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Nonlocal effective gravitational field equations and the running of Newton's constant G

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Nonperturbative studies of quantum gravity have recently suggested the possibility that the strength of gravitational interactions might slowly increase with distance. Here a set of generally covariant effective field equations are proposed, which are intended to incorporate the gravitational, vacuum-polarization induced, running of Newton's constant G. One attractive feature of this approach is that, from an underlying quantum gravity perspective, the resulting long-distance (or large time) effective gravitational action inherits only one adjustable parameter ξ, having the units of a length, arising from dimensional transmutation in the gravitational sector. Assuming the above scenario to be correct, some simple predictions for the long-distance corrections to the classical standard model Robertson-Walker metric are worked out in detail, with the results formulated as much as possible in a model-independent framework. It is found that the theory, even in the limit of vanishing renormalized cosmological constant, generally predicts an accelerated power-law expansion at later times $t \sim \xi \sim 1/H$.

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I. INTRODUCTION

Nonperturbative studies of quantum gravity have recently suggested the possibility that gravitational couplings might be weakly scale dependent due to nontrivial renormalization-group effects. This would introduce a new gravitational scale, unrelated to Newton's constant, required in order to parametrize the gravitational running in the infrared region. If one is willing to accept such a scenario, then it seems difficult to find a compelling theoretical argument for why the nonperturbative scale entering the coupling evolution equations should be very small, comparable to the Planck length. One possibility put forward recently is that the relevant nonperturbative scale is related to the curvature and therefore macroscopic in size, which could have observable consequences. One key ingredient in this argument is the relationship, to some extent supported by Euclidean lattice results combined with renormalization-group arguments, between the scaling violation parameter and the scale of the average curvature. Irrespective of the specific details of a gravitational theory at very short distances, such results would bring gravitation more in line with the rest of the standard model, where all gauge couplings are in fact known to run.

In this paper we investigate the effects of a running gravitational coupling G at large distances, with as few assumptions as possible about the ultimate behavior of the theory at extremely short distances, where several possible scenarios include a string cutoff at length scales $\lambda_S = (2\pi\alpha')^{1/2}$ [1], the appearance of higher derivative terms (either as direct contributions or as radiative corrections), or perhaps a—somewhat less appealing—explicit ultraviolet cutoff at the Planck scale. The running of the gravitational coupling will generally be assumed to be driven by graviton vacuum-polarization effects, which produce an antiscreening effect some distance away from the primary source, and therefore tend to increase the strength of the gravitational coupling. The above scenario is quite different from what one would expect, for example, in supergravity theories, where significant cancellations arise in perturbation theory between graviton and matter loops [2], and in contrast to ordinary gravity where in weak field perturbation theory L loops contribute $L + 1$ powers of the curvature tensor to the effective action [3]. Instead, the running of Newton's constant is thought to arise due to the presence of a nontrivial, genuinely nonperturbative, ultraviolet fixed point [4–6] (a phase transition in statistical mechanics parlance [4]).

In this paper a power-law (as opposed to a logarithmic) running of G will be implemented via manifestly covariant nonlocal terms in the effective gravitational action and field equations. It ultimately will involve the inverse of the covariant d'Alembertian raised to some fractional power $1/2\nu$, which in the framework of the present paper remains largely unspecified, although nonperturbative models for quantum gravity have recently put forward some rather specific predictions.

Let us recall here, to provide some degree of motivation, the recent discussions of [7,8] as a possible theoretical framework for the running of Newton's G. The above results suggest that the gravitational constant G cannot be regarded a constant as in the classical theory, but instead changes slowly with scale due to the presence of weak

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gravitational vacuum-polarization effects, in a way described by

$$G(r) = G(0)[1 + c_\xi (r/\xi)^{1/\nu} + O((r/\xi)^{2/\nu})]. \tag{1.1}$$

The exponent ν, generally related to the derivative of the beta function for pure gravity evaluated at the nontrivial ultraviolet fixed point via the relation $\beta'(G_c) = -1/\nu$, is supposed to universally characterize the long-distance properties of quantum gravitation, and is therefore expected to be independent of the specifics related to the nature of the ultraviolet regulator, or other detailed short-distance features of the theory.[1]

Recent estimates for the value of the universal scaling dimension $\nu^{-1} = -\beta'(G_c)$ derived from nonperturbative studies of gravity vary from $\nu^{-1} \approx 3.0$ [8] in the Euclidean Regge lattice case, to $\nu^{-1} \approx 3.8$ in the $2 + \epsilon$ expansion [10] about two dimensions carried to two loops [11–14], to $\nu^{-1} \approx 2.7$ [15] and $\nu^{-1} \approx 1.7$ [16] in an approximate renormalization-group treatment a la Wilson based on an Einstein-Hilbert truncation, with some significant uncertainties in all three approaches. More details, as well as a systematic comparison of the various methods and estimates, can be found in [17], where we argued, based on geometric arguments, in favor of the exact value of the exponent $\nu = 1/3$ for pure gravity in four dimensions, and $O(1/(d-1))$ for large d. It is perhaps a testament to how far these calculations have progressed that actual numbers have emerged which can meaningfully be compared between different (lattice and continuum) approaches. It should also be noted that, from a quantum gravity perspective, there are really no adjustable parameters in Eq. (1.1), except for the new nonperturbative curvature scale ξ: both c_ξ and ν are in principle finite and calculable numbers.

The mass scale $m = \xi^{-1}$ in Eq. (1.1) is supposed to determine the magnitude of quantum deviations from the classical theory, and separates the short-distance, ultraviolet regime with characteristic momentum scale $\mu \ll m$, where nonperturbative quantum corrections are negligible, from the long-distance regime where quantum corrections become significant. The magnitude of ξ itself involves, in a rather nontrivial way, the dimensionless bare coupling G, the fixed point value G_c, and the ultraviolet cutoff Λ,

$$\xi^{-1} \propto \Lambda \exp\left(-\int^G \frac{dG'}{\beta(G')}\right) \sim_{G\to G_c} \Lambda |G - G_c|^{-1/\beta'(G_c)}. \tag{1.2}$$

Ultimately to make progress and determine the actual physical value for the nonperturbative scale ξ some physical input is needed, as the underlying theory *cannot* fix it (the ratio of the physical Newton's constant to ξ^2 can be as small as one desires, provided the bare coupling G is very close to its fixed point value G_c). It seems natural to identify $1/\xi^2$ with either some very large average spatial curvature scale, or perhaps more appropriately with the Hubble constant (as measured today) determining the macroscopic expansion rate of the Universe via the correspondence

$$\xi = 1/H, \tag{1.3}$$

in a system of units for which the speed of light equals one.[2]

Let us briefly digress here, and recall that in non-Abelian $SU(N)$ gauge theories a similar set of results is known to hold for the renormalization-group induced running of the gauge coupling g, so it will be instructive to draw further on the analogy with QCD, and non-Abelian gauge theories in general. Of course one crucial difference between gravity and ordinary gauge theories lies in the fact that, in the latter case, the evolution of the coupling constant can be systematically computed in perturbation theory due to asymptotic freedom, a statement which is known to reflect the fact that such theories become noninteracting at short distances, up to logarithmic corrections, making perturbation theory consistently applicable. It is well known that for weak enough gauge coupling in $SU(N)$ gauge theories one has

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(\Lambda_{\overline{MS}})} + 2\beta_0 \log\left(\frac{\mu}{\Lambda_{\overline{MS}}}\right) + \cdots \tag{1.4}$$

with β_0 the coefficient of the lowest order term in the beta function, $\mu = 1/r$ an arbitrary momentum scale, $\Lambda_{\overline{MS}} \approx 220$ MeV a nonperturbative scale parameter, and the dots denoting higher loop effects. Instead of the $\Lambda_{\overline{MS}}$ parameter one could just as well use some other physical scale, such as the inverse of the gauge correlation length, $m_{0^{++}} = \xi^{-1}$, where the 0^{++} denotes the lowest glueball state (the Slavnov-Taylor identities prevent of course the gluon from acquiring a mass to any order in perturbation theory). For the purpose of comparing to gravity, one should perhaps emphasize that confining non-Abelian gauge theories such as QCD do not, and cannot, directly determine the scale $\Lambda_{\overline{MS}}$, which needs to be ultimately fixed by experiment from say a direct measurement of the size of scaling violations. Its magnitude involves in a nontrivial way the bare gauge coupling g and the ultraviolet cutoff Λ,

$$\Lambda_{\overline{MS}} \propto \Lambda \exp\left(-\int^g \frac{dg'}{\beta(g')}\right) \tag{1.5}$$

which is very much analogous to Eq. (1.2). The correspon-

[1]Already in ordinary Einstein gravity one finds for very short distances $r \sim l_P$ corrections to the static potential, which can be computed perturbatively [9]. In general for such short distances string corrections and/or higher derivative terms should be considered as well.

[2]A possible scenario is one in which $\xi^{-1} = H_\infty = \lim_{t\to\infty} H(t) = \sqrt{\Omega_\Lambda} H_0$ with $H_\infty^2 = \frac{8\pi G}{3}\lambda = \frac{\Lambda}{3}$, where λ is the observed cosmological constant, and for which the horizon radius is $R_\infty = H_\infty^{-1}$.

dence with QCD and non-Abelian gauge theories would therefore suggest $\xi^{-1} \leftrightarrow \Lambda_{\overline{MS}}$, with the gravitational ξ a new nonperturbative scale, ultimately also to be determined from experiment.

Although not always necessarily advantageous (most perturbative calculations, being based on Feynman diagrams, are eventually done in momentum space and do not seem to benefit significantly from this approach), the running of the gauge coupling g can be reformulated in terms of an effective action, involving the d'Alembertian acting on functions of the field strength. One sets

$$\frac{1}{g^2(\Box)} = \frac{1}{g^2(\Lambda_{\overline{MS}})} + \beta_0 \log\left(\frac{\Box}{\Lambda^2_{\overline{MS}}}\right) + \cdots \tag{1.6}$$

with $2\beta_0 = (11N - 2n_f)/(24\pi^2)$ for non-Abelian $SU(N)$ gauge theories with n_f massless fermion flavors, and with the log of the d'Alembertian $\Box$ suitably defined, for example, via

$$\log\left(\frac{\Box}{\mu^2}\right) = \int_0^\infty dm^2 \left\{\frac{1}{m^2 + \mu^2} - \frac{1}{m^2 + \Box}\right\} \tag{1.7}$$

leading to a one-loop corrected effective action of the form

$$I_{\text{eff}} = \frac{1}{4} \int dx F_{\mu\nu}(x) \left(\frac{1}{g_0^2} + \beta_0 \log\left(\frac{\Box}{\mu^2}\right) + \cdots\right) F^{\mu\nu}(x) \tag{1.8}$$

with μ an appropriately chosen mass scale [18].

In the gravitational case the corrections described by Eq. (1.1) have a more complicated structure, and, in particular, are no longer logarithmic. But they can be viewed, for example, as arising from a resummation of an infinite number of loop logarithms, as in the expansion

$$\sum_{n=0}^\infty \frac{1}{n!}\left(-\frac{1}{2\nu}\right)^n (\log \xi^2 \Box)^n = \left(\frac{1}{\xi^2 \Box}\right)^{1/2\nu}. \tag{1.9}$$

In the next section we shall describe how the renormalization-group induced running of the gravitational constant can be implemented in a simple way via a nonlocal set of manifestly covariant correction terms arising in the effective, long-distance gravitational field equations. These effective equations can then be used as a basis for a systematic discussion of various quantum corrections to the standard solutions of the classical field equations.

II. EFFECTIVE GRAVITATIONAL ACTION AND EFFECTIVE FIELD EQUATIONS

In general terms, a quantum-mechanical running of the gravitational coupling implies the replacement

$$G \to G(r) \tag{2.1}$$

in classical physical observables. This is easier said than done, as in gravity the r in the running coupling $G(r)$ is coordinate dependent, and as such can lead to considerable ambiguities regarding the interpretation of exactly which distance r is involved. A more satisfactory approach would replace $G(r)$ in the gravitational action

$$I = \frac{1}{16\pi G} \int dx \sqrt{g} R \tag{2.2}$$

with a manifestly covariant object, intended to correctly represent an invariant distance, and incorporating the running of G as expressed in Eq. (1.1),

$$\to \frac{1}{16\pi G} \int dx \sqrt{g} \left(1 - c_\Box \left(\frac{1}{\xi^2 \Box}\right)^{1/2\nu} + O((\xi^2 \Box)^{-1/\nu})\right) R \tag{2.3}$$

with the covariant d'Alembertian operator $\Box$ defined through an appropriate combination of covariant derivatives

$$\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu. \tag{2.4}$$

Multiplication by the coordinate r gets therefore replaced by the action of $\Box$, whose Green's function in D space-time dimensions is known to behave as

$$< x | \frac{1}{\Box} | y > \delta(r - d(x, y|g)) \sim \frac{1}{r^{D-2}}. \tag{2.5}$$

Here d would be the distance along a minimal path $z^\mu(\tau)$ connecting the points x and y in a fixed background geometry characterized by the metric $g_{\mu\nu}$, and given by

$$d(x, y|g) = \int_{\tau(x)}^{\tau(y)} d\tau \sqrt{g_{\mu\nu}(z) \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau}}. \tag{2.6}$$

As a result $1/\Box$ can be envisioned as a coordinate independent way of defining consistently what is meant by r in the running of $G(r)$,

$$G(r) \to G(\Box). \tag{2.7}$$

The above prescription has in fact been used successfully for some time to systematically incorporate the effects of radiative corrections in an effective action formalism [19–21]. It should be noted that the coefficient c_ξ in Eq. (1.1) is expected to be a calculable number of order one, but not necessarily the same as the coefficient $c_\Box$, as r and $1/\sqrt{\Box}$ are clearly rather different entities to begin with.[3]

One should recall here that in general the form of the covariant d'Alembertian operator $\Box$ depends on the specific tensor nature of the object it is acting on,

$$\Box T^{\alpha\beta\ldots}{}_{\gamma\delta\ldots} = g^{\mu\nu} \nabla_\mu (\nabla_\nu T^{\alpha\beta\ldots}{}_{\gamma\delta\ldots}). \tag{2.8}$$

[3]In the lattice theory c_ξ was originally estimated from the invariant curvature correlations at around $c_\xi \approx 0.01$, while more recently it was estimated at $c_\xi \approx 0.06$ from the correlation of Wilson lines [17]. It is important to note that while the exponent ν is universal, c_ξ in general depends on the specific choice of regularization scheme (i.e. lattice regularization versus dimensional regularization or momentum subtraction scheme).

Thus on scalar functions one obtains the fairly simple result

$$\Box S(x) = \frac{1}{\sqrt{g}} \partial_\mu g^{\mu\nu} \sqrt{g} \partial_\nu S(x) \tag{2.9}$$

whereas on second rank tensors one has the significantly more complicated expression $\Box T_{\alpha\beta} \equiv g^{\mu\nu} \nabla_\mu (\nabla_\nu T_{\alpha\beta})$. Furthermore one should recognize that the form for the effective gravitational action of Eq. (2.3) is possibly not unique. A more integration-by-parts symmetric expression would be, for example,

$$I = \frac{1}{16\pi G} \int dx \sqrt{g} \sqrt{R} \left(1 - c\left(\frac{1}{\xi^2 \Box}\right)^{1/2\nu} + \ldots\right) \sqrt{R}. \tag{2.10}$$

In general the covariant operator appearing in the above expression, namely

$$A(\Box) = c\left(\frac{1}{\xi^2 \Box}\right)^{1/2\nu} \tag{2.11}$$

has to be suitably defined by analytic continuation from positive integer powers. The latter can be done either by computing $\Box^n$ for positive integer n and then analytically continuing to $n \to -1/2\nu$, or alternatively by making use of the identity

$$\frac{1}{\Box^n} = \frac{(-1)^n}{\Gamma(n)} \int_0^\infty ds\, s^{n-1} \exp(is\Box) \tag{2.12}$$

and subsequent use of the Schwinger-DeWitt representation for the kernel $\exp(is\Box)$ of the massless operator $\Box$. Within the limited scope of this paper, we will be satisfied with computing the effects of positive integer powers n of the covariant d'Alembertian $\Box$, and then analytically continue the answer to fractional $n = -1/2\nu$. In the following the above analytic continuation from positive integer n will always be understood.[4]

It should be stressed here that the action in Eq. (2.10) should be treated as a *classical* effective action, with dominant radiative corrections at short distances ($r \ll \xi$) already automatically built in, and for which a restriction to generally smooth field configurations does make some sense. In particular one would expect that in most instances it should be possible, as well as meaningful, to neglect terms involving large numbers of derivatives of the metric in order to compute the effects of the new contributions appearing in the effective action.[5]

A number of useful results can already be obtained from the form of the effective action in Eq. (2.10). In particular, once a specific metric is chosen, the running of G can be readily expressed in terms of the coordinates appropriate for that metric. Later in this work we will illustrate extensively this statement for the specific, and physically relevant, case of the Robertson-Walker (RW) metric.

The next major step involves a derivation of the effective field equations, incorporating the running of G. As will be shown below this is not entirely straightforward, as the variation of the nonlocal effective action is complicated by the presence of a differential operator raised to a fractional power, acting on what are rather complicated functions of the metric to begin with. We shall therefore postpone a discussion of this aspect to the Appendix, which focuses on this specific topic.

Had one *not* considered the action of Eq. (2.10) as a starting point for constructing the effective theory, one would naturally be led (following Eq. (2.7)) to consider the following effective field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G (1 + A(\Box)) T_{\mu\nu}, \tag{2.13}$$

the argument again being the replacement $G(r) \to G(1 + A(\Box))$ involving the invariant object $\Box$. Here, following common notation, Λ is the scaled cosmological constant, not to be confused with the ultraviolet cutoff. Being manifestly covariant, these expressions at least satisfy some of the requirements for a set of consistent field equations incorporating the running of G. The above effective field equation can then be easily recast in a form similar to the classical field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G \tilde{T}_{\mu\nu} \tag{2.14}$$

with $\tilde{T}_{\mu\nu} = (1 + A(\Box)) T_{\mu\nu}$ defined as an effective, or *gravitationally dressed*, energy-momentum tensor. Just like the ordinary Einstein gravity case, in general $\tilde{T}_{\mu\nu}$ might not be covariantly conserved *a priori*, $\nabla^\mu \tilde{T}_{\mu\nu} \neq 0$, but ultimately the consistency of the effective field equations *demands* that it be exactly conserved in consideration of the Bianchi identity satisfied by the Riemann tensor. The ensuing new covariant conservation law

$$\nabla^\mu \tilde{T}_{\mu\nu} \equiv \nabla^\mu [(1 + A(\Box)) T_{\mu\nu}] = 0 \tag{2.15}$$

can be then be viewed as a *constraint* on $\tilde{T}_{\mu\nu}$ (or $T_{\mu\nu}$)

[4]We notice in passing that in this approach it is not obvious how to formulate a running cosmological constant, as the d'Alembertian $\Box$ in $\lambda(r) \int dx \sqrt{g} \to \lambda \int dx \sqrt{g} (1 - c(1/\xi^2 \Box)^\gamma)$ has no function of the metric left to act on [22]. This situation is not entirely surprising as, lacking derivatives, the effect of the λ term is just to control the overall scale. In pure lattice gravity the bare λ is trivially scaled out and does not run [8,17]. In this scenario the physical long-distance cosmological constant $\sim 1/\xi^2$, being related to an average curvature, is considered a physical quantity to be kept fixed as the gravitational coupling $G(r)$ slowly evolves with scale.

[5]Dominant contributions to the original Feynman path integral for the underlying quantum gravity theory are, on the other hand, presumably nowhere differentiable, the smooth configurations having ultimately zero measure in the gravitational functional integral [23]. Furthermore, issues related to causality, unitarity, and positivity are better referred to the original, local microscopic action, which presumably shares all of these properties.

which, for example, in the specific case of a perfect fluid, will imply again a definite relationship between the density $\rho(t)$, the pressure $p(t)$, and the RW scale factor $R(t)$, just as it does in the standard case.

This point is sufficiently important that we wish to elaborate on it further. In *ordinary* Einstein gravity the energy-momentum tensor is *defined* via the variation of the matter action

$$\delta I_M = \frac{1}{2}\int dx\sqrt{g}\delta g_{\mu\nu}T^{\mu\nu}. \tag{2.16}$$

But when the above arbitrary variation $\delta g_{\mu\nu}$ is taken to be a gauge variation,

$$\delta g_{\mu\nu} = g_{\mu\lambda}\partial_\nu\epsilon^\lambda + g_{\lambda\nu}\partial_\mu\epsilon^\lambda + \epsilon^\lambda\partial_\lambda g_{\mu\nu} \tag{2.17}$$

integration-by-parts in Eq. (2.16) immediately yields the covariant conservation law $\nabla^\mu T_{\mu\nu} = 0$, as a direct consequence of the gauge invariance of the matter action.

On the other hand, in the modified field equations of Eq. (2.13), the object which will be required to be conserved by the consistency of the field equations is the gravitationally dressed energy-momentum tensor, namely $(1 + A(\Box))T_{\mu\nu}$, and not the original bare $T_{\mu\nu}$ itself. Referring therefore to the original $T_{\mu\nu}$ as "the energy-momentum tensor" would appear to be improper, since, for the consistency of the effective field equations of Eq. (2.13), the latter is no longer required to be covariantly conserved.[6] In a sense, the effective field equations of Eq. (2.13) can be seen simply as a consequence of having changed the expression in Eq. (2.16) to

$$\delta I'_M = \frac{1}{2}\int dx\sqrt{g}\delta g_{\mu\nu}(1 + A(\Box))T^{\mu\nu}. \tag{2.18}$$

Let us make a few more comments regarding the above effective field equations, in which we will set the cosmological constant $\Lambda = 0$ from now on. One simple observation is that the trace equation only involves the (simpler) scalar d'Alembertian, acting on the trace of the energy-momentum tensor

$$R = 8\pi G(1 + A(\Box))T_\mu{}^\mu. \tag{2.19}$$

Furthermore, to the order one is working here, the above effective field equations should be equivalent to

$$(1 - A(\Box) + O(A(\Box)^2))\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 8\pi G T_{\mu\nu} \tag{2.20}$$

where the running of G has been moved over to the "gravitational" side. Indeed it has recently been claimed [22] that equations similar to the above effective field equations (at least for positive integer power n, including the classical case $n = 0$) can be derived from a nonlocal extension of the Einstein-Hilbert action. In the classical case ($A(\Box) = 0$) one writes a new nonlocal action

$$I = \frac{1}{16\pi G}\int dx\sqrt{g}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\frac{1}{\Box}R_{\mu\nu} \tag{2.21}$$

whose variation, it is argued, gives the correct field equations up to curvature squared terms

$$\sqrt{g}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + O(R_{\mu\nu}^2)\right) = 0. \tag{2.22}$$

For nonvanishing $A(\Box)$ the above construction can then be generalized to

$$I = \frac{1}{16\pi G}\int dx\sqrt{g}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)(1 - A(\Box) + O(A(\Box)^2))\frac{1}{\Box}R_{\mu\nu} \tag{2.23}$$

whose variation can now be shown to give

$$\sqrt{g}(1 - A(\Box) + O(A(\Box)^2))\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + O(R_{\mu\nu}^2) = 0 \tag{2.24}$$

and which would coincide with the previously proposed effective field equations, again up to higher order curvature terms.

III. COVARIANT D'ALEMBERTIAN ON SCALAR FUNCTIONS

As a first step in solving the new set of effective field equations, consider first the *trace* of the field equation in Eq. (2.19), written as

$$(1 - A(\Box) + O(A(\Box)^2))R = 8\pi G T_\mu{}^\mu \tag{3.1}$$

where R is the scalar curvature. Here we have made the choice to move the operator $A(\Box)$ over on the gravitational side, so that it now acts on functions of the metric only, using the binomial expansion of $1/(1 + A(\Box))$. A discussion of the full tensor equations and their added complexity will be postponed to the next section. To proceed further, one needs to compute the effect of $A(\Box)$ on the scalar curvature. The d'Alembertian operator acting on scalar functions $S(x)$ is given by

[6] This can be illustrated further by the specific case of the perfect fluid, for which the energy-momentum tensor is usually written as $T_{\mu\nu} = (p(t) + \rho(t))u_\mu u_\nu + g_{\mu\nu}p(t)$. In general its covariant divergence is not zero, but consistency of the Einstein field equations demands $\nabla^\mu T_{\mu\nu} = 0$, which for the RW metric forces a definite relationship between $R(t)$, $\rho(t)$, and $p(t)$, namely $\dot\rho(t) + 3(\rho(t) + p(t))(\dot R(t)/R(t)) = 0$, irrespective of the equation-of-state relating ρ to p. In the effective field equations of Eq. (2.13) the perfect fluid form for $T_{\mu\nu}$ can still be used (as it still satisfies all the original symmetry requirements), but the covariant conservation law has the new form displayed in Eq. (2.15), which imposes a *new* constraint on the scale factor $R(t)$, as well as on the underlying $\rho(t)$ and $p(t)$.

$$\frac{1}{\sqrt{g}}\partial_\mu g^{\mu\nu}\sqrt{g}\partial_\nu S(x) \tag{3.2}$$

and for the RW metric, acting on functions of t only, one obtains a fairly simple result in terms of the scale factor $R(t)$

$$-\frac{1}{R^3(t)}\frac{\partial}{\partial t}\left[R^3(t)\frac{\partial}{\partial t}\right]F(t). \tag{3.3}$$

As a next step one computes the action of $\Box$ on the scalar curvature R, which gives

$$-6(-2k\ddot{R}(t) - 5\dot{R}^2(t)\ddot{R}(t) + R(t)\ddot{R}^2(t) + 3R(t)\dot{R}(t)R^{(3)}(t) + R^2(t)R^{(4)}(t))/R^3(t) \tag{3.4}$$

and then $\Box^2$ on R which gives

$$6(-6k\dot{R}^2(t)\ddot{R}(t) - 15\dot{R}^4(t)\ddot{R}(t) + 6kR(t)\ddot{R}^2(t) + 45R(t)\dot{R}^2(t)\ddot{R}^2(t) - 12R^2(t)\ddot{R}^3(t) + 6kR(t)\dot{R}(t)R^{(3)}(t) + 15R(t)\dot{R}^3(t)R^{(3)}(t) - 41R^2(t)\dot{R}(t)\ddot{R}(t)R^{(3)}(t) + 5R^3(t)R^{(3)2}(t) - 2kR^2(t)R^{(4)}(t) - 9R^2(t)\dot{R}^2(t)R^{(4)}(t) + 7R^3(t)\ddot{R}(t)R^{(4)}(t) + 4R^3(t)\dot{R}(t)R^{(5)}(t) + R(t)^4R^{(6)}(t))/R(t)^5, \tag{3.5}$$

etc. It should already become clear at this point that the computed expressions are rapidly becoming quite complicated. Nevertheless some of the higher order terms can, for example, be interpreted as higher derivative curvature contributions, since for Riemann squared, Ricci squared, and scalar curvature squared, one has respectively

$$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} = 12(k^2 + 2k\dot{R}^2(t) + \dot{R}^4(t) + R^2(t)\ddot{R}^2(t))/R(t)^4, \tag{3.6}$$

$$R_{\mu\nu}R^{\mu\nu} = 12(k^2 + 2k\dot{R}^2(t) + kR(t)\ddot{R}(t) + \dot{R}^4(t) + R^2(t)\ddot{R}^2(t) + R(t)\dot{R}^2(t)\ddot{R}(t))/R(t)^4, \tag{3.7}$$

$$R^2 = 36(k + \dot{R}^2(t) + R(t)\ddot{R}(t))^2/R(t)^4, \tag{3.8}$$

with

$$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - \frac{1}{6}R_{\mu\nu}R^{\mu\nu} - \frac{1}{2}R^2 = 0 \tag{3.9}$$

for arbitrary scale factor $R(t)$. But in the following we will just simply set $R(t) = R_0 t^\alpha$, in which case

$$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} = \frac{12\alpha^2(2\alpha^2 - 2\alpha + 1)}{t^4}, \tag{3.10}$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{12\alpha^2(3\alpha^2 - 3\alpha + 1)}{t^4}, \tag{3.11}$$

$$R^2 = \frac{36\alpha^2(2\alpha - 1)^2}{t^4}, \tag{3.12}$$

and for the scalar curvature (here allowing for $k \neq 0$, see Eq. (A9) in Appendix A)

$$6\left(\frac{k}{R_0^2 t^{2\alpha}} + \frac{\alpha(-1 + 2\alpha)}{t^2}\right). \tag{3.13}$$

Acting with $\Box^n$ on the above scalar curvature now gives for $k = 0$

$$6\alpha(-1 + 2\alpha)t^{-2}, \tag{3.14}$$

$$36(-1 + \alpha)\alpha(-1 + 2\alpha)t^{-4}, \tag{3.15}$$

$$144(-1 + \alpha)\alpha(-1 + 2\alpha)(-5 + 3\alpha)t^{-6}, \tag{3.16}$$

$$864(-1 + \alpha)\alpha(-1 + 2\alpha)(-7 + 3\alpha)(-5 + 3\alpha)t^{-8}, \tag{3.17}$$

for $n = 0$, 1, 2, and 3, respectively, and therefore for arbitrary power n

$$c_n 6\alpha(-1 + 2\alpha)t^{-2-2n} \tag{3.18}$$

with the coefficient c_n given by

$$c_n = 4^n\frac{\Gamma(n+1)\Gamma(\frac{3\alpha-1}{2})}{\Gamma(\frac{3\alpha-1}{2} - n)}. \tag{3.19}$$

Here use has been made of the relationship

$$\left(\frac{d}{dz}\right)^\alpha (z - c)^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)}(z - c)^{\beta - \alpha} \tag{3.20}$$

to analytically continue the above expressions to negative fractional n [24]. For $n = -1/2\nu$ the correction on the scalar curvature term R is therefore of the form

$$(1 - c_\nu(t/\xi)^{1/\nu}) \cdot 6\alpha(-1 + 2\alpha)t^{-2} \tag{3.21}$$

with

$$c_\nu = 2^{-(1/\nu)}\frac{\Gamma(1 - \frac{1}{2\nu})\Gamma(\frac{3\alpha-1}{2})}{\Gamma(\frac{3\alpha-1}{2} + \frac{1}{2\nu})}. \tag{3.22}$$

In particular for $\alpha = 2/3$ (the classical value for a pressureless perfect fluid) and $\nu = 1/3$ one has

$$c_\nu = 2^{-3}\frac{\Gamma(-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} = -\frac{\pi}{4} \tag{3.23}$$

whereas, for example, for $\alpha = 1/2$ and $\nu = 1/3$ one obtains $c_\nu = -\sqrt{\pi}\Gamma(\frac{5}{4})/\Gamma(\frac{7}{4})$. Putting everything together, one then obtains for the trace part of the effective field equations

$$\left(1 - c_\Box c_\nu\left(\frac{t}{\xi}\right)^{1/\nu} + O((t/\xi)^{2/\nu})\right)\frac{6\alpha(2\alpha-1)}{t^2} = 8\pi G\rho(t). \tag{3.24}$$

The new term can now be moved back over to the matter side (since the correction is assumed to be small), in accordance with the structure of the original effective field equations Eqs. (2.13) and (2.19), and thus avoids the problem of having to deal with the binomial expansion of $1/(1 + A(\Box))$. One then has

$$\frac{6\alpha(2\alpha-1)}{t^2} = 8\pi G\left(1 + c_\Box c_\nu\left(\frac{t}{\xi}\right)^{1/\nu} + O((t/\xi)^{2/\nu})\right)\rho(t) \tag{3.25}$$

which is the RW metric form of Eq. (2.19). If one assumes for the matter density $\rho(t) \sim \rho_0 t^\beta$, then matching powers when the new term starts to take over at larger distances gives the first result

$$\beta = -2 - 1/\nu. \tag{3.26}$$

Thus the density decreases *faster* in time than the classical value ($\beta = -2$) would indicate. The expansion appears therefore to be accelerating, but before reaching such a conclusion one needs to determine the time dependence of the scale factor $R(t)$ (or α) as well.

One might be troubled by the fact that some of the Gamma functions appearing in the expression for c_ν can diverge for specific choices of ν, e.g. when $\nu = 1/2(n + 1)$ as in Eq. (3.22) for n integer. But further thought reveals that this is not necessarily a concern here, as the coefficient c_ν actually has to be divided out and then multiplied by c_ξ (which, as discussed in the introduction and in [17], is expected to be a number of order one) to get the correct magnitude for the correction. One has therefore

$$c_\Box c_\nu = c_\xi \tag{3.27}$$

so that the correction eventually ends up as $(1 + c_\xi(t/\xi)^{1/\nu})$, as it should, in accordance with Eq. (1.1) for $G(r)$ (the "t" here is like "r" there).

Having completed the calculation of the quantum correction term acting on the scalar curvature, as in Eq. (3.1), one can alternatively pursue the following exercise in order to check the overall consistency of the approach. Consider $\Box^n$ acting on $T_\mu{}^\mu = -\rho(t)$ instead, as in the trace of the effective field equation Eq. (2.19)

$$R = 8\pi G(1 + A(\Box))T_\mu{}^\mu \tag{3.28}$$

for $\Lambda = 0$ and $p(t) = 0$. For $\rho(t) = \rho_0 t^\beta$ and $R(t) = R_0 t^\alpha$ one finds in this case

$$\Box^n(-\rho(t)) \to 4^n(-1)^{n+1} \times \frac{\Gamma(\frac{\beta}{2}+1)\Gamma(\frac{\beta+3\alpha+1}{2})}{\Gamma(\frac{\beta}{2}+1-n)\Gamma(\frac{\beta+3\alpha+1}{2}-n)}\rho_0 t^{\beta-2n} \tag{3.29}$$

which again implies $\beta = -2 - 1/\nu$ as in Eq. (3.26) for large(r) times, when the quantum correction starts to become important (since the left-hand side of Einstein's equation always goes like $1/t^2$, no matter what the value for α is, at least for $k = 0$).

IV. COVARIANT D'ALEMBERTIAN ON TENSOR FUNCTIONS

Next we will examine the full effective field equations (as opposed to just their trace part) of Eq. (2.13) with $\Lambda = 0$,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G(1 + A(\Box))T_{\mu\nu}. \tag{4.1}$$

Here the d'Alembertian operator

$$\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu \tag{4.2}$$

acts on a second rank tensor,

$$\begin{aligned}\nabla_\nu T_{\alpha\beta} &= \partial_\nu T_{\alpha\beta} - \Gamma^\lambda_{\alpha\nu}T_{\lambda\beta} - \Gamma^\lambda_{\beta\nu}T_{\alpha\lambda} \equiv I_{\nu\alpha\beta},\\ \nabla_\mu(\nabla_\nu T_{\alpha\beta}) &= \partial_\mu I_{\nu\alpha\beta} - \Gamma^\lambda_{\nu\mu}I_{\lambda\alpha\beta} - \Gamma^\lambda_{\alpha\mu}I_{\nu\lambda\beta} - \Gamma^\lambda_{\beta\mu}I_{\nu\alpha\lambda}\end{aligned} \tag{4.3}$$

and would thus seem to require the calculation of 1920 terms, of which fortunately many vanish by symmetry. Next assume that $T_{\mu\nu}$ has the perfect fluid form, for which one obtains

$$\begin{aligned}(\Box T_{\mu\nu})_{tt} &= 6[\rho(t) + p(t)]\left(\frac{\dot R(t)}{R(t)}\right)^2 - 3\dot\rho(t)\frac{\dot R(t)}{R(t)} - \ddot\rho(t),\\ (\Box T_{\mu\nu})_{rr} &= \frac{1}{1-kr^2}\{2[\rho(t) + p(t)]\dot R(t)^2 - 3\dot p(t)R(t)\dot R(t) - \ddot p(t)R(t)^2\},\\ (\Box T_{\mu\nu})_{\theta\theta} &= r^2(1-kr^2)(\Box T_{\mu\nu})_{rr},\\ (\Box T_{\mu\nu})_{\varphi\varphi} &= r^2(1-kr^2)\sin^2\theta(\Box T_{\mu\nu})_{rr}\end{aligned} \tag{4.4}$$

with the remaining components equal to zero. Note that a nonvanishing pressure contribution is generated in the effective field equations, even if one assumes initially a pressureless fluid, $p(t) = 0$. As before, repeated applications of the d'Alembertian $\Box$ to the above expressions leads to rapidly escalating complexity (for example, eighteen distinct terms are generated by $\Box^2$ for each of the above contributions), which can only be tamed by introducing some further simplifying assumptions. In the following we will therefore assume that $T_{\mu\nu}$ has the perfect fluid form appropriate for nonrelativistic matter, with a power-law behavior for the density, $\rho(t) = \rho_0 t^\beta$, and

$p(t) = 0$. Thus all components of $T_{\mu\nu}$ vanish in the fluid's rest frame, except the tt one, which is simply $\rho(t)$. Setting $k = 0$ and $R(t) = R_0 t^\alpha$ one then finds

$$(\Box T_{\mu\nu})_{tt} = (6\alpha^2 - \beta^2 - 3\alpha\beta + \beta)\rho_0 t^{\beta-2}, \quad (\Box T_{\mu\nu})_{rr} = 2R_0^2 t^{2\alpha}\alpha^2\rho_0 t^{\beta-2} \tag{4.5}$$

which again shows that the tt and rr components get mixed by the action of the $\Box$ operator, and that a nonvanishing rr component gets generated, even though it was not originally present.

Higher powers of the d'Alembertian $\Box$ acting on $T_{\mu\nu}$ can then be computed as well, but it is easier to introduce the slightly more general auxiliary diagonal tensor $V_{\mu\nu}$ with components $V_{tt} = \rho_0 t^\beta$, $V_{rr} = \rho_1 t^\gamma$, $V_{\theta\theta} = r^2 V_{rr}$, and $V_{\varphi\varphi} = r^2 \sin^2\theta V_{rr}$, with γ an arbitrary power. One then finds

$$(\Box V_{\mu\nu})_{tt} = (6\alpha^2 - \beta^2 - 3\alpha\beta + \beta)\rho_0 t^{\beta-2} + \frac{6\alpha^2}{R_0^2 t^{2\alpha}}\rho_1 t^{\gamma-2},$$
$$(\Box V_{\mu\nu})_{rr} = 2R_0^2 t^{2\alpha}\alpha^2\rho_0 t^{\beta-2} + (4\alpha^2 + \alpha(\gamma-2) - \gamma(\gamma-1))\rho_1 t^{\gamma-2} \tag{4.6}$$

as well as

$$(\Box V_{\mu\nu})_{\theta\theta} = r^2(\Box V_{\mu\nu})_{rr}, \quad (\Box V_{\mu\nu})_{\varphi\varphi} = r^2\sin^2\theta(\Box V_{\mu\nu})_{rr}, \tag{4.7}$$

and zero for the remaining components. The above expressions can then be used conveniently to generate $\Box^n$ acting on $T_{\mu\nu}$ to any desired power n. But since the problem at each step involves a two by two matrix acting on the energy-momentum tensor, it would seem rather complicated to get a closed form solution for arbitrary n. But a comparison with the left-hand (gravitational) side of the effective field equation, which always behaves like $\sim 1/t^2$ for $k = 0$, shows that in fact a solution can only be achieved at order $\Box^n$ provided the exponent β satisfies $\beta = -2 + 2n$, or since $n = -1/(2\nu)$,

$$\beta = -2 - 1/\nu \tag{4.8}$$

as was found previously from the trace equation, Eqs. (2.19) and (3.26). As a result one obtains a much simpler set of expressions, which now read

$$(\Box T_{\mu\nu})_{tt} \rightarrow 6\alpha^2\rho_0 t^{-2}, \tag{4.9}$$

$$(\Box^2 T_{\mu\nu})_{tt} \rightarrow 12\alpha^2(\alpha-1)(4\alpha+1)\rho_0 t^{-2}, \tag{4.10}$$

$$(\Box^3 T_{\mu\nu})_{tt} \rightarrow 48\alpha^2(\alpha-1)(4\alpha+1)(2\alpha^2-3\alpha-3)\rho_0 t^{-2}, \tag{4.11}$$

$$(\Box^4 T_{\mu\nu})_{tt} \rightarrow 96\alpha^2(\alpha-1)(4\alpha+1)(2\alpha^2-3\alpha-3) \times (4\alpha^2-9\alpha-15)\rho_0 t^{-2}, \tag{4.12}$$

$$(\Box^5 T_{\mu\nu})_{tt} \rightarrow 768\alpha^2(\alpha-1)(4\alpha+1)(2\alpha^2-3\alpha-3) \times (4\alpha^2-9\alpha-15)(\alpha^2-3\alpha-7)\rho_0 t^{-2}, \tag{4.13}$$

etc., here for powers $n = 1$ to $n = 5$, respectively, and with β changing with n in accordance with Eq. (4.8). For general n one can then write

$$(\Box^n T_{\mu\nu})_{tt} \rightarrow c_{tt}(\alpha, \nu)\rho_0 t^{-2} \tag{4.14}$$

and similarly for the rr component

$$(\Box^n T_{\mu\nu})_{rr} \rightarrow c_{rr}(\alpha, \nu) R_0^2 t^{2\alpha}\rho_0 t^{-2}. \tag{4.15}$$

But remarkably (see also Eq. (4.4)) one finds for the two coefficients the simple identity

$$c_{rr}(\alpha, \nu) = \frac{1}{3} c_{tt}(\alpha, \nu) \tag{4.16}$$

as well as $c_{\theta\theta} = r^2 c_{rr}$ and $c_{\varphi\varphi} = r^2\sin^2\theta c_{rr}$. Then for large times, when the quantum correction starts to become important, the tt and rr field equations reduce to

$$3\alpha^2 t^{-2} = 8\pi G c_{tt}(\alpha, \nu)\rho_0 t^{-2} \tag{4.17}$$

and

$$-\alpha(3\alpha-2)R_0^2 t^{2\alpha-2} = 8\pi G c_{rr}(\alpha, \nu) R_0^2 t^{2\alpha}\rho_0 t^{-2}, \tag{4.18}$$

respectively. But the identity $c_{rr} = \frac{1}{3} c_{tt}$ implies, simply from the consistency of the tt and rr effective field equations at large times,

$$\frac{c_{rr}(\alpha, \nu)}{c_{tt}(\alpha, \nu)} \equiv \frac{1}{3} = -\frac{3\alpha-2}{3\alpha} \tag{4.19}$$

whose only possible solution finally gives the second sought-for result, namely

$$\alpha = \frac{1}{2}. \tag{4.20}$$

For the specific value of $\alpha = \frac{1}{2}$ one can then show that the coefficients c_{tt} obey the recursion relation

$$(c_{tt})_n = -(4n^2 - 7n + 1)(c_{tt})_{n-1} \tag{4.21}$$

with initial condition $(c_{tt})_{n=1} = 3/2$. Consequently a closed form expression for c_{tt} and $c_{rr} = c_{tt}/3$ can be written down, either in terms of the Pochhammer symbol $(x)_n = x(x+1)\ldots(x+n-1) = \Gamma(x+n)/\Gamma(x)$, or more directly in terms of ratios of Gamma functions as

$$c_{tt}\left(\alpha = \frac{1}{2}, n = -1/2\nu\right) = 3(-1)^{n+1}2^{-3+2n}\frac{\Gamma(\frac{1-\sqrt{33}}{8}+n)\Gamma(\frac{1+\sqrt{33}}{8}+n)}{\Gamma(\frac{9-\sqrt{33}}{8})\Gamma(\frac{9+\sqrt{33}}{8})}. \tag{4.22}$$

Still, the above expression does not seem to be particularly illuminating at this point, except for providing an explicit proof that the coefficients c_{tt} and c_{rr} exist and are finite for specific values of n, such as $n = -1/2\nu = -3/2$.

One might worry at this point whether the above solution is consistent with covariant energy conservation. With the assumed form for $T_{\mu\nu}$ it is easy to check that energy conservation yields for the t component

$$(\nabla^\mu(\Box^n T_{\mu\nu}))_t \rightarrow -((3\alpha+\beta+1/\nu)c_{tt}+3\alpha c_{rr})\rho_0 t^{\beta+1/\nu-1} = 0 \tag{4.23}$$

when evaluated for $n = -1/2\nu$, and zero for the remaining three spatial components. But from the solution for the matter density $\rho(t)$ at large times one has $\beta = -2 - 1/\nu$, so the above zero condition gives again $c_{rr}/c_{tt} = -(3\alpha - 2)/3\alpha$, exactly the same relationship previously implied by the consistency of the tt and rr field equations.

In conclusions the values for $\alpha = 1/2$ of Eq. (4.20) and $\beta = -2 - 1/\nu$ of Eq. (4.8), determined from the effective field equations at large times, are found to be consistent with *both* the field equations *and* covariant energy conservation. More importantly, the above solution is also consistent with what was found previously by looking at the trace of the effective field equations, Eq. (2.19), which also implied the result $\beta = -2 - 1/\nu$, Eq. (3.26).

Together these results imply that for sufficiently large times the scale factor $R(t)$ behaves as

$$R(t) \sim t^\alpha \sim t^{1/2} \tag{4.24}$$

and the density $\rho(t)$ as

$$\rho(t) \sim t^\beta \sim t^{-2-1/\nu} \sim (R(t))^{-2(2+1/\nu)}. \tag{4.25}$$

Thus the density decreases significantly faster in time than the classical value ($\rho(t) \sim t^{-2}$), again a signature of an accelerating expansion at later times.

It is amusing to note that the vacuum-polarization term we have been discussing so far behaves very much like a positive pressure term, as should already have been clear from the fact that the covariant d'Alembertian $g^{\mu\nu}\nabla_\mu\nabla_\nu$ causes, in the RW metric case, a mixing of the tt and rr components in the field equations. Furthermore, within the classical FRW model, the value $\alpha = 1/2$ corresponds to an equation-of-state parameter $\omega = 1/3$ in Eq. (A20), with

$$\alpha = \frac{2}{3(1+\omega)}, \tag{4.26}$$

where $p(t) = \omega\rho(t)$, and which is therefore characteristic of radiation. Thus one can visualize the covariant gravitational vacuum-polarization contribution as behaving to some extent like classical radiation, here in the form of a dilute gas of virtual gravitons.

V. CONCLUSIONS

The main results of this paper are the effective field equations of Eq. (2.13),

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G(1 + A(\Box))T_{\mu\nu}, \tag{5.1}$$

their trace in (Eq. (2.19)), and the solution for the trace and full equations for the specific case of the RW metric and $\Lambda = 0$ outlined in Sections III and IV, respectively.

The combined results for the density $\rho(t) \sim \rho_0 t^\beta$, namely $\beta = -2 - 1/\nu$ for large times (Eqs. (3.26) and (4.8)), and for the scale factor $R(t) \sim R_0 t^\alpha$, namely $\alpha = 1/2$ (Eq. (4.20)) again for large times, imply that for $\Lambda = 0$ and for sufficiently large times the density falls off as

$$\rho(t) \sim t^{-2-1/\nu} \sim (R(t))^{-2(2+1/\nu)}. \tag{5.2}$$

Thus the matter density decreases significantly faster in time than predicted by the classical value ($\rho(t) \sim t^{-2}$), a signature of an accelerating expansion at later times.

Within the Friedmann-Robertson-Walker (FRW) framework the gravitational vacuum-polarization term behaves in many ways like a positive pressure term. The value $\alpha = 1/2$ corresponds to $\omega = 1/3$ in Eq. (A20),

$$\alpha = \frac{2}{3(1+\omega)}, \tag{5.3}$$

where we have taken the pressure and density to be related by $p(t) = \omega\rho(t)$, and which is therefore characteristic of radiation. One can therefore visualize the gravitational vacuum-polarization contribution as behaving like ordinary radiation, in the form of a dilute virtual graviton gas. It should be emphasized though that the relationship between density $\rho(t)$ and scale factor $R(t)$ is very different from the classical case.

The results of Section IV show that the effective Friedmann equations for a universe filled with nonrelativistic matter ($p = 0$) have the following appearance

$$\frac{k}{R^2(t)} + \left(\frac{\dot{R}(t)}{R(t)}\right)^2 = \frac{8\pi G(t)}{3}\rho(t) + \frac{1}{3}\Lambda = \frac{8\pi G}{3}[1 + c_\xi(t/\xi)^{1/\nu} + O((t/\xi)^{2/\nu})]\rho(t) + \frac{1}{3}\Lambda, \tag{5.4}$$

$$\frac{k}{R^2(t)} + \left(\frac{\dot{R}(t)}{R(t)}\right)^2 + \frac{2\ddot{R}(t)}{R(t)} = -\frac{8\pi G}{3}[c_\xi (t/\xi)^{1/\nu} + O((t/\xi)^{2/\nu})]\rho(t) + \Lambda \tag{5.5}$$

with the running G appropriate for the RW metric appearing explicitly in the first equation,[7]

$$G(t) = G[1 + c_\xi (t/\xi)^{1/\nu} + O((t/\xi)^{2/\nu})] \tag{5.6}$$

and used, in the second equation, the result $c_{tt} = 3c_{rr}$ of Eq. (4.16). We have also restored the cosmological constant term, with a scaled cosmological constant $\Lambda \sim 1/\xi^2$. One can therefore sensibly talk about an effective density

$$\rho_{\text{eff}}(t) = \frac{G(t)}{G}\rho(t) = [1 + c_\xi (t/\xi)^{1/\nu} + \cdots]\rho(t) \tag{5.7}$$

and an effective pressure

$$p_{\text{eff}}(t) = \frac{1}{3}\left(\frac{G(t)}{G} - 1\right)\rho(t) = \frac{1}{3}[c_\xi (t/\xi)^{1/\nu} + \cdots]\rho(t) \tag{5.8}$$

with $p_{\text{eff}}(t)/\rho_{\text{eff}}(t) = \frac{1}{3}(G(t) - G)/G(t)$. Strictly speaking, the above results can only be proven if one assumes that the pressure's time dependence is given by a power-law (as discussed in Section IV).

[7]Corrections to the above formulae are expected to be fixed by higher order terms in the renormalization-group β-function. In the vicinity of the fixed point at G_c one writes

$$\beta(G) = \beta_0(G - G_c) + \beta_1(G - G_c)^2 + \cdots$$

and obtains then by integration

$$\xi^{-1} = C_m \Lambda \left| \exp\left\{-\int^{G(\Lambda)} \frac{dG'}{\beta(G')}\right\}\right| \sim_{G(\Lambda)\to G_c} C_m \Lambda |G(\Lambda) - G_c|^{-1/\beta'(G_c)}$$

with an exponent ν given by $\beta'(G_c) = \beta_0 = -1/\nu$, C_m a numeric constant and Λ the ultraviolet cutoff. After replacing $\Lambda \to 1/r$ and $G(\Lambda) \to G(r)$ one finds for the scale dependence of G

$$G(r) = G_c\left[1 + \frac{1}{(1 + \beta_1 G_c \nu) C_m^{1/\nu}}\left(\frac{r}{\xi}\right)^{1/\nu} - \frac{\beta_1 G_c \nu}{(1 + \beta_1 G_c \nu)^2 C_m^{2/\nu}} \times \left(\frac{r}{\xi}\right)^{2/\nu} + \cdots\right].$$

Note that $\beta_0 = -1/\nu < 0$, and that for $\beta_1 < 0$ the second correction term is positive as well. If one restricts oneself to the lowest order term, valid in the vicinity of the ultraviolet fixed point, then for a given static source of mass M one has for the gravitational potential the additional contribution $\delta V(r) \sim (2MG/\xi^3) r^2$ for $\nu = 1/3$, as discussed in [17].

Equivalently, substituting $t \approx \alpha R(t)/\dot{R}(t)$, one can, as an example, rewrite the first Friedman equation as

$$\frac{k}{R^2(t)} + \left(\frac{\dot{R}(t)}{R(t)}\right)^2 = \frac{8\pi G}{3}\left[1 + c_\xi (\alpha/\xi)^{1/\nu}\left(\frac{\dot{R}(t)}{R(t)}\right)^{-1/\nu} + \cdots\right] \times \rho(t) + \frac{1}{3}\Lambda. \tag{5.9}$$

The effective Friedman equations of Eqs. (5.4) and (5.5) also bear a superficial degree of resemblance to what might be obtained in scalar-tensor theories of gravity [25–27] (for recent reviews and further references see [28,29]),

$$S = \int dx \sqrt{g}\left[\frac{1}{16\pi G} f(\phi) R - \frac{1}{2} g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi - V(\phi)\right] + S_{\text{matter}}, \tag{5.10}$$

where the gravitational Lagrangian is some arbitrary function of the scalar curvature [30]. It is also well known that often these theories can be reformulated in terms of ordinary Einstein gravity coupled appropriately to a scalar field [31]. In the FRW case one has for the scalar curvature in terms of the scale factor

$$R = 6(k + \dot{R}^2(t) + R(t)\ddot{R}(t))/R^2(t) \tag{5.11}$$

and therefore for $k = 0$ and $R(t) = R_0 t^\alpha$,

$$R = \frac{6\alpha(2\alpha - 1)}{t^2}. \tag{5.12}$$

The quantum correction in Eq. (5.4) is therefore, at this level, indistinguishable from an inverse curvature term of the type $(\xi^2 R^2)^{-1/2\nu}$. But the resemblance is seen here merely as an artifact due to the particularly simple form of the RW metric, with the coincidence of several curvature invariants (see for example, Eqs. (3.8) and (3.12)) not expected to be true in general.

Finally let us note that the effective field equations incorporating a vacuum-polarization-driven running of G, Eq. (2.13)), could potentially run into serious difficulties with experimental constraints on the time variability of G. These have recently been summarized in [32–36], where it is argued on the basis of detailed studies of the cosmic background anisotropy that the variation of G at the recombination epoch is constrained as $|G(z = z_{\text{rec}}) - G_0|/G_0 < 0.05(2\sigma)$. Solar system measurements also severely restrict the time variation of Newton's constant to $|\dot{G}/G| < 10^{-12}\ \text{yr}^{-1}$ [32]. It would seem though that these constraints can still be accommodated provided the scale ξ entering the effective field equations of Eq. (2.13) is chosen to be sufficiently large, at least of the order of $\xi > 3H^{-1}$, given that in the present model one has $|\dot{G}/G| \sim \frac{1}{\nu} c_\xi t^{1/\nu - 1}/\xi^{1/\nu}$ and therefore $|\dot{G}/G| \sim 3c_\xi t^2/\xi^3$ for $\nu = 1/3$.

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APPENDIX A: CLASSICAL FIELD EQUATIONS AND CONVENTIONS

This appendix is mostly about notation, but also collects a few simple results used extensively in the rest of the paper. We will write the Robertson-Walker (RW) metric as

$$ds^2 = -dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right\} \tag{A1}$$

and note that with this choice of signature (i.e. a minus sign for the dt^2 term), $\Box$ is a positive operator (on functions of t). Also $\sqrt{g} \equiv \sqrt{-\det(g)} = +r^2 \sin\theta R^3(t)/\sqrt{1-kr^2}$.

The energy-momentum tensor for a perfect fluid is

$$T_{\mu\nu} = (p(t) + \rho(t))u_\mu u_\nu + g_{\mu\nu}p(t) \tag{A2}$$

giving in the fluid's rest frame $T_{\mu\nu} = \mathrm{diag}(\rho, pR^2/(1-kr^2), r^2pR^2, r^2\sin^2\theta pR^2)$, with trace

$$T_\mu{}^\mu = 3p(t) - \rho(t). \tag{A3}$$

The field equations are then written as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}. \tag{A4}$$

The tt component of the Einstein tensor reads

$$3(k + \dot{R}^2(t))/R^2(t), \tag{A5}$$

while the rr component is

$$\frac{-1}{1-kr^2}(k + \dot{R}^2(t) + 2R(t)\ddot{R}(t)), \tag{A6}$$

and the $\theta\theta$ component

$$-r^2(k + \dot{R}^2(t) + 2R(t)\ddot{R}(t)), \tag{A7}$$

and finally the $\varphi\varphi$ component

$$-r^2\sin^2\theta(k + \dot{R}^2(t) + 2R(t)\ddot{R}(t)). \tag{A8}$$

The scalar curvature is simply

$$6(k + \dot{R}^2(t) + R(t)\ddot{R}(t))/R^2(t). \tag{A9}$$

Thus the tt component of the Einstein equation becomes

$$3(k + \dot{R}^2(t))/R^2(t) = 8\pi G\rho(t) \tag{A10}$$

while the rr component reads

$$\frac{-1}{1-kr^2}(k + \dot{R}^2(t) + 2R(t)\ddot{R}(t)) = 8\pi G\frac{1}{1-kr^2}p(t)R^2(t). \tag{A11}$$

The trace equation is

$$6(k + \dot{R}^2(t) + R(t)\ddot{R}(t))/R^2(t) = 8\pi G(\rho(t) - 3p(t)). \tag{A12}$$

Covariant conservation of the energy-momentum tensor, $\nabla^\mu T_{\mu\nu} = 0$ implies a definite relationship between $R(t)$, $\rho(t)$, and $p(t)$, which reads

$$\dot{\rho}(t) + 3(\rho(t) + p(t))(\dot{R}(t)/R(t)) = 0 \tag{A13}$$

(and which the tensor of Eq. (A2) in its most general form does not satisfy).

Next consider the case $k = 0$ (spatially flat) and $p = 0$ (nonrelativistic matter). If $R(t) = R_0 t^\alpha$ and $\rho(t) = \rho_0 t^\beta$, then the tt field equation

$$\frac{3\alpha^2}{t^2} = 8\pi G\rho_0 t^\beta \tag{A14}$$

implies $\beta = -2$ and $\alpha^2 = 8\pi G\rho_0/3$, while the rr field equation

$$-\alpha(3\alpha - 2)R_0^2 t^{2\alpha-2} = 0 \tag{A15}$$

implies $\alpha = 2/3$. Also both of these together imply

$$\rho(t) \sim t^{-2} \sim (t^{2/3})^{-3} \sim 1/R(t)^3. \tag{A16}$$

The trace equation now reads

$$\frac{6\alpha(2\alpha - 1)}{t^2} = 8\pi G\rho_0 t^\beta \tag{A17}$$

and implies again $\beta = -2$ and $6\alpha(2\alpha - 1) = 8\pi G\rho_0$. The latter combined with the tt equation gives $3\alpha^2 = 6\alpha(2\alpha - 1)$, or again $\alpha = 2/3$. Finally covariant energy conservation implies

$$(3\alpha + \beta)\rho_0 t^\beta = 0 \tag{A18}$$

or $3\alpha + \beta = 0$, which does not add to what already comes out of the tt and rr (or, equivalently, tt and trace) equations, but is consistent with it. In conclusion the tt and rr (or tt and trace) equations are sufficient to determine both α and β .

The case of nonvanishing pressure can be dealt with in the same way. In most instances one is interested in a fairly simple equation-of-state $p(t) = \omega\rho(t)$, with ω a constant. For nonrelativistic matter $\omega = 0$, for radiation $\omega = 1/3$, while the cosmological term can be modeled by $\omega = -1$. The consistency of the tt and rr equations now requires

$$\frac{\alpha(3\alpha - 2)}{3\alpha^2} = -\omega \tag{A19}$$

which gives

$$\alpha = \frac{2}{3(1+\omega)} \tag{A20}$$

for $-1 < \omega \leq 1/3$. Furthermore from the covariant energy conservation law one has

$$3(1+\omega)\alpha + \beta = 0 \tag{A21}$$

which implies $\beta = -2$ again. Therefore

$$R(t) \sim t^{2/3(1+\omega)}, \qquad \rho(t) \sim [R(t)]^{-3(1+\omega)}. \tag{B.22}$$

These results are well known and have been collected here for convenient reference.

APPENDIX B: SCALE TRANSFORMATIONS AND GRAVITATIONAL FUNCTIONAL INTEGRAL

Consider the (Euclidean) Einstein-Hilbert action with a cosmological term

$$I_E = \lambda_0 \Lambda^4 \int dx \sqrt{g} - \kappa_0 \Lambda^2 \int dx \sqrt{g} R. \tag{B1}$$

Here λ_0 is the bare cosmological constant, $\kappa_0 = 1/16\pi G_0$ with G_0 the bare Newton's constant. Also, and in this section only, we follow customary notation used in cutoff field theories and denote by Λ an ultraviolet cutoff, not to be confused with the scaled cosmological constant. The natural expectation is for the bare microscopic, dimensionless couplings to have magnitudes of order one in units of the cutoff, $\lambda_0 \sim \kappa_0 \sim O(1)$. Next one can rescale the metric

$$g'_{\mu\nu} = \sqrt{\lambda_0} g_{\mu\nu}, \qquad g'^{\mu\nu} = \frac{1}{\sqrt{\lambda_0}} g^{\mu\nu} \tag{B2}$$

to obtain

$$I_E = \Lambda^4 \int dx \sqrt{g'} - \frac{\kappa_0}{\sqrt{\lambda_0}} \Lambda^2 \int dx \sqrt{g'} R'. \tag{B3}$$

Next consider the fact that the (Euclidean) Feynman path integral

$$Z = \int d\mu[g] \exp\left\{ - \int dx \sqrt{g} \left(\lambda_0 \Lambda^4 - \frac{\Lambda^2}{16\pi G_0} R \right) \right\} \tag{B4}$$

includes a functional integration over all metrics, with functional measure [37,38]

$$\int d\mu[g] = \int \prod_x (\det G)^{1/2} \prod_{\mu\geq\nu} dg_{\mu\nu}(x) = \int \prod_x (g(x))^{(D-4)(D+1)/8} \prod_{\mu\geq\nu} dg_{\mu\nu}(x) \underset{D=4}{\longrightarrow} \int \prod_x \prod_{\mu\geq\nu} dg_{\mu\nu}(x) \tag{B5}$$

with the supermetric over metric deformations given by

$$G^{\mu\nu,\alpha\beta}(g(x)) = \frac{1}{2}(g(x))^{1/2}[g^{\mu\alpha}(x)g^{\nu\beta}(x) + g^{\mu\beta}(x)g^{\nu\alpha}(x) + \lambda g^{\mu\nu}(x)g^{\alpha\beta}(x)]. \tag{B6}$$

For our purposes it will be sufficient to note that under a rescaling of the metric the functional measure only picks up an irrelevant multiplicative constant. Such a constant automatically drops out when computing averages. Equivalently one can view a rescaling of the metric as simply a redefinition of the ultraviolet cutoff Λ, $\Lambda \rightarrow \lambda_0^{1/4}\Lambda$. As a consequence, the nontrivial part of the functional integral over metrics only depends on λ_0 and κ_0 through the dimensionless combination $\kappa_0/\sqrt{\lambda_0} = 1/(16\pi G_0 \sqrt{\lambda_0})$. The existence of an ultraviolet fixed point is then entirely controlled by this dimensionless parameter only, both on the lattice [8,17] and in the continuum [11,15]. It is the only relevant (as opposed to marginal or irrelevant, in statistical mechanics parlance) scaling variable in the pure gravity case, in the sense of having only one positive (growing) eigenvalue of the linearized renormalization-group transformation in the vicinity of the fixed point.

The parameter λ_0 controls the overall scale of the problem (the volume of space-time), while the κ_0 term provides the necessary derivative or coupling term. Since the total volume of space-time can hardly be considered a physical observable, quantum averages are computed by dividing out by the total space-time volume. For example, for the quantum expectation value of the Ricci scalar one writes

$$\frac{< \int dx \sqrt{g(x)} R(x) >}{< \int dx \sqrt{g(x)} >}. \tag{B7}$$

Without any loss of generality one can therefore fix the overall scale in terms of the ultraviolet cutoff, and set the bare cosmological constant λ_0 equal to one in units of the ultraviolet cutoff.[8]

The addition of matter field prompts one to do some further rescalings. Thus for a scalar field with action

[8]These considerations are not dissimilar from the case of a self-interacting scalar field where one might want to introduce three couplings for the kinetic term, the mass term, and the quartic coupling term, respectively. A simple rescaling of the field would then reveal that only two coupling ratios are in fact relevant.

$$I_S = \frac{1}{2}\int dx\sqrt{g}\{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m_0^2\phi^2 + R\phi^2\} \quad \text{(B8)}$$

and functional measure

$$\int d\mu[\phi] = \int\prod_x (g(x))^{1/2} d\phi(x), \quad \text{(B9)}$$

the metric rescaling is to be followed by a field rescaling

$$\phi'(x) = \phi(x)/\lambda_0^{1/4} \quad \text{(B10)}$$

with the only surviving change being a rescaling of the bare mass $m_0 \to m_0/\lambda_0^{1/4}$. The scalar functional measure acquires an irrelevant multiplicative factor which does not affect quantum averages. The bare mass rescaling is of course ineffectual if the fields are massless to begin with.

The same set of considerations apply as well to the Euclidean lattice [39,40] regularized version of Eq. (B1), which now reads [41,42]

$$I_L = \lambda_0\sum_h V_h(l^2) - 2\kappa_0\sum_h \delta_h(l^2)A_h(l^2) \quad \text{(B11)}$$

and

$$Z_L = \int d\mu[l^2]\exp\{-\lambda_0\sum_h V_h(l^2) + 2\kappa_0\sum_h \delta_h(l^2)A_h(l^2)\}, \quad \text{(B12)}$$

where, as is customary, the lattice ultraviolet cutoff is set equal to one (i.e. all lengths and masses are measured in units of the cutoff). It is known that convergence of the Euclidean lattice functional integral requires a *positive* bare cosmological constant $\lambda_0 > 0$ [41–43].

The coupling λ should really not be allowed to "run," as the overall space-time volume is intended to be *fixed*, not to be itself rescaled under a renormalization-group transformation. Indeed, in the spirit of Wilson [4], a renormalization-group transformation allows a description of the original physical system in terms of a new course grained Hamiltonian, whose new operators are interpreted as describing averages of the original system on the finest scale, but within the *same* physical volume. This new effective Hamiltonian is still supposed to describe the original physical system, but does so more economically in terms of a reduced set of degrees of freedom.

The pure gravity theory depends only on one coupling (the dimensionless G), and only that coupling is allowed to run. This is also, to some extent, implicit in the correct definition of gravitational averages, for example in Eq. (B7). Physical observable averages such as the one in Eq. (B7) in general have some rather nontrivial dependence on the bare coupling G_0, more so in the presence of an ultraviolet fixed point. Renormalization in the vicinity of the ultraviolet fixed point invariably leads to the introduction of a new dynamically generated, nonperturbative scale for $G > G_c$

$$m = \xi^{-1} \equiv \Lambda \exp\left(-\int^G \frac{dG'}{\beta(G')}\right) \sim_{G\to G_c} \Lambda|G - G_c|^{-1/\beta'(G_c)} \quad \text{(B13)}$$

with an exponent related to the derivative of the beta function at the fixed point

$$\beta'(G_c) = -1/\nu. \quad \text{(B14)}$$

The overall size of this new scale ξ is controlled by the distance from the fixed point $G - G_c$, which can be made arbitrarily small (in the Regge lattice theory one finds for the critical coupling, in units of the ultraviolet cutoff, $G_c \approx 0.626$, and for the exponent $\nu \approx 0.33$).

Thus a result such as

$$\frac{<\int dx\sqrt{g(x)}R(x)>}{<\int dx\sqrt{g(x)}>} \sim \Lambda^2(G - G_c)^{\gamma\nu} \sim \Lambda^{2-\gamma}\frac{1}{\xi^\gamma} \quad \text{(B15)}$$

referring here to an average curvature on the largest observable scales (with ν and γ some positive exponents) does not presumably allow one to state whether the average curvature is large or small at large distances (that would clearly depend on the choice of $G - G_c$ and the cutoff Λ).[9] But it does establish a definite relationship between the fundamental scale ξ in Eq. (B13) and say the scale of the curvature at the largest scales, Eq. (B15), as well as with any other observable involving $G - G_c$ or ξ. It is the latter curvature that most likely should be identified with a physical, astrophysically measurable, macroscopic cosmological constant (and not in any way with λ_0). While it is natural to assume for the curvature measured on the largest distance scales (for example via the parallel transport of vectors along very large loops) that $R \sim 1/\xi^2$, and therefore $\gamma = 2$, it has proven difficult so far to establish such a result in the lattice theory, due to the great technical difficulties involved in measuring small invariant correlations at large geodesic distances [44].

APPENDIX C: EFFECTIVE ACTION VARIATION

In this section we will consider the effective gravitational action of Eq. (2.10),

$$I = \frac{1}{16\pi G}\int dx\sqrt{g}\sqrt{R}(1 - A(\Box))\sqrt{R} \quad \text{(C1)}$$

and compute its variation. One needs the following elementary variations

[9]Pursuing the analogy with quantum chromodynamics, we note that there the nonperturbative gluon condensate depends in a nontrivial way on the corresponding confinement scale parameter, $\alpha_S < F_{\mu\nu} \cdot F^{\mu\nu} > \approx (250\ \text{MeV})^4 \sim \xi^{-4}$ with $\xi_{QCD}^{-1} \sim \Lambda_{\overline{MS}}$.

$$\delta\sqrt{g}\cdot\sqrt{R}\cdot(1-A(\Box))\cdot\sqrt{R}+\sqrt{g}\cdot\delta\sqrt{R}\cdot(1-A(\Box))\cdot\sqrt{R}+\sqrt{g}\cdot\sqrt{R}\cdot\delta(1-A(\Box))\cdot\sqrt{R}+\sqrt{g}\cdot\sqrt{R}\cdot(1-A(\Box))\cdot\delta\sqrt{R}. \quad (C2)$$

Using the identity

$$\delta\sqrt{g}=-\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu} \quad (C3)$$

as well as $\nabla_\lambda g_{\mu\nu}=0$ one then has

$$-\frac{1}{2}\sqrt{g}\delta g^{\mu\nu}g_{\mu\nu}\sqrt{R}(1-A(\Box))\sqrt{R}+\sqrt{g}\delta\sqrt{R}(1-A(\Box))\sqrt{R}-n\sqrt{g}\sqrt{R}A(\Box)\frac{1}{\Box}(\delta\Box)\sqrt{R}+\sqrt{g}\sqrt{R}(1-A(\Box))\delta\sqrt{R}. \quad (C4)$$

Next use is made of the definition of the Ricci scalar,

$$\delta R=g^{\mu\nu}\delta R_{\mu\nu}+R_{\mu\nu}\delta g^{\mu\nu}. \quad (C5)$$

For the variation of the affine connection one has

$$\delta\Gamma^\alpha_{\mu\nu}=\frac{1}{2}g^{\alpha\beta}[\nabla_\mu\delta g_{\beta\nu}+\nabla_\nu\delta g_{\beta\mu}-\nabla_\beta\delta g_{\mu\nu}] \quad (C6)$$

or, equivalently,

$$\delta\Gamma^\alpha_{\mu\nu}=-\frac{1}{2}[\nabla_\mu(g_{\nu\lambda}\delta g^{\alpha\lambda})+\nabla_\nu(g_{\mu\lambda}\delta g^{\alpha\lambda})-\nabla_\beta(g_{\mu\kappa}g_{\nu\lambda}g^{\alpha\beta}\delta g^{\kappa\lambda})], \quad (C7)$$

and therefore for the variation of the Ricci tensor

$$\delta R_{\mu\nu}=\nabla_\alpha(\delta\Gamma^\alpha_{\mu\nu})-\nabla_\mu(\delta\Gamma^\alpha_{\alpha\nu}) \quad (C8)$$

from which it follows that

$$g^{\mu\nu}\delta R_{\mu\nu}=\nabla_\mu\nabla_\nu(-\delta g^{\mu\nu}+g^{\mu\nu}g_{\alpha\beta}\delta g^{\alpha\beta})=g_{\alpha\beta}\Box\delta g^{\alpha\beta}-\nabla_{(\mu}\nabla_{\nu)}\delta g^{\mu\nu}, \quad (C9)$$

which is one of the required variations in Eq. (C4). The second term on the right hand side of the last equation is a total derivative in the ordinary Einstein case, but it needs to be kept here. Note also that in general $\Box\nabla_\mu\neq\nabla_\mu\Box$, and that $\Box g_{\mu\nu}=0$ but $\Box\delta g_{\mu\nu}\neq 0$. For the variation of the covariant d'Alembertian

$$\delta\Box=\delta g^{\mu\nu}\nabla_\mu\nabla_\nu-g^{\mu\nu}\delta\Gamma^\sigma_{\mu\nu}\nabla_\sigma, \quad (C10)$$

one needs the variation of $\Gamma^\sigma_{\mu\nu}$ given by Eq. (C6), which then gives

$$\delta\Box=\delta g^{\mu\nu}\nabla_\mu\nabla_\nu+\nabla_\mu\delta g^{\mu\nu}\nabla_\nu-\frac{1}{2}\nabla_\mu g^{\mu\nu}g_{\alpha\beta}\delta g^{\alpha\beta}\nabla_\nu. \quad (C11)$$

Here (or at the end) one also needs to properly symmetrize the result for the variation of $\Box$,

$$\delta(\Box^n)\to\sum_{k=1}^{n}\Box^{k-1}(\delta\Box)\Box^{n-k}. \quad (C12)$$

Next several integrations by parts, involving both the operator $\Box^n$ (with integer n) as well as the operator $g_{\mu\nu}\Box-\nabla_{(\mu}\nabla_{\nu)}$, have to be performed in order to isolate the $\delta g^{\mu\nu}$ term. This follows from $\int\sqrt{g}\nabla_\mu V^\mu=\int\sqrt{g}(1/\sqrt{g})\partial_\mu\sqrt{g}V^\mu=0$ which allows us to repeatedly integrate by parts and move some covariant derivatives around. In general one has to be careful about the ordering of covariant derivatives, whose commutator is in general nonzero in accord with the Ricci identity

$$[\nabla_\mu,\nabla_\nu]T^{\alpha_1\alpha_2\ldots}{}_{\beta_1\beta_2\ldots}=-\sum_i R_{\mu\nu\sigma}{}^{\alpha_i}T^{\alpha_1\ldots\sigma\ldots}{}_{\beta_1\ldots}-\sum_j R_{\mu\nu\beta_j}{}^{\sigma}T^{\alpha_1\ldots}{}_{\beta_1\ldots\sigma\ldots} \quad (C13)$$

with the σ index in T in the ith position in the first term, and in the jth position in the second term. The term involving the variation of the covariant d'Alembertian $\Box$ then gives

$$-n(\nabla_\mu\nabla_\nu\sqrt{R})\left(\frac{A(\Box)}{\Box}\sqrt{R}\right)-n(\nabla_\mu\sqrt{R})\left(\nabla_\nu\frac{A(\Box)}{\Box}\sqrt{R}\right)+\frac{1}{2}ng_{\mu\nu}(\nabla_\alpha\sqrt{R})g^{\alpha\beta}\left(\nabla_\beta\frac{A(\Box)}{\Box}\sqrt{R}\right) \quad (C14)$$

which again needs to be symmetrized with respect to $\frac{A(\Box)}{\Box}\sqrt{R}\leftrightarrow\sqrt{R}$, in the way described above. After adding the remaining terms, the effective field equations become

$$\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)\left(1-\frac{1}{\sqrt{R}}A(\Box)\sqrt{R}\right)-(g_{\mu\nu}\Box-\nabla_{(\mu}\nabla_{\nu)})\left(\frac{1}{\sqrt{R}}A(\Box)\sqrt{R}\right)-n(\nabla_\mu\nabla_\nu\sqrt{R})\left(\frac{A(\Box)}{\Box}\sqrt{R}\right)-n(\nabla_\mu\sqrt{R})\left(\nabla_\nu\frac{A(\Box)}{\Box}\sqrt{R}\right)+\frac{1}{2}ng_{\mu\nu}(\nabla_\alpha\sqrt{R})g^{\alpha\beta}\left(\nabla_\beta\frac{A(\Box)}{\Box}\sqrt{R}\right)=8\pi GT_{\mu\nu}, \quad (C15)$$

where again the last three terms need to be properly symmetrized in $\frac{A(\Box)}{\Box}\sqrt{R}\leftrightarrow\sqrt{R}$, as described above.

Taking the covariant divergence of the left-hand side (l.h.s.) gives zero for some of the terms, while the remaining terms give

$$-\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)\nabla^{\mu}\left[\frac{1}{\sqrt{R}}\Box^{n}\sqrt{R}\right]-n\nabla^{\mu}[(\nabla_{\mu}\nabla_{\nu}\sqrt{R})(\Box^{n-1}\sqrt{R})+(\nabla_{\mu}\sqrt{R})(\nabla_{\nu}\Box^{n-1}\sqrt{R})$$
$$-\frac{1}{2}g_{\mu\nu}(\nabla_{\alpha}\sqrt{R})g^{\alpha\beta}(\nabla_{\beta}\Box^{n-1}\sqrt{R})] \qquad \text{(C16)}$$

which has to vanish due to the invariance of the original nonlocal action. (Again the last term needs to be symmetrized in $\Box^{n-1}\sqrt{R}\leftrightarrow\sqrt{R}$).

The above derivation can be slightly generalized to an action of the form

$$I=\frac{1}{16\pi G}\int dx\sqrt{g}R^{1-\alpha}(1-A(\Box))R^{\alpha} \qquad \text{(C17)}$$

with α a parameter between zero and one (with the previous case corresponding to $\alpha=1/2$). Then for the field equations one obtains an expression of the type

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\frac{1}{2}g_{\mu\nu}RR^{-\alpha}A(\Box)R^{\alpha}-R_{\mu\nu}((1-\alpha)R^{-\alpha}A(\Box)R^{\alpha}+\alpha R^{\alpha-1}A(\Box)R^{1-\alpha})$$
$$-(g_{\mu\nu}\Box-\nabla_{(\mu}\nabla_{\nu)})((1-\alpha)R^{-\alpha}A(\Box)R^{\alpha}+\alpha R^{\alpha-1}A(\Box)R^{1-\alpha})-n(\nabla_{\mu}\nabla_{\nu}R^{\alpha})\left(\frac{A(\Box)}{\Box}R^{1-\alpha}\right)$$
$$-n(\nabla_{\mu}R^{\alpha})\left(\nabla_{\nu}\frac{A(\Box)}{\Box}R^{1-\alpha}\right)+\frac{1}{2}ng_{\mu\nu}(\nabla_{\sigma}R^{\alpha})g^{\sigma\rho}\left(\nabla_{\rho}\frac{A(\Box)}{\Box}R^{1-\alpha}\right)=8\pi GT_{\mu\nu} \qquad \text{(C18)}$$

(where again the last term needs to be symmetrized) which shows that the choice of either $\alpha=1$ or $\alpha=0$ is a bit problematic.

One final question remains, namely, what is the relationship between the above effective field equations, Eq. (C15) or Eq. (C18), and the clearly more economical field equations proposed in Eq. (2.13). Obviously the equations obtained above from the variational principle are much more complicated. They contain a number of nontrivial terms, some of which are reminiscent of the $1+A(\Box)$ term, and others with a completely different structure (such as the $g_{\mu\nu}\Box-\nabla_{(\mu}\nabla_{\nu)}$ term). It is of course possible that when restricted to specific metrics, such as the RW one, the two sets of equations will ultimately give similar results, but in general this remains a largely open question. One possibility is that both sets of effective field equations describe the same running of the gravitational coupling, up to curvature squared (higher derivative) terms, which become irrelevant at very large distances.

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