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On the measure in simplicial gravity

Herbert W. Hamber* and Ruth M. Williams†
Theoretical Physics Division, CERN, CH-1211 Geneva 23, Switzerland
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Functional measures for lattice quantum gravity should agree with their continuum counterparts in the weak field, low momentum limit. After showing that the standard simplicial measure satisfies the above requirement, we prove that a class of recently proposed non-local measures for lattice gravity do not satisfy such a criterion, already to lowest order in the weak field expansion. We argue therefore that the latter cannot represent acceptable discrete functional measures for simplicial geometries. [S0556-2821(99)00206-4]

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I. INTRODUCTION

In the simplicial formulation of quantum gravity one approximates the functional integration over continuous metrics by a discretized sum over piecewise linear simplicial geometries. In such a model the role of the continuum metric is played by the edge lengths of the simplices, while curvature is naturally described by a set of deficit angles which can be computed as functions of the given edge lengths. It has been known for some time that the simplicial lattice formulation of gravity is locally gauge invariant, and that it contains perturbative gravitons in the lattice weak field expansion, making it an attractive lattice regularization of the continuum theory.

Recent evidence seems to indicate that simplicial quantum gravity in four dimensions exhibits a phase transition between a smooth and a rough phase. Only the smooth, small curvature phase appears to be physically acceptable [1]. The existence of a phase transition implies non-trivial and calculable non-perturbative scaling properties for the coupling constants of the theory and, in particular, Newton's constant. All calculations so far have been performed in the Euclidean formulation. As usual, the starting point for a non-perturbative study of quantum gravity is a suitable definition of the path integral. In the simplicial lattice approach one starts from the discretized Euclidean path integral for pure gravity, with the squared edge lengths as fundamental variables:

$$Z_L=\int_0^\infty \prod_s \, [V_d(s)]^\sigma \prod_{ij} \, dl_{ij}^2 \Theta[l_{ij}^2] \times \exp\left\{-\sum_h \left(\lambda V_h - k\,\delta_h A_h + a\,\frac{\delta_h^2 A_h^2}{V_h}+\cdots\right)\right\}. \tag{1.1}$$

The above expression represents a suitable discretization of the continuum Euclidean path integral for pure quantum gravity:

$$Z_C=\int \prod_x \, [\sqrt{g(x)}]^\sigma \prod_{\mu \geqslant \nu} dg_{\mu\nu}(x) \times \exp\left\{-\int d^4x \sqrt{g}\left(\lambda - \frac{k}{2}R + \frac{a}{4}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}+\cdots\right)\right\}, \tag{1.2}$$

with $k^{-1}=8\pi G$. The δA term in the lattice action is the well-known Regge term [2], which reduces to the Einstein-Hilbert action in the lattice weak field limit [3]. A cosmological constant term is needed for convergence of the path integral, while the curvature squared term allows one to control the fluctuations in the curvature. In the discrete case the integration over metrics is replaced by integrals over the elementary lattice degrees of freedom, the squared edge lengths, as discussed in [4–6]. The higher derivative terms eventually become irrelevant at distances much larger than the Planck length, $r \gg \sqrt{aG}$. For phenomenological reasons one is therefore mostly interested in the limit $a \to 0$, and in this limit the theory depends, in the absence of matter and after a suitable rescaling of the metric, only on one bare parameter, the dimensionless coupling k^2/λ.

The two phases of quantized gravity found in [1] can loosely be described as having, in one phase ($G<G_c$, the rough, branched-polymer-like phase),

$$\langle g_{\mu\nu}\rangle = 0, \tag{1.3}$$

while, in the other phase ($G>G_c$, the smooth phase),

$$\langle g_{\mu\nu}\rangle \approx c\,\eta_{\mu\nu}, \tag{1.4}$$

with a small negative average curvature (anti–de Sitter space) in the vicinity of the critical point at G_c, which then vanishes as the critical point is approached from above. It appears that only the phase $G>G_c$ is physically acceptable, since in the complementary phase the simplicial lattice degenerates into a lower-dimensional branched-polymer-like manifold, with a proliferation of sharp curvature singularities and no physically acceptable continuum limit. The challenge of course lies in extracting accurate physical predictions

*Permanent address: University of California, Irvine, California 92717. Email address: hamber@cern.ch
†Permanent address: DAMTP, Silver Street, Cambridge CB3 9EW, England. Email address: rmw7@damtp.cam.ac.uk

from the theory as one approaches the lattice continuum limit by taking $G \rightarrow G_c$ from the smooth, negative curvature phase side. It is only in the physical, smooth phase that the simplicial lattice theory leads to a prediction for the nonperturbative scale dependence of Newton's constant, which can be cast in the simple form [1]

$$G(r) = G(0)[1 + c(r/R_0)^{1/\nu} + O((r/R_0)^{2/\nu})]. \quad (1.5)$$

Here the critical exponent $1/\nu = 2.8(3)$ and c a numerical constant of order 1; the scale R_0^{-1} plays a role similar to the scaling violation parameter $\Lambda_{\overline{MS}}$ in QCD, with $R_0 \approx c H_0^{-1}$. A more detailed discussion of the properties of the two phases characterizing four-dimensional quantum gravity, and of the computation of the associated critical exponents, can be found in [1]. A description of earlier work on simplicial gravity can be found in [7]. For related work on simplicial gravity see also the references in [8], where the same two-phase structure for four-dimensional simplicial gravity has been observed. An up-to-date description of work in classical simplicial gravity and the discrete time evolution problem can be found in [11]. For results with an alternative and complementary approach to problems in quantum gravity based on dynamical triangulations, we shall point the reader to the references in [12].

The functional measure over metrics is an essential ingredient in the quantum theory of gravity. In this paper we address the issue of whether the lattice gravitational measure is unique and, if not, how to decide among a set of different possible lattice measures. It is sometimes stated that the universal character of long distance critical behavior will wash out the difference between similar actions and measures. While this statement might be true for action terms that contain higher derivatives, and are therefore potentially irrelevant in the lattice continuum limit, it is less clear that it applies to the functional measure. In this paper we focus on a comparison of different approaches to the functional measure in simplicial quantum gravity, by examining both the traditional local measure [4–6] and highly non-local measures which have recently been proposed in the literature [13,14]. Throughout the paper we shall make use of the fact that in the continuum the functional measure for quantized gravity is well known and understood. We then point out the obvious, and natural, requirement that the lattice functional measure should agree with the continuum functional measure in the weak field, low momentum limit. A straightforward lattice perturbative calculation will then show that this key requirement *is* satisfied by a class of local measures currently used in the numerical simulations, but that, on the other hand, it is *not* satisfied by another set of non-local measures which have been recently proposed in the literature. We will conclude therefore that the latter do not represent acceptable functional measures for simplicial geometries.

A. Standard measure

As the edge lengths play the role of the metric in the continuum, one expects the discrete measure to involve an integration over the squared edge lengths [4–6]. Indeed the induced metric at a simplex is related to the squared edge lengths within that simplex, via the expression for the invariant line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. After choosing coordinates along the edges emanating from a vertex, the relation between metric perturbations and squared edge length variations for a given simplex based at 0 in d dimensions is

$$\delta g_{ij}(l^2) = \frac{1}{2}(\delta l_{0i}^2 + \delta l_{0j}^2 - \delta l_{ij}^2). \quad (1.6)$$

For one d-dimensional simplex labeled by s the integration over the metric is thus equivalent to an integration over the edge lengths, and one has the well-known identity

$$\left(\frac{1}{d!}\sqrt{\det g_{ij}(s)}\right)^{\sigma} \prod_{i \geq j} dg_{ij}(s) = \left(-\frac{1}{2}\right)^{d(d-1)/2} [V_d(l^2)]^{\sigma} \prod_{k=1}^{d(d+1)/2} dl_k^2. \quad (1.7)$$

There are $d(d+1)/2$ edges for each simplex, just as there are $d(d+1)/2$ independent components for the metric tensor in d dimensions. Here one is ignoring temporarily the triangle inequality constraints, which will further require all subdeterminants of g_{ij} to be positive, including the obvious restriction $l_k^2 > 0$. The extension to many simplices glued together at their common faces is then immediate. For this purpose one first needs to identify edges $l_k(s)$ and $l_{k'}(s')$ which are shared between simplices s and s':

$$\int_0^\infty dl_k^2(s) \int_0^\infty dl_{k'}^2(s') \, \delta(l_k^2(s) - l_{k'}^2(s')) = \int_0^\infty dl_k^2(s). \quad (1.8)$$

After summing over all simplices one derives, up to an irrelevant numerical constant, the unique functional measure for simplicial geometries:

$$\int d\mu[l^2] = \int_0^\infty \prod_s [V_d(s)]^{\sigma} \prod_{ij} dl_{ij}^2 \Theta[l_{ij}^2]. \quad (1.9)$$

Here $\Theta[l_{ij}^2]$ is a (step) function of the edge lengths, with the property that it is equal to 1 whenever the triangle inequalities and their higher dimensional analogue are satisfied, and zero otherwise. In four dimensions the lattice analogue of the DeWitt measure ($\sigma = 0$) takes on a particularly simple form: namely

$$\int d\mu[l^2] = \int_0^\infty \prod_{ij} dl_{ij}^2 \Theta[l_{ij}^2]. \quad (1.10)$$

The above lattice measure over the space of squared edge lengths has been used extensively in numerical simulations of simplicial quantum gravity [1,4,7–10].

The derivation of the above lattice measure closely parallels the analogous procedure in the continuum. There, following DeWitt [15,16], one defines an invariant norm for metric fluctuations,

$$\|\delta g\|^2 = \int d^dx [g(x)]^{\omega/2} G^{\mu\nu,\alpha\beta}[g(x);\omega]\, \delta g_{\mu\nu}(x)\, \delta g_{\alpha\beta}(x), \tag{1.11}$$

with the inverse of the super-metric G given by

$$G^{\mu\nu,\alpha\beta}[g(x);\omega] = \frac{1}{2}[g(x)]^{(1-\omega)/2}[g^{\mu\alpha}(x)g^{\nu\beta}(x) + g^{\mu\beta}(x)g^{\nu\alpha}(x) + \lambda g^{\mu\nu}(x)g^{\alpha\beta}(x)]. \tag{1.12}$$

DeWitt originally considered the case $\omega=0$, but it will be useful later to consider other values for ω, such as $\omega=1$. The resulting functional measure in the continuum is then given by

$$\int d\mu[g] = \int \prod_x [\det G(g(x))]^{1/2} \prod_{\mu \geqslant \nu} dg_{\mu\nu}(x). \tag{1.13}$$

Since the super-metric $G^{\mu\nu,\alpha\beta}(g(x))$ is ultra-local, one expects its determinant to be a local function of x as well. Up to an irrelevant multiplicative constant, one has for the determinant of G the simple result

$$\det G(g(x)) \propto \left(1 + \frac{1}{2} d\lambda\right)[g(x)]^{(d+1)[(1-\omega)d-4]/4}. \tag{1.14}$$

One also needs to impose the condition $\lambda \neq -2/d$ in order to avoid the vanishing of the determinant of G. As a result, one obtains the local measure for the functional integration over metrics:

$$\int d\mu[g] = \int \prod_x [\sqrt{g(x)}]^\sigma \prod_{\mu \geqslant \nu} dg_{\mu\nu}(x), \tag{1.15}$$

with $\sigma = (d+1)[(1-\omega)d-4]/4$. For $\omega=0$ one obtains the DeWitt measure for pure gravity, which takes on a particularly simple form in $d=4$,

$$\int \prod_x [g(x)]^{(d-4)(d+1)/8} \prod_{\mu \geqslant \nu} dg_{\mu\nu}(x) \xrightarrow[d=4]{} \int \prod_x \prod_{\mu \geqslant \nu} dg_{\mu\nu}(x), \tag{1.16}$$

and which obviously corresponds to the lattice measure in Eq. (1.10). In general the volume factors are absent ($\sigma=0$) if one choses $\omega=(d-4)/d$. On the other hand, for $\omega=1$ one recovers the Misner measure[1] [17,18].

There is no clear way of deciding between these two choices ($\omega=0$ or 1), or any intermediate one for that matter, and one should consider σ as an arbitrary parameter of the model, to be constrained only by the requirement that the path integral be well defined (which incidentally rules out singular measures). Note that the volume term in the measure is completely local and contains no derivatives. In perturbation theory it does not therefore affect the propagation properties of gravitons, and contributes $\delta^d(0)$ terms to the effective action; to some extent these can be regarded as similar to a renormalization of the cosmological constant, affecting only the distribution of local volumes. Numerical simulations in the lattice model show very little sensitivity of the critical exponents to either σ or a [1].

There is no obstacle in defining a discrete analogue of the supermetric, as a way of introducing an invariant notion of distance between simplicial manifolds. It leads to an alternative way of deriving the lattice measure in Eq. (1.10), by considering the discretized distance between induced metrics $g_{ij}(s)$ [20],

$$\|\delta g(s)\|^2 = \sum_s G^{ijkl}(g(s))\, \delta g_{ij}(s)\, \delta g_{kl}(s), \tag{1.19}$$

with the inverse of the lattice DeWitt supermetric now given by the expression

$$G^{ijkl}[g(s)] = \frac{1}{2}\sqrt{g(s)}[g^{ik}(s)g^{jl}(s) + g^{il}(s)g^{jk}(s) + \lambda g^{ij}(s)g^{kl}(s)], \tag{1.20}$$

and with again $\lambda \neq -2/d$. This procedure defines a metric on the tangent space of positive real symmetric matrices $g_{ij}(s)$. After computing the determinant of G, the resulting functional measure is

$$\int d\mu[l^2] = \int \prod_s [\det G(g(s))]^{1/2} \prod_{i \geqslant j} dg_{ij}(s), \tag{1.21}$$

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[1] It is easy to show [18] that the continuum measure of Eq. (1.15) is invariant under coordinate transformations, irrespective of the value of σ. Under a change of coordinates $x'^\mu = x^\mu + \epsilon^\mu(x)$,

$$\prod_x [g(x)]^{\sigma/2} \prod_{\mu \geqslant \nu} dg_{\mu\nu}(x) \to \prod_x \left(\det \frac{\partial x'^\beta}{\partial x^\alpha}\right)^\gamma [g(x)]^{\sigma/2} \prod_{\mu \geqslant \nu} dg_{\mu\nu}(x). \tag{1.17}$$

For infinitesimal coordinate transformations the additional factor is equal to 1:

$$\prod_x \left(\det \frac{\partial x'^\beta}{\partial x^\alpha}\right)^\gamma = \prod_x [\det(\delta_\alpha^\beta + \partial_\alpha \epsilon^\beta)]^\gamma = \exp\left\{\gamma \delta^d(0) \int d^dx\, \partial_\alpha \epsilon^\alpha\right\} = 1. \tag{1.18}$$

In many respects σ can be thought of as a gauge parameter. We should caution the reader that some authors regard the above manipulations as somewhat formal [16].

with the determinant of the super-metric $G^{ijkl}(g(s))$ given by the local expression

$$\det G(g(s)) \propto \left(1+\frac{1}{2}d\lambda\right)[g(s)]^{(d-4)(d+1)/4}, \quad (1.22)$$

Using Eq. (1.7) and up to irrelevant constants, one obtains again the standard lattice measure of Eq. (1.9). Of course the same procedure can be followed for the Misner-like measure, leading to a similar result for the lattice measure, but with a different power σ. For a related discussion see also [19].

B. Alternative approach

The previous derivation of the standard lattice functional measure is based on the direct and obvious correspondence between the induced lattice metric within a simplex and the continuum metric at a point. It leads to an essentially unique local measure over the squared edge lengths, in close analogy to the continuum expression. In particular it is clear from the derivation that the lattice and continuum measures agree with each other in the weak field expansion, essentially by construction.

Still, one might be tempted to try to find an alternative lattice measure by looking directly at the discrete form for the supermetric, written as a quadratic form in the squared edge lengths (instead of the metric components), and then evaluating the resulting determinant. The main idea, inspired by work described in an unpublished paper by Lund and Regge [21] on the $3+1$ formulation of simplicial gravity, can be found in some detail in a recent paper [6]; see also another recent paper [22], which discusses somewhat different issues, not directly related to the measure. First one considers a lattice analogue of the DeWitt supermetric, by writing

$$\|\delta l^2\|^2 = \sum_{ij} G_{ij}(l^2)\,\delta l_i^2\,\delta l_j^2, \quad (1.23)$$

with $G_{ij}(l^2)$ playing a role analogous to the DeWitt supermetric, but defined now on the space of squared edge lengths. The next step is to find an appropriate form for $G_{ij}(l^2)$ expressed in terms of known geometric objects. One simple way of constructing the explicit form for $G_{ij}(l^2)$, in any dimension, is to first focus on one simplex, and write the squared volume of a given simplex in terms of the induced metric components within the *same* simplex s,

$$V^2(s) = \left(\frac{1}{d!}\right)^2 \det g_{ij}(l^2(s)). \quad (1.24)$$

One computes, to linear order,

$$\frac{1}{V(l^2)}\sum_i \frac{\partial V^2(l^2)}{\partial l_i^2}\,\delta l_i^2 = \frac{1}{d!}\sqrt{\det(g_{ij})}\,g^{ij}\,\delta g_{ij} \quad (1.25)$$

and, to quadratic order,

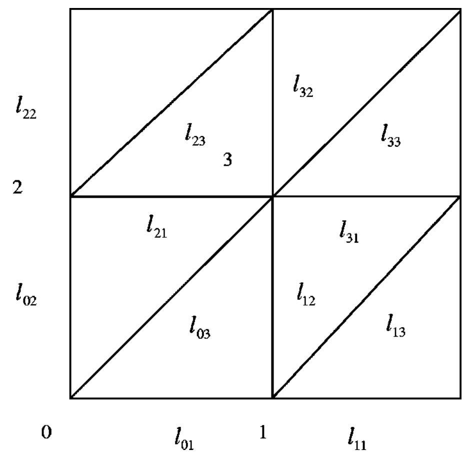

FIG. 1. Notation for the weak-field expansion about the rigid square lattice.

$$\frac{1}{V(l^2)}\sum_{ij} \frac{\partial^2 V^2(l^2)}{\partial l_i^2 \partial l_j^2}\,\delta l_i^2\,\delta l_j^2 = \frac{1}{d!}\sqrt{\det(g_{ij})}[g^{ij}g^{kl}\,\delta g_{ij}\,\delta g_{kl} - g^{ij}g^{kl}\,\delta g_{jk}\,\delta g_{li}]. \quad (1.26)$$

The right hand side of this equation contains precisely the expression appearing in the continuum supermetric of Eq. (1.12), for the specific choice of the parameter $\lambda = -2$. One is led therefore to the obvious identification

$$G_{ij}(l^2) = -d!\sum_s \frac{1}{V(s)}\,\frac{\partial^2 V^2(s)}{\partial l_i^2 \partial l_j^2} \quad (1.27)$$

and therefore, for the norm,

$$\|\delta l^2\|^2 = \sum_s V(s)\left\{-\frac{d!}{V^2(s)}\sum_{ij}\frac{\partial^2 V^2(s)}{\partial l_i^2 \partial l_j^2}\,\delta l_i^2\,\delta l_j^2\right\}. \quad (1.28)$$

One could be tempted (as already discussed in [6]) at this point to write down a lattice measure, in parallel with Eq. (1.12), and write

$$\int d\mu[l^2] = \int \prod_i \sqrt{G_{ij}^{(\omega')}(l^2)}\,dl_i^2, \quad (1.29)$$

with

$$G_{ij}^{(\omega')}(l^2) = -d!\sum_s \frac{1}{[V(s)]^{1+\omega'}}\,\frac{\partial^2 V^2(s)}{\partial l_i^2 \partial l_j^2}, \quad (1.30)$$

Again we have allowed here for a parameter ω', which is possibly different from zero, and interpolates between apparently equally acceptable measures. As in the continuum, different edge length measures, here parametrized by ω', are obtained, depending on whether the local volume factor $V(s)$ is included in the supermetric or not. Irrespective of the value chosen for ω', we will show below that the measure of

Eq. (1.29) disagrees with the continuum measure of Eq. (1.15) already to lowest order in the weak field expansion, and does not therefore describe an acceptable lattice measure.

An obviously undesirable (and puzzling) feature of the measure of Eq. (1.29) is that in general it is non-local, in spite of the fact that the original continuum measure of Eq. (1.15) *is* completely local (although it is clear that for some special choices of ω' and d, one *does* recover a local measure; thus in two dimensions and for $\omega' = -1$ one obtains again the simple result $\int d\mu[l^2] = \int_0^\infty \Pi_i dl_i^2$). It was already pointed out in [6] that the above procedure also fails to give the correct measure already in one dimension.

Let us now turn to the calculation of the determinant $\det G(l^2)$. In general it is given by a rather formidable expression, which can be simplified though by considering its lattice weak field expansion, and which will allow us to make a direct comparison with the continuum answer of Eq. (1.14). In order to discuss the weak field expansion of the lattice measure of Eq. (1.29), we shall focus here for simplicity on the two-dimensional case, for which an explicit answer can readily be obtained; although our arguments are general, the algebraic complexity is significantly reduced in two dimensions. Also for definiteness we will consider the case $\omega' = 0$ in Eq. (1.30). It is clear that the determinant, being a non-local function of the edge lengths, will couple edges which are arbitrarily far apart on the lattice. For a square lattice made rigid by the introduction of diagonals, $G(l^2)$ will be a $3N_0 \times 3N_0$ matrix, with N_0 denoting the total number of sites in the lattices. It will be sufficient in the following to examine the form of $\det G(l^2)$ for a square lattice with 12 edges (see Fig. 1), with the usual imposition of periodic boundary conditions to minimize edge effects.

For such a lattice $G(l^2)$ is given by the symmetric 12×12 matrix

$$G(l^2) = \frac{1}{4} \begin{pmatrix}
\frac{1}{A_{01}} + \frac{1}{A_{22}} & 0 & -\frac{1}{A_{01}} & 0 & -\frac{1}{A_{01}} & 0 & \cdots \\
0 & \frac{1}{A_{02}} + \frac{1}{A_{11}} & -\frac{1}{A_{02}} & -\frac{1}{A_{11}} & 0 & -\frac{1}{A_{11}} & \cdots \\
-\frac{1}{A_{01}} & -\frac{1}{A_{02}} & \frac{1}{A_{01}} + \frac{1}{A_{02}} & 0 & -\frac{1}{A_{01}} & 0 & \cdots \\
0 & -\frac{1}{A_{11}} & 0 & \frac{1}{A_{11}} + \frac{1}{A_{32}} & 0 & -\frac{1}{A_{11}} & \cdots \\
-\frac{1}{A_{01}} & 0 & -\frac{1}{A_{01}} & 0 & \frac{1}{A_{01}} + \frac{1}{A_{12}} & -\frac{1}{A_{12}} & \cdots \\
0 & -\frac{1}{A_{11}} & 0 & -\frac{1}{A_{11}} & -\frac{1}{A_{12}} & \frac{1}{A_{11}} + \frac{1}{A_{12}} & \cdots \\
0 & -\frac{1}{A_{02}} & -\frac{1}{A_{02}} & 0 & 0 & 0 & \cdots \\
-\frac{1}{A_{22}} & 0 & 0 & 0 & 0 & 0 & \cdots \\
-\frac{1}{A_{22}} & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & -\frac{1}{A_{12}} & -\frac{1}{A_{12}} & \cdots \\
0 & 0 & 0 & -\frac{1}{A_{32}} & 0 & 0 & \cdots \\
0 & 0 & 0 & -\frac{1}{A_{32}} & 0 & 0 & \cdots
\end{pmatrix} \tag{1.31}$$

where A_{i1} and A_{i2} denote the areas of the two triangles based at site i. The area of a triangle with arbitrary edge lengths l_1, l_2, and l_3 is given here as usual in terms of the edge lengths by

$$A_T(l_1,l_2,l_3) = \frac{1}{4}\sqrt{2(l_1^2 l_2^2 + l_2^2 l_3^2 + l_3^2 l_1^2) - l_1^4 - l_2^4 - l_3^4}. \tag{1.32}$$

After expanding it in terms of the edge lengths, the determinant $\det G(l^2)$ is in the general case given by a rather complicated expression. To make progress, one can further expand it for small fluctuations in the edge lengths. It is convenient for this purpose to use a binary notation [3] for the vertices, and introduce small edge length fluctuations ϵ_i, by writing

$$l_i = l_i^0 (1+\epsilon_i), \tag{1.33}$$

with $l_1^0 = l_2^0 = 1$ and $l_3^0 = \sqrt{2}$ for a square background lattice (see again Fig. 1). The individual triangle areas can in turn be expanded in term of the ϵ's, to give, for example,

$$A_{01}(\epsilon) = \frac{1}{2} + \frac{1}{2}(\epsilon_{01}+\epsilon_{12}) + \frac{1}{4}(\epsilon_{01}\epsilon_{03}+\epsilon_{03}\epsilon_{12}-\epsilon_{01}^2-\epsilon_{12}^2-4\epsilon_{03}^2)+O(\epsilon^3) \tag{1.34}$$

and similarly for the remaining triangle areas. Our notation here is that the first index labels the site and the second one the lattice direction. It can be shown that the expansion needs to be carried out to fourth order in ϵ in order to get a non-vanishing result for the determinant of $G(l^2)$. The resulting expressions are then inserted into the formula for the determinant and give, for the square lattice,

$$\begin{aligned}\det G(\epsilon) = {}& \frac{1}{2}(\epsilon_{01}+\epsilon_{11}-\epsilon_{21}-\epsilon_{31})(\epsilon_{02}-\epsilon_{12}+\epsilon_{22}-\epsilon_{32})(2\epsilon_{01}\epsilon_{03}+2\epsilon_{02}\epsilon_{03}-4\epsilon_{03}^2+\epsilon_{02}\epsilon_{11}-\epsilon_{01}\epsilon_{12}+2\epsilon_{03}\epsilon_{12}-2\epsilon_{02}\epsilon_{13}-2\epsilon_{11}\epsilon_{13}\\ & -2\epsilon_{12}\epsilon_{13}+4\epsilon_{13}^2-\epsilon_{02}\epsilon_{21}+2\epsilon_{03}\epsilon_{21}+\epsilon_{01}\epsilon_{22}-2\epsilon_{01}\epsilon_{23}-2\epsilon_{21}\epsilon_{23}-2\epsilon_{22}\epsilon_{23}+4\epsilon_{23}^2+\epsilon_{12}\epsilon_{31}-2\epsilon_{13}\epsilon_{31}-\epsilon_{22}\epsilon_{31}-\epsilon_{11}\epsilon_{32}\\ & +\epsilon_{21}\epsilon_{32}-2\epsilon_{23}\epsilon_{32}+2\epsilon_{11}\epsilon_{33}+2\epsilon_{22}\epsilon_{33}+2\epsilon_{31}\epsilon_{33}+2\epsilon_{32}\epsilon_{33}-4\epsilon_{33}^2)+O(\epsilon^5).\end{aligned} \tag{1.35}$$

As expected, the result is indeed non-local, and involves to this order contributions from all the edges on the 4-site lattice. It is in fact easy to see that this will be the case for any size lattice, due to the general non-locality of the determinant. As a check of the calculation, one can verify that as the ϵ's approach zero, one recovers the zero eigenvalues of the matrix G for the square lattice, with the correct multiplicity (the eigenvalues for G in this case are $-1, 3\times 0, 3\times 1, 5\times 2$). A somewhat simpler and more symmetric expression is obtained in the case of an equilateral lattice, for which one can show that

$$\begin{aligned}\det G(\epsilon) = {}& \frac{2^{15}}{3^9}(\epsilon_{01}+\epsilon_{11}-\epsilon_{21}-\epsilon_{31})(\epsilon_{02}-\epsilon_{12}+\epsilon_{22}-\epsilon_{32})\\ & \times(\epsilon_{03}-\epsilon_{13}-\epsilon_{23}+\epsilon_{33})+O(\epsilon^4),\end{aligned} \tag{1.36}$$

reflecting the permutation symmetry under the interchange of the three coordinate directions in this case. Note also that for this choice of background lattice the determinant is now of cubic order in the ϵ's. In this case one can verify again that, as the ϵ's approach zero, one recovers correctly the 3 zero eigenvalues of the matrix G for the equilateral lattice. In general of course the determinant does not vanish, as can be verified explicitly from the original expression for

$$\det G(l^2).$$

The above expression for the determinant on the square lattice case can be simplified a bit by going to momentum space. Here we shall take the ϵ_i's to be plane waves. When transforming to momentum space, one assumes that the fluctuation ϵ_i at the point i, j steps in one coordinate direction and k steps in the other coordinate direction from the origin, is related to the corresponding ϵ_i at the origin by

$$\epsilon_i^{(j+k)} = \omega_1^j \omega_2^k \epsilon_i^{(0)}, \tag{1.37}$$

where $\omega_i = e^{-ik_i}$ and k_i is the momentum in the direction i. Inserting the above expression into the weak-field expression for the determinant, Eq. (1.35), one obtains (still in the weak field limit)

$$\begin{aligned}\det G(\epsilon) & \\ = {}& (e^{ik_1}-1)^2(e^{ik_1}+1)^2(e^{ik_2}-1)^2(e^{ik_2}+1)^2\epsilon_1^{(0)}(k)\epsilon_2^{(0)}\\ & \times(k)\epsilon_3^{(0)}(k)[\epsilon_1^{(0)}(k)+\epsilon_2^{(0)}(k)-2\epsilon_3^{(0)}(k)],\end{aligned} \tag{1.38}$$

which can formally be expanded for small momenta to give

$$\begin{aligned}\det G(\epsilon) = {}& 2^4\epsilon_1^{(0)}(k)\epsilon_2^{(0)}(k)\epsilon_3^{(0)}(k)[\epsilon_1^{(0)}(k)+\epsilon_2^{(0)}(k)\\ & -2\epsilon_3^{(0)}(k)]k_1^2k_2^2+O(k^5).\end{aligned} \tag{1.39}$$

If the lattice periodicity is imposed on the momenta, then the expression in Eq. (1.38) vanishes identically for plane waves, while in general the expressions of Eqs. (1.35) and (1.36) do not.

The above expression for the determinant can be transformed into an equivalent form involving the metric field, using the fact that the edge lengths on the lattice correspond to the metric degrees of freedom in the continuum. Given the choice of edges in Fig. 2, one writes, for the induced metric

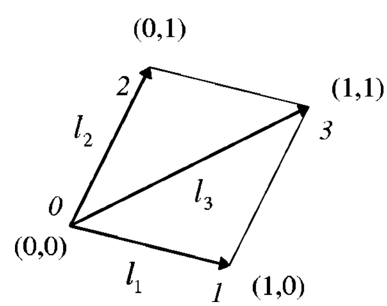

FIG. 2. Edge lengths and metric components.

at the origin,

$$g_{ij}(l^2)=\begin{pmatrix} l_1^2 & \frac{1}{2}(l_3^2-l_1^2-l_2^2) \\ \frac{1}{2}(l_3^2-l_1^2-l_2^2) & l_2^2 \end{pmatrix}. \quad (1.40)$$

One can then relate the edge lengths l_i (or, equivalently, the fluctuations ϵ_i) to the metric components in the continuum, which in the weak field limit are more conveniently written as

$$g_{\mu\nu}=\delta_{\mu\nu}+h_{\mu\nu}. \quad (1.41)$$

One then obtains the obvious correspondence between squared edge lengths and metric components at each lattice vertex,

$$l_1^2=(1+\epsilon_1)^2=1+h_{11}$$

$$l_2^2=(1+\epsilon_2)^2=1+h_{22}$$

$$\frac{1}{2}l_3^2=(1+\epsilon_3)^2=1+\frac{1}{2}(h_{11}+h_{22})+h_{12}, \quad (1.42)$$

which can be inverted to give the small edge length fluctuations in terms of the metric components:

$$\epsilon_1(h)=\frac{1}{2}h_{11}-\frac{1}{8}h_{11}^2+O(h_{11}^3)$$

$$\epsilon_2(h)=\frac{1}{2}h_{22}-\frac{1}{8}h_{22}^2+O(h_{22}^3)$$

$$\epsilon_3(h)=\frac{1}{4}(h_{11}+h_{22}+2h_{12})-\frac{1}{32}(h_{11}+h_{22}+2h_{12})^2 +O(h^3), \quad (1.43)$$

at each point. It is also known that this relationship is the correct one for relating edge lengths and continuum metric components in the weak field expansion for the lattice action, as shown in detail in Refs. [10,6]. Inserting then these expressions into the weak-field lattice formula for the determinant of Eq. (1.39) one obtains

$$\det(G(h))=-h_{11}(k)h_{12}(k)h_{22}(k)[h_{11}(k)+2h_{12}(k) +h_{22}(k)]k_1^2k_2^2+O(k^5). \quad (1.44)$$

At this point, one is ready to compare the resulting expression for the lattice functional measure to the continuum result, as given in Eq. (1.15). In the continuum case one has, in the weak field expansion,

$$\det g(x)=1+h_{11}(x)+h_{22}(x)+h_{11}(x)h_{22}(x)-h_{12}^2(x) +O(h^3) \quad (1.45)$$

and therefore the functional measure is given by [see Eq. (1.15)]

$$\int d\mu[g] =\int\prod_x [1+h_{11}(x)+h_{22}(x)+\cdots]^{\sigma/2}\prod_{\mu\geqslant\nu} dh_{\mu\nu}(x). \quad (1.46)$$

On the simplicial lattice this last expression obviously becomes

$$=2^{3N_0}\int\prod_{n=1}^{N_0}(1+2\epsilon_1^{(n)}+2\epsilon_2^{(n)}+\cdots)^{\sigma/2}\prod_{i=1}^{3}d\epsilon_i^{(n)} \quad (1.47)$$

which is clearly very different from the measure of Eq. (1.29), with the determinant $\det G$ given (for $\omega'=0$) either by the general weak-field answer of Eq. (1.35) or, for plane waves, by Eqs. (1.38) and (1.44).

One concludes therefore that the nonlocal measure of Eq. (1.29) taken from Ref. [6], which was proposed in [13] as a ‘‘new’’ measure for simplicial gravity, disagrees with the continuum measure already to leading order in the weak field expansion.

II. CONCLUSIONS

In this paper we have compared different approaches to the functional measure in simplicial quantum gravity. We have pointed out that the obvious requirement that the lattice measure agree with the continuum measure in the weak field, low momentum limit is satisfied by a class of local measures used extensively for numerical simulations. It is well known that a similar requirement is satisfied by the measure used for lattice gauge fields in discretized non-Abelian gauge theories [23]. Dropping this requirement leads one to enter largely unknown territory, by discussing a discrete theory whose weak field lattice Feynman rules do not reduce to those of continuum quantum theory in the small lattice spacing limit. The resulting theory then might or might not be related to gravity. We have also shown in this paper that the above requirement is not satisfied by another set of non-local measures, recently proposed in the literature. The latter do not therefore in our opinion represent acceptable functional measures for simplicial geometries. In general we believe that the criterion that lattice operators should agree with their continuum counterparts in the weak field, low momentum limit is an important one, and that it should be checked systematically for any proposed variant action or measure. A closely related but perhaps weaker requirement is that the lattice theory reproduce well established physical quantities of continuum perturbation theory, such as the universal long-distance quantum correction to the Newtonian potential [24] and the conformal anomaly discussed in [25].

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