the kind of observational data available from the study of large galactic clusters. This requires, as expected, a careful consideration of the relative length scales that come into play. On the one hand, one length scale is given by the size of the largest clusters reached by observation, typically of the order of a few Mpcs. On the other hand it should involve the absolute reference scale given by $\xi = \sqrt{3/\lambda} \simeq$ 4890 Mpc. The comparison between theory and observation would then seem straightforward, were it not for the fact that this ratio generally comes in to a certain power, whose detailed knowledge is necessary in order to eventually reduce the quantitative uncertainties. Eventually, these could be bracketed by a more systematic study of the solutions to the $\delta(a)$ equation, and the corresponding growth exponents γ. We are referring here, in particular, to a study of the sensitivity of the results to the specific choices of the exponent γ_ν, appearing in $\delta G(a)$ and determined in part by the relationship between the variables t and $a(t)$, which we discussed earlier. In addition, there is still perhaps a certain level of uncertainty in the actual coefficients c_0 and c_t entering the theoretical predictions, which we have also described above in some detail. The latter could be reduced further by improved nonperturbative lattice computations. Nevertheless, the value of the present calculations lies in our opinion in the fact that so far a discernible trend seems to emerge from the results. The trend we have found seems to suggest that the correction to the growth exponent γ is initially rather small for small clusters, negative in sign, and then slowly increasing in magnitude, close to linearly with scale.

It is clear that the effects discussed in this paper are only relevant for very large scales, much bigger than those usually considered, and well constrained, by laboratory, solar, or galactic dynamics tests [1,28–30]. Furthermore, the effects we have described here are quite different from what one would expect in $f(R)$ theories [31,32], which also tend to predict some level of deviation from classical GR in the growth exponents [33–35]. Future more accurate astrophysical observations might make it possible to see the difference in the predictions of various models [26,27,36–38].

ACKNOWLEDGMENTS

One of the authors (H. W. H.) wishes to thank Thibault Damour and Gabriele Veneziano for inspiration and discussions leading to the present work, and Alexey Vikhlinin for correspondence regarding astrophysical measurements of structure growth indices. He also wishes to thank Hermann Nicolai and the Max Planck Institut für Gravitationsphysik (Albert-Einstein-Institut) in Potsdam for warm hospitality. Thanks also go to Salvatore Capozziello and Paolo Serra for providing us with a number of useful references. The work of H. W. H. was supported in part by the Max Planck Gesellschaft zur Förderung der Wissenschaften, and by the University of California. The work of R. T. was supported in part by the DOE.

APPENDIX A: NONRELATIVISTIC (NEWTONIAN) TREATMENT OF MATTER DENSITY PERTURBATIONS

In this section, we discuss the Newtonian theory of small matter fluctuations, first by recalling the relevant equations in the usual treatment, and then by presenting what changes need to be implemented in order to account for the running of G. Later these equations will be solved, so that a comparison can be made with the results in the absence of a running G.

When discussing a nonrelativistic Hubble flow it is customary to define coordinates in the following way:

$$\mathbf{x} = \frac{\mathbf{r}}{a(t)} \qquad \mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{\dot{a}}{a}\mathbf{r} \tag{A1}$$

where $\mathbf{x}$ is attached to the comoving frame, while $\mathbf{r}$ is the flat Minkowski space coordinate, such that in the comoving frame $\mathbf{x}$ one has, by construction, $d\mathbf{x}/dt = 0$.

In the following, some simplification will arise due to the fact that we shall consider a nonrelativistic fluid with the negligible pressure, $p \simeq 0$ or $w = 0$. The relevant equations are then the continuity equation, the Euler equation, and the gravitational field equations. These will be listed below to zeroth and first order in the matter density (ρ), pressure (p), velocity field ($\mathbf{v}$), and gravitational field $\mathbf{g}$.

1. Newtonian treatment without the running of G

After decomposing the fields into a background and a fluctuation contribution, $\rho = \bar{\rho} + \delta\rho$, $p = \bar{p} + \delta p$, and $\mathbf{v} = \bar{\mathbf{v}} + \delta\mathbf{v}$, one obtains from the continuity equation, to zeroth and first order, respectively,

$$\begin{gathered} \dot{\bar{\rho}} + \nabla \cdot (\bar{\rho}\mathbf{v}) = 0 \\ \dot{\delta\rho} + 3\frac{\dot{a}}{a}\delta\rho + \frac{\dot{a}}{a}(\mathbf{r}\cdot\nabla)\delta\rho + \bar{\rho}\nabla\cdot\delta\mathbf{v} = 0. \end{gathered} \tag{A2}$$

When the effect of the Hubble flow is included, i.e., Eq. (A1), the above zeroth order equation reduces to

$$\dot{\bar{\rho}}(t) + 3\frac{\dot{a}(t)}{a(t)}\bar{\rho}(t) = 0 \tag{A3}$$

with solution $\bar{\rho}(t) = \bar{\rho}_0(a_0/a(t))^3$, where $\bar{\rho}_0$ and a_0 are the two integration constants corresponding to the present matter density and to the present scale factor (usually taken to be $a_0 = 1$). We note here that Eq. (A3), and hence Eq. (3.14), will continue to hold for a running G, as these equations are derived from the kinematics and the continuity equations in the RW background metric given in Eq. (A2), which is not affected by the running of $G \to G(\Box)$.

To zeroth and first order in the fluctuations, the Euler equations for a fluid in the RW background are given, respectively, by

$$\begin{gathered} \dot{\mathbf{v}} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{g} \\ \dot{\delta\mathbf{v}} + \frac{\dot{a}}{a}\delta\mathbf{v} + \frac{\dot{a}}{a}(\mathbf{r}\cdot\nabla)\delta\mathbf{v} = -\frac{1}{\bar{\rho}}\nabla\delta p + \delta\mathbf{g}. \end{gathered} \tag{A4}$$

Finally, the gravitational field equations are given to zeroth and first order in the fluctuations by

$$\nabla\times\mathbf{g} = 0 \qquad \nabla\cdot\mathbf{g} = -4\pi G_0\bar{\rho} \tag{A5}$$

$$\nabla\times\delta\mathbf{g} = 0 \qquad \nabla\cdot\delta\mathbf{g} = -4\pi G_0\delta\rho \tag{A6}$$

incorporating Gauss' law and the constraint that the gravitational fields are longitudinal. Only the last set of equations contain the gravitational constant G. Hence, in the framework of the Newtonian treatment, the modification of a running $G \to G(\Box)$ only affects the gravitational Poisson equation.

It is customary at this stage to introduce Fourier components of the fluctuations, and write

$$\delta\rho(\mathbf{r}, t) = \delta\rho_{\mathbf{q}}(t)\exp\left[\frac{i\mathbf{r}\cdot\mathbf{q}}{a(t)}\right] \tag{A7}$$

and similarly for $\delta\mathbf{v}$, $\delta\mathbf{g}$, and possibly δp. For an adiabatic fluctuation one can also set $\delta p = v_s^2\delta\rho$, with v_s the speed of sound.

Then to first order in the fluctuations, the continuity equation, Euler equation, and the gravitational field equations take on the form, for each mode $\mathbf{q}$,

$$\dot{\delta\rho}_{\mathbf{q}}(t) + 3\frac{\dot{a}(t)}{a(t)}\delta\rho_{\mathbf{q}}(t) + \frac{i\mathbf{q}\cdot\delta\mathbf{v}_{\mathbf{q}}(t)}{a(t)}\bar{\rho}(t) = 0 \tag{A8}$$

$$\dot{\delta\mathbf{v}}_{\mathbf{q}}(t) + \frac{\dot{a}(t)}{a(t)}\delta\mathbf{v}_{\mathbf{q}}(t) = -\frac{i\mathbf{q}}{a(t)}\frac{v_s^2}{\bar{\rho}(t)}\delta\rho_{\mathbf{q}}(t) + \delta\mathbf{g}_{\mathbf{q}}(t) \tag{A9}$$

$$\delta\mathbf{g}_{\mathbf{q}}(t) = \frac{4\pi i\mathbf{q}}{\mathbf{q}^2}a(t)G_0\delta\rho_{\mathbf{q}}(t). \tag{A10}$$

Subsequent elimination of the gravitational and velocity fields then leads to a single second order differential equation for the matter density contrast $\delta_{\mathbf{q}}(t) \equiv \delta\rho_{\mathbf{q}}(t)/\bar{\rho}(t)$ describing the physics of compressional modes:

$$\ddot{\delta}_{\mathbf{q}}(t) + 2\frac{\dot{a}(t)}{a(t)}\dot{\delta}_{\mathbf{q}}(t) + \left(\frac{v_s^2\mathbf{q}^2}{a(t)^2} - 4\pi G_0\bar{\rho}(t)\right)\delta_{\mathbf{q}}(t) = 0. \tag{A11}$$

In the limit of very long wavelength fluctuations, $\mathbf{q} \to 0$, the above equation simplifies to

$$\ddot{\delta}(t) + 2\frac{\dot{a}(t)}{a(t)}\dot{\delta}(t) - 4\pi G_0\bar{\rho}(t)\delta(t) = 0. \tag{A12}$$

A solution can then be found, using $\bar{\rho}(t) = 1/6\pi G t^2$ and $\dot{a}(t)/a(t) \equiv H(t) = 2/3t$, such that the general form for $\delta(t)$ is given by a linear combination of either $\sim t^{2/3}$ or $\sim t^{-1}$. The latter corresponds to a decaying (as opposed to growing) solution and is usually discarded, giving finally the standard Newtonian result $\delta(a) \propto a$. We note here that the above nonrelativistic equation and solution applies to the case of nonrelativistic matter only; in particular, it excludes the presence of a cosmological constant.

2. Newtonian treatment with running $G(\Box)$

The next step is a modification of the nonrelativistic equations in Eqs. (A2) and (A4)–(A6) to incorporate a suitable running of G. Since only the latter set of equations, Eqs. (A5) and (A6), contain G it is only these that need to be suitably modified. In the presence of a scale-dependent coupling, one has

$$\delta \mathbf{g} = -\nabla \delta \phi \tag{A13}$$

with the perturbing potential $\delta\phi$ given by a solution to Poisson's equation

$$\nabla^2 \delta\phi(\mathbf{r}, t) = -\nabla \cdot \delta \mathbf{g}(\mathbf{r}, t) = 4\pi G(\Box)\delta\rho(\mathbf{r}, t) \tag{A14}$$

and $G(\Box)$ given in Eq. (2.28). Following Eq. (A7), as it applies here to $\delta \mathbf{g}$ and $\delta\rho$, we Fourier transform the spatial components of the above Poisson equation, which requires the Fourier transform of $G(\Box)$ as obtained from Eq. (2.28), namely,

$$G(\mathbf{q}^2, \partial_t^2) = G_0\left\{1 + c_0 \frac{\xi^{-1/\nu}}{[-\partial_t^2 - \mathbf{q}^2/a^2(t)]^{1/2\nu}} + \ldots\right\}. \tag{A15}$$

As a result, the gravitational field perturbation is of the form

$$\delta \mathbf{g}_{\mathbf{q}}(t) = \frac{4\pi i \mathbf{q}}{\mathbf{q}^2} a(t) \cdot \exp\left[\frac{-i\mathbf{r}\cdot\mathbf{q}}{a(t)}\right] G(\mathbf{q}^2, \partial_t^2) \times \left(\delta\rho_{\mathbf{q}}(t) \exp\left[\frac{i\mathbf{r}\cdot\mathbf{q}}{a(t)}\right]\right). \tag{A16}$$

Since we are mainly interested in the long wavelength limit, it suffices here to evaluate the above expression in the limit $\mathbf{q} \to 0$,

$$\begin{aligned}\delta \mathbf{g}_{\mathbf{q}}(t) &= \frac{4\pi i \mathbf{q}}{\mathbf{q}^2} a(t)\left[1 - \frac{i\mathbf{r}\cdot\mathbf{q}}{a(t)} + \ldots\right] G(\mathbf{q}^2, \partial_t^2) \\ &\quad \times \left(\delta\rho_{\mathbf{q}}(t)\left[1 + \frac{i\mathbf{r}\cdot\mathbf{q}}{a(t)} + \ldots\right]\right) \\ &\simeq \frac{4\pi i \mathbf{q}}{\mathbf{q}^2} a(t)\left[G(\mathbf{q}^2, \partial_t^2)\delta\rho_{\mathbf{q}}(t) \right. \\ &\quad - \frac{i\mathbf{r}\cdot\mathbf{q}}{a(t)} G(\mathbf{q}^2, \partial_t^2)\delta\rho_{\mathbf{q}}(t) + G(\mathbf{q}^2, \partial_t^2)\delta\rho_{\mathbf{q}}(t)\frac{i\mathbf{r}\cdot\mathbf{q}}{a(t)} \\ &\quad \left. + \ldots\right],\end{aligned} \tag{A17}$$

and for $\mathbf{q} = 0$ only the first term survives. Furthermore, when $G(\Box) = G(\mathbf{q}^2, \partial_t^2)$ acts on a function of t, which we will assume here is of the form of a power (e.g., t^α, with the power α a number of order 1) one obtains

$$G(\mathbf{q}^2, \partial_t^2) \cdot t^\alpha \to G(t) \cdot t^\alpha. \tag{A18}$$

Here the running coupling $G(t)$ is given by the expression in Eq. (2.18), with $t_0 \equiv \xi$, and the coefficient

$$c_t = \left|\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+1/\nu)}\right| c_0. \tag{A19}$$

Thus, for example, for $\alpha = -4/3$ (the standard Newtonian result for matter density perturbations) one has $c_t = (27/10)c_0$; in the following, it will be safe to assume that the coefficient c_t in Eq. (2.18) is a number of the same order of magnitude as the original c_0 in Eq. (2.28).

Consequently, when acting on a density perturbation $\delta\rho_{\mathbf{q}}(t)$ in the form of a power law in t, to leading order in $\mathbf{q}$, one obtains simply

$$\delta \mathbf{g}_{\mathbf{q}}(t) = \frac{4\pi i \mathbf{q}}{\mathbf{q}^2} a(t) G_0\left[1 + c_t\left(\frac{t}{t_0}\right)^{1/\nu} + \ldots\right]\delta\rho_{\mathbf{q}}(t). \tag{A20}$$

This last result can be compared with Eq. (A10) for the case of a constant G.

As stated previously, the continuity equation for the fluctuations, Eq. (A8), and the corresponding Euler equation for the fluctuations, Eq. (A9), are not modified by the presence of a running $G(\Box)$, as given in Eqs. (A16) and (A20). To solve the resulting equations of motion for the fluctuations, it is now customary to decompose the velocity perturbation $\delta \mathbf{v}$ into parts perpendicular and parallel to $\mathbf{q}$

$$\delta \mathbf{v}_{\mathbf{q}}(t) = \delta \mathbf{v}_{\mathbf{q}\perp}(t) + i\mathbf{q}\epsilon_{\mathbf{q}}(t) \tag{A21}$$

with

$$\mathbf{q} \cdot \delta \mathbf{v}_{\mathbf{q}\perp} = 0 \qquad \epsilon_{\mathbf{q}} \equiv -\frac{i\mathbf{q}\cdot\delta\mathbf{v}_{\mathbf{q}}}{\mathbf{q}^2}. \tag{A22}$$

The fractional change in the matter density δ is then defined as

$$\delta_{\mathbf{q}}(t) \equiv \frac{\delta\rho_{\mathbf{q}}(t)}{\bar\rho(t)}. \tag{A23}$$

With the above decomposition of the velocity field $\delta\mathbf{v}$ and the expression for the density contrast δ inserted into the first order continuity equation, Eq. (A8), one obtains the unmodified result

$$\dot\delta_{\mathbf{q}}(t) = \frac{\mathbf{q}^2}{a(t)}\epsilon_{\mathbf{q}}(t), \tag{A24}$$

so that there is no change in the relationship between δ and ϵ when $G \to G(\Box)$. In turn, the Euler equation for the fluctuation, Eq. (A9), now becomes the two sets of equations

$$\begin{aligned}&\mathrm{Re:}\ \delta\dot{\mathbf{v}}_{\mathbf{q}\perp}(t) + \frac{\dot a}{a}\delta\mathbf{v}_{\mathbf{q}\perp}(t) = 0 \\ &\mathrm{Im:}\ i\mathbf{q}\dot\epsilon_{\mathbf{q}}(t) + \frac{\dot a}{a} i\mathbf{q}\epsilon_{\mathbf{q}}(t) = -\frac{i\mathbf{q}}{a} v_s^2\delta_{\mathbf{q}}(t) + \delta\mathbf{g}_{\mathbf{q}}\end{aligned} \tag{A25}$$

with the gravitational field fluctuation $\delta\mathbf{g}_{\mathbf{q}}$ now given by the expression in Eq. (A16). From the real part (corresponding to rotational modes), one concludes

$$\delta\mathbf{v}_{\mathbf{q}\perp} \propto a^{-1}(t), \tag{A26}$$

which is of the same form as in the case of a constant G. From the imaginary part (corresponding to compressional modes), in Eq. (A25) one obtains, using Eq. (A24),

$$\ddot{\delta}_{\mathbf{q}}(t) + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{q}}(t) + \frac{\mathbf{q}^2}{a^2}v_s^2\delta_{\mathbf{q}}(t) - 4\pi\exp\left[\frac{-i\mathbf{r}\cdot\mathbf{q}}{a(t)}\right]$$
$$G(\mathbf{q}^2, \partial_t^2)\left(\exp\left[\frac{i\mathbf{r}\cdot\mathbf{q}}{a(t)}\right]\bar{\rho}(t)\delta_{\mathbf{q}}(t)\right) = 0. \tag{A27}$$

The latter can be recast into the slightly simpler form

$$\ddot{\delta}_{\mathbf{q}}(t) + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{q}}(t) + \left(\frac{\mathbf{q}^2}{a^2}v_s^2 - 4\pi\mathcal{G}(\mathbf{q}^2, \partial_t^2)\right)\delta_{\mathbf{q}}(t) = 0 \tag{A28}$$

by defining a modified source term

$$\mathcal{G}(\mathbf{q}^2, \partial_t^2) \equiv \frac{1}{\delta_{\mathbf{q}}(t)}\left\{\exp\left[\frac{-i\mathbf{r}\cdot\mathbf{q}}{a(t)}\right]G(\mathbf{q}^2, \partial_t^2)\right.$$
$$\left.\times\left(\exp\left[\frac{i\mathbf{r}\cdot\mathbf{q}}{a(t)}\right]\bar{\rho}(t)\delta_{\mathbf{q}}(t)\right)\right\}. \tag{A29}$$

In the limit $\mathbf{q}\to 0$ one obtains immediately

$$\ddot{\delta}(t) + 2\frac{\dot{a}}{a}\dot{\delta}(t) - 4\pi G(t)\bar{\rho}(t)\delta(t) = 0. \tag{A30}$$

The last two equations can now be compared with the corresponding results for a constant G, given in Eqs. (A11) and (A12).

3. Computation of the nonrelativistic (Newtonian) growth index with $G(\Box)$

The next step requires a solution of the differential equation for the density perturbations $\delta_{\mathbf{q}}(t)$, in the Newtonian approximation and in the limit $\mathbf{q}\to 0$, as in Eq. (A30). It is convenient and customary at this point to change variables from t to the scale factor $a(t)$, so that $\delta_{\mathbf{q}}(t)\to\delta_{\mathbf{q}}(a) = \tilde{\delta}_{\mathbf{q}}\cdot\delta(a)$. From Eq. (3.90) one has

$$\dot{\delta}(t) = aH(a)\frac{\partial\delta(a)}{\partial a}$$
$$\ddot{\delta}(t) = a^2H^2(a)\left[\frac{\partial\ln H(a)}{\partial a} + \frac{1}{a}\right]\frac{\partial\delta(a)}{\partial a} + a^2H^2(a)\frac{\partial^2\delta(a)}{\partial a^2}. \tag{A31}$$

Here $H(a)$ is defined as the Hubble "constant" $H(a)\equiv\dot{a}(t)/a(t)$, as it appears in the equations of motion for a background FLRW geometry

$$H(a) = \sqrt{\frac{8\pi}{3}G(a)\bar{\rho}(a) + \frac{\lambda}{3}}, \tag{A32}$$

but with a running Newton's constant $G(a)$ [see Eq. (2.18)]

$$G(a) = G_0\left[1 + \frac{\delta G(a)}{G_0}\right] = G_0\left[1 + c_a\left(\frac{a}{a_0}\right)^{\gamma_\nu} + \ldots\right]. \tag{A33}$$

Here the index is $\gamma_\nu = 3/2\nu$, since from Eq. (2.18) one has for nonrelativistic matter $a(t)/a_0\approx(t/t_0)^{2/3}$. In the above expression $c_a\approx c_t$, if a_0 is identified with a scale factor corresponding to a universe of size ξ; to a good approximation this corresponds to the Universe today, with the relative scale factor customarily normalized to $a/a_0 = 1$. As a consequence, the constant c_a in Eq. (A33) can be taken to be of the same order as the constant c_0 appearing in the original expressions for $G(\Box)$ in Eqs. (2.5) and (2.28). Note also that by the use of Eq. (A32) for the scale factor, we have allowed for a nonvanishing cosmological constant in our otherwise Newtonian (nonrelativistic) treatment.

After these substitutions, one finally obtains the differential equation for the matter density contrast, Eq. (A30), in the variable $a(t)$

$$\frac{d^2\delta(a)}{da^2} + \left(\frac{d\ln H(a)}{da} + \frac{3}{a}\right)\frac{d\delta(a)}{da} - \frac{4\pi G(a)\bar{\rho}(a)}{a^2H^2(a)}\delta(a) = 0. \tag{A34}$$

Note that in order to compute the leading, in $\delta G(a)/G_0$, correction to the density contrast $\delta(a)$, one only needs $\bar{\rho}(a)$ to lowest order as given in Eq. (3.14), and $H(a)$ as given in Eq. (A32).

With the aid of the parameter θ [see Eq. (3.86)]

$$\theta\equiv\frac{1-\Omega}{\Omega}, \tag{A35}$$

where Ω is the matter density fraction and $1-\Omega$ the cosmological constant fraction as measured today, one obtains the following differential equation for the density contrast $\delta(a)$

$$\frac{\partial^2\delta}{\partial a^2} + \frac{3(1+2a^3\theta)}{2a(1+a^3\theta)}\left(1 + c_a\frac{\gamma_\nu a^{\gamma_\nu} + (\frac{1}{3}\gamma_\nu - 1)a^{3+\gamma_\nu}\theta}{(1+a^3\theta)(1+2a^3\theta)}\right)\frac{\partial\delta}{\partial a}$$
$$-\frac{3}{2a^2(1+a^3\theta)}\left(1 + c_a\frac{a^{3+\gamma_\nu}\theta}{1+a^3\theta}\right)\delta = 0 \tag{A36}$$

for a reference scale $a_0 = 1$; the latter can always be reintroduced later by the trivial replacement $a\to a/a_0$.

Without a scale-dependent G [$c_a = 0$ in Eq. (A33)], the growing solution to the above equation is given by

$$\delta_0(a)\propto a\cdot{}_2F_1(\tfrac{1}{3}, 1; \tfrac{11}{6}; -a^3\theta), \tag{A37}$$

where ${}_2F_1$ is the Gauss hypergeometric function. To evaluate the correction to $\delta_0(a)$ coming from the terms proportional to c_a, one sets

$$\delta(a)\propto a\cdot{}_2F_1(\tfrac{1}{3}, 1; \tfrac{11}{6}; -a^3\theta)[1 + c_a\mathcal{F}(a)], \tag{A38}$$

then inserts the resulting expression in Eq. (A36), and finally expands the resulting expression to lowest order in c_a to find the correction $\mathcal{F}(a)$. The resulting differential equation can then be solved for $\mathcal{F}(a)$, giving the density contrast $\delta(a)$ as a function of the two parameters [γ_ν and Ω or $\theta\equiv(1-\Omega)/\Omega$] appearing in Eq. (A36). In the following, we will focus on the specific choice $\nu = \frac{1}{3}$ obtained from the lattice theory of gravity [8], which leads to the $G(a)$ exponent $\gamma_\nu = \frac{3}{2\nu} = 9/2$. It is customary at this point to define the growth index $f(a) \equiv \frac{\partial \ln\delta(a)}{\partial \ln a}$ and the related growth index parameter γ via $\gamma \equiv \frac{\ln f}{\ln\Omega}\big|_{a=a_0}$. Then the solution to Eq. (A36) gives an explicit expression for the growth index γ parameter, as a function of the matter fraction Ω.

Based on observational constraints, one is mostly interested in the case $\Omega \approx 0.25$, therefore in the following we will limit our discussion to this choice only. In the absence of a running G ($G \to G_0$, thus $c_a = 0$) one has $f(a = a_0) = 0.4625$ and $\gamma = 0.5562$ for $\Omega = 0.25$ [17]. On the other hand, when the running of G is taken into account one finds from the solution to Eq. (A36) for the growth index parameter γ at $\Omega = 0.25$

$$\gamma = 0.5562 - 0.0142 c_a + O(c_a^2). \tag{A39}$$

In the end it would seem therefore that at least in the Newtonian treatment the correction comes out rather small. Note that both the Newtonian and the relativistic treatment, described in the main body, give a negative sign for the correction arising from the running of G.

To estimate quantitatively the actual size of the correction in Eq. (A39), one needs an estimate for the coefficient $c_0 \approx 33.3$ in Eq. (2.28), as obtained from the lattice gravity calculations of invariant correlation functions at fixed geodesic distance [19]. In addition, one uses the fact that $c_a \approx c_t \approx 2.7 c_0$ [see the previous discussion related to Eq. (A33)]. From this, one would then get the estimate $\gamma = 0.5562 - 1.28$ on the largest scales, which looks like a significant $O(1)$ correction to γ.

[1] T. Damour, J. Phys. G **33**, 1 (2006); update in http://pdg.lbl.gov/2009/reviews/rpp2009-rev-gravity-tests.pdf (November 2009).
[2] T. Damour and K. Nordtvedt, Phys. Rev. Lett. **70**, 2217 (1993); T. Damour and G. Esposito-Farese, Phys. Rev. D **53**, 5541 (1996).
[3] G. Veneziano, arXiv:hep-th/0002094; M. Gasperini and G. Veneziano, arXiv:hep-th/0703055.
[4] K. G. Wilson, Phys. Rev. Lett. **28**, 548 (1972); Phys. Rev. D **7**, 2911 (1973); see, also, G. Parisi, in *The Proceedings of the 1976 Cargèse Summer Institute* (Plenum Press, New York, 1977), Vol. 26; E. Brezin and J. Zinn-Justin, Phys. Rev. Lett. **36**, 691 (1976).
[5] S. Weinberg, in *General Relativity-An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
[6] H. W. Hamber, *Quantum Gravitation* (Springer Publishing, Berlin and New York, 2009), and references therein.
[7] H. W. Hamber and R. M. Williams, Nucl. Phys. **B248**, 392 (1984); **B260**, 747 (1985); **B269**, 712 (1986); Phys. Lett. **157B**, 368 (1985); Nucl. Phys. **B435**, 361 (1995); Phys. Rev. D **59**, 064014 (1999).
[8] H. W. Hamber, Phys. Rev. D **45**, 507 (1992); Nucl. Phys. **B400**, 347 (1993); Phys. Rev. D **61**, 124008 (2000).
[9] H. Kawai and M. Ninomiya, Nucl. Phys. **B336**, 115 (1990); H. Kawai, Y. Kitazawa, and M. Ninomiya, Nucl. Phys. **B393**, 280 (1993); **B404**, 684 (1993); Y. Kitazawa and M. Ninomiya, Phys. Rev. D **55**, 2076 (1997); T. Aida and Y. Kitazawa, Nucl. Phys. **B491**, 427 (1997).
[10] M. Reuter, Phys. Rev. D **57**, 971 (1998); M. Reuter and H. Weyer, Gen. Relativ. Gravit. **41**, 983 (2009); E. Manrique, M. Reuter, and F. Saueressig, arXiv:1006.0099, and references therein.
[11] G. A. Vilkovisky, in *Quantum Theory of Gravity*, edited by S. Christensen (Hilger, Bristol, 1984); Nucl. Phys. **B234**, 125 (1984).
[12] T. R. Taylor and G. Veneziano, Nucl. Phys. **B345**, 210 (1990).
[13] H. W. Hamber and R. M. Williams, Phys. Rev. D **72**, 044026 (2005).
[14] D. Lopez Nacir and F. D. Mazzitelli, Phys. Rev. D **75**, 024003 (2007).
[15] H. W. Hamber and R. M. Williams, Phys. Rev. D **73**, 044031 (2006); **76**, 084008 (2007); **81**, 084048 (2010).
[16] H. W. Hamber and R. M. Williams, Phys. Lett. B **643**, 228 (2006); Phys. Rev. D **75**, 084014 (2007).
[17] P. J. E. Peebles, *Principles of Physical Cosmology* (Princeton University Press, Princeton, New Jersey, 1993).
[18] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, New York, 1972).
[19] H. W. Hamber, Phys. Rev. D **50**, 3932 (1994).
[20] F. Schmidt, A. Vikhlinin, and W. Hu, Phys. Rev. D **80**, 083505 (2009).
[21] A. Vikhlinin *et al.*, arXiv:0903.5320; arXiv:0903.2297.
[22] D. Rapetti, S. W. Allen, A. Mantz, and H. Ebeling, arXiv:0911.1787.
[23] C.-P. Ma and E. Bertschinger, Astrophys. J. **455**, 7 (1995).
[24] F. Bernardeau, S. Colombi, E. Gaztanaga, and R. Scoccimarro, Phys. Rep. **367**, 1 (2002).
[25] H. W. Hamber and R. Toriumi (unpublished).
[26] L. Amendola, M. Kunz, and D. Sapone, J. Cosmol. Astropart. Phys. 04 (2008) 013.
[27] S. F. Daniel *et al.*, Phys. Rev. D **80**, 023532 (2009).
[28] J. P. Uzan, Rev. Mod. Phys. **75**, 403 (2003).
[29] J. P. Uzan, arXiv:0908.2243.
[30] E. G. Adelberger, B. R. Heckel, and A. E. Nelson, Annu. Rev. Nucl. Part. Sci. **53**, 77 (2003).
[31] S. Capozziello, M. De Laurentis, and V. Faraoni, arXiv:0909.4672, and references therein.
[32] S. M. Carroll, V. Duvvuri, M. Trodden, and M. S. Turner, Phys. Rev. D **70**, 043528 (2004).

[33] H. Motohashi, A. A. Starobinsky, and J. Yokoyama, arXiv:1005.1171.
[34] P. Zhang, Phys. Rev. D **73**, 123504 (2006).
[35] R. Gannouji, B. Moraes, and D. Polarski, J. Cosmol. Astropart. Phys. 02 (2009) 034; S. Tsujikawa, R. Gannouji, B. Moraes, and D. Polarski, Phys. Rev. D **80**, 084044 (2009).
[36] A. S. Goldhaber and M. M. Nieto, Rev. Mod. Phys. **82**, 939 (2010).
[37] O. Bertolami, J. M. Mourao, and J. Perez-Mercader, Phys. Lett. B **311**, 27 (1993); O. Bertolami and J. Garcia-Bellido, Int. J. Mod. Phys. D **5**, 363 (1996).
[38] G. Robbers, N. Afshordi, and M. Doran, Phys. Rev. Lett. **100**, 111101 (2008).