Galaxies **2014**, *2*, 275-291; doi:10.3390/galaxies2020275

OPEN ACCESS

galaxies

ISSN 2075-4434

www.mdpi.com/journal/galaxies

Article

Quantum Gravity and Cosmological Density Perturbations

Herbert W. Hamber [1,2,*] and Reiko Toriumi [2]

[1] Max Planck Institute for Gravitational Physics, Albert Einstein Institute, D-14476 Potsdam, Germany

[2] Department of Physics and Astronomy, University of California, Irvine, CA 92697-4575, USA; E-Mail: rtoriumi@uci.edu

* Author to whom correspondence should be addressed; E-Mail: herbert.hamber@aei.mpg.de; Tel.: +1-949-856-5596.

Received: 14 February 2014; in revised form: 10 May 2014 / Accepted: 27 May 2014 / Published: 11 June 2014

Abstract: We explore the possible cosmological consequences of a running Newton's constant, $G(\Box)$, as suggested by the non-trivial ultraviolet fixed point scenario for Einstein gravity with a cosmological constant term. Here, we examine what possible effects a scale-dependent coupling might have on large-scale cosmological density perturbations. Starting from a set of manifestly covariant effective field equations, we develop the linear theory of density perturbations for a non-relativistic perfect fluid. The result is a modified equation for the matter density contrast, which can be solved and thus provides an estimate for the corrections to the growth index parameter, γ.

Keywords: quantum gravity; quantum cosmology; renormalization group; quantum infrared effects

---

1. Introduction

Recent years have seen the development of a bewildering variety of alternative theories of gravity, in addition to the more traditional ones, such as scalar-tensor, higher derivative and dilaton gravities, just to mention a few examples. Many of these theories eventually predict some level of deviation from classical gravity, which is often parametrized either by a suitable set of post-Newtonian parameters or, more recently, by the introduction of a slip function [1–4]. The latter has been quite useful in describing deviations from classical General Relativity (GR) and specifically from the standard ΛCDM

(Lambda Cold Dark Matter) model, when analyzing the latest cosmological CMB (Cosmic Microwave Background), weak lensing, supernovae and galaxy clustering data.

In this work, we will focus on the systematic analysis of departures from GR in the growth history of matter perturbations arising from a quantum running of G, within the narrow context of the non-trivial ultraviolet fixed point scenario for Einstein gravity with a cosmological term. Thus, instead of looking at deviations from GR at very short distances, due to new interactions, such as the ones suggested by string theories [5,6], we will be considering here infrared effects, which could therefore become manifest at very large distances. We will argue here that such effects are in principle calculable and could therefore be confronted with present and future astrophysical observations. The classical theory of small density perturbations is by now well developed in standard textbooks, and the resulting theoretical predictions for the growth exponents are simple to state and well understood. Except possibly on the very largest scales, where the data so far is still rather limited, the predictions agree quite well with current astrophysical observations. Here, we will be interested in computing and predicting possible small deviations in the growth history of matter perturbations and specifically in the values of the growth exponents, arising from a very specific scenario, namely a weakly scale-dependent gravitational coupling, whose value very gradually increases with distance. The specific nature of the scenario we will be investigating here is motivated by the treatment of field-theoretic models of quantum gravity, based on the Einstein action with a bare cosmological term. Its long-distance scaling properties are derived from the existence of a non-trivial ultraviolet fixed point of the renormalization group in Newton's constant, G [7–21].

The first step in analyzing the consequences of a running of G is thus to re-write the expression for $G(k^2)$ in a coordinate-independent way, either by the use of a non-local Vilkovisky-type effective gravity action [22–25] or by the use of a set of consistent effective field equations. In going from momentum to position space, one employs $k^2 \to -\Box$, which gives for the quantum-mechanical running of the gravitational coupling the replacement $G \quad \to \quad G(\Box)$. One then finds that the running of G is given, in the vicinity of the Ultra Violet (UV) fixed point, by:

$$G(\Box) \;=\; G_0 \left[1 \,+\, c_0 \left(\frac{1}{\xi^2 \Box} \right)^{1/2\nu} \,+\, \dots \right] \tag{1}$$

where $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the covariant d'Alembertian, and the dots represent higher order terms in an expansion in $1/(\xi^2 \Box)$. Current evidence from Euclidean lattice quantum gravity points toward $c_0 > 0$ (implying infrared growth) and $\nu = \frac{1}{3}$ [15–17]. It is well known that for theories with a non-trivial ultraviolet fixed point [26–31], the long distance (and thus infrared) universal scaling properties are uniquely determined, up to subleading correction to exponents and scaling amplitudes, by the (generally nontrivial) scaling dimensions obtained by renormalization group methods in the vicinity of the UV fixed point [32–36]. These sets of results form the basis for universal predictions in the non-linear sigma model [37–39], which provides today the second most accurate test of quantum field theory [40], after the $g-2$ prediction for QED (Quantum Electro-Dynamics) (for a comprehensive set of references, see [8,34], and the references therein). It is also an established fact of modern RG (Renormalization Group) theory that in lattice QCD (Quantum Chromo-Dynamics) the scaling behavior of the theory in the vicinity of the asymptotic freedom UV fixed point unambiguously determines the universal non-perturbative long distance scaling properties of the theory [41], as quantified by hadron masses, vacuum condensates, decay amplitudes, anomalous baryon magnetic moments and the QCD string tension [42,43].

Within the quantum-field-theoretic renormalization group treatment, the quantity, ξ, arises as an integration constant of the Callan–Symanzik renormalization group equations. One challenging issue, and of great relevance to the physical interpretation of the results, is a correct identification of the renormalization group invariant scale, ξ. A number of arguments can be given in support of the suggestion that the infrared scale, ξ (analogous to the $\Lambda_{\overline{MS}}$ of QCD), can, in fact, be very large, even cosmological, in the gravity case (see, for example, [8] and references therein). From these arguments, one would then first infer that the constant, G_0, can, to a very close approximation, be identified with the laboratory value of Newton's constant, $\sqrt{G_0} \sim 1.6 \times 10^{-33}$ cm. The appearance of the d'Alembertian $\Box$ in the running of G then naturally leads to a set of non-local field equations; instead of the ordinary Einstein field equations with constant G, one is now led to consider the modified effective field equations:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G(\Box) T_{\mu\nu} \tag{2}$$

with a new non-local term due to the $G(\Box)$. By being manifestly covariant, they still satisfy some of the basic requirements for a set of consistent field equations incorporating the running of G. Not unexpectedly, though, the new nonlocal equations are much harder to solve than the original classical field equations for constant G.

As stated above, physically, it would seem at first, based on the perturbative treatment alone [7,18], that the non-perturbative scale, ξ, could take any value (including perhaps a very small one), which could then possibly preclude any observable quantum effects in the foreseeable future. In perturbation theory, the reason for this is that the non-perturbative scale, ξ, appears, as in gauge theories, as an integration constant of the renormalization group equations and is therefore not fixed by perturbation theory alone. However, a number of recent non-perturbative results for the gravitational Wilson loop on the Euclidean lattice at strong coupling, giving an area law, and their subsequent interpretation in light of the observed large-scale semiclassical curvature [8,44–46], would suggest otherwise: namely that the non-perturbative scale, ξ, appears, in fact, to be related to macroscopic curvature. From astrophysical observation, the average curvature on very large scales, or, stated in somewhat better terms, the measured cosmological constant, λ, is very small. This would then suggest that the new scale, ξ, can be very large, even cosmological, and comparable to the Hubble scale, $1/\xi^2 \simeq \lambda/3$. This would then give a more concrete semi-quantitative estimate for the scale in the $G(\Box)$ of Equation (1), namely $\xi \sim 1/\sqrt{\lambda/3} \sim 1.51 \times 10^{28}$ cm. Note that in common astronomical units (Mpcs), the reference scale appearing in $G(\Box)$ is then of the order of $\xi \simeq 4890$ Mpc.

A scale-dependent Newton's constant is already expected to lead to small modifications of the standard cosmological solutions to the Einstein field equations. The starting point is the quantum effective field equations of Equation (2), with $G(\Box)$ defined in Equation (1). In the Friedmann–Lemaître–Robertson–Walker (FLRW) framework, these are applied to the standard homogeneous isotropic metric. In the following, we will mainly consider the case $k = 0$ (a spatially flat universe). The next step, therefore, is a systematic examination of the nature of the solutions to the full effective field equations, with $G(\Box)$ involving the relevant covariant d'Alembertian operator $\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu$. To start the process, one assumes, for example, that the matter, $T_{\mu\nu}$, has the perfect fluid form:

$$T_{\mu\nu} = [\,p(t) + \rho(t)\,]\, u_\mu\, u_\nu + g_{\mu\nu}\, p(t) \tag{3}$$

for which one needs to compute the action of $\Box^n$ on $T_{\mu\nu}$ and, then, analytically continue the answer to negative fractional values of $n = -1/2\nu$. Even in the simplest case, with $G(\Box)$ acting on a scalar, such as the trace of the energy-momentum tensor, $T_\lambda{}^\lambda$, one finds a rather unwieldy expression.

A more general calculation [47] shows that a non-vanishing pressure contribution is generated in the effective field equations, even if one initially assumes a pressureless fluid, $p(t) = 0$. After a somewhat lengthy derivation, one obtains, for a universe filled with non-relativistic matter ($p = 0$), a set of effective Friedmann equations incorporating the running of G. It was also noted in [47] that the effective field equations with a running G can be recast in an equivalent, but slightly more appealing, form by defining a vacuum polarization pressure, p_{vac}, and density, ρ_{vac}, such that, for the FLRW background, one has:

$$\rho_{vac}(t) = \frac{\delta G(t)}{G_0}\, \rho(t), \qquad\qquad p_{vac}(t) = \frac{1}{3}\, \frac{\delta G(t)}{G_0}\, \rho(t) \tag{4}$$

with $G(t)$ given by:

$$G(t) \;\equiv\; G_0 \left(1 + \frac{\delta G(t)}{G_0} \right) \;=\; G_0 \left[1 + c_t \left(\frac{t}{t_0} \right)^{1/\nu} + \, \dots \right] \tag{5}$$

The explicit computations also shows that c_t is of the same order as c_0 in Equation (1), and $t_0 = \xi$ [47]; in the quoted reference, it was estimated $c_t = 0.450\; c_0$ for the tensor box operator.

Then, the source term in the effective tt field equation can be regarded as a combination of the two density terms $\rho(t) + \rho_{vac}(t)$, while the effective rr equation involves the new vacuum polarization pressure term, $p_{vac}(t)$. Just as one introduces the parameter, w, describing the matter equation of state, $p(t) = w\,\rho(t)$, with $w = 0$ for non-relativistic matter, one can do the same for the remaining contribution by setting $p_{vac}(t) = w_{vac}\; \rho_{vac}(t)$. This more compact notation allows one to finally re-write the field equations for the FLRW background (and $k = 0$) as:

$$\begin{aligned} 3\, \frac{\dot a^2(t)}{a^2(t)} &= 8\pi\, G_0 \left(1 + \frac{\delta G(t)}{G_0} \right) \bar\rho(t) + \lambda \\ \frac{\dot a^2(t)}{a^2(t)} + 2\, \frac{\ddot a(t)}{a(t)} &= -8\pi\, G_0 \left(w + w_{vac} \frac{\delta G(t)}{G_0} \right) \bar\rho(t) + \lambda \end{aligned} \tag{6}$$

2. Relativistic Treatment of Matter Density Perturbations

Besides the modified cosmic scale factor evolution just discussed, the running of $G(\Box)$ given in Equation (1) also affects the nature of matter density perturbations on very large scales. In computing these effects, it is customary to introduce a perturbed metric of the form:

$$d\tau^2 = dt^2 - a^2\, (\delta_{ij} + h_{ij})\, dx^i dx^j \tag{7}$$

with $a(t)$ the unperturbed scale factor and $h_{ij}(\mathbf{x}, t)$ a small metric perturbation; and $h_{00} = h_{i0} = 0$ by the choice of coordinates. As will become clear later, we will mostly be concerned here with the trace mode, $h_{ii} \equiv h$, which determines the nature of matter density perturbations. After decomposing the matter fields into background and fluctuation contribution, $\rho = \bar{\rho} + \delta\rho$, $p = \bar{p} + \delta p$, and $\mathbf{v} = \bar{\mathbf{v}} + \delta\mathbf{v}$, it is customary in these treatments to expand the density, pressure and metric trace perturbation modes in spatial Fourier modes, $\delta\rho(\mathbf{x},t) = \delta\rho_{\mathbf{q}}(t)\, e^{i\,\mathbf{q}\cdot\mathbf{x}}$ and similarly for $\delta p(\mathbf{x},t)$, $\delta\mathbf{v}(\mathbf{x},t)$ and $h_{ij}(\mathbf{x},t)$, with $\mathbf{q}$ the comoving wavenumber.

The first equation one obtains is the zeroth (in the fluctuations) order energy conservation in the presence of $G(\Box)$, which reads:

$$3\,\frac{\dot{a}(t)}{a(t)}\left[(1+w)+(1+w_{vac})\,\frac{\delta G(t)}{G_0}\right]\bar{\rho}(t)+\frac{\delta\dot{G}(t)}{G_0}\,\bar{\rho}(t)+\left(1+\frac{\delta G(t)}{G_0}\right)\dot{\bar{\rho}}(t)=0 \tag{8}$$

It will be convenient in the following to solve the energy conservation equation not for $\bar{\rho}(t)$, but instead for $\bar{\rho}(a)$. This requires that, instead of using the expression for $G(t)$ in Equation (5), one uses the equivalent expression for $G(a)$:

$$G(a) = G_0\left(1+\frac{\delta G(a)}{G_0}\right), \qquad \text{with} \quad \frac{\delta G(a)}{G_0} \equiv c_a\left(\frac{a}{a_0}\right)^{\gamma_\nu} + \dots \tag{9}$$

In this last expression, the power is $\gamma_\nu = 3/2\nu$, since from Equation (5), one has for non-relativistic matter $a(t)/a_0 \approx (t/t_0)^{2/3}$ in the absence of a running G. In the following, we will almost exclusively consider the case $\nu = \frac{1}{3}$ [15–17] for which, therefore, $\gamma_\nu = 9/2$. Then, in the above expression, $c_a \approx c_t$ if a_0 is identified with a scale factor appropriate for a universe of size ξ; to a good approximation, this should correspond to the universe "today", with the relative scale factor customarily normalized at such a time to $a/a_0 = 1$. Consequently, and with the above proviso, the constant c_a in Equation (9) can safely be taken to be of the same order as the constant, c_0, appearing in the original expressions for $G(\Box)$ in Equation (1). The solution to Equation (8) for $w_{vac} = \frac{1}{3}$ can then be written as:

$$\bar{\rho}(a) = \bar{\rho}_0\,\left(\frac{a_0}{a}\right)^3\left(\frac{1+c_a}{1+c_a\left(\frac{a}{a_0}\right)^{\gamma_\nu}}\right)^{(1+\gamma_\nu)/\gamma_\nu} \tag{10}$$

with $\bar{\rho}(a)$ normalized, so that $\bar{\rho}(a=a_0) = \bar{\rho}_0$. For $c_a = 0$, the above expression reduces, of course, to the usual result for non-relativistic matter.

The zeroth order field equations with the running of G included were already given in Equation (6). The next step consists in obtaining the equations that govern the effects of small field perturbations. These equations will involve, apart from the metric perturbation, h_{ij}, the matter and vacuum polarization contributions. The latter arise from:

$$\left(1+\frac{\delta G(\Box)}{G_0}\right)T_{\mu\nu} \;=\; T_{\mu\nu} + T_{\mu\nu}^{vac} \tag{11}$$

with a nonlocal $T_{\mu\nu}^{vac}$. Fortunately, to zeroth order in the fluctuations, the results of [47–49] indicated that the modifications from the nonlocal vacuum polarization term could simply be accounted for by the substitution $\bar{\rho}(t) \rightarrow \bar{\rho}(t) + \bar{\rho}_{vac}(t)$ and $\bar{p}(t) \rightarrow \bar{p}(t) + \bar{p}_{vac}(t)$. Here, we will apply this last result to the small field fluctuations, as well, and set:

$$\delta\rho_{\mathbf{q}}(t) \rightarrow\; \delta\rho_{\mathbf{q}}(t) + \delta\rho_{\mathbf{q}\,vac}(t), \qquad\qquad \delta p_{\mathbf{q}}(t) \rightarrow\; \delta p_{\mathbf{q}}(t) + \delta p_{\mathbf{q}\,vac}(t) \tag{12}$$

The underlying assumptions is, of course, that the equation of state for the vacuum fluid still remains roughly correct when a small perturbation is added. Furthermore, just like we had $\bar{p}(t) = w\,\bar{\rho}(t)$ and $\bar{p}_{vac}(t) = w_{vac}\,\bar{\rho}_{vac}(t)$ with $w_{vac} = \frac{1}{3}$, we now write for the fluctuations:

$$\delta p_{\mathbf{q}}(t) = w\,\delta\rho_{\mathbf{q}}(t), \qquad \delta p_{\mathbf{q}\,vac}(t) = w_{vac}\,\delta\rho_{\mathbf{q}\,vac}(t) \tag{13}$$

at least to leading order in the long wavelength limit, $\mathbf{q} \to 0$. In this limit, we then have simply:

$$\delta p(t) = w\,\delta\rho(t), \qquad \delta p_{vac}(t) = w_{vac}\,\delta\rho_{vac}(t) \equiv w_{vac}\,\frac{\delta G(t)}{G_0}\delta\rho(t) \tag{14}$$

with $G(t)$ given in Equation (5), and we have used Equation (4), now applied to the fluctuation, $\delta\rho_{vac}(t)$,

$$\delta\rho_{vac}(t) \;=\; \frac{\delta G(t)}{G_0}\,\delta\rho(t) + \ldots \tag{15}$$

where the dots indicate possible additional $O(h)$ contributions. Indeed, a bit of thought reveals that the above treatment is incomplete, since $G(\Box)$ in the effective field equation of Equation (2) contains, for the perturbed RW (Robertson-Walker) metric of Equation (7), terms of order h_{ij}, which need to be accounted for in the effective $T^{\mu\nu}_{vac}$. Consequently, the covariant d'Alembertian has to be Taylor expanded in the small field perturbation, h_{ij}, $\Box(g) \;=\; \Box^{(0)} + \Box^{(1)}(h) + O(h^2)$, and similarly for $G(\Box)$:

$$G(\Box) \;=\; G_0\left[1 + \frac{c_0}{\xi^{1/\nu}}\left(\frac{1}{\Box^{(0)} + \Box^{(1)}(h) + O(h^2)}\right)^{1/2\nu} + \ldots\right] \tag{16}$$

To compute the correction of $O(h)$ to $\delta\rho_{vac}(t)$, one needs to consider the relevant term in the expansion of $(1 + \delta G(\Box)/G_0)\,T_{\mu\nu}$, which we write as:

$$-\,\frac{1}{2\,\nu}\,\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h)\cdot\frac{\delta G(\Box^{(0)})}{G_0}\cdot T_{\mu\nu} \tag{17}$$

This last form allows us to use the results obtained previously for the FLRW case in [47–49], namely:

$$\frac{\delta G(\Box^{(0)})}{G_0}\,T_{\mu\nu} \;=\; T^{vac}_{\mu\nu} \tag{18}$$

with here $T^{vac}_{\mu\nu} \;=\; [p_{vac}(t) + \rho_{vac}(t)]\,u_\mu\,u_\nu + g_{\mu\nu}\,p_{vac}(t)$, and (see Equation (4)), to zeroth order in h,

$$\rho_{vac}(t) = \frac{\delta G(t)}{G_0}\,\bar{\rho}(t) \qquad p_{vac}(t) = w_{vac}\,\frac{\delta G(t)}{G_0}\,\bar{\rho}(t) \tag{19}$$

with $w_{vac} = \frac{1}{3}$. Therefore, in light of the results of [47–49], the problem has been dramatically reduced to just computing the much more tractable expression:

$$-\,\frac{1}{2\,\nu}\,\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h)\cdot T^{vac}_{\mu\nu} \tag{20}$$

Still, in general, the resulting expression for $\frac{1}{\Box^{(0)}}\cdot\Box^{(1)}(h)$ is rather complicated if evaluated for arbitrary functions. Here, we will resort, for lack of better insights, to a treatment where one assumes a harmonic time dependence for the metric trace fluctuation $h(t) \;=\; h_0\,e^{i\omega t}$ and similarly for $a(t) \;=\; a_0\,e^{i\Gamma t}$ and $\rho(t) \;=\; \rho_0\,e^{i\Gamma t}$. In the following, we will assume that the complex functions, $\omega(t)$ and $\Gamma(t)$, can be treated as slowly varying, with the time fluctuation scale in h comparatively smaller than the time scales relevant for the background quantities, a and ρ. One should then be allowed to do a derivative (or Wentzel-Kramers-Brillouin (WKB)) expansion on $\omega(t)$ and $\Gamma(t)$, neglecting in a first approximation the derivative terms. In this way, one can separate fast (h) from slow (a, ρ, δG) modes, so that, for example, $\dot{h}/h \gg \dot{a}/a$. The specific nature of the time dependence of $h(t)$ will then depend on whether the slowly varying functions, Λ and ω, are purely real or imaginary. One useful result in this context is the fact that, as shown below, the combination that is relevant here, namely $(\dot{a}/a)\,(h/\dot{h}$, can be traded for the growth exponent $f(a) = dlogh/dloga$. In the limit $\omega \gg \Gamma$, corresponding to $\dot{h}/h \gg \dot{a}/a$, one finds for the fluctuation, $\delta\rho_{vac}(t)$:

$$\delta\rho_{vac}(t) = \frac{\delta G(t)}{G_0}\,\delta\rho(t) + \frac{1}{2\,\nu}\,c_h\,\frac{\delta G(t)}{G_0}\,h(t)\,\bar{\rho}(t) \tag{21}$$

The $O(h)$ correction factor, c_h, for the tensor box is then found to be:

$$c_h = \frac{11}{3}\,\frac{\dot{a}}{a}\,\frac{h}{\dot{h}} \tag{22}$$

with all other off-diagonal matrix elements vanishing. Furthermore, one finds to this order, but only for the specific choice $w_{vac} = \frac{1}{3}$ in the zeroth order $T_{\mu\nu}^{vac}$, $\delta p_{vac}(t) = \frac{1}{3}\,\delta\rho_{vac}(t)$, *i.e.*, the $O(h)$ correction preserves the original result $w_{vac} = \frac{1}{3}$.

As far as the magnitude of the correction, c_h, in Equation (22), one can argue that from Equation (23), one can relate the combination $(\dot{h}/h)(a/\dot{a})$ to the growth index, $f(a)$,

$$\frac{\dot{h}}{h}\,\frac{a}{\dot{a}} = \frac{\partial \log h(a)}{\partial \log a} = \frac{\partial \log \delta(a)}{\partial \log a} \equiv f(a) \tag{23}$$

where $\delta(a)$ is the matter density contrast and $f(a)$ the known density growth index [50]. Then, in the absence of a running G (which is all that is needed here, to the order one is working), an explicit form for $f(a)$ is known in terms of suitable derivatives of a Gauss hypergeometric function. These can then be inserted into Equation (22). Alternatively, one can make use again of the fact that for a scale factor referring to "today" $a/a_0 \approx 1$ and for a matter fraction $\Omega \approx 0.25$, one knows that $f(a = a_0) \simeq 0.4625$, and thus, in Equation (21), $c_h \simeq (11/3) \times 2.1621 = +7.927$. Furthermore, as an exercise, one can redo the whole calculation in the much simpler scalar box acting on the $T_\lambda{}^\lambda$ case, where one finds the smaller value $c_h \simeq +2.162$.

Finally, one can do the same analysis in the opposite, but less physical, limit $\omega \ll \Gamma$ or $\dot{h}/h \ll \dot{a}/a$. However, this second limit is, in our opinion, less physical, because of the fact that, now, the background is assumed to be varying more rapidly in time than the metric perturbation itself, $\dot{a}/a \gg \dot{h}/h$. Furthermore, one disturbing, but not entirely surprising, general aspect of the whole calculation in this second $\omega \ll \Gamma$ limit is its extreme sensitivity as far as magnitudes and signs of the results are concerned, to the set of assumptions initially made about the time development of the background. For the reasons mentioned, in the following, we will no longer consider this limit of rapid background fluctuations any further.

To summarize, the results for a scalar box and for a very slowly varying background, $\dot{h}/h \gg \dot{a}/a$, give the $O(h)$ corrected expression for $\delta\rho_{vac}(t)$ in Equation (21) and $\delta p_{vac}(t) = w_{vac}\,\delta\rho_{vac}(t)$ with

$c_h \simeq +2.162$, while the tensor box calculation, under essentially the same assumptions, gives the somewhat larger result $c_h \simeq +7.927$. From now on, these will be the only two choices we shall consider here.

The next step in the analysis involves the derivation of the energy-momentum conservation to first order in the fluctuations and a derivation of the relevant field equations to the same order. After that, energy conservation is used to eliminate the h field entirely and, thus, obtain a single equation for the matter density fluctuation, δ. First, we will look here at the implications of energy-momentum conservation, $\nabla^\mu \left(T_{\mu\nu} + T_{\mu\nu}^{vac} \right) = 0$, to first order in the fluctuations. After defining the matter density contrast, $\delta(t)$, as the ratio $\delta(t) \equiv \delta\rho(t)/\bar{\rho}(t)$, the energy conservation equation to first order in the perturbations is found to be:

$$\left[-\frac{1}{2} \left((1+w) + (1+w_{vac}) \frac{\delta G(t)}{G_0} \right) - \frac{1}{2\nu} c_h \frac{\delta G(t)}{G_0} \right] \quad \dot{h}(t)$$
$$+ \quad \left[\frac{1}{2\nu} c_h \left(3\,(w - w_{vac}) \frac{\dot{a}(t)}{a(t)} \frac{\delta G(t)}{G_0} - \frac{\delta \dot{G}(t)}{G_0} \right) \right] \quad h(t) \quad = \quad \left[1 + \frac{\delta G(t)}{G_0} \right] \dot{\delta}(t) \tag{24}$$

In the absence of a running G ($\delta G(t) = 0$), this reduces simply to $-\frac{1}{2}\,(1+w)\,\dot{h}(t) = \dot{\delta}(t)$. This last result then allows us to solve explicitly, at the given order, *i.e.*, to first order in the fluctuations and to first order in δG, for the metric perturbation, $\dot{h}(t)$, in terms of the matter density fluctuation, $\delta(t)$ and $\dot{\delta}(t)$.

Furthermore, to first order in the perturbations, the tt and ii effective field equations become, respectively,

$$\frac{\dot{a}(t)}{a(t)} \dot{h}(t) - 8\pi\, G_0 \frac{1}{2\nu} c_h \frac{\delta G(t)}{G_0} \bar{\rho}(t)\, h(t) = 8\pi\, G_0 \left(1 + \frac{\delta G(t)}{G_0} \right) \bar{\rho}(t)\, \delta(t) \tag{25}$$

and:

$$\ddot{h}(t) + 3\frac{\dot{a}(t)}{a(t)} \dot{h}(t) + 24\pi\, G_0 \frac{1}{2\nu} c_h\, w_{vac} \frac{\delta G(t)}{G_0} \bar{\rho}(t)\, h(t) = -\,24\pi\, G_0 \left(w + w_{vac} \frac{\delta G(t)}{G_0} \right) \bar{\rho}(t)\, \delta(t) \tag{26}$$

In the second ii equation, the zeroth order ii field equation of Equation (6) has been used to achieve some simplification. It is easy to check the overall consistency of the first order energy conservation equation of Equation (24) and of the two field equations given in Equations (25) and (26).

To obtain an equation for the matter density contrast $\delta(t) = \delta\rho(t)/\bar{\rho}(t)$, one needs to eliminate the metric trace field, $h(t)$, from the field equations. This is first done by taking a suitable linear combination of the two field equations in Equations (25) and (26), to get the equivalent equation:

$$\ddot{h}(t) + 2\frac{\dot{a}(t)}{a(t)} \dot{h}(t) \quad + \quad 8\pi\, G_0 \frac{1}{2\nu} c_h \; (1 + 3\, w_{vac}) \frac{\delta G(t)}{G_0} \bar{\rho}(t)\, h(t)$$
$$= \quad -\, 8\pi\, G_0 \left[(1+3\,w) + (1+3\,w_{vac}) \; \frac{\delta G(t)}{G_0} \right] \bar{\rho}(t)\, \delta(t) \tag{27}$$

Then, the first order energy conservation equations to zeroth and first order in δG allow one to completely eliminate the h, $\dot{h}$ and $\ddot{h}$ field in terms of the matter density perturbation, $\delta(t)$, and its derivatives. The resulting equation reads, for $w = 0$ and $w_{vac} = \frac{1}{3}$,

$$\ddot{\delta}(t) + \left[\left(2\,\frac{\dot{a}(t)}{a(t)} - \frac{1}{3}\,\frac{\delta\dot{G}(t)}{G_0} \right) - \frac{1}{2\nu}\cdot 2\,c_h \cdot \left(\frac{\dot{a}(t)}{a(t)}\,\frac{\delta G(t)}{G_0} + 2\,\frac{\delta\dot{G}(t)}{G_0} \right) \right] \dot{\delta}(t)$$
$$+ \left[-4\pi\,G_0 \left(1 + \frac{7}{3}\,\frac{\delta G(t)}{G_0} - \frac{1}{2\nu}\cdot 2\,c_h \cdot \frac{\delta G(t)}{G_0} \right) \bar{\rho}(t) \right.$$
$$\left. - \frac{1}{2\nu}\cdot 2\,c_h \cdot \left(\frac{\dot{a}^2(t)}{a^2(t)}\,\frac{\delta G(t)}{G_0} + 3\,\frac{\dot{a}(t)}{a(t)}\,\frac{\delta\dot{G}(t)}{G_0} + \frac{\ddot{a}(t)}{a(t)}\,\frac{\delta G(t)}{G_0} + \frac{\delta\ddot{G}(t)}{G_0} \right) \right] \delta(t) = 0 \tag{28}$$

This last equation then describes matter density perturbations to linear order, taking into account the running of $G(\Box)$ and is, therefore, the main result of this work. The terms proportional to c_h, which can be clearly identified in the above equation, describe the feedback of the metric fluctuations, h, on the vacuum density, $\delta\rho_{vac}$, and pressure, δp_{vac}, fluctuations.

The above equation can now be compared with the corresponding, much simpler, equation obtained for constant G, *i.e.*, for $G \to G_0$ and still $w = 0$ (see, for example, [50,51]):

$$\ddot{\delta}(t) + 2\,\frac{\dot{a}}{a}\,\dot{\delta}(t) - 4\pi\,G_0\,\bar{\rho}(t)\,\delta(t) = 0 \tag{29}$$

It is common practice at this point to write an equation for the density contrast, $\delta(a)$, as a function not of t, but of the scale factor, $a(t)$. This is done by utilizing simple derivative identities to relate derivatives with respect to t to derivatives with respect to $a(t)$, with $H \equiv \dot{a}(t)/a(t)$ the Hubble constant. This last quantity can be obtained from the zeroth order tt field equation, sometimes written in terms of current density fractions:

$$H^2(a) \equiv \left(\frac{\dot{a}}{a} \right)^2 = \left(\frac{\dot{z}}{1+z} \right)^2 = H_0^2 \left[\Omega\,(1+z)^3 + \Omega_R\,(1+z)^2 + \Omega_\lambda \right] \tag{30}$$

with $a/a_0 = 1/(1+z)$, where z is the red shift and a_0 the scale factor "today". Then, H_0 is the Hubble constant evaluated today, Ω the (baryonic and dark) matter density, Ω_R the space curvature contribution corresponding to a curvature, k, term and Ω_λ the dark energy or cosmological constant part, all again measured today. In the absence of spatial curvature $k = 0$, one has today:

$$\Omega_\lambda \equiv \frac{\lambda}{3\,H_0^2}, \qquad \Omega \equiv \frac{8\,\pi\,G_0\,\bar{\rho}_0}{3\,H_0^2}, \qquad \Omega + \Omega_\lambda = 1 \tag{31}$$

It is convenient at this stage to introduce a parameter, θ, describing the cosmological constant fraction as measured today, $\theta \equiv \Omega_\lambda/\Omega$. While the following discussion will continue with some level of generality, in practice, one is mostly interested in the observationally favored case of a current matter fraction $\Omega \approx 0.25$, for which $\theta \approx 3$. In terms of the parameter, θ, the growing solution to the differential equation for the density contrast, $\delta(a)$, for constant G is:

$$\delta_0(a) \propto a \cdot {}_2F_1 \left(\frac{1}{3}, 1; \frac{11}{6}; -a^3\,\theta \right) \tag{32}$$

where ${}_2F_1$ is the Gauss hypergeometric function. The subscript, 0, in $\delta_0(a)$ is to remind us that this solution is appropriate for the case of constant $G = G_0$. To evaluate the correction to $\delta_0(a)$ coming from the terms proportional to c_a, one sets:

$$\delta(a) \propto \delta_0(a)\,\left[\,1 + c_a\,\mathcal{F}(a)\,\right] \tag{33}$$

and inserts the resulting expression in Equation (28), written now as a differential equation in $a(t)$. One only needs to determine the differential equations for density perturbations, δ, up to first order in the fluctuations, so it will be sufficient to obtain an expression for Hubble constant, H, from the tt component of the effective field equation to zeroth order in the fluctuations, namely the first of Equations (6). One has:

$$H(a) = \sqrt{\frac{8\pi}{3} G_0 \left(1 + \frac{\delta G(a)}{G_0}\right) \bar{\rho}(a) + \frac{\lambda}{3}} \tag{34}$$

with $G(a)$ given in Equation (9) and $\bar{\rho}(a)$ in Equation (10). In this last expression, the exponent is $\gamma_\nu = 3/2\nu \simeq 9/2$ for a matter dominated background universe, although more general choices, such as $\gamma_\nu = 3(1+w)/2\nu$, are possible and should be explored (see the discussion later). Furthermore, $c_a \approx c_t$ if a_0 is identified with a scale factor corresponding to a universe of size ξ; to a good approximation, this corresponds to the universe "today", with the relative scale factor customarily normalized at that time to $a/a_0 = 1$. In [47], it was found that in Equation (5), $c_t \simeq 0.785\, c_0$ in the scalar box case and $c_t \simeq 0.450\, c_0$ in the tensor box case; in the following, we will use the more relevant tensor box value.

After the various substitutions and insertions have been performed, one obtains, after expanding to linear order in a_0, a second order linear differential equation for the correction, $\mathcal{F}(a)$ to $\delta(a)$, as defined in Equation (33). Since this equation looks rather complicated for general $\delta G(a)$, it will not be recorded here, but it is easily obtained from Equation (28) by a sequence of straightforward substitutions and expansions. The resulting equation can then be solved for $\mathcal{F}(a)$, giving the desired density contrast, $\delta(a)$, as a function of the parameter, Ω.

To obtain an explicit solution to the $\delta(a)$ equation, one needs to know the coefficient, c_a, and the exponent, γ_ν, in Equation (9), whose likely values are discussed above and right after the quoted expression for $G(a)$. For the exponent, ν, one has $\nu \simeq \frac{1}{3}$, whereas for the value for c_h, one finds, according to the discussion in the previous section, $c_h \simeq 7.927$ for the tensor box case. Furthermore, one needs at some point to insert a value for the matter density fraction parameter, θ, which, based on current observation, is close to $\theta = (1-\Omega)/\Omega \simeq 3$.

3. Relativistic Growth Index with G($\Box$)

When discussing the growth of density perturbations in classical General Relativity, it is customary at this point to introduce a scale factor-dependent growth index, $f(a)$, defined as:

$$f(a) \equiv \frac{\partial \ln \delta(a)}{\partial \ln a} \tag{35}$$

which is, in principle, obtained from the differential equation for any scale factor, $a(t)$. Nevertheless, here, one is mainly interested in the neighborhood of the present era, $a(t) \approx a_0$. One therefore introduces today's growth index parameter, γ, via:

$$f(a = a_0) \equiv \left.\frac{\partial \ln \delta(a)}{\partial \ln a}\right|_{a=a_0} \equiv \Omega^\gamma \tag{36}$$

The solution of the above differential equation for $\delta(a)$ then determines an explicit value for the growth index, γ, parameter, for any value of the current matter fraction, Ω. In the end, because of observational constraints, one is mostly interested in the range $\Omega \approx 0.25$, so the following discussion will be limited to this case only, although from the original differential equation for $\delta(a)$, one can, in principle, obtain a solution for any sensible Ω.

It is known that in the absence of a running Newton's constant G ($G \to G_0$, thus $c_a = 0$), one has $f(a = a_0) = 0.4625$ and $\gamma = 0.5562$ for the standard ΛCDM scenario with $\Omega = 0.25$ [50]. On the other hand, when the running of $G(\Box)$ is taken into account, one finds from the solution to Equation (28) for the growth index parameter, γ, at $\Omega = 0.25$ the following set of results. For γ, one has:

$$\gamma = 0.5562 - (0.703 + 25.04\, c_h)\, c_a + O(c_a^2) \tag{37}$$

with $c_h = (11/3) \times 2.1621 = 7.927$ in the tensor box case (see Equation (21)) and $c_h = 2.1621$ in the scalar box case. In the Newtonian (non-relativistic) treatment, one finds the much smaller correction:

$$\gamma = 0.5562 - 0.0142\, c_a + O(c_a^2) \tag{38}$$

Among these last expressions, the tensor box case is supposed to give ultimately the correct answer; the scalar box case only serves as a qualitative comparison. The c_h term is responsible for the feedback of the metric fluctuations h on the vacuum density, $\delta\rho_{vac}$, and pressure, δp_{vac}, fluctuations.

To quantitatively estimate the actual size of the correction in the above expressions for the growth index parameter, γ, and to make some preliminary comparison to astrophysical observations, some additional information is needed. The first item is the coefficient $c_0 \approx 8.02 \pm 0.55$ in Equation (1), as obtained from lattice gravity calculations of invariant correlation functions at a fixed geodesic distance [52]. We have re-analyzed and extended the results of [52], which involve rather large uncertainties for this particular quantity; nevertheless, it would seem very difficult to accommodate values for c_0 that are more than an order of magnitude smaller than the quoted value.

The next item that is needed here is a quantitative estimate for the magnitude of the coefficient, c_a, in Equation (9) in terms of c_t in Equation (5) and, therefore, in terms of c_0 in the original Equation (1). First of all, one has $c_a = c_t\,(t_0/\xi)^3$, where t_0 represents the current age of the universe (about 13.82 Gyrs), giving $c_a = 0.646\, c_t$. The factor $(t_0/\xi)^3 \approx 0.646$ accounts for the fact that the two time scales, t_0 ("today") and ξ/c, do not coincide exactly and differ instead by a rather small, but relevant, amount. Regarding the numerical value of the coefficient, c_t, itself, it was found in [47] that in Equation (5), $c_t \simeq 0.785\, c_0$ in the scalar box case and $c_t \simeq 0.450\, c_0$ in the tensor box case. In both cases, these estimates refer to values obtained from the zeroth order covariant effective field equations. In the following, we will take for concreteness the tensor box value, thus $c_t \approx 0.450\, c_0$; then, for all three covariant calculations recorded above $c_a \approx 0.450 \times 8.02 \times (t_0/\xi)^3 \approx 2.33$, a rather large coefficient. From all of these considerations, one would tend to get estimates for the growth parameter, γ, with rather large corrections! For example, in the tensor box case, the corrections would add up to $-199.20\, c_a \approx -464.1$.

Nevertheless, it would seem that one should account somewhere for the fact that the largest galaxy clusters and superclusters studied today up to redshifts $z \simeq 1$ extend for only about, at the very most, $1/20$ the overall size of the visible universe. This would suggest then that the corresponding scale for the running coupling, $G(t)$ or $G(a)$, in Equations (5) and (9), respectively, should be reduced by a suitable ratio of the two relevant length scales, one for the largest observed galaxy clusters or superclusters and the second for the very large, cosmological scale $\xi \sim 1/\sqrt{\lambda/3} \sim 1.51 \times 10^{28}$ cm, entering the expression for $\delta G(\Box)$ in Equations (1) and (2). This would then reduce the overall magnitude of the quantum correction by a factor of $(\mathrm{scale}/\xi)^3$ on slightly smaller scales, with the correction factor attaining unity only on the largest accessible scales, comparable to ξ. Note that this cubic scaling is already implicit in the form of $G(k)$ or $G(\Box)$ given in Equation (1) with $\nu = \frac{1}{3}$. When this suppression is taken properly into account, one finds that the quantum correction to the growth exponent, γ, drops to about 10% on scales of 240 Mpc, to 5% on scales of 190 Mpc and to a rather small 1% effect on scales of about 110 Mpc.

It should also be emphasized here that, so far, all of the above results have been obtained by solving the differential equation for $\delta(a)$ with $G(a)$ given in Equation (9) and exponent $\gamma_\nu = 3/2\nu \simeq 9/2$ relevant for a matter dominated background universe. However, it is this last choice that needs to be critically analyzed, as it might give rise to a definite bias. Our value for γ_ν, so far, reflects our choice of a matter dominated background. More general choices, such as an "effective" $\gamma_\nu = 3(1+w)/2\nu$ with an "effective" w, are, in principle, possible. Then, although Equation (28) for $\delta(t)$ remains unchanged, the differential equation for $\delta(a)$ needs to be solved with new parameters. Therefore, we will discuss here a number of options that should allow one to refine the accuracy of the above result and, in particular, correct the possible shortcomings coming so far from the specific choice of the exponent, γ_ν.

Before we used $a(t) \sim a_0(t/t_0)^{2/3}$ in relating $G(a)$ in Equation (9) to $G(t)$ in Equation (5). In general, if w is not small, one should use the more general equation relating the variable, t, to $a(t)$. The problem here is that, loosely speaking, for $w \neq 0$, at least two ws are involved, $w = 0$ (matter) and $w = -1$ (λ term). This issue considerably complicates the problem of relating $\delta G(t)$ to $\delta G(a)$ and, therefore, the solution to the resulting differential equation for $\delta(a)$ [53–55]. As a tractable approximation, we will set here instead $a(t) \sim a_0(t/t_0)^{2/3(1+w)}$ and, then, use an "effective" value of $w \approx -7/9$, which would seem more appropriate for the final target value of a matter fraction $\Omega \approx 0.25$. For this choice, one then obtains a significantly reduced power in Equation (9), namely $\gamma_\nu = 3(1+w)/2\nu = 1$. Furthermore, the resulting differential equation for $\delta(a)$ is still relatively easy to solve, by the same methods used in the previous section.

One then obtains for the growth index parameter, γ, at $\Omega = 0.25$ the following set of results:

$$\gamma = 0.5562 - (0.920 + 7.70\, c_h)\, c_a + O(c_a^2) \tag{39}$$

with, again as before, $c_h = (11/3)\times 2.1621 = 7.927$ in the tensor box case. Using again $c_a = (t_0/\xi)^3 c_t = 0.646 c_t$ and $c_t \simeq 0.450\, c_0$ for the tensor box case, with $c_0 \approx 8.02$ from the lattice calculation, one has, as before, $c_a \approx 2.33$. One then obtains for the growth parameter, γ, a quantum correction of magnitude $-61.96\, c_a \approx -144.4$. The latter number is the correction on the largest possible scales, comparable to the length scale, ξ. On smaller scales, a suppression factor, $(\mathrm{scale}/\xi)^3$, needs to be taken into account. When this factor is included, one finds that the quantum correction to the growth exponent, γ, drops to about 10% on scales of 360 Mpc, to 5% on scales of 280 Mpc, and to a rather small 1% effect on scales of about 170 Mpc. Note that these numbers are not very different from what was obtained earlier for the case of pure matter $w = 0$. When the cosmological constant term is taken into account in Ω, one needs to go to slightly larger scales to get that same size correction to the growth exponent.

As a practical example, consider the galaxy clusters studied recently in [56–60], which typically involve comoving radii of ~8.5 Mpc and viral radii of ~1.4 Mpc. For these, one would obtain an approximate overall scale reduction factor of $(8.5/4890)^3 \approx 5.3 \times 10^{-9}$. This would give for the tensor box ($c_h = 7.927$) correction to the growth index, γ, in Equation (39) the still rather small order of magnitude estimate, $\sim 10^{-6}$. To clearly see the corrections, larger clusters will be needed. Nevertheless this last case is suggestive of a trend, quite independent of the specific value of c_h and, therefore, of the overall numerical coefficient of the correction in Equation (39): namely, that the correction to the growth index parameter is expected to be negative and will gradually and quite rapidly increase with the size of the cluster. In view of these results, it would seem rather worthwhile to further investigate, from an observational point of view, what happens on very large astrophysical scales, comparable to either H_0^{-1} or ξ.

4. Conclusions

In this work, we have presented some preliminary results on the evolution of matter density perturbations in quantum-gravity motivated models, where Newton's constant, $G(\Box)$, is scale dependent, due to strong infrared quantum radiative corrections. Such a scale dependence arises naturally in the non-perturbative treatment of Euclidean quantum gravity via the covariant path integral approach, in close analogy to what happens in other perturbatively non-renormalizable theories, such as the non-linear sigma model. The relevant scaling violation scale, ξ, similar to the $\Lambda_{\overline{MS}}$ of QCD, is seen here to be naturally related to the infrared cutoff, H_0^{-1}, or perhaps, more appropriately (since the latter is time dependent), to the scaled cosmological constant, λ, via $\xi \sim 1/\sqrt{\lambda/3}$. As in non-abelian gauge theories and QCD, the infrared cutoff scale is, in turn, expected to be connected, by scaling considerations, to a non-perturbative gravitational condensate:

$$\langle R \rangle = 4\lambda = \frac{12}{\xi^2} \tag{40}$$

From all of the above considerations, it should be clear that the results presented in this paper are completely different from what one would expect based on a naive perturbative treatment of quantum gravity.

Acknowledgments

One of the authors (Herbert W. Hamber) wishes to thank Thibault Damour and Gabriele Veneziano for inspiration and discussions leading to the present work and Alexey Vikhlinin for correspondence regarding the astrophysical measurements of growth indices. He also wishes to thank Hermann Nicolai and the Max Planck Institut für Gravitationsphysik (Albert Einstein Institut) in Potsdam for warm hospitality. The work of Herbert W. Hamber was supported in part by the Max Planck Gesellschaft zur Förderung der Wissenschaften and by the University of California. The work of Reiko Toriumi was supported in part by a Department of Education fellowship.

Author Contributions

Each author has contributed in a significant way to the reported research and to the writing of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

References

1. Damour, T. Experimental Tests of Gravitational Theory. Available online: http://pdg.lbl.gov/2009/reviews/rpp2009-rev-gravity-tests.pdf (accessed on 11 November 2009).
2. Yao, W.-M.; Particle Data Group. Review of Particle Physics. *J. Phys. G Nucl. Part. Phys.* **2006**, *33*, doi:10.1088/0954-3899/33/1/001.
3. Damour, T.; Nordtvedt, K. General relativity as a cosmological attractor of tensor scalar theories. *Phys. Rev. Lett.* **1993**, *70*, 2217–2219.
4. Damour, T.; Esposito-Farese, G. Testing gravity to second postNewtonian order: A Field theory approach. *Phys. Rev. D* **1996**, *53*, doi.org/10.1103/PhysRevD.53.5541.
5. Veneziano, G. String cosmology: The Pre-Big Bang Scenario. *ArXiv E-Prints*, **2000**, arXiv:hep-th/0002094.
6. Gasperini, M.; Veneziano, G. String Theory and Pre-Big Bang Cosmology. *ArXiv E-Prints*, **2007**, arXiv:hep-th/0703055.
7. Israel, W.; Hawking, S.W. *General Relativity—An Einstein Centenary Survey*; Cambridge University Press: Cambridge, UK, 1979.
8. Hamber, H.W. *Quantum Gravitationm - The Feynman Path Integral Approach*; Springer: Berlin, Germany, 2009.
9. Hamber, H.W. Quantum Gravity on the Lattice. *Gen. Relativ. Gravit.* **2009**, *41*, 817–876.
10. Hamber, H.W. Ultraviolet Divergences and Scale-Dependent Gravitational Couplings. In Proceedings of the 12th Marcel Grossmann Conference on Recent Developments in General Relativity, Astrophysics and Relativistic Field Theories (MG-12), Paris, France, 12–18 July 2009.
11. Hamber, H.W.; Williams, R.M. Higher Derivative Quantum Gravity on A Simplicial Lattice. *Nucl. Phys. B* **1984**, *248*, 392–414.
12. Simplicial Quantum Gravity with Higher Derivative Terms: Formalism and Numerical Results in Four-Dimensions. *Nucl. Phys. B* **1986**, *269*, 712–743.
13. Hamber, H.W. Nonperturbative Simplicial Quantum Gravity. *Phys. Lett. B* **1985**, *157*, 368–374.
14. Hamber, H.W.; Williams, R.M. Newtonian potential in quantum Regge gravity. *Nucl. Phys. B* **1995**, *435*, 361–397.
15. Hamber, H.W. Phases of four-dimensional simplicial quantum gravity. *Phys. Rev. D* **1992**, *45*, doi:10.1103/PhysRevD.45.507.
16. Hamber, H.W. Phases of Simplicial Quantum Gravity in Four Dimensions: Estimates for the Critical Exponents. *Nucl. Phys. B* **1993**, *400*, 347–389.
17. Hamber, H.W. Gravitational scaling dimensions. *Phys. Rev. D* **2000**, *61*, doi:10.1103/PhysRevD.61.124008.
18. Kawai, H. Renormalization group and quantum gravity. *Nucl. Phys. B* **1990**, *336*, 115–145.
19. Aida, T.; Kitazawa, Y. Two-loop prediction for scaling exponents in $(2 + \epsilon)$-dimensional quantum gravity. *Nucl. Phys. B* **1997**, *491*, 427–458.

20. Reuter, M. Nonperturbative Evolution Equation for Quantum Gravity. *Phys. Rev. D* **1998**, *57*, doi:10.1103/PhysRevD.57.971.
21. Manrique, E.; Reuter, M.; Saueressig, F. Bimetric Renormalization Group Flows in Quantum Einstein Gravity. *Ann. Phys.* **2011**, *326*, 463–485.
22. Vilkovisky, G.A. *Quantum Theory of Gravity*; Christensen, S., Ed.; Adam Hilger: Bristol, UK, 1984.
23. Vilkovisky, G.A. The Unique Effective Action in Quantum Field Theory. *Nucl. Phys. B* **1984**, *234*, 125–137.
24. Taylor, T.R. ; Veneziano, G. Quantum Gravity at Large Distances and the Cosmological Constant. *Nucl. Phys. B* **1990**, *345*, 210–230.
25. Taylor, T.R. ; Veneziano, G. Quenching the Cosmological Constant. *Phys. Lett. B* **1989**, *228*, 311–316.
26. Wilson, K.G. Renormalization group and critical phenomena. I. Renormalization group and the Kadanoff scaling picture. *Phys. Rev. B* **1971**, *4*, doi:10.1103/PhysRevB.4.3174.
27. Wilson, K.G. Renormalization group and critical phenomena. II. Phase space cell analysis of critical behavior. *Phys. Rev. B* **1971**, *4*, doi:10.1103/PhysRevB.4.3184.
28. Wilson, K.G. Feynman-graph expansion for critical exponents. *Phys. Rev. Lett.* **1972**, *28*, doi:10.1103/PhysRevLett.28.548.
29. Wilson, K.G. Quantum field-theory models in less than 4 dimensions. *Phys. Rev. D* **1973**, *7*, doi:10.1103/PhysRevD.7.2911.
30. Wilson, K.G. The Renormalization Group: Critical Phenomena and the Kondo Problem. *Rev. Mod. Phys.* **1975**, *47*, doi:10.1103/RevModPhys.47.773.
31. Wilson, K.G. The renormalization group and critical phenomena. *Rev. Mod. Phys.* **1983**, *55*, 583–600.
32. Parisi, G. *Statistical Field Theory*; Benjamin Cummings: San Francisco, CA, USA, 1981.
33. Itzykson, C.; Drouffe, J.-M. *Statistical Field Theory*; Cambridge University Press: Cambridge, UK, 1991.
34. Zinn-Justin, J. *Quantum Field Theory and Critical Phenomena*, 4th ed.; Oxford University Press: Oxford, UK, 2002.
35. Cardy, J. *Scaling and Renormalization in Statistical Physics (Cambridge Lecture Notes in Physics)*; Cambridge University Press: Cambridge, UK, 1996.
36. Brezin, E. *Introduction to Statistical Field Theory*; Cambridge University Press: Cambridge, UK, 2010.
37. Brezin, E.; Zinn-Justin, J. Renormalization of the nonlinear σ model in $2 + \epsilon$ dimensions—Application to the Heisenberg ferromagnets. *Phys. Rev. Lett.* **1976**, *36*, doi:10.1103/PhysRevLett.36.691.
38. Brezin, E.; Zinn-Justin, J.; le Guillou, J.C. Renormalization of the Nonlinear σ Model in $2 + \epsilon$ Dimensions. *Phys. Rev. D* **1976**, *14*, doi:10.1103/PhysRevD.14.2615 .
39. Brezin, E.; le Guillou, J.C.; Zinn-Justin, J. *Phase Transitions and Critical Phenomena*; Domb, C., Green, M.S., Eds.; Academic Press: Waltham, MA, USA, 1976; Volume 6.

40. Guida, R.; Zinn-Justin, J. Critical Exponents of the *N*-vector Model. *J. Phys. A Math. Gen.* **1998**, *31*, doi:10.1088/0305-4470/31/40/006.
41. Gross, D.J. Applications of the Renormalization Group to High-energy Physics. In Proceedings of the Methods in Field Theory, Les Houches, France, 28 July–6 September 1975.
42. Hägler, P. Hadron Structure from Lattice Quantum Chromodynamics. *Phys. Rep.* **2010**, *490*, 49–175.
43. Fodor, Z.; Hoelbling, C. Light Hadron Masses from Lattice QCD. *Rev. Mod. Phys.* **2012**, *84*, doi:10.1103/RevModPhys.84.449.
44. Hamber, H.W.; Williams, R.M. Quantum Gravity in Large Dimensions. *Phys. Rev. D* **2006**, *73*, doi:10.1103/PhysRevD.73.044031.
45. Hamber, H.W.; Williams, R.M. Gravitational Wilson Loop and Large Scale Curvature. *Phys. Rev. D* **2007**, *76*, doi:10.1103/PhysRevD.76.084008.
46. Hamber, H.W.; Williams, R.M. Gravitational Wilson Loop in Discrete Quantum Gravity. *Phys. Rev. D* **2010**, *81*, doi:10.1103/PhysRevD.81.084048.
47. Hamber, H.W.; Williams, R.M. Nonlocal Effective Gravitational Field Equations and the Running of Newton's Constant *G*. *Phys. Rev. D* **2005**, *72*, doi:10.1103/PhysRevD.72.044026.
48. Hamber, H.W.; Williams, R.M. Constraints on Gravitational Scaling Dimensions from Non-Local Effective Field Equations. *Phys. Lett. B* **2006**, *643*, 228–234.
49. Hamber, H.W.; Williams, R.M. Renormalization Group Running of Newton's G: The Static Isotropic Case. *Phys. Rev. D* **2007** *75*, doi:10.1103/PhysRevD.75.084014.
50. Peebles, P.J.E. *Principles of Physical Cosmology*; Princeton University Press: Princeton, NJ, USA, 1993.
51. Weinberg, S. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*; John Wiley & Sons: Hoboken, NJ, USA, 1972.
52. Hamber, H.W. Invariant Correlations in Simplicial Gravity. *Phys. Rev. D* **1994**, *50*, 3932–3941.
53. Hamber, H.W.; Toriumi, R. Cosmological Density Perturbations with a Scale-Dependent Newton's constant *G*. *Phys. Rev. D* **2010**, *82*, doi:10.1103/PhysRevD.82.043518.
54. Hamber, H.W.; Toriumi, R. Scale-Dependent Newton's Constant *G* in the Conformal Newtonian Gauge. *Phys. Rev. D*, **2011**, *84*, doi:10.1103/PhysRevD.84.103507.
55. Hamber, H.W.; Toriumi, R. Inconsistencies from a Running Cosmological Constant. *Int. J. Mod. Phys. D* **2013**, *22*, doi:10.1142/S0218271813300231.
56. Schmidt, F.; Vikhlinin. A.; Hu, W. Cluster Constraints on f(R) Gravity. *Phys. Rev. D* **2009**, *80*, doi:10.1103/PhysRevD.80.083505.
57. Vikhlinin, A.; Murray, S.; Gilli, R.; Tozzi, P.; Paolillo, M.; Brandt, N.; Tagliaferri, G.; Bautz, M.; Allen, S.; Donahue, M.; *et al.* X-ray Cluster Cosmology. *ArXiv E-Prints*, **2009**, arXiv:0903.5320.
58. Vikhlinin, A.; Allen, S.W.; Arnaud, M.; Bautz, M.; Boehringer, H.; Bonamente, M.; Burns, J.; Evrard, A.; Henry, J.P.; Jones, C.; *et al.* Cosmological Studies With a Large-Area X-ray Telescope. *ArXiv E-Prints*, **2009**, arXiv:0903.2297.

59. Rapetti, D.; Allen, S.W.; Mantz, A.; Ebeling, H. The Observed Growth of Massive Galaxy Clusters III: Testing General Relativity on Cosmological Scales. *ArXiv E-Prints*, **2010**, arXiv:0911.1787.
60. Uzan, J.-P. Tests of General Relativity on Astrophysical Scales. *ArXiv E-Prints*, **2009**, arXiv:0908.2243.