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## $O(n)$ Heisenberg Model Close to $n=d=2$

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The critical behavior of the  $O(n)$  classical Heisenberg model in  $d$  dimensions close to  $n=d=2$  is discussed, assuming analyticity in  $n$  and  $d$  of the renormalization-group equations. In the  $(d, n)$  plane there is a line passing through  $(2, 2)$  across which the critical exponents are nonanalytic. For  $d=2$  and  $-2 \leq n \leq 2$ , a conjecture is presented for the exact form of the leading and subdominant thermal eigenvalues.

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Recently, a scaling theory of the  $q$ -state two-dimensional Potts model close to  $q=4$  was proposed.<sup>1,2</sup> While this was motivated by approximate renormalization-group (RG) analyses,<sup>3</sup> it was possible to show that the results in fact followed from minimal assumptions on the analyticity of the RG equations, together with previously known exact results. It is possible to apply the same methods to the  $O(n)$  Heisenberg model in  $2+\epsilon$  dimensions close to the special case of the planar model, which corresponds to  $n=2$ ,  $\epsilon=0$ , for which some presumably exact results are known.<sup>4-6</sup>

The RG equations take the form

$$dg/dl = -\epsilon g + (n-2)f(g) + 4\pi^3 y^2 + \dots, \quad (1)$$

$$dy^2/dl = (4 - 2\pi/g)y^2 + \dots \quad (2)$$

For  $n=2$  these reduce to the equations studied by Nelson and Fisher<sup>7</sup> for the XY model in  $2+\epsilon$  dimensions: They are a simple extension of the Kosterlitz<sup>5</sup> equations. When  $y^2=0$ , Eqs. (1) and (2) reduce to those obtained by Polyakov<sup>8</sup> and Brézin and Zinn-Justin,<sup>9</sup> with  $f(g)=g^2/2\pi+O(g^3)$ . The physical interpretation of  $y$  (which for the planar model is a fugacity for the vortices) is ambiguous when  $(n, d) \neq (2, 2)$ , but it is presumably connected with the compactness of the  $O(n)$  space,

a feature neglected in Refs. 8 and 9. However, it is sufficient to point out that (1) and (2) are the only equations compatible with the assumption of analyticity in  $y^2$ ,  $\epsilon$ , and  $n$ , and the two limits described above. They can, in fact, also be derived from first principles.<sup>10</sup> The corrections to these equations are  $O(\epsilon^2)$ , where  $\epsilon$ ,  $n-2$ , and  $y^2$  are of the same order. This last condition is justified since the fixed-point values of  $y^2$  will turn out to be  $O(\epsilon)$ .

The fixed points and thermal eigenvalues of Eqs. (1) and (2) are

$$\text{I: } g = g_0 + O(\epsilon), \quad f(g_c)/g_c = \epsilon/(n-2),$$

$$y^2 = O(\epsilon g_c),$$

$$y_{\text{I}} = -\epsilon + (n-2)f'(g_c) + O(\epsilon^2/(n-2)) + O(\epsilon^2),$$

$$y_{\text{I}}' = 4 - 2\pi/g_c + O(\epsilon);$$

$$\text{II: } g = \pi/2 + O(\epsilon),$$

$$y^2 = \Delta/4\pi^3 + O(\epsilon^2), \quad \Delta = \epsilon\pi/2 + (2-n)f(\pi/2),$$

$$y_{\text{II}} = (8\Delta/\pi)^{1/2} + O(\epsilon),$$

$$y_{\text{II}}' = -(8\Delta/\pi)^{1/2} + O(\epsilon).$$

In the limit  $\epsilon/(n-2) \rightarrow 0$ , I reduces to the fixed point found in Ref. 9, while as  $\epsilon \rightarrow 0$ ,  $(n-2) \rightarrow 0$ , II goes over into the Kosterlitz-Thouless point.<sup>4-6</sup>

Now assume that  $f(g)/g$  is a monotonic increasing function of  $g$ . For  $\epsilon < 0$  and  $n < 2$ , neither fixed point is real, and there is no transition, as expected. For  $n < 2$ , only II is real and this determines the critical behavior. In the remaining quadrant, for  $\Delta < 0$ , II is unphysical and I determines the critical behavior. When  $\Delta = 0$ , the fixed points collide, so that  $y_{I'}$  and  $y_{II'}$  become marginal, and  $y_I = y_{II}$ . For  $\Delta > 0$ , both fixed points exist, but I is an unstable tricritical fixed point, the critical behavior being determined by II.

For  $d = 2$  and  $n \leq 2$ , the leading and subdominant thermal eigenvalues correspond to different branches of the same analytic function, with a square-root branch point at  $n = 2$ . This is similar to the situation for critical and tricritical exponents in the  $q$ -state Potts model.<sup>3</sup> It is possible to conjecture an exact form for this eigenvalue in terms of the quantity  $x = (2/\pi) \cos^{-1}[(2+n)^{1/2}]$ , which is rational for integer  $n$  between  $-2$  and  $+2$ , and incorporates the branch point. The conjecture is

$$\nu^{-1} = 4x/(1+x).$$

This agrees with the known exact results<sup>11,12</sup>  $\nu = 1$  and  $\frac{1}{2}$  for  $n = 1$  and  $-2$ , respectively, and gives  $\nu = \frac{3}{4}$  for the  $n = 0$  polymer problem, in agreement with series and real-space RG calculations<sup>13,14</sup>  $\nu = 0.750 \pm 0.005$ . The other branch gives nonleading eigenvalues  $y' = -2$  and  $-\infty$  for  $n = 1$  and  $-2$ , respectively. This agrees with the known spectrum of negative integers for these cases. Note

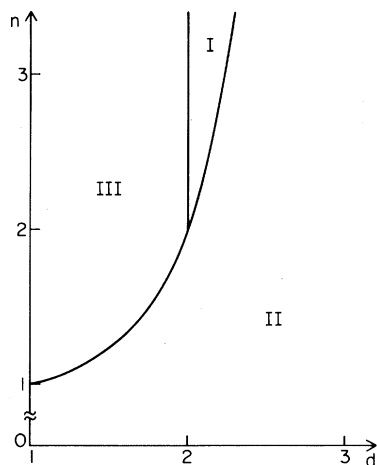


FIG. 1. Schematic diagram of regions of the  $(d, n)$  plane. In I the  $2 + \epsilon$  expansion of Brézin and Zinn-Justin (Ref. 9) applies. The critical exponents are continuous but nonanalytic across the boundary between I and II. In III there is no transition at finite temperatures.

that for  $n \leq 1$ ,  $y'$  is no longer the next-to-leading eigenvalue.

Close to  $n = 2$  the conjecture implies that  $\nu^{-1} \sim (4/\pi)(2-n)^{1/2}$  which means that  $f(\pi/2) = 2/\pi$ . The validity of the conjecture has also been confirmed for several noninteger values of  $n$  with series expansions.<sup>14</sup>

The magnetic exponent  $y_H$  in the region  $\Delta > 0$  has the form  $y_H = \frac{15}{8} + O(\epsilon)$ . It does not have the double-valued feature of the thermal exponent, unlike the case of the Potts model.<sup>2</sup>

Although the fixed-point structure found above is strictly valid only to first order in  $\epsilon$  and  $(n - 2)$ , by continuity it implies the existence of a boundary in the  $(d, n)$  plane, passing through  $(2, 2)$ , across which the exponents are continuous but nonanalytic. This boundary is plotted schematically in Fig. 1. It is reasonable to suppose that it passes through  $n = 1$  at  $d = 1$ , the lower critical dimensionality for the Ising model. It would appear to pass  $n = 3$  somewhat below  $d = 3$ . This possibly explains why the  $O(\epsilon^2)$  calculations of Brézin and Zinn-Justin,<sup>9</sup> which were made about the fixed point I, give poor results for  $n = 3$  and  $\epsilon = 1$  compared to the  $4 - \epsilon$  expansion ( $\nu = 0.5$ , vs  $\nu = 0.717 \pm 0.007$  from series expansion,<sup>15</sup> and  $0.71$  from the  $4 - \epsilon$  expansion<sup>16</sup>). To first order in  $\sqrt{\epsilon}$ , the fixed point II gives  $\nu \approx 0.65$ . What happens to this boundary at large  $n$ , and why is it not seen in the  $1/n$  expansion? Effects of the compactness of the group, such as vortices, typically give contributions of the form  $\exp[-\text{const}/g]$  in the free energy. For large  $n$ , the critical value of  $g$  is  $O(1/n)$ , so these are effects which never appear to any finite order in the  $1/n$  expansion.

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## Effective Potential for a Renormalized $d$ -Dimensional $g\varphi^4$ Field Theory in the Limit $g \rightarrow \infty$

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The first four coefficients in the effective potential  $V_{\text{eff}}(\varphi_R) = \sum_{n=1}^{\infty} V_{2n} \varphi_R^{2n}$  are calculated for a mass- and wave-function-renormalized  $g\varphi^4$  field theory in  $d$  space-time dimensions in the limit where the unrenormalized coupling  $g \rightarrow \infty$  with the renormalized mass  $M$  held fixed. The accuracy of these numerical results is verified by exact analytical calculations of the effective potential performed in 0 and 1 space-time dimensions.

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In the previous papers<sup>1-3</sup> we have formulated a simple procedure for expanding the Green's functions of a  $g\varphi^4$  theory in  $d$ -dimensional space-time in inverse powers of the bare coupling constant  $g$ . In this paper we indicate how such expansions can be mass and wave-function renormalized, and report the results of extensive calculations of the renormalized  $n$ -point Green's functions.

The Lagrangian density describing the  $g\varphi^4$  theory in Euclidean space is  $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{1}{4}g\varphi^4$ , where  $m$  is the bare-mass parameter. To calculate the connected  $n$ -point Green's functions  $W_n(x_1, \dots, x_n)$  one introduces a source function  $J(x)$  in the functional-integral representation

for the vacuum persistence function  $Z$ :

$$Z[J] = \int \mathcal{D}\varphi \exp\left\{-\int d^d x [\mathcal{L} + J(x)\varphi(x)]\right\}. \quad (1)$$

We obtain  $W_n$  from  $Z[J]$  by

$$W_n(x_1, \dots, x_n) = \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \ln Z[J] \Big|_{J=0}. \quad (2)$$

The Green's functions are renormalized in three steps: First, the wave-function renormalization constant  $Z_3$  is computed from the two-point Green's function by  $Z_3^{-1} = [dW_2^{-1}(p^2)/dp^2]_{p^2=0}$ . Second, the renormalized mass  $M$  is defined by  $M^2 = Z_3[W_2^{-1}(p^2)]_{p^2=0}$ . Third, the renormalized  $n$ -point Green's functions are obtained by multiply-